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Ulam-Hyers stability of nonlinear integro-differential equations having complex fractional order

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Abstract. This article discusses the two-step Adomian decomposition method for solving nonlinear integro-differential equations of complex fractional order to obtain an analytical solution using a simple algorithm. We obtain the analytical solution using only one iteration. We find the results without using the approximation and discretization tools, which generally generate the round-off error. The obtained solution has higher accuracy than other numerical methods. We consider two examples to prove the applicability of the method and compare the results with other numerical methods. Here, we use the Caputo definition for complex fractional order operators. We provide new results such as the existence, uniqueness, and stability of the solution of the complex fractional nonlinear integro-differential equations (CF-NL-IDEs) employing the fixed point theory and Ulam-Hyers stability. Additionally, we solve examples using two popular numerical methods: the Adomian decomposition method and the modified Adomian decomposition method. We also compare the solution obtained from the proposed method and the solutions obtained from these two numerical methods. It is observed via examples that our method provides more accurate results with faster convergence for the CF-NL-IDEs.

§1 Introduction

For many years, the study of fractional differential equations (FDEs) has attracted researchers from different fields who have focused on developing this theory and improving and obtaining accurate definitions. The FDEs have strong physical meaning in modeling real-world problems [12]. In COVID-19 situations [1, 5, 6, 15, 27, 28], it becomes an essential aspect of achieving new theory and higher-level information. Several definitions in the literature define the fractional operators (derivative and integral), but there is no accurate definition due to their nature. Researchers have developed their definitions of fractional operators using singular and non-singular kernels. They have also used special functions such as the Mittag-Leffler function.

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The fractional definitions developed, such as Caputo and Riemann-Liouville definitions, are famous definitions that use singular kernels. The Atangana-Baleanu and the Caputo-Fabrizio definitions are recent fractional derivatives that use a non-singular kernel [7, 8, 16, 19].

An important aspect of FDEs is to develop the theory regarding their solution, whether their solution of the FDEs exists or not. If a solution exists, then whether it is unique or not. The proof of the analysis is provided under some assumptions in [28–30]. There are also some extended versions of the FDEs, such as variable fractional differential equations (VFDEs) in which the order is in the form of a function (either bounded or continuous), and complex fractional order differential equations (CFODEs) where the order is in the form of a complex number [23, 32, 36]. The order of the differential equation is imaginary if the real part of the complex fractional order is zero, and if the imaginary part becomes zero, then the considered problem represents the FDEs. There are few studies on CFDEs, and an analytical solutions of the differential equations with complex fractional order have not been obtained until now.

The motivation for studying CFODEs comes from viscoelasticity with constitutive equations containing complex order derivatives. Using complex fractional order derivatives leads to one additional degree of freedom in modeling physical phenomena. Those problems are transformed into approximate systems of integer order equations; their convergence to initial boundary value problems is illustrated by a numerical procedure based on the approximate systems of integer order equations. Many authors have provided theoretical results on existence, uniqueness, and stability using various operators (derivative/integral) of complex order [9,14,18,21,22,32,33,37]. Among these previously considered problems based on complex fractional order derivatives, some problems are solved with the help of numerical schemes that give approximate solutions [2, 3, 13, 17, 25, 35]. In general, finding an analytical solution is a significant challenge, and it becomes difficult to obtain the analytical solution of CFODEs via existing numerical tools. In this paper, we give a simple method to find the analytical solution of CFODEs.

This study presents an analytical solution for the complex fractional nonlinear integro-differential equations (CF-NL-IDEs) in fewer iterations, without using traditional techniques in numerical methods, such as converting the problem into a system of algebraic equations, discretization, and other steps that can cause errors in the solution. Further, to improve accuracy, we use additional techniques to reduce the error of the solutions. Our proposed method does not require traditional procedures, and the resulting solution is error-free. The method always provides the analytical solution to the proposed problem and uses the initial term of the series involved in the Adomian decomposition method (ADM). The initial term is the sum of all the associated (initial/boundary) conditions and the integration of the source/forcing terms. This method is called the two-step Adomian decomposition method (TSADM), and it is efficient when applicable to the problem [29–31].

Further, we provide an analysis of the proposed problem and present new results on the existence and uniqueness of a solution under certain hypotheses, using the Banach contraction principle and Krasnoselskii's fixed point theorem, respectively. Also, we discuss the stability analysis of the CF-NL-IDEs following the Ulam-Hyers (U-H) and Ulam-Hyers-Rassias (U-H-R) stability definitions, which are based on fixed-point theory. In the study of differential equations and dynamical systems, the concept of stability is crucial for understanding the robustness of solutions under perturbations. Stability concepts help determine how small changes in initial conditions or system parameters affect solution behavior over time. Among the various stability concepts, U-H stability, generalized U-H stability, U-H-R stability, and generalized U-H-R stability provide distinct perspectives on how perturbations influence system stability. The progression from U-H stability to generalized U-H-R stability represents an increasing flexibility in how perturbations can affect the system's stability. U-H stability assumes simple, bounded perturbations, while its generalized and Rassias forms accommodate more complex, variable perturbations, allowing for a broader analysis of stability across different scenarios.

Nonlinear dynamics and systems theory have made substantial advances in recent years, particularly in the modeling and analysis of complex systems where traditional integer-order calculus falls short. In this context, fractional calculus has emerged as a powerful tool for capturing intricate behaviors of systems with memory and hereditary effects. One significant area of interest is the solution of nonlinear integro-differential equations (IDEs) with complex fractional orders, which present unique challenges and opportunities for research. Integro-differential equations involving fractional derivatives can model a wide range of phenomena across various fields, including control systems, signal processing, and materials science. These equations provide a more nuanced representation of systems exhibiting anomalous diffusion or memory effects than classical integer-order models. However, solving nonlinear integro-differential equations with complex fractional orders remains a challenging task due to their inherent complexity and the difficulties associated with conventional numerical methods.

We have used the Caputo definition to describe the fractional operators of complex fractional order (derivative and integral). Examples are solved using ADM and the modified ADM. Finally, we compare the solutions obtained with the results obtained by TSADM. The convergence of the method is also discussed, and an error estimate is provided. The error estimate confirms the accuracy of the solution. The convergence of the method is discussed based on the partial sum of the iterations obtained from the methods, and the results show that the TSADM method is more efficient and provides an accurate solution. Further, we explain the convergence of the method through examples and compare the results obtained using ADM and modified Adomian decomposition (MADM).

The outline of the rest of the paper is arranged as follows: In Section 2, we collect some basic knowledge of the complex fractional operators and their properties. In Section 3, we present the algorithm for the adopted method. In Section 4, we establish new results on the existence and uniqueness of the CF-NL-IDEs using fixed point theorems. Section 5 is devoted to the stability analysis using the fixed point theory. Finally, we discuss the conclusion of the study and provide our final observations in Section 6.

§2 Preliminaries

This section contains definitions and lemmas of fractional calculus that help prove our results [23, 27, 32, 36].

Let $C^k([0,\mathcal{T}],\mathbb{C})$ be the spaces of complex valued functions having smooth derivatives up to order $k-1\in\mathbb{N}\cup\{0\}$ so that their kth derivative is continuous (integrable on $[0,\mathcal{T}]$). The fractional integral of complex fractional order $\vartheta\in\mathbb{C}, Re(\vartheta)>0$, is defined for a function $\mathcal{G}\in L^1([0,\mathcal{T}],\mathbb{C})$ (the space of integrable functions on $[0,\mathcal{T}]$) [2,3,25,32,33].

Definition 1. The Riemann-Liouville fractional integral operator of complex fractional order $\vartheta\in\mathbb{C},\{Re(\vartheta)>0\}$, of a function $\mathcal{G}:\mathbb{R}^+\to\mathbb{R}$ is defined as

$$\begin{aligned}&{}^{RL}_{0+}J^{\vartheta}_{\xi}\mathcal{G}(\xi)=\frac{1}{\Gamma(\vartheta)}\int_0^{\xi}(\xi-y)^{\vartheta-1}\mathcal{G}(y)dy,\quad Re(\vartheta)>0,\xi>0,\\&{}^{RL}_{0+}J^{\vartheta}_{\xi}\mathcal{G}(\xi)=\mathcal{G}(\xi).\end{aligned}\tag{1}$$

The operator ${}^{RL}_{0+}J^{\vartheta}_{\xi}\mathcal{G}(\xi)$ satisfies the following properties [11,25,32–34]

$$(i)\ {}^{RL}_{0+}J^{\vartheta}_{\xi}\,{}^{RL}_{0+}J^{\vartheta_1}_{\xi}\mathcal{G}(\xi)={}^{RL}_{0+}J^{\vartheta+\vartheta_1}_{\xi}\mathcal{G}(\xi),\tag{2}$$

$$(ii)\ {}^{RL}_{0+}J^{\vartheta}_{\xi}\,{}^{RL}_{0+}J^{\vartheta_1}_{\xi}\mathcal{G}(\xi)={}^{RL}_{0+}J^{\vartheta_1}\,{}^{RL}_{0+}J^{\vartheta}_{\xi}\mathcal{G}(\xi),\tag{3}$$

$$(iii)\ {}^{RL}_{0+}J^{\vartheta}_{\xi}\xi^{\zeta}=\frac{\Gamma(\zeta+1)}{\Gamma(\vartheta+\zeta+1)}\xi^{\vartheta+\zeta}.\tag{4}$$

Definition 2. The Caputo fractional derivative of complex fractional order $\vartheta\in\mathbb{C},\{Re(\vartheta)>0\}$, of a function $\mathcal{G}:\mathbb{R}^+\to\mathbb{R}$ is defined as

$${}^{C}_{0+}D^{\vartheta}_{\xi}\mathcal{G}(\xi)={}^{RL}_{0+}J^{m-\vartheta}_{\xi}\,{}^{C}_{0+}D^{m}_{\xi}\mathcal{G}(\xi)=\frac{1}{\Gamma(m-\vartheta)}\int_0^{\xi}(\xi-y)^{m-\vartheta-1}\mathcal{G}^m(y)dy,\tag{5}$$

where $m=[Re(\vartheta)]+1$ and $[Re(\vartheta)]$ denotes the integer part of the real number ϑ, and

$${}^{C}_{0+}D^{\vartheta}_{\xi}\mathcal{K}=0.\tag{6}$$

The Caputo fractional operators are linear operators, described as

$${}^{C}_{0+}D^{\vartheta}_{\xi}(\mathcal{K}s(\xi)+\mathcal{J}h(\xi))=\mathcal{K}\,{}^{C}_{0+}D^{\vartheta}_{\xi}s(\xi)+\mathcal{J}\,{}^{C}_{0+}D^{\vartheta}_{\xi}h(\xi),\tag{7}$$

where $\mathcal{K}$ and $\mathcal{J}$ are constants.

