Appl. Math. J. Chinese Univ.
2026, 41(1): 430-454

Optimal distributed control of Cahn-Hilliard-Brinkman system in three dimension

XIAO Xiang-yu   PU Zhi-lin

Abstract. In this paper, we consider the distributed optimal control of the three-dimensional Cahn-Hilliard-Brinkman system with a more general potential. We obtain the regularity results for the weak solution which are essential for an associated optimal control problem. We then show that the control-to-state operator is Fréchet differentiable and we derive the first-order necessary optimality conditions in terms of a variational inequality involving the adjoint state variables.

§1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$ and let Ω_T denote $\Omega \times [0, T]$. We consider the Cahn-Hilliard-Brinkman system as follows:

$$\begin{cases} \frac{\partial\varphi(x,t)}{\partial t} + \nabla\cdot(\mathbf{u}\varphi)(x,t) = \nabla\cdot(M\nabla\mu)(x,t), & (x,t)\in\Omega_T, \\ \mu(x,t) = -\varepsilon\Delta\varphi(x,t) + \frac{1}{\varepsilon}f(\varphi(x,t)), & (x,t)\in\Omega_T, \\ -\nu\Delta\mathbf{u}(x,t) + \eta\mathbf{u}(x,t) = -\nabla p(x,t) - \gamma\varphi\nabla\mu(x,t) + \mathbf{v}(x,t), & (x,t)\in\Omega_T, \\ \nabla\cdot\mathbf{u}(x,t) = 0, & (x,t)\in\Omega_T, \\ \mathbf{u}(x,t) = 0,\ \frac{\partial\varphi(x,t)}{\partial\mathbf{n}} = \frac{\partial\mu(x,t)}{\partial\mathbf{n}} = 0, & (x,t)\in\partial\Omega_T, \\ \varphi(x,0) = \varphi_0(x), & x\in\Omega. \end{cases} \tag{1}$$

$M > 0$ represents the mobility; $\varepsilon > 0$ is associated with the diffuse interface thickness; $\nu > 0$ denotes the viscosity; $\eta > 0$ denotes the fluid permeability; and $\gamma > 0$ is the surface tension parameter. $p(x,t)$ represents the fluid pressure; $\varphi(x,t)$ indicates the difference in relative concentrations between the two phases; $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the fluid velocity; and $\mu(x,t)$ refers to chemical potential. $\varphi_0(x)$ provides the initial data; $\partial\mathbf{n}$ represents the derivative in the direction of the outward unit normal $\mathbf{n}$; and $\mathbf{v}(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ acts as the distributed control. For simplicity, we assume $M = \varepsilon = \gamma = \nu = \eta = 1$.

Received: 2024-03-22.   Revised: 2025-06-13.
MR Subject Classification: 49J20, 49J50, 49K20, 92C50.
Keywords: distributed optimal control, more general potential, first-order necessary optimality conditions, the adjoint state variables.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-026-5185-5.
Supported by the Scientific Research Fund of the Science and Technology Department of Sichuan Province (22CXTD0029).

System (1) comprises a convective Cahn-Hilliard equation coupled with the Brinkman equation via the surface tension force $\gamma\varphi\nabla\mu$. This diffuse interface model is closely related to the behavior of multi-phase fluids [1]. We refer to a series of papers where well-posedness and regularity have been established [2, 3]. Additionally, the long-time behavior and global attractor have been analyzed in [4, 5]. Furthermore, Xiao and Pu [6] demonstrated the existence of the global attractors in $H^s(\Omega)$ $(s = 1, 2, 3, 4)$ for Cahn-Hilliard-Brinkman system with a more general potential.

It is well known that control problems are essential in many technical applications. The optimal control problem for this model with a regular potential was studied in [7,8]. The authors in [9,10,11,12] investigated optimal control for other similar models of the local Cahn-Hilliard-Brinkman system and the nonlocal Cahn-Hilliard-Brinkman system can be found in [13,14,15]. For further theoretical insights, a series of papers are available [16,17,18,19,20,21, 22].