Definition 3. The Stirling asymptotic formula of the Gamma function for $\mathcal{Z}\in\mathbb{C}$ is

$$\Gamma(\mathcal{Z})=(2\pi)^{\frac{1}{2}}\mathcal{Z}^{\frac{\mathcal{Z}-1}{2}}\left[\exp\big(-\mathcal{Z}\big)\left(1+\mathcal{O}\left(\frac{1}{\mathcal{Z}}\right)\right)\right],(|\arg(\mathcal{Z})<\pi;|\mathcal{Z}|\to\infty|).\tag{8}$$

Further, for $\mathcal{U},\mathcal{V}\in\mathbb{R}$,

$$|\Gamma(\mathcal{U}+i\mathcal{V})|=(2\pi)^{\frac{1}{2}}|\mathcal{V}|^{\mathcal{U}-\frac{1}{2}}\exp\left(-\mathcal{U}-\frac{\pi|\mathcal{V}|}{2}\right)\left[1+\mathcal{O}\left(\frac{1}{\mathcal{V}}\right)\right],(\mathcal{V}\to\infty).\tag{9}$$

Lemma 1 ([25,32–34]): If $\vartheta\in\mathbb{C},\ m\in\mathbb{N}$, then

$${}^{C}_{0+}D^{\vartheta}_{\xi}\,{}^{RL}_{0+}J^{\vartheta}_{\xi}\mathcal{G}(\xi)=\mathcal{G}(\xi),\tag{10}$$

$${}^{RL}_{0+}J^{\vartheta}_{\xi}\,{}^{C}_{0+}D^{\vartheta}_{\xi}\mathcal{G}(\xi)=\mathcal{G}(\xi)-\sum_{\ell=0}^{m-1}\mathcal{G}^{\ell}(0^+)\frac{\xi^{\ell}}{\ell!},\ \ \xi>0.\tag{11}$$

Here is a brief overview of these types of stability

Definition 4. Ulam-Hyers stability (U-H stability): A function $x(t)$ is said to be U-H stable if, for any given $\epsilon>0$, there exists a $\delta>0$ such that whenever the initial condition x_0 of the system is perturbed by an amount less than δ, the solution $x(t)$ of the perturbed system remains within ϵ of the solution of the unperturbed system for all t in the domain.

$$\text{For any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } \|x_0 - \tilde{x}_0\| < \delta, \text{ then } \|x(t) - \tilde{x}(t)\| < \epsilon \text{ for all } t \geq 0. \tag{12}$$

Definition 5. Generalized U-H stability: This is a generalization of U-H stability. A system is generally U-H stable if the perturbations are not just bounded by a constant but are governed by a more complex functional relationship. It may involve varying perturbation bounds depending on the state of the system.

$$\begin{aligned}&\text{For any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } \|x_0 - \tilde{x}_0\| < \delta \text{ and } \tilde{x}(t) \text{ satisfies a certain}\\ &\text{generalized condition, then } \|x(t) - \tilde{x}(t)\| < \epsilon \text{ for all } t \geq 0.\end{aligned} \tag{13}$$

Definition 6. Ulam-Hyers-Rassias stability (U-H-R stability): This concept combines Ulam-Hyers stability with the Rassias stability approach, which involves a more flexible perturbation model. Rassias's stable allows perturbations that are not necessarily uniformly bounded but may grow in a controlled manner over time.

$$\begin{aligned}&\text{For any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } \|x_0 - \tilde{x}_0\| < \delta \text{ and the perturbation grows in a}\\ &\text{specific manner, then } \|x(t) - \tilde{x}(t)\| < \epsilon \text{ for all } t \geq 0.\end{aligned} \tag{14}$$

Definition 7. Generalized U-H-R stability: This extends U-H-R stability by allowing for even more general forms of perturbations and their growth over time. It combines the flexibility of Rassias stable with the idea of generalized perturbation bounds.

$$\begin{aligned}&\text{For any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } \|x_0 - \tilde{x}_0\| < \delta \text{ and the perturbations are}\\ &\text{governed by generalized conditions, then } \|x(t) - \tilde{x}(t)\| < \epsilon \text{ for all } t \geq 0.\end{aligned} \tag{15}$$

These concepts help analyze the robustness of solutions to differential equations under various types of perturbations.

Note: In essence, functional equations seek specific functions that satisfy a relationship, while inequalities identify the ranges of values or conditions where a relationship holds. The methods and types of solutions for each are adapted to their respective problems.

§3 The TSADM Algorithm for the CF-NL-IDEs

In this section, we present the description of TSADM along with the advantages and limitations of the method.

Consider the complex fractional nonlinear integro-differential equations (CF-NL-IDEs), described as

$$ {}_{0+}^{C}D_{\xi}^{\vartheta}\varpi(\xi) + \int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi) = \Upsilon(\xi), \varpi(0) = \varpi_0,\ \xi \in \mathcal{I} := (0, \tau], \vartheta = u + iv, \tag{16}$$

where ${}_{0+}^{C}D_{\xi}^{\vartheta}$ is the Caputo fractional derivative of order $\vartheta \in \mathbb{C}$. Consider $u \in (0, 1], v \in \mathbb{R}$ and $\Upsilon : \mathcal{I} \to \mathbb{R}$ be a continuous function, $\zeta_1(\xi)$ and $\zeta_2(\xi)$ are known variable coefficients. It should be noted that the operator ${}_{0+}^{C}D_{\xi}^{\vartheta}$ has a complex fractional order but it is not a complex valued function.

The algorithm consists of five steps [24].

Step 1. On applying the inverse operator ${}^{RL}_{0+}J^{\vartheta}_{\xi}$ of ${}^{C}_{0+}D^{\vartheta}_{\xi}$ on both sides of equation (16), we obtain

$$\varpi(\xi) = \varpi_0 + {}^{RL}_{0+}J^{\vartheta}_{\xi}\bigg(\Upsilon(\xi) - \Big(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\Big)\bigg). \tag{17}$$

Step 2 ([24]). Write the recursion formula for the TSADM from (17) in step 1 as

$$\varpi_0(\xi) = \varpi_0 + {}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(\Upsilon(\xi)\Big), \tag{18}$$

and

$$\varpi_{\ell+1}(\xi) = -{}^{RL}_{0+}J^{\vartheta}_{\xi}\bigg(\int_0^{\xi} \varpi_{\ell}(s)ds + \zeta_1(\xi)\varpi'_{\ell}(\xi) + \zeta_2(\xi)\varpi_{\ell}(\xi)\bigg), \tag{19}$$

where $\ell = 1, 2, \cdots$.

Step 3. The first iteration (zeroth term) in equation (18) can be split into several components as

$$\varpi_0(\xi) = \Xi = \Xi_1 + \Xi_2 + \Xi_3 + \cdots + \Xi_M, \tag{20}$$

where $\Xi_1, \Xi_2, \Xi_3, \cdots$, and Ξ_M are the terms obtained by integrating the source term $\Upsilon(\xi)$ and from the associated initial/boundary conditions.

Step 4 ([24]). Select the first component of ϖ_0, that is, Ξ_1. Then Ξ_1 satisfies (16) and that the associated initial conditions, if Ξ_1 is the exact solution of (16). If the first component does not satisfy either (16) or the initial conditions, then we proceed to the next component and repeat the same process. If any component in ϖ_0 satisfies (16) and the initial conditions, then that component becomes our exact solution of (16) with the initial conditions. If all the components involved in ϖ_0 do not satisfy (16) or the initial conditions, then we go to the next step.

Step 5. Apply the ADM to obtain the solution by choosing $\varpi_0(\xi) = \Xi$ and iterate the solution by using equation (19) in Step 2.

Advantages: The advantages of the TSADM for the considered problem are listed as follows.

(1) If the TSADM is applicable to the problem, then the solution obtained is an exact solution.

(2) The TSADM gives the exact solution in just one iteration and reduces the computational effort.

(3) The TSADM is a powerful and efficient method for such types of problems in comparison to other existing numerical methods, which use linearization, discretization, and Adomian polynomial. Hence, the TSADM reduces the memory space and cost. Thus, the TSADM is superior to ADM, MADM, and other numerical methods.

Limitations: The proposed method will fail under the condition stated as follows.
The first term/zeroth term of the series solution obtained should contain a term that satisfies the considered equation and the related initial/boundary conditions. If there is no such zeroth term in the series, then we obtain a semi-analytical solution.

Choice of Parameters for TSADM: The specific design of the two-step Adomian decomposition method, including parameter choices and iteration strategies, can impact its ability to find an exact solution near the approximate one. Proper tuning of these parameters is necessary for better accuracy.

§4 Convergence analysis and error estimate

In this section, we focus on the convergence of the ADM used in the TSADM. Here, we present the sufficient conditions for convergence of the method and error estimate for equations (16) and (17).

Theorem 1 ([20]). Let $\varpi_n(\xi)$ and $\varpi(\xi)$ be defined in the Banach space $C(\mathcal{I}, \|\cdot\|)$ and $\varpi(\xi)$ be the exact solution of (16)-(17). Then the series solution $\{\varpi_n\}_{n=0}^{\infty}$ defined by

$$\varpi = \sum_{\ell=0}^{\infty} \varpi_\ell(\xi) = \varpi_0 + \varpi_1 + \varpi_2 + \cdots \tag{21}$$

converges to the solution of $\varpi(\xi)$, and $|\varpi_0| < \infty$ when $\exists\ 0 \leq \gamma < 1, \|\varpi_{n+1}\| \leq \gamma \|\varpi_n\|\ \forall\ n \in N \cup \{0\}$.

Proof. Assume that $(C(\mathcal{I}), \|\cdot\|)$ is a Banach space, i.e., the space of all continuous functions on $\mathcal{I}$ with the norm

$$\|\varpi(\xi)\| = \max_{\xi \in \mathcal{I}} |\varpi(\xi)|. \tag{22}$$

Define $\{\mathcal{S}_n\}$ as the sequence of partial sums of the series $\sum_{n=0}^{\infty} \varpi_n(\xi)$ by,

$$\begin{aligned}
\mathcal{S}_0 &= \varpi_0(\xi),\\
\mathcal{S}_1 &= \varpi_0(\xi) + \varpi_1(\xi),\\
\mathcal{S}_2 &= \varpi_0(\xi) + \varpi_1(\xi) + \varpi_2(\xi),\\
&\vdots\\
\mathcal{S}_n &= \varpi_0(\xi) + \varpi_1(\xi) + \varpi_2(\xi) + \cdots + \varpi_n(\xi).
\end{aligned} \tag{23}$$

We need to show that $\{\mathcal{S}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C(\mathcal{I}), \|\cdot\|)$. For this purpose, we consider

$$\|\mathcal{S}_{n+1} - \mathcal{S}_n\| = \|\varpi_{n+1}(\xi)\| \leq \gamma \|\varpi_n(\xi)\| \leq \gamma^2 \|\varpi_{n-1}(\xi)\| \leq \cdots \leq \gamma^{n+1} \|\varpi_0(\xi)\|. \tag{24}$$

For every, $n, m \in \mathbb{N}$ with $n \geq m$, from (24), we have

$$\begin{aligned}
\|\mathcal{S}_n - \mathcal{S}_m\| &= \|(\mathcal{S}_n - \mathcal{S}_{n-1}) + (\mathcal{S}_{n-1} - \mathcal{S}_{n-2}) + \cdots + (\mathcal{S}_{m+1} - \mathcal{S}_m)\|,\\
&\leq \|\mathcal{S}_n - \mathcal{S}_{n-1}\| + \|\mathcal{S}_{n-1} - \mathcal{S}_{n-2}\| + \cdots + \|\mathcal{S}_{m+1} - \mathcal{S}_m\|,\\
&\leq \gamma^n \|\varpi_0(\xi)\| + \gamma^{n-1} \|\varpi_0(\xi)\| + \cdots + \gamma^m \|\varpi_0(\xi)\|,\\
&= \frac{1 - \gamma^{n-m}}{1 - \gamma} \gamma^{m+1} \|\varpi_0(\xi)\|.
\end{aligned} \tag{25}$$

Since $0 < \gamma < 1$, it follows that $1 - \gamma^{n-m} < 1$. Thus

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1 - \gamma} \max_{\xi \in \mathcal{I}} |\varpi_0(\xi)|. \tag{26}$$

Since $\varpi_0(\xi)$ is bounded, we have $\lim_{n,m \to \infty} \|\mathcal{S}_n - \mathcal{S}_m\| = 0$, i.e., $\{\mathcal{S}_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Banach space $C(\mathcal{I}, \|\cdot\|)$. Therefore, there exists $\mathcal{S} \in C(\mathcal{I}, \|\cdot\|)$ such that $\lim_{n \to +\infty} \mathcal{S}_n = \mathcal{S}$. This implies that the series solution $\sum_{\ell=0}^{\infty} \varpi_\ell(\xi)$ converges. □

Theorem 2 ([20]). The maximum absolute truncation error of the series solution in (21) for (16) and (17) is estimated to be

$$\max_{\xi\in\mathcal{I}}\left|\varpi(\xi)-\sum_{\ell=0}^{m}\varpi_\ell(\xi)\right|\leq\frac{M\gamma^{m+1}}{1-\gamma}. \tag{27}$$

Proof. From Theorem 1 and (25), we have

$$\|\mathcal{S}_n-\mathcal{S}_m\|\leq\frac{\gamma^{m+1}(1-\gamma^{n-m})}{1-\gamma}\,\|\varpi_0(\xi)\|, \tag{28}$$

for $n\geq m$.

As $n\to\infty$, we have $\mathcal{S}_n\to\varpi(\xi)$, and suppose that $\max_{\xi\in\mathcal{I}}|\varpi_0(\xi)|\leq M$. Thus,

$$\|\varpi(\xi)-\mathcal{S}_m\|\leq\frac{\gamma^{m+1}}{1-\gamma}\|\varpi_0(\xi)\|. \tag{29}$$

Since $0<\gamma<1$ and $1-\gamma^{n-m}<1$, the above inequality becomes

$$\begin{aligned}\max_{\xi\in\mathcal{I}}\left|\varpi(\xi)-\sum_{\ell=0}^{m}\varpi_\ell(\xi)\right|&\leq\frac{\gamma^{m+1}}{(1-\gamma)}\|\varpi_0(\xi)\|\\&\leq\frac{M\gamma^{m+1}}{(1-\gamma)}\end{aligned} \tag{30}$$

Hence, the proof is complete. □

§5 Existence and uniqueness for the solution

This section is devoted to the proof of the new existence and uniqueness conditions of the solution by employing some fixed point theorems (see Theorems 1-3 in [27]) of the problem (16).

By $C(\mathcal{I},\mathbb{R})$, we denote the space of all continuous functions from $\mathcal{I}$ into $\mathbb{R}$, and this space is a Banach space when endowed with the norm $\|\cdot\|$, defined as $\|\varpi\|=\max\{|\varpi(\xi)|:\ \xi\in\mathcal{I}\}$.

The following hypotheses are required to prove our main results.

$[\digamma_1]$**:** $\Upsilon:\mathbb{I}\to\mathbb{R}$ is a continuous function.

$[\digamma_2]$**:** There exists a constant M such that

$$|\Upsilon(\xi)|\leq M$$

for any $\xi\in\mathcal{I}$.

$[\digamma_3]$**:** There exists a constant L such that

$$|\varpi'(\xi)|\leq L$$

for any $\xi\in\mathcal{I}$.

$[\digamma_4]$**:** There exist constants $K,A_1,A_2>0$ such that

$$\begin{aligned}|\varpi(\xi_1)-\varpi(\xi_2)|\leq A_1|\xi_1-\xi_2|,|\varpi'(\xi_1)-\varpi'(\xi_2)|\leq A_2|\xi_1-\xi_2|\text{ and }|\varpi_1'(\xi)-\varpi_2'(\xi)|\\ \leq K\,|\varpi_1(\xi)-\varpi_2(\xi)|,\end{aligned}$$

for any $\xi_1,\xi_2\in\mathcal{I},\varpi\in C(\mathcal{I},\mathbb{R})$.

The following results are based on Theorems 1 and 2 in [27].

Theorem 3. Assume that the hypotheses $[\digamma_1]$-$[\digamma_4]$ are satisfied. If

$$\mathbb{L} = \left[\frac{\tau^u}{u|\Gamma(\vartheta)|} \times \left(\tau + K\|\zeta_1\| + \|\zeta_2\|\right)\right] < 1,$$

then the proposed problem (16) has a unique solution in $C(\mathcal{I}, \mathbb{R})$.

Proof. Consider the operator $\Delta : C(\mathcal{I}, \mathbb{R}) \to C(\mathcal{I}, \mathbb{R})$, described as

$$\Delta(\varpi(\xi)) := \varpi_0 + {}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\Upsilon(\xi) - \left(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\right)\right). \tag{31}$$

It can be written as

$$\Delta(\varpi(\xi)) := \varpi_0 + \frac{1}{\Gamma(\vartheta)}\int_0^{\xi}(\xi - s)^{\vartheta-1}\left(\Upsilon(s) - \left(\int_0^{s}\varpi(r)dr + \zeta_1(s)\varpi'(s) + \zeta_2(s)\varpi(s)\right)\right)ds. \tag{32}$$

It is obvious that a fixed point of the operator (31) is a solution of (16).
Let $\varpi_1, \varpi_2 \in C(\mathcal{I}, \mathbb{R})$ and $\xi \in \mathcal{I}$. Then

$$\begin{aligned}
&|\Delta(\varpi_1(\xi)) - \Delta(\varpi_2(\xi))| \\
=&\left|\frac{1}{\Gamma(\vartheta)}\int_0^{\xi}(\xi - s)^{\vartheta-1}\left(\Upsilon(s) - \left(\int_0^{s}\varpi_1(r)dr + \zeta_1(s)\varpi_1'(s) + \zeta_2(s)\varpi_1(s)\right)\right)ds\right. \\
&\left. - \left(\frac{1}{\Gamma(\vartheta)}\int_0^{\xi}(\xi - s)^{\vartheta-1}\left(\Upsilon(s) - \left(\int_0^{s}\varpi_2(r)dr + \zeta_1(s)\varpi_2'(s) + \zeta_2(s)\varpi_2(s)\right)\right)ds\right)\right| \\
\leq&\left|\frac{1}{\Gamma(\vartheta)}\int_0^{\xi}(\xi - s)^{\vartheta-1}\left(\int_0^{s}\left(\varpi_1(r) - \varpi_2(r)\right)dr\right.\right. \\
&\left.\left. + \zeta_1(s)(\varpi_1'(s) - \varpi_2'(s)) + \zeta_2(s)(\varpi_1(s) - \varpi_2(s))\right)ds\right| \\
\leq&\frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}\left|(\xi - s)^{\vartheta-1}\right|\left(\int_0^{s}\left|\varpi_1(r) - \varpi_2(r)\right|dr\right. \\
&\left. + |\zeta_1(s)||\varpi_1'(s) - \varpi_2'(s)| + |\zeta_2(s)||\varpi_1(s) - \varpi_2(s)|\right)ds \\
\leq&\frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}(\xi - s)^{u-1}\left(\int_0^{s}\left|\varpi_1(r) - \varpi_2(r)\right|dr\right. \\
&\left. + K|\zeta_1(s)||\varpi_1(s) - \varpi_2(s)| + |\zeta_2(s)||\varpi_1(s) - \varpi_2(s)|\right)ds \\
\leq&\frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}(\xi - s)^{u-1}\left(s + K|\zeta_1(s)| + |\zeta_2(s)|\right)|\varpi_1(s) - \varpi_2(s)|ds \\
\leq&\frac{\tau^u}{u|\Gamma(\vartheta)|}\left(\tau + K|\zeta_1(\xi)| + |\zeta_2(\xi)|\right)|\varpi_1 - \varpi_2|.
\end{aligned}$$