Generally speaking, one uses $H^k(\Omega)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k(\Omega)}$ and $L^p(\Omega)$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p(\Omega)}$. We also use $\mathbf{L}^2(\Omega)$ to denote $\left(L^2(\Omega)\right)^3$ with the norms $\|\mathbf{u}\|_{L^2(\Omega)} = \|(u_1, u_2, u_3)\|_{L^2(\Omega)} = \left(\sum\limits_{i=1}^{3} \|u_i\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ and use $\mathbf{H}^1(\Omega)$ to denote $\left(H^1(\Omega)\right)^3$ with the norms $\|\mathbf{u}\|_{H^1(\Omega)} = \|(u_1, u_2, u_3)\|_{H^1(\Omega)} = \left(\sum\limits_{i=1}^{3} \|u_i\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}$.

We recall that

$$\mathbf{H} = \overline{\left\{\mathbf{u} \in (C_0^\infty(\Omega))^3 : \nabla\cdot\mathbf{u} = 0\right\}}^{\mathbf{L^2}(\Omega)}$$

and

$$\mathbf{V} = \overline{\left\{\mathbf{u} \in (C_0^\infty(\Omega))^3 : \nabla\cdot\mathbf{u} = 0\right\}}^{\mathbf{H^1}(\Omega)}$$

are the classical Hilbert spaces for the incompressible Navier-Stokes equations with no-slip boundary conditions [23] equipped with the norms in $\mathbf{L^2}(\Omega)$ and $\mathbf{H^1}(\Omega)$.

In this paper, we primarily consider the following control problem.

(**CP**) Minimize the cost functional[24]

$$\begin{aligned} J(\varphi, \mathbf{u}, \mu, \mathbf{v}) = &\frac{\lambda_0}{2}\int_0^T\int_\Omega |\varphi(x,t) - \varphi_{\Omega_T}(x,t)|^2 dxdt + \frac{\lambda_1}{2}\int_\Omega |\varphi(x,T) - \varphi_\Omega(x)|^2 dx \\ &+ \frac{\lambda_2}{2}\int_0^T\int_\Omega |\mathbf{u}(x,t) - \mathbf{u}_{\Omega_T}(x,t)|^2 dxdt + \frac{\lambda_3}{2}\int_0^T\int_\Omega |\mathbf{v}(x,t)|^2 dxdt \end{aligned} \tag{2}$$

subjecting to the control constraint

$$\mathbf{v} \in U_{ad} = \left\{\mathbf{v} \in L^2(0,T;\mathbf{L}^2(\Omega)) : \mathbf{v}_{\min} \le \mathbf{v} \le \mathbf{v}_{\max},\ a.e.(x,t) \in \Omega_T\right\}, \tag{3}$$

where $\varphi, \mathbf{u}$ and μ respectively represent the concentration difference, fluid velocity and chemical potential in system (1).

To analyze the state system (1)-(3), we assume the following conditions.

(H_1) $\lambda_0,\ \lambda_1,\ \lambda_2$ and λ_3 are nonnegative but not all zero.

(H_2) U is a nonempty bounded open subset of $L^2(0,T;\mathbf{H})$ containing U_{ad} and there exists a constant $R > 0$ satisfying

$$\|\mathbf{v}\|_{L^2(0,T;\mathbf{H})} \le R,\ \forall \mathbf{v} \in U.$$

(H_3) $\varphi_{\Omega_T} \in L^2(\Omega_T),\ \varphi_\Omega \in L^2(\Omega),\ \mathbf{u}_{\Omega_T} \in L^2(0,T;\mathbf{L}^2(\Omega)),\ \mathbf{v}_{\min},\ \mathbf{v}_{\max} \in L^\infty(\Omega_T)$ with $\mathbf{v}_{\min} \le \mathbf{v}_{\max}$ for almost everywhere in Ω_T.

Here, suppose f satisfy the following hypothesis, which will be assumed throughout this paper, $f \in C^3(\mathbb{R})$, $F(t) = \int f(t)dt$, and

$$|f(t)| \leq c(1+|t|^3), \tag{4}$$

$$|f'(s) - f'(t)| \leq c\,|s-t|\,(1+|s|+|t|), \tag{5}$$

$$|f''(t)| \leq c(1+|t|), \tag{6}$$

$$F(t) \geq -c, c \geq 0. \tag{7}$$

We observe that $f(t) = t^3 - t$ satisfies the above conditions.