Finally, we obtain

$$\|\Delta(\varpi_1(\xi)) - \Delta(\varpi_2(\xi))\| \leq \mathbb{L}\|\varpi_1(\xi) - \varpi_2(\xi)\|, \tag{33}$$

where $\mathbb{L} = \frac{\tau^u}{u|\Gamma(\vartheta)|} \times \left(\tau + K\|\zeta_1\| + \|\zeta_2\|\right)$.
Since $\mathbb{L} < 1$, therefore Δ is a strict contraction. As a consequence of the Banach contraction principle, there exists a unique fixed point which is a unique solution of the problem (16) in $\mathcal{I}$. □

Theorem 4. Assume that the hypotheses $[\mathcal{F}_1]$-$[\mathcal{F}_4]$ hold. Then the problem (16) has at least one fixed point in $\mathcal{I}$.

Proof. Choose $t \geq \varpi_0 + \frac{t\tau^u}{u|\Gamma(\vartheta)|} \times \left(M+\tau+L\|\zeta_1\|+\|\zeta_2\|\right)$ and let $\mathbb{H}_t = \{\varpi \in C(\mathcal{I},\mathbb{R}) : \|\varpi\| \leq t\}$. Suppose $\Delta = \Delta_1 + \Delta_2$, where Δ_1 and Δ_2 are the two operators from (31) defined on $\mathbb{H}_t$ by

$$\Delta_1(\varpi(\xi)) := {}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\Upsilon(\xi) - \left(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\right)\right) \tag{34}$$

and

$$\Delta_2(\varpi(\xi)) := \varpi_0, \tag{35}$$

respectively.

Since $\varpi \in \mathbb{H}_t$, therefore $\Delta_1\varpi + \Delta_2\varpi \in \mathbb{H}_t$. Thus,

$$\begin{aligned}
|\Delta_1\varpi(\xi) + \Delta_2\varpi(\xi)| &= \left|\varpi_0 + {}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\Upsilon(\xi) - \left(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\right)\right)\right| \\
&\leq |\varpi_0| + \frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}\left|(\xi-s)^{\vartheta-1}\right||\Upsilon(\xi)| + \left(\int_0^{s}|\varpi(r)|dr\right. \\
&\quad \left. + |\zeta_1(s)||\varpi'(s)| + |\zeta_2(s)||\varpi(s)|\right)ds \\
&\leq |\varpi_0| + \frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}(\xi-s)^{u-1}\left(|\Upsilon(\xi)| + \int_0^{s}|\varpi(r)|dr\right. \\
&\quad \left. + |\zeta_1(s)||\varpi'(s)| + |\zeta_2(s)||\varpi(s)|\right)ds \\
&\leq |\varpi_0| + \frac{\xi^u}{u|\Gamma(\vartheta)|}\left(M + \xi|\varpi| + L|\zeta_1(s)||\varpi| + |\zeta_2(s)||\varpi(s)|\right) \\
&\leq |\varpi_0| + \frac{t\tau^u}{u|\Gamma(\vartheta)|}\left(M + \tau + L|\zeta_1| + |\zeta_2|\right) \\
&\leq t.
\end{aligned}$$

Hence,

$$\Delta_1\varpi(\xi) + \Delta_2\varpi(\xi) \in \mathbb{H}_t.$$

Next, we show that Δ_1 is continuous and compact, while Δ_2 is a contraction. The operator $\Delta(\varpi(\xi))$ is continuous in accordance with $[\digamma_1]$. Clearly,

$$\begin{aligned}
|\Delta_1(\varpi(\xi))| &= \left|{}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\Upsilon(\xi) - \left(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\right)\right)\right| \\
&= \frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}\left|(\xi-s)^{\vartheta-1}\left(\Upsilon(\xi) - \left(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\right)\right)ds\right| \\
&\leq \frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}(\xi-s)^{u-1}\left(|\Upsilon(\xi)| + \left|\left(\int_0^{\xi} \varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi)\right)\right)\right|ds \\
&\leq \frac{\xi^u}{u|\Gamma(\vartheta)|} \times \left(M + \xi + L|\zeta_1(\xi)| + |\zeta_2(s)|\right)|\varpi(\xi)|.
\end{aligned}$$

Finally,

$$\|\Delta_1(\varpi(\xi))\| \leq \frac{t\tau^u}{u|\Gamma(\vartheta)|} \times \left(M + \tau + L\|\zeta_1\| + \|\zeta_2\|\right).$$

Thus Δ_1 is uniformly bounded in $\mathbb{H}_t$. Now, we prove that the operator Δ_1 is equicontinuous.

Consider $\xi_1, \xi_2 \in \mathcal{I}, \xi_2 \leq \xi_1$ and $\varpi \in \mathbb{H}_t$.
Then □

$$
\begin{aligned}
&|\Delta_1\varpi(\xi_1) - \Delta_1\varpi(\xi_2)| \\
&= \Bigg| {}^{RL}_{0+}J^{\vartheta}_{\xi_1}\Bigg(\Upsilon(\xi_1) - \Bigg(\int_0^{\xi_1}\varpi(\xi_1)d\xi_1 + \zeta_1(\xi_1)\varpi'(\xi_1) + \zeta_2(\xi_1)\varpi(\xi_1)\Bigg)\Bigg) \\
&- \Bigg({}^{RL}_{0+}J^{\vartheta}_{\xi_2}\Bigg(\Upsilon(\xi_2) - \Bigg(\int_0^{\xi_2}\varpi(\xi_2)d\xi_2 + \zeta_1(\xi_2)\varpi'(\xi_2) + \zeta_2(\xi_2)\varpi(\xi_2)\Bigg)\Bigg)\Bigg)\Bigg| \\
&\leq \frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}(\xi - s)^{u-1}\Bigg(\int_0^{\xi}|\varpi(\xi_1) - \varpi(\xi_2)| + |\zeta_1||\varpi'(\xi_1) - \varpi'(\xi_2)| \\
&+ |\zeta_2|\varpi(\xi_1) - \varpi(\xi_2)|\Bigg) \\
&\leq \frac{1}{|\Gamma(\vartheta)|}\int_0^{\xi}(\xi - s)^{u-1}\Bigg(\xi A_1 + A_2|\zeta_1| + A_1|\zeta_2|\Bigg)|\xi_1 - \xi_2|.
\end{aligned}
$$

Hence

$$\|\Delta_1\varpi(\xi_1) - \Delta_1\varpi(\xi_2)\| \leq \frac{t\tau^u}{u|\Gamma(\vartheta)|}\Bigg((\tau + \|\zeta_2\|)A_1 + A_2\|\zeta_1\|\Bigg)\|\xi_1 - \xi_2\|.$$

If $\xi_1 \to \xi_2$, then $\|\Delta_1\varpi(\xi_1) - \Delta_1\varpi(\xi_2)\| \to 0$.
Thus, the operator Δ_1 is equicontinuous. So, $\Delta_1(\mathbb{H}_t)$ is relatively compact. Therefore, Δ_1 is compact (Theorem 2, see in [27]). Therefore, the conditions of Theorem 3 in [27] are satisfied. Hence, (16) has at least one fixed point in $\mathcal{I}$.

§6 Stability analysis

In the present section, we discuss the Ulam-Hyers (U-H) stability of the CF-NL-IDEs given by equation (16). Further, we also discuss the generalized U-H stability, U-H-R stability, and generalized U-H-R stability [34] of equation (16) using the Banach contraction principle (see [27]).

Definition 4. Equation (16) is U-H stable if there exists a real number $\mathcal{F}_\Upsilon$ such that for each $\epsilon > 0$ and for each solution $\varpi \in C(\mathcal{I}, \mathbb{R})$ of the inequality

$$\left|{}^{C}_{0+}D^{\vartheta}_{\xi}\varpi(\xi) + \int_0^{\xi}\varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi) - \Upsilon(\xi)\right| \leq \epsilon,\ \xi \in \mathcal{I}, \tag{36}$$

there exists a solution $\varpi^* \in C(\mathcal{I}, \mathbb{R})$ of the equation (16) which satisfies

$$|\varpi(\xi) - \varpi^*(\xi)| \leq \mathcal{F}_\Upsilon\epsilon,\ \xi \in \mathcal{I}. \tag{37}$$

Definition 5. Equation (16) is generalized U-H stable if there exists a function $p' \in \mathbb{C}([0, \infty), [0, \infty))$ which satisfies $p'(0) = 0$ such that for each solution $\varpi \in C(\mathcal{I}, \mathbb{R})$ of the inequality

$$\left|{}^{C}_{0+}D^{\vartheta}_{\xi}\varpi(\xi) + \int_0^{\xi}\varpi(s)ds + \zeta_1(\xi)\varpi'(\xi) + \zeta_2(\xi)\varpi(\xi) - \Upsilon(\xi)\right| \leq \epsilon,\ \xi \in \mathcal{I}, \tag{38}$$

there exists a solution $\varpi^* \in C(\mathcal{I}, \mathbb{R})$ of the equation (16) which satisfies

$$|\varpi(\xi) - \varpi^*(\xi)| \leq p'\epsilon,\ \xi \in \mathcal{I}. \tag{39}$$

Definition 6. Equation (16) is U-H-R stable with respect to $\aleph \in C(\mathcal{I}, \mathbb{R})$ if there exists a real number $\mathcal{F}_\Upsilon$ such that for each $\epsilon > 0$ and for each solution $\varpi \in C(\mathcal{I}, \mathbb{R})$ of the inequality

$$\left|{}_{0+}^{C}D_{\xi}^{\vartheta}\varpi(\xi)+\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)-\Upsilon(\xi)\right|\leq\epsilon\aleph(\xi),\ \xi\in\mathcal{I},\tag{40}$$

there exists a solution $\varpi^{*}\in C(\mathcal{I},\mathbb{R})$ of the equation (16) which satisfies

$$|\varpi(\xi)-\varpi^{*}(\xi)|\leq\mathcal{F}_{\Upsilon}\epsilon\aleph(\xi),\ \xi\in\mathcal{I}.\tag{41}$$

Definition 7. Equation (16) is generalized U-H-R stable with respect to $\aleph\in C(\mathcal{I},\mathbb{R})$ if there exists a real number $\mathcal{F}_{\Upsilon,\aleph}>0$ such that for each solution $\varpi\in C(\mathcal{I},\mathbb{R})$ of the inequality

$$\left|{}_{0+}^{C}D_{\xi}^{\vartheta}\varpi(\xi)+\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)-\Upsilon(\xi)\right|\leq\aleph(\xi),\ \xi\in\mathcal{I},\tag{42}$$

there exists a solution $\varpi^{*}\in C(\mathcal{I},\mathbb{R})$ of equation (16) which satisfies

$$|\varpi(\xi)-\varpi^{*}(\xi)|\leq\mathcal{F}_{\Upsilon,\aleph}\aleph(\xi),\ \xi\in\mathcal{I}.\tag{43}$$

Remark 1. A function $\varpi\in C(\mathcal{I},\mathbb{R})$ is a solution of the inequality (36) if and only if there exists a function $\varpi^{*}\in C(\mathcal{I},\mathbb{R})$ (which depends on ϖ) such that

(i) $|\varpi^{*}(\xi)|\leq\epsilon,\ \xi\in\mathcal{I}$,

(ii) ${}_{0+}^{C}D_{\xi}^{\vartheta}\varpi(\xi)+\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)=\Upsilon(\xi)+\varpi^{*}(\xi),\ \xi\in\mathcal{I}.$

Remark 2. A function $\varpi\in C(\mathcal{I},\mathbb{R})$ is a solution of the inequality (40) if and only if there exists a function $\varpi^{*}\in C(\mathcal{I},\mathbb{R})$ (which depends on ϖ) such that

(i) $|\varpi^{*}(\xi)|\leq\epsilon\aleph(\xi),\ \xi\in\mathcal{I}$,

(ii) ${}_{0+}^{C}D_{\xi}^{\vartheta}\varpi(\xi)+\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)=\Upsilon(\xi)+\varpi^{*}(\xi),\ \xi\in\mathcal{I}.$

Remark 3. Clearly,

(i) If equation (16) is U-H stable, then it is generalized U-H stable.

(ii) If equation (16) is U-H-R stable, then it is generalized U-H-R stable.

Theorem 5. If the conditions $[F_1]$-$[F_4]$ hold and $\mathbb{L}<1$, then (16) is U-H stable and generalized U-H stable.

Proof. Let $\epsilon>0$, $\varpi\in C(\mathcal{I},\mathbb{R})$ be a function which satisfies the inequality (36) and $\varpi\in C(\mathcal{I},\mathbb{R})$ be the unique solution of the following problem

$${}_{0+}^{C}D_{\xi}^{\vartheta}\varpi(\xi)+\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)=\Upsilon(\xi)+\varpi^{*}(\xi),\tag{44}$$

$$\varpi(0)=\varpi_{0},\ \xi\in\mathcal{I}:=(0,\tau],\tag{45}$$

where $\vartheta=u+iv,u\in(0,1]$ and $v\in\mathbb{R}$.

Then by Lemma 1 and Remark 1, we have

$$\varpi(\xi)=\varpi_{0}+{}_{0+}^{RL}J_{\xi}^{\vartheta}\left(\Upsilon(\xi)+\varpi^{*}(\xi)-\left(\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)\right)\right).\tag{46}$$

By integration the inequality (36) and using Remark 1, we get

$$\begin{aligned}&|\varpi(\xi)-\Delta\varpi(\xi)|\\&=\left|\varpi(\xi)-\left(\varpi_{0}+{}_{0+}^{RL}J_{\xi}^{\vartheta}\left(\Upsilon(\xi)-\left(\int_{0}^{\xi}\varpi(s)ds+\zeta_{1}(\xi)\varpi'(\xi)+\zeta_{2}(\xi)\varpi(\xi)\right)\right)\right)\right|\\&\leq\frac{\epsilon\tau^{u}}{u|\Gamma(\vartheta)|}.\end{aligned}$$

Thus,

$$\begin{aligned}&|\varpi(\xi)-\varpi^*(\xi)|\\ \leq&|\varpi(\xi)-\Delta\varpi(\xi)|+|\Delta\varpi(\xi)-\varpi^*(\xi)|\\ \leq&\frac{\epsilon\tau^u}{u|\Gamma(\vartheta)|}+|\Delta\varpi(\xi)-\varpi^*(\xi)|.\end{aligned}$$

Finally, we obtain the following result using Theorem 3

$$\begin{aligned}&|\varpi(\xi)-\varpi^*(\xi)|\\ \leq&\frac{\epsilon\tau^u}{u|\Gamma(\vartheta)|}+\frac{\tau^u}{u|\Gamma(\vartheta)|}\Big(\tau+K|\zeta_1(\xi)|+|\zeta_2(\xi)|\Big)|\varpi(\xi)-\varpi^*(\xi)|\\ \leq&\frac{\frac{\epsilon\tau^u}{u|\Gamma(\vartheta)|}}{1-\mathbb{L}}.\end{aligned}$$

Hence, the problem (16) is U-H stable. If we set $\mathcal{D}_\Upsilon(\epsilon)=\mathcal{F}_\Upsilon\epsilon$; $\mathcal{D}_\Upsilon(0)=0$, then the problem (16) is generalized U-H stable.

Theorem 6. If the conditions $[\digamma_1]$-$[\digamma_4]$ hold and $\mathbb{L}<1$, then (16) is U-H-R stable and also generalized U-H-R stable.

Proof. Let $\epsilon>0$, and $\varpi\in C(\mathcal{I},\mathbb{R})$ be a function which satisfies the inequality (40) and $\varpi\in C(\mathcal{I},\mathbb{R})$ be the unique solution of the following problem

$${}^{C}_{0+}D^{\vartheta}_{\xi}\varpi(\xi)+\int_0^{\xi}\varpi(s)ds+\zeta_1(\xi)\varpi'(\xi)+\zeta_2(\xi)\varpi(\xi)=\Upsilon(\xi)+\varpi^*(\xi), \tag{47}$$

$$\varpi(0)=\varpi_0,\ \xi\in\mathcal{I}:=(0,\tau], \tag{48}$$

where $\vartheta=u+iv, u\in(0,1]$ and $v\in\mathbb{R}$.

Then by Lemma 1 and Remark 2, we have

$$\varpi(\xi)=\varpi_0+{}^{RL}_{0+}J^{\vartheta}_{\xi}\Bigg(\Upsilon(\xi)+\varpi^*(\xi)-\bigg(\int_0^{\xi}\varpi(s)ds+\zeta_1(\xi)\varpi'(\xi)+\zeta_2(\xi)\varpi(\xi)\bigg)\Bigg). \tag{49}$$

By integrating the inequality (40) and using Remark 2, we get

$$\begin{aligned}|\varpi(\xi)-\Delta\varpi(\xi)|=&\left|\varpi(\xi)-\Bigg(\varpi_0+{}^{RL}_{0+}J^{\vartheta}_{\xi}\bigg(\Upsilon(\xi)-\Big(\int_0^{\xi}\varpi(s)ds+\zeta_1(\xi)\varpi'(\xi)+\zeta_2(\xi)\varpi(\xi)\Big)\bigg)\Bigg)\right|\\ &\leq\frac{\epsilon\aleph(\xi)\tau^u}{u|\Gamma(\vartheta)|}.\end{aligned}$$

Thus,

$$\begin{aligned}|\varpi(\xi)-\varpi^*(\xi)|&=|\varpi(\xi)-\Delta\varpi(\xi)|+|\Delta\varpi(\xi)-\varpi^*(\xi)|\\ &\leq\frac{\epsilon\aleph(\xi)\tau^u}{u|\Gamma(\vartheta)|}+|\Delta\varpi(\xi)-\varpi^*(\xi)|.\end{aligned}$$

Finally, we obtain the following result using Theorem 3

$$\begin{aligned}|\varpi(\xi)-\varpi^*(\xi)|&\leq\frac{\epsilon\aleph(\xi)\tau^u}{u|\Gamma(\vartheta)|}+\frac{\tau^u}{u|\Gamma(\vartheta)|}\Big(\tau+K|\zeta_1(\xi)|+|\zeta_2(\xi)|\Big)|\varpi-\varpi^*|\\ &\leq\frac{\frac{\epsilon\aleph(\xi)\tau^u}{u|\Gamma(\vartheta)|}}{1-\mathbb{L}}.\end{aligned}$$

Hence, the problem (16) is U-H-R stable as well as generalized U-H-R stable.

§7 Applications

This section solves two examples using TSADM and compares it with other existing numerical methods: ADM and MADM.