In this paper, we concentrate on the optimal control problem for the Cahn-Hilliard-Brinkman system. Indeed, solving an optimal control problem necessitates a higher degree of solution regularity. Leveraging the well-posedness results established in [6], we extend the work of [7] to consider a more general potential. The structure of this paper is outlined as follows. In Section 2, we derive the higher regularity. In Section 3, we prove the existence of a solution for the optimal control problem (**CP**), then we get the Fréchet differentiability of the control-to-state operator and the fist-order necessary optimality conditions for (**CP**).

§2 The regularity of weak solutions

Definition 2.1. ([6]) Let $T > 0$, $\mathbf{v} \in L^2(0,T;\mathbf{H})$ and $\varphi_0 \in H^1(\Omega)$ be given. A triple $(\varphi, \mathbf{u}, \mu)$ is called a weak solution to the state system (1) on $[0,T]$, if

$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)),$$
$$\varphi_t \in L^{\frac{8}{5}}(0,T;(H^1(\Omega))^*),$$
$$\mu \in L^2(0,T;H^1(\Omega)),$$
$$\mathbf{u} \in L^2(0,T;\mathbf{V}),$$

and it satisfies

$$\begin{cases} (\varphi_t, \psi) + (\nabla\mu, \nabla\psi) = (\mathbf{u}\varphi, \nabla\psi), \quad \forall\psi \in H^1(\Omega), \\ (\nabla\mathbf{u}, \nabla w) + (\mathbf{u}, w) = (\mu\nabla\varphi, w) + (\mathbf{v}, w), \quad \forall w \in \mathbf{V}, \\ \varphi(x,0) = \varphi_0(x), \quad x \in \Omega. \end{cases}$$

In the following, we will prove the continuous dependence of weak solution for the state system (1) with respect to the control term $\mathbf{v}$.

Theorem 2.1. *Assume $\varphi_0 \in H^1(\Omega)$ be given. Then for any $\mathbf{v} \in U$, there exists a triple $(\varphi, \mathbf{u}, \mu)$ which is a unique weak solution to the state system (1) on $[0,T]$. If $(\varphi_1, \mathbf{u}_1, \mu_1)$ and $(\varphi_2, \mathbf{u}_2, \mu_2)$ are two weak solutions to the Cahn-Hilliard-Brinkman system for any $\mathbf{v_1}, \mathbf{v_2} \in U$, there exists a constant $C_1 > 0$ depending only on T, R and the initial data of the system for every $T > 0$ such that the following continuous dependence estimate holds*

$$\begin{aligned} &\max_{t\in[0,T]} \|\nabla\varphi_1(t) - \nabla\varphi_2(t)\|_{L^2(\Omega)} + \|\nabla\mu_1 - \nabla\mu_2\|_{L^2(0,T;L^2(\Omega))} + \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(0,T;\mathbf{V})} \\ &+\|\varphi_1 - \varphi_2\|_{L^2(0,T;H^3(\Omega))} + \|\partial_t\varphi_1 - \partial_t\varphi_2\|_{L^2(0,T;(H^1(\Omega))^*)} \\ &\leq C_1\|\mathbf{v_1} - \mathbf{v_2}\|_{L^2(0,T;\mathbf{H})}. \end{aligned}$$

Proof. As said in [6], the well-posedness of weak solution for the state system (1) can be established using the Faedo-Galerkin method. In the next step, we will provide some estimates of weak solution for the state system.

By multiplying the first equation of (1) by 1 and then integrating the resulting equation, we obtain the following result

$$\int_\Omega \varphi(t)dx = \int_\Omega \varphi_0 dx.$$

Multiplying the first equation and the third equation of (1) by μ and $\mathbf{u}$, respectively, and integrating the resulting equations over the domain Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi)\right)dx\right) + \|\nabla\mu\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{H^1(\Omega)}\\ &= -\int_\Omega \nabla\cdot(\mathbf{u}\varphi)\mu dx - \int_\Omega (\mathbf{u}\varphi)\cdot\nabla\mu dx + \int_\Omega \mathbf{v}\cdot\mathbf{u}dx\\ &= \int_\Omega \mathbf{v}\cdot\mathbf{u}dx.\end{aligned}$$

Subsequently, by applying Hölder's inequality and Young's inequality, we derive the following result

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi)\right)dx\right) + \|\nabla\mu\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{H^1(\Omega)}\\ &\le \frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)},\end{aligned}$$

which entails that

$$\frac{d}{dt}\left(\int_\Omega \left(|\nabla\varphi|^2 + 2F(\varphi)\right)dx\right) + 2\|\nabla\mu\|^2_{L^2(\Omega)} + 2\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{L^2(\Omega)} \le \|\mathbf{v}\|^2_{L^2(\Omega)}. \quad (8)$$

For any $t\in[0,T]$, integrating the inequality (8) from 0 to t, we get

$$\begin{aligned}&\|\nabla\varphi(t)\|^2_{L^2(\Omega)} + 2\int_\Omega F(\varphi(x,t))dx + \int_0^t (2\|\nabla\mu(s)\|^2_{L^2(\Omega)} + 2\|\nabla\mathbf{u}(s)\|^2_{L^2(\Omega)} + \|\mathbf{u}(s)\|^2_{L^2(\Omega)})ds\\ &\le \int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)}ds + \|\nabla\varphi_0\|^2_{L^2(\Omega)} + 2\int_\Omega F(\varphi_0(x))dx.\end{aligned} \quad (9)$$

From equation (4), we derive

$$\begin{aligned}\left|\int_\Omega \mu(x)dx\right| &= \left|-\int_\Omega \Delta\varphi(x)dx + \int_\Omega f(\varphi(x))dx\right|\\ &\le c\int_\Omega (1+|\varphi(x)|^3)dx \le c(1+\|\varphi\|^3_{H^1(\Omega)}) \le c(R),\end{aligned}$$

such that

$$\begin{aligned}\|\mu(t)\|_{H^1(\Omega)} &\le \|\nabla\mu(t)\|_{L^2(\Omega)} + \left\|\mu(t) - \overline{\mu(t)}\right\|_{L^2(\Omega)} + \left|\overline{\mu(t)}\right||\Omega|^{\frac{1}{2}}\\ &\le c(1+\|\nabla\mu(t)\|_{L^2(\Omega)} + \|\varphi\|^3_{H^1(\Omega)}),\end{aligned}$$

where $\overline{\mu(t)} = \frac{1}{|\Omega|}\int_\Omega \mu(x)dx$. Furthermore, to determine the norms of μ and φ_t, we utilize equation (9) and the following result

$$\begin{aligned}&\|\varphi_t\|_{(H^1(\Omega))^*}\\ &\le \|\nabla\mu\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\\ &\le \|\nabla\mu\|_{L^2(\Omega)} + c\|\mathbf{u}\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)}\end{aligned}$$

to derive

$$
\begin{aligned}
&\int_0^t (\|\mu(s)\|_{H^1(\Omega)}^2 + \|\varphi_t\|_{(H^1(\Omega))^*}^2) ds \\
&\leq c\int_0^T \Big(1 + \|\nabla\mu(s)\|_{L^2(\Omega)}^2 + \|\varphi(s)\|_{H^1}^6 + \|\mathbf{u}(s)\|_{H^1(\Omega)}^2 \|\varphi(s)\|_{H^1(\Omega)}^2\Big) ds \\
&\leq c(R,T).
\end{aligned} \tag{10}
$$

Applying the estimate

$$
\begin{aligned}
&\|\nabla\Delta\varphi\|_{L^2(\Omega)} \\
&\leq \|\nabla\mu\|_{L^2(\Omega)} + \|f'(\varphi)\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq \|\nabla\mu\|_{L^2(\Omega)} + c(1+\|\varphi\|_{L^\infty(\Omega)}^2)\|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq \|\nabla\mu\|_{L^2(\Omega)} + c\|\nabla\varphi\|_{L^2(\Omega)} + c\,\|\nabla\varphi\|_{L^2(\Omega)}^{\frac{3}{2}}\,\|\nabla\Delta\varphi\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\leq \|\nabla\mu\|_{L^2(\Omega)} + c\|\nabla\varphi\|_{L^2(\Omega)} + \frac{1}{2}\|\nabla\Delta\varphi\|_{L^2(\Omega)} + c\,\|\nabla\varphi\|_{L^2(\Omega)}^3,
\end{aligned}
$$

we derive

$$
\int_0^T \|\varphi(t)\|_{H^3(\Omega)}^2 dt \leq c(R,T). \tag{11}
$$

Moreover, we employ interpolation inequality to derive

$$
\int_0^T \|\Delta\varphi(t)\|_{L^2(\Omega)}^4 dt \leq \int_0^T \|\nabla\varphi(t)\|_{L^2(\Omega)}^2 \|\nabla\Delta\varphi(t)\|_{L^2(\Omega)}^2\, dt \leq c(R,T).
$$