Example 1. *Let the nonlinear complex fractional integro-differential equation be given by*

$$ {}^{C}_{0+}D^{\vartheta}_{\xi}\varpi(\xi)+6\int_0^{\xi}\varpi(s)ds+2\xi\varpi'(\xi)+\varpi(\xi)=\Upsilon(\xi), 0<\xi\le\tau, \vartheta=u+iv, u\in(0,1], v\in\mathbb{R}, \tag{50}$$

$$\varpi(0)=0, \tag{51}$$

where $\Upsilon(\xi)=\frac{10\xi^{2-\vartheta}}{\Gamma(3-\vartheta)}+\frac{15\xi^{1-\vartheta}}{\Gamma(2-\vartheta)}+10\xi^3+70\xi^2+45\xi$.

The analytical solution of the example (50) *is* $\varpi(\xi)=5\xi^2+15\xi$.

Applying the TSADM algorithm, we apply the inverse operator ${}^{RL}_{0+}J^{\vartheta}_{\xi}$ *into equation* (50). *Therefore*

$$\varpi(\xi)=\varpi(0)+{}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(\Upsilon(\xi)-\Big(6\int_0^{\xi}\varpi(s)ds+2\xi\varpi'(\xi)+\varpi(\xi)\Big)\Big). \tag{52}$$

The iteration formulae of the TSADM, from equation (52), *are*

$$\varpi_0(\xi)={}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(\Upsilon(\xi)\Big), \tag{53}$$

and

$$\varpi_{\ell+1}(\xi)=-{}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(6\int_0^{\xi}\varpi_{\ell}(s)ds+2\xi\varpi'_{\ell}(\xi)+\varpi_{\ell}(\xi)\Big), \tag{54}$$

where $\ell=0,1,2,\cdots$.

From (53), *we obtain*

$$\begin{aligned}\varpi_0(\xi)&={}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(\frac{10\xi^{2-\vartheta}}{\Gamma(3-\vartheta)}+\frac{15\xi^{1-\vartheta}}{\Gamma(2-\vartheta)}+10\xi^3+70\xi^2+45\xi\Big)\\&=5\xi^2+15\xi+\frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta}+\frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}+\frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}.\end{aligned} \tag{55}$$

From equation (55), *the initial term* ϖ_0 *of the TSADM can be expressed into three components as*

$$\varpi_0=\Sigma_0+\Sigma_1+\Sigma_2, \tag{56}$$

where

$$\Sigma_0=5\xi^2+15\xi, \tag{57}$$

$$\Sigma_1=\frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta}+\frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}, \tag{58}$$

and

$$\Sigma_2=\frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}. \tag{59}$$

Now, we obtain an analytical solution of (50).

From the TSADM process, we select the initial term as shown in (57). *If the component satisfies* (50), *then we terminate the process and find the analytical solution of* (50).

Let $\varpi_0=\Sigma_0$ *and verify that the assumption of* ϖ_0 *satisfies* (50). *If this choice of* ϖ_0 *is valid, we confirm that the selected term is an analytical solution of* (50).

Putting $\varpi = \Sigma_0$ *as a solution of* (50)*, we observe that it satisfies* (50)*.*
That is, $\varpi = \Sigma_0$ *is the analytical solution of equation* (50)*. We substitute* $\varpi = \Sigma_0$ *in the left-hand side (LHS) of* (50)*, and hence we have*

$$ {}^{C}_{0+}D^{\vartheta}_{\xi}\Sigma_0 + 6\int_0^{\xi} \Sigma_0(s)ds + 2\xi\Sigma_0'(\xi) + \Sigma_0. \tag{60}$$

Now, we solve the expression given in (60) *to obtain*

$$ {}^{C}_{0+}D^{\vartheta}_{\xi}\Sigma_0 = \frac{10\xi^{2-\vartheta}}{\Gamma(3-\vartheta)} + \frac{15\xi^{1-\vartheta}}{\Gamma(2-\vartheta)}, \tag{61}$$

and

$$ 6\int_0^{\xi} \Sigma_0(s)ds + 2\xi\Sigma_0'(\xi) + \Sigma_0 = 10\xi^3 + 70\xi^2 + 45\xi. \tag{62}$$

Adding equations (61) *and* (62)*, we obtain the left-hand side of* (50)*, that is* Υ*,*

$$ {}^{C}_{0+}D^{\vartheta}_{\xi}\Sigma_0 + 6\int_0^{\xi} \Sigma_0(s)ds + 2\xi\Sigma_0'(\xi) + \Sigma_0 = \Upsilon(\xi). \tag{63}$$

From (63)*, we observe that the LHS of equation* (50) *is equal to the RHS, that is* $\Upsilon(\xi)$*. This shows that the assumption* $\varpi_0 = \Sigma_0$ *satisfies* (50)*. Thus,* Σ_0 *is the analytical solution of* (50) *by employing one iteration of the TSADM.*
Further, we solve Example 1 using ADM and MADM. Moreover, we compare the solutions obtained from the results obtained by applying the TSADM.

Using the ADM:

The iteration formulae of the ADM (see [31]) from the equation (52) *are*

$$\begin{aligned} \varpi_0(\xi) &= {}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(\Upsilon(\xi)\Big), \\ &= 5\xi^2 + 15\xi + \frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta} + \frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}, \end{aligned} \tag{64}$$

and

$$ \varpi_{\ell+1}(\xi) = -{}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(6\int_0^{\xi} \varpi_\ell(s)ds + 2\xi\varpi_\ell'(\xi) + \varpi_\ell(\xi)\Big), \tag{65}$$

where $\ell = 0, 1, 2, \cdots$ *.*
Now, we find the values of the further iterations using ϖ_0 *from* (64) *and putting* $\ell = 0, 1, \cdots$ *in* (65)*.*
Putting $\ell = 0$*, we get*

$$\begin{aligned} \varpi_1(\xi) &= -{}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(6\int_0^{\xi} \varpi_0(s)ds + 2\xi\varpi_0'(\xi) + \varpi_0(\xi)\Big), \\ &= -{}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(20\xi^3 + 80\xi^2 + 15\xi + \Big(\frac{60\Gamma(4)}{\Gamma(5+\vartheta)} + \frac{20\Gamma(4)}{\Gamma(4+\vartheta)}\Big)\xi^{4+\vartheta} \\ &\quad + \Big(\frac{1120}{\Gamma(4+\vartheta)} + \frac{140\Gamma(3)}{\Gamma(3+\vartheta)}\Big)\xi^{3+\vartheta} \\ &\quad + \Big(\frac{90\Gamma(2)}{\Gamma(2+\vartheta)} + \frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\Big)\xi^{2+\vartheta} + \frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}\Big) \end{aligned}$$

$$=\frac{20\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta}+\frac{80\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}+\frac{15\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}+\frac{80\Gamma(4)(4+\vartheta)}{\Gamma(5+2\vartheta)}\xi^{4+2\vartheta}$$
$$+\frac{1120+140\Gamma(3)(3+\vartheta)}{\Gamma(4+2\vartheta)}\xi^{3+2\vartheta}+\frac{90(2+\vartheta)+70\Gamma(3)}{\Gamma(3+2\vartheta)}\xi^{2+2\vartheta}+\frac{45\Gamma(2)}{\Gamma(2+2\vartheta)}\xi^{1+2\vartheta}. \tag{66}$$

Putting $\ell=1$, we get

$$\varpi_2(\xi)=-{}^{RL}_{0+}J^{\vartheta}_{\xi}\left(6\int_0^{\xi}\varpi_1(\xi)d\xi+2\xi\varpi_1'(\xi)+\varpi_1(\xi)\right),$$
$$=\left(\frac{20\Gamma(4)}{\Gamma(4+2\vartheta)}+\frac{40\Gamma(4)(3+\vartheta)}{\Gamma(4+2\vartheta)}+\frac{480\Gamma(3)}{\Gamma(4+2\vartheta)}\right)\xi^{3+2\vartheta}$$
$$+\left(\frac{160\Gamma(3)(2+\vartheta)}{\Gamma(2+2\vartheta)}+\frac{80\Gamma(3)}{\Gamma(2+2\vartheta)}\right)\xi^{2+2\vartheta}+\frac{120\Gamma(4)}{\Gamma(5+2\vartheta)}\xi^{4+2\vartheta}$$
$$+\left(\frac{480\Gamma(3)+20\Gamma(4)}{\Gamma(4+2\vartheta)}+\frac{40\Gamma(4)(3+\vartheta)}{\Gamma(4+2\vartheta)}\right)\xi^{3+2\vartheta}+\frac{480\Gamma(4)(4+\vartheta)}{\Gamma(6+3\vartheta)}\xi^{5+3\vartheta}$$
$$+\left(\frac{6720+840\Gamma(3)+80\Gamma(4)(4+\vartheta)}{\Gamma(5+3\vartheta)}+\frac{160\Gamma(4)(4+2\vartheta)}{\Gamma(5+3\vartheta)}\right)\xi^{4+3\vartheta}$$
$$+\left(\frac{540(2+\vartheta)+420\Gamma(3)+1120+140\Gamma(3)(3+\vartheta)}{\Gamma(3+3\vartheta)}+\frac{180(2+\vartheta)(2+2\vartheta)+140\Gamma(3)}{\Gamma(3+3\vartheta)}\right)$$
$$\times\xi^{2+3\vartheta}$$
$$+\left(\frac{45\Gamma(2)}{\Gamma(2+3\vartheta)}+\frac{90\Gamma(2)}{\Gamma(2+3\vartheta)}\right)\xi^{2+3\vartheta}. \tag{67}$$

The solution is obtained using the ADM by truncating the series up to the second iteration, that is

$$\varpi=\sum_{\ell=0}^{2}\varpi_{\ell}. \tag{68}$$

From (67)*, we observe that the series solution obtained using the ADM, involving the first three terms, is not convergent to the solution via Theorem 1. This implies* $|\mathcal{S}_0-\mathcal{S}_1|\nrightarrow 0$ *or* $|\mathcal{S}_1-\mathcal{S}_2|\nrightarrow 0,\cdots$ *or* $|\mathcal{S}_n-\mathcal{S}_m|\nrightarrow 0$*. Therefore,* $\{\mathcal{S}\}_{n=0}$ *is not a Cauchy sequence. Overall, the ADM, with the first three terms considered here, does not converge to the exact solution of problem* (50)*.*

The solution of* (50) *using MADM:

The iteration formulae of the MADM [31] from the equation (52) *are*

$$\varpi_0(\xi)=5\xi^2+15\xi, \tag{69}$$

$$\varpi_1(\xi)=\frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta}+\frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}+\frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}-{}^{RL}_{0+}J^{\vartheta}_{\xi}\left(6\int_0^{\xi}\varpi_0(s)ds+2\xi\varpi_0'(\xi)+\varpi_0(\xi)\right), \tag{70}$$

and

$$\varpi_{\ell+1}(\xi)=-{}^{RL}_{0+}J^{\vartheta}_{\xi}\left(6\int_0^{\xi}\varpi_{\ell}(s)ds+2\xi\varpi_{\ell}'(\xi)+\varpi_{\ell}(\xi)\right), \tag{71}$$

where $\ell=0,1,2,\cdots$.

Now, we find the values of the further iterations using ϖ_0 *from* (64) *and putting* $\ell=0,1,\cdots$

in (65),

$$
\begin{aligned}
\varpi_1(\xi) =&\frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta} + \frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}\\
&- {}_{0+}^{RL}J_\xi^\vartheta\left(6\int_0^\xi \varpi_0(\xi)d\xi + 2\xi\varpi_0'(\xi) + \varpi_0(\xi)\right),\\
=&\frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta} + \frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}\\
&- {}_{0+}^{RL}J_\xi^\vartheta\left(10\xi^3 + 45\xi^2 + 20\xi^2 + 30\xi + 5\xi^2 + 15\xi\right),\\
=&\frac{10\Gamma(4)}{\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{70\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta} + \frac{45\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta} - {}_{0+}^{RL}J_\xi^\vartheta\left(10\xi^3 + 70\xi^2 + 45\xi\right),\\
=&0,
\end{aligned} \tag{72}
$$

using the value ϖ_1, *we obtain*

$$
\varpi_{\ell+1}(\xi) = -{}_{0+}^{RL}J_\xi^\vartheta\left(6\int_0^\xi \varpi_\ell(s)ds + 2\xi\varpi_\ell'(\xi) + \varpi_\ell(\xi)\right) = 0, \tag{73}
$$

where $\ell = 0, 1, 2, \cdots$.

Finally, we have the solution of (50) *after truncating the series up to* m*th terms*

$$
\begin{aligned}
\varpi(\xi) &= \sum_{j=0}^{m} \varpi_j(\xi),\\
&= \varpi_0(\xi) + \varpi_1(\xi) + \cdots + \varpi_m(\xi),\\
&= 5\xi^2 + 15\xi.
\end{aligned} \tag{74}
$$

The solution obtained in (74), *using MADM, is the exact solution of the problem* (50). *Thus, the MADM provides exact solutions in two iterations.*

Hence, the proposed method (TSADM) is more efficient and applicable for obtaining the exact solution with fewer iterations than ADM and MADM.

Table 1. Comparison between TSADM, MADM and ADM for Example 1.

Remark	TSADM	MADM	ADM
Number of required iterations	One	Two	–
Solution	Exact solution	Exact solution	Does not converge to the solution from Theorem 1

Example 2. *Consider the following nonlinear complex fractional integro-differential equation,*

$$
\begin{aligned}
&{}_{0+}^{C}D_\xi^\vartheta\varpi(\xi) + e^\xi\int_0^\xi \varpi(s)ds + \sin(\xi)\varpi'(\xi) + (1-\xi)\varpi(\xi) = \Upsilon(\xi),\\
&0 < \xi \le \tau, \vartheta = u + iv, u \in (0,1], v \in \mathbb{R}, &(75)\\
&\varpi(0) = 1, &(76)
\end{aligned}
$$

$\Upsilon(\xi) = \frac{2\xi^{2-\vartheta}}{\Gamma(3-\vartheta)} - \frac{\xi^{1-\vartheta}}{\Gamma(2-\vartheta)} + e^{\xi}\left(\frac{1}{3}\xi^3 - \frac{1}{2}\xi^2 + \xi\right) + (2\xi - 1)\sin(\xi) + (\xi - 1)(\xi^2 - \xi + 1).$

The analytical solution of the example (75) *is* $\varpi(\xi) = \xi^2 - \xi + 1$.

Applying the TSADM algorithm, we apply the inverse operator ${}^{RL}_{0+}J^{\vartheta}_{\xi}$ *into equation* (75), *we obtain*

$$\varpi(\xi) = \varpi(0) + {}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\Upsilon(\xi) - \left(e^{\xi}\int_0^{\xi}\varpi(s)ds + \sin(\xi)\varpi'(\xi) + (1-\xi)\varpi(\xi)\right)\right). \tag{77}$$

The iteration formulae of the TSADM from the equation (77) *are*

$$\varpi_0(\xi) = 1 + {}^{RL}_{0+}J^{\vartheta}_{\xi}\Big(\Upsilon(\xi)\Big), \tag{78}$$

and

$$\varpi_{\ell+1}(\xi) = -{}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\left(e^{\xi}\int_0^{\xi}\varpi_{\ell}(s)ds + \sin(\xi)\varpi'_{\ell}(\xi) + (1-\xi)\varpi_{\ell}(\xi)\right)\right), \tag{79}$$

where $\ell = 0, 1, 2, \cdots$.

We approximate the variable coefficients involved in (85) *by first order polynomials for computational simplicity, that is,* $e^{\xi} \approx \xi$ *and* $\sin(\xi) \approx \xi$. *From* (78), *we obtain*

$$\begin{aligned}\varpi_0(\xi) =& 1 + {}^{RL}_{0+}J^{\vartheta}_{\xi}\left(\frac{2\xi^{2-\vartheta}}{\Gamma(3-\vartheta)} - \frac{\xi^{1-\vartheta}}{\Gamma(2-\vartheta)} + \xi \times \left(\frac{1}{3}\xi^3 - \frac{1}{2}\xi^2 + \xi\right)\right.\\ &\left. + \xi(2\xi - 1) + (\xi - 1)(\xi^2 - \xi + 1)\right),\\ =& 1 + \xi^2 - \xi + \frac{\Gamma(5)}{3\Gamma(5+\vartheta)}\xi^{4+\vartheta} - \frac{\Gamma(4)}{2\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}\\ &+ \frac{\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta} - \frac{1}{\Gamma(1+\vartheta)}\xi^{\vartheta}.\end{aligned} \tag{80}$$

From (80), *the initial term* ϖ_0 *of the TSADM can be expressed into three components as*

$$\varpi_0 = \Sigma_0 + \Sigma_1 + \Sigma_2, \tag{81}$$

where

$$\Sigma_0 = 1 + \xi^2 - \xi, \tag{82}$$

$$\Sigma_1 = \frac{\Gamma(5)}{3\Gamma(5+\vartheta)}\xi^{4+\vartheta} - \frac{\Gamma(4)}{2\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}, \tag{83}$$

and

$$\Sigma_2 = \frac{\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta} - \frac{1}{\Gamma(1+\vartheta)}\xi^{\vartheta}. \tag{84}$$

Here, we obtain an analytical solution of (75).

Following the TSADM process, we select the initial term shown in (82). *If the component satisfies* (75), *then we terminate the process, because the chosen component is the analytical solution of* (75).

Let $\varpi_0 = \Sigma_0$ *and verify whether assumption of* ϖ_0 *satisfies* (75). *If this choice of* ϖ_0 *is valid, then we confirm that the selected term is an analytical solution of* (75). *We substitute* $\varpi = \Sigma_0$ *in the left-hand side (LHS) of* (75). *Hence, we have*

$${}^{C}_{0+}D^{\vartheta}_{\xi}\Sigma_0 + e^{\xi}\int_0^{\xi}\Sigma_0(s)ds + \sin(\xi)\Sigma'_0(\xi) + (1-\xi)\Sigma_0. \tag{85}$$

Now, we solve the expression given in (75) *to obtain*

$$ {}_{0+}^{C}D_{\xi}^{\vartheta}\Sigma_0 = \frac{2\xi^{2-\vartheta}}{\Gamma(3-\vartheta)} - \frac{\xi^{1-\vartheta}}{\Gamma(2-\vartheta)}, \tag{86}$$

and

$$ e^{\xi}\int_0^{\xi}\Sigma_0(s)ds+\sin(\xi)\Sigma_0'(\xi)+(1-\xi)\Sigma_0 = e^{\xi}\left(\frac{1}{3}\xi^3-\frac{1}{2}\xi^2+\xi\right)+(2\xi-1)\sin(\xi)+(\xi-1)(\xi^2-\xi+1). \tag{87}$$

Adding equations (86) *and* (87)*, we obtain the LHS of* (75)*, that is,* Υ,

$$ {}_{0+}^{C}D_{\xi}^{\vartheta}\Sigma_0 + e^{\xi}\int_0^{\xi}\Sigma_0(s)ds + \sin(\xi).\Sigma_0' + (1-\xi)\Sigma_0 = \Upsilon(\xi). \tag{88}$$

From equation (88) *we observe that the LHS of* (75) *is equal to the RHS, that is* $\Upsilon(\xi)$*. Thus, the assumption* $\varpi_0 = \Sigma_0$ *satisfies equation* (75)*. Hence,* Σ_0 *is the analytical solution of* (75) *which is obtained by employing only one iteration of the TSADM.*

Now, we solve Example 2 by ADM and MADM. Further, we compare the solutions obtained by ADM, MADM, and TSADM.