Suppose $(\varphi_1, \mathbf{u}_1, \mu_1)$ and $(\varphi_2, \mathbf{u}_2, \mu_2)$ be two weak solutions for the system with control term $(\mathbf{v}_1, \mathbf{v}_2)$, respectively. Denote by

$$
\varphi = \varphi_1 - \varphi_2, \mu = \mu_1 - \mu_2,
$$

and

$$
\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2.
$$

Then $(\varphi, \mathbf{u}, \mu)$ satisfies

$$
\begin{cases}
\varphi_t(x,t) = \Delta\mu(x,t) - \nabla\cdot(\mathbf{u}\varphi_1)(x,t) - \nabla\cdot(\mathbf{u}_2\varphi)(x,t), & (x,t)\in\Omega_T, \\
\mu(x,t) = -\Delta\varphi(x,t) + f(\varphi_1)(x,t) - f(\varphi_2)(x,t), & (x,t)\in\Omega_T, \\
-\Delta\mathbf{u}(x,t) + \mathbf{u}(x,t) = -\varphi_1\nabla\mu(x,t) - \varphi\nabla\mu_2(x,t) + \mathbf{v}(x,t), & (x,t)\in\Omega_T, \\
\mathbf{u}(x,t) = 0,\ \frac{\partial\varphi(x,t)}{\partial\mathbf{n}} = \frac{\partial\mu(x,t)}{\partial\mathbf{n}} = 0, & (x,t)\in\partial\Omega_T, \\
\varphi(x,0) = 0, & x\in\Omega.
\end{cases} \tag{12}
$$

Multiplying the first equation and the third equation of (12) by $-\Delta\varphi$ and $\mathbf{u}$, respectively, and integrating the resulting equation over Ω, we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\mathbf{u}\|_{H^1(\Omega)}^2 \\
&= -\int_\Omega (\Delta\mu - \nabla\cdot(\mathbf{u}\varphi_1) - \nabla\cdot(\mathbf{u}_2\varphi))\Delta\varphi + \int_\Omega \mathbf{u}\cdot\nabla\varphi\mu_2 dx \\
&\quad + \int_\Omega \mathbf{u}\cdot\nabla\varphi_1(-\Delta\varphi + f(\varphi_1) - f(\varphi_2)) + \mathbf{v}\cdot\mathbf{u} dx \\
&= -\int_\Omega \Delta\mu\Delta\varphi dx + \mathbf{u}_2\cdot\nabla\varphi\Delta\varphi dx + \int_\Omega \mathbf{u}\cdot\nabla\varphi\mu_2 dx
\end{aligned}
$$

$$+\int_\Omega \mathbf{u}\cdot\nabla\varphi_1(f(\varphi_1)-f(\varphi_2))+\mathbf{v}\cdot\mathbf{u}dx. \tag{13}$$

Next, we estimate the terms on the right-hand side of (13)

$$-\int_\Omega \Delta\mu\Delta\varphi dx \le -\|\nabla\Delta\varphi\|^2_{L^2(\Omega)}+\|\nabla(f(\varphi_1)-f(\varphi_2))\|_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)},$$

and

$$\begin{aligned}
&\int_\Omega \mathbf{u}_2\cdot\nabla\varphi\Delta\varphi dx+\int_\Omega \mathbf{u}\cdot\nabla\varphi\mu_2 dx+\int_\Omega \mathbf{u}\cdot\nabla\varphi_1(f(\varphi_1)-f(\varphi_2))dx\\
\le&\|\mathbf{u}_2\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}+\|\mathbf{u}\|_{L^3(\Omega)}\|\nabla\mu_2\|_{L^2(\Omega)}\|\varphi\|_{L^6(\Omega)}\\
&+\|\mathbf{u}\|_{L^3(\Omega)}\|\varphi_1\|_{L^6(\Omega)}\|\nabla(f(\varphi_1)-f(\varphi_2))\|_{L^2(\Omega)}\\
\le& c\|\mathbf{u}_2\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}+\|\mathbf{u}\|_{H^1(\Omega)}\|\nabla\mu_2\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\\
&+\|\mathbf{u}\|_{H^1(\Omega)}\|\varphi_1\|_{H^1(\Omega)}\|\nabla(f(\varphi_1)-f(\varphi_2))\|_{L^2(\Omega)}.
\end{aligned}$$