Solution of the Example* (75) *by the ADM:

The iteration formulae of the ADM (see [26]) from the equation (52) *are*

$$\begin{aligned} \varpi_0(\xi) =& 1 + {}_{0+}^{RL}J_{\xi}^{\vartheta}\Big(\Upsilon(\xi)\Big), \\ =& 1 + \xi^2 - \xi + \frac{\Gamma(5)}{3\Gamma(5+\vartheta)}\xi^{4+\vartheta} - \frac{\Gamma(4)}{2\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta} \\ &+ \frac{\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta} - \frac{1}{\Gamma(1+\vartheta)}\xi^{\vartheta}, \end{aligned} \tag{89}$$

and

$$ \varpi_{\ell+1}(\xi) = -{}_{0+}^{RL}J_{\xi}^{\vartheta}\left(\left(e^{\xi}\int_0^{\xi}\varpi(s)ds + \sin(\xi)\varpi'(\xi) + (1-\xi)\varpi(\xi)\right)\right), \tag{90}$$

where $\ell = 0, 1, 2, \cdots$.

Now, we find the values of the further iterations using ϖ_0 *from* (64) *and putting* $\ell = 0, 1, \cdots$ *in* (65).

Putting $\ell = 0$*, we get*

$$\begin{aligned} \varpi_1 =& (1-\xi)\times\left(1 + \xi^2 - \xi + \frac{\Gamma(5)}{3\Gamma(5+\vartheta)}\xi^{4+\vartheta} - \frac{\Gamma(4)}{2\Gamma(4+\vartheta)}\xi^{3+\vartheta} + \frac{\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}\right. \\ &\left. + \frac{\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta} - \frac{1}{\Gamma(1+\vartheta)}\xi^{\vartheta}\right) + (-1 + 2\xi - (\xi^{-1+\vartheta}\vartheta)\Gamma(1+\vartheta) + (\xi^{\vartheta}(1+\vartheta))\Gamma(2+\vartheta) \\ &+ (2\xi^{1+\vartheta}(2+\vartheta))\Gamma(3+\vartheta) - (3\xi^{2+\vartheta}(3+\vartheta))\Gamma(4+\vartheta) \\ &+ (8\xi^{3+\vartheta}(4+\vartheta))\Gamma(5+\vartheta)\sin(\xi) + \frac{e^{\xi}\xi}{6\Gamma(6+\vartheta)}\left(-720\frac{1+\vartheta}{1+2\vartheta}\xi^{2\vartheta} + 360\frac{2+\vartheta}{2+2\vartheta}\xi^{1+2\vartheta}\right. \\ &+ 240\frac{3+\vartheta}{3+2\vartheta}\xi^{2+2\vartheta} - 90\frac{4+\vartheta}{4+2\vartheta}\xi^{3+2\vartheta} + 48\frac{5+\vartheta}{5+2\vartheta}\xi^{4+2\vartheta} - 924\frac{1+\vartheta}{1+2\vartheta}\xi^{2\vartheta}\vartheta \\ &+ 282\frac{2+\vartheta}{2+2\vartheta}\xi^{1+2\vartheta}\vartheta + 108\frac{3+\vartheta}{3+2\vartheta}\xi^{2+2\vartheta}\vartheta - 18\frac{4+\vartheta}{4+2\vartheta}\xi^(3+2\vartheta)\vartheta - 426\frac{1+\vartheta}{1+2\vartheta}\xi^{2\vartheta}\vartheta^2 \end{aligned}$$

$$
\begin{aligned}
&+72\frac{2+\vartheta}{2+2\vartheta}\xi^{1+2\vartheta}\vartheta^2+12\frac{3+\vartheta}{3+2\vartheta}\xi^{2+2\vartheta}\vartheta^2-84\frac{1+\vartheta}{1+2\vartheta}\xi^{2\vartheta}\vartheta^3+6\frac{2+\vartheta}{2+2\vartheta}\xi^{1+2\vartheta}\vartheta^3\\
&-6\frac{1+\vartheta}{1+2\vartheta}\xi^{2\vartheta}\vartheta^4+\frac{6\Gamma(6+\vartheta)}{1+\vartheta}\xi^{\vartheta}-\frac{3\Gamma[6+\vartheta]}{1+\vartheta}\xi^{\vartheta+1}+\frac{6\Gamma(6+\vartheta)}{3+\vartheta}\xi^{2+\vartheta}\Big).
\end{aligned}
\tag{91}
$$

The solution is obtained using the ADM by truncating the series up to the second iteration, that is

$$\varpi=\sum_{\ell=0}^{1}\varpi_\ell. \tag{92}$$

From (91)*, we observe that the series solution obtained using the ADM involving the first three terms is not convergent to the solution by Theorem 1. This implies* $|\mathcal{S}_0-\mathcal{S}_1|\nrightarrow 0$ *or* $|\mathcal{S}_1-\mathcal{S}_2|\nrightarrow 0\cdots$ *or* $|\mathcal{S}_n-\mathcal{S}_m|\nrightarrow 0$*. Therefore,* $\{\mathcal{S}\}_{n=0}$ *is not a Cauchy sequence. Overall, the ADM does not provide the solution and does not converge to the solution of* (50)*.*

The solution of the example (50) ***using MADM:***

The iteration formulae of the MADM [10] from equation (52) *are*

$$\varpi_0(\xi)=1+\xi^2-\xi, \tag{93}$$

$$
\begin{aligned}
\varpi_1(\xi)=&\frac{\Gamma(5)}{3\Gamma(5+\vartheta)}\xi^{4+\vartheta}-\frac{\Gamma(4)}{2\Gamma(4+\vartheta)}\xi^{3+\vartheta}+\frac{\Gamma(3)}{\Gamma(3+\vartheta)}\xi^{2+\vartheta}+\frac{\Gamma(2)}{\Gamma(2+\vartheta)}\xi^{1+\vartheta}-\frac{1}{\Gamma(1+\vartheta)}\xi^{\vartheta}\\
&-{}^{RL}_{0+}J^{\vartheta}_{\xi}\bigg(\bigg(e^{\xi}\int_0^{\xi}\varpi(s)ds+\sin(\xi)\varpi'(\xi)+(1-\xi)\varpi(\xi)\bigg)\bigg),
\end{aligned}
\tag{94}
$$

and

$$\varpi_{\ell+1}(\xi)=-{}^{RL}_{0+}J^{\vartheta}_{\xi}\bigg(\bigg(e^{\xi}\int_0^{\xi}\varpi(s)ds+\sin(\xi)\varpi'(\xi)+(1-\xi)\varpi(\xi)\bigg)\bigg), \tag{95}$$

where $\ell=0,1,2,\cdots$.

The MADM is not convergent to the solution of the problem (75)*. The MADM is inefficient to provide the solution for* (75) *using Theorem 1.*

Clearly, we can observe that the proposed method TSADM is more efficient and applicable to give the exact solution with fewer iterations as compared to ADM and MADM.

Table 2. Comparison between TSADM, MADM, and ADM for Example 2.

Remark	TSADM	MADM	ADM
Number of iterations required	One	−	−
Solution	Exact solution	Not convergent to solution by Theorem 1	Not convergent to solution by Theorem 1

Remark: Fractional calculus extends traditional calculus to model systems with memory and hereditary effects, presenting unique challenges, particularly in solving nonlinear integro-differential equations (IDEs) with complex fractional orders. Advanced methods, such as the two-step Adomian decomposition method, are employed to address these challenges. This method aims to provide accurate and efficient solutions to such complex equations. Finding exact solutions is an excellent way to see how accurate the numerical method is because we can directly compare the results from the two-step Adomian decomposition method with exact solutions. Demonstrating how well the two-step Adomian decomposition method performs compared to exact solutions highlights its strengths, including accuracy, convergence, and computational efficiency. It validates the method's accuracy and showcases its advantages in practical applications, providing a comprehensive assessment of its effectiveness in solving complex fractional nonlinear integro-differential equations.

§8 Conclusion

In this study, we introduce the analytical approach to solve integro-differential equations for complex fractional orders without using approximations and discretization. Also, the proposed method do not use the Adomian polynomial. The method possesses straightforward steps and is easy to understand. In general, we always try to find such methods that solve our difficult problems without any complexity and provide accurate results. The TSADM is very efficient and fulfills our desire. The contribution of this study can be summarized in the following points:

(1) We introduce the CF-NL-IDEs with the initial condition using the complex fractional Caputo derivative.

(2) An accurate solution for the CF-NL-IDEs is obtained using the Caputo derivative, in one iteration.

(3) There is no need to calculate the computation time and cost of the procedure.

(4) The algorithm of the proposed method is very simple and easy to implement on the problem considered.

(5) Using fixed point theorems, we establish new results for the existence and uniqueness conditions for the solution of the considered problem, under some assumptions.

(6) We discuss the stability analysis of the proposed problem via Ulam-Hyers type stabilities.

(7) Besides, we solve examples with the help of the most valuable numerical methods, such as ADM and MADM, and provide the analysis of the solutions obtained via these methods. We also discuss which numerical method is most suitable for the CF-NL-IDEs.

(8) The commitment of the solution is proven by using the convergence of the method. Also, the maximum error analysis is provided for the method to facilitate the decision on which step the solution yields a good result.

Future work: This study is also helpful for the considered problem with variable fractional order, which is considered in [4], where the authors used a numerical method and found the approximate solution for variable fractional order (VFO) IDEs. They convert the VFO-IDEs into a system of algebraic equations, and it is obvious that such a type of conversion is not always possible or easy. If we use numerical methods, the solution obtained always comes with an error due to approximations used in the existing numerical methods. Thus, the solution becomes the approximate solution to the problem. Here, our adopted method, known as the TSADM, is superior in these cases because it provides the analytical solution without converting the problems into a system of differential equations, and the solution is obtained in just one iteration. For more details, see the references [29,30]. Moreover, the problem considered is onedimensional. Suppose we extend it to more than two dimensions. Then our adopted method can be easily applied to the extended problems. It is also fascinating and good news to solve these kinds of problems and obtain an analytical solution.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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