Furthermore, we apply (5), (6) and (9) to derive

$$\begin{aligned}
&\|\nabla(f(\varphi_1)-f(\varphi_2))\|_{L^2(\Omega)}\\
=&\|f'(\varphi_1)\nabla\varphi+(f'(\varphi_1)-f'(\varphi_2))\nabla\varphi_2\|_{L^2(\Omega)}\\
\le& c\|f'(\varphi_1)\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}+c\|\varphi\|_{L^6(\Omega)}\left(1+\|\varphi_1\|_{L^6(\Omega)}+\|\varphi_2\|_{L^6(\Omega)}\right)\|\nabla\varphi_2\|_{L^6(\Omega)}\\
\le& c(1+\|\varphi_1\|^2_{L^\infty(\Omega)})\|\nabla\varphi\|_{L^2(\Omega)}\\
&+c\|\nabla\varphi\|_{L^2(\Omega)}\left(1+\|\varphi_1\|_{L^6(\Omega)}+\|\varphi_2\|_{L^6(\Omega)}\right)\|\varphi_2\|_{H^2(\Omega)}\\
\le& c\|\nabla\varphi\|_{L^2(\Omega)}(1+\|\Delta\varphi_1\|+\|\varphi_2\|_{H^2(\Omega)}).
\end{aligned}$$

Therefore, we can transform (13) into the following form

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\varphi\|^2_{L^2(\Omega)}+\|\mathbf{u}\|^2_{H^1(\Omega)}+\|\nabla\Delta\varphi\|^2_{L^2(\Omega)}\\
=&-\int_\Omega \Delta\mu\Delta\varphi dx+\mathbf{u}_2\cdot\nabla\varphi\Delta\varphi dx+\int_\Omega \mathbf{u}\cdot\nabla\varphi\mu_2 dx\\
&+\int_\Omega \mathbf{u}\cdot\nabla\varphi_1(f(\varphi_1)-f(\varphi_2))+\mathbf{v}\cdot\mathbf{u}dx\\
\le& c\|\nabla\varphi\|^2_{L^2(\Omega)}(1+\|\Delta\varphi_1\|^2_{L^2(\Omega)}+\|\varphi_2\|^2_{H^2(\Omega)}+\|\mathbf{u}_2\|^2_{H^1(\Omega)}+\|\nabla\mu_2\|^2_{L^2(\Omega)})+\|\mathbf{v}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Integrating the result on $[0,t]$, we derive

$$\begin{aligned}
&\|\nabla\varphi(t)\|^2_{L^2(\Omega)}+\int_0^t(\|\nabla\Delta\varphi(s)\|^2_{L^2(\Omega)}+\|\mathbf{u}(s)\|^2_{H^1(\Omega)})ds\\
\le&(\int_0^t\|\mathbf{v}(s)\|^2_{L^2(\Omega)}ds)e^{\int_0^T(1+\|\Delta\varphi_1(s)\|^2_{L^2(\Omega)}+\|\varphi_2(s)\|^2_{H^2(\Omega)}+\|\mathbf{u}_2(s)\|^2_{H^1(\Omega)}+\|\nabla\mu_2(s)\|^2_{L^2(\Omega)})ds} \qquad (14)\\
\le&\frac{C_1}{3}\int_0^t\|\mathbf{v}(s)\|^2_{L^2(\Omega)}ds.
\end{aligned}$$

Furthermore, we get estimates for $\|\nabla\mu\|^2_{L^2(\Omega)}$ and $\|\varphi_t\|^2_{(H^1(\Omega))^*}$

$$\int_0^t\|\nabla\mu(s)\|^2_{L^2(\Omega)}ds\le\int_0^t\|\nabla\Delta\varphi(s)\|^2_{L^2(\Omega)}ds+\int_0^t\|\nabla(f(\varphi_1(s))-f(\varphi_2(s)))\|^2_{L^2(\Omega)}ds$$

$$\leq \frac{C_1}{3}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)}ds,$$

and

$$\int_0^t \|\varphi_t(s)\|^2_{(H^1(\Omega))^*}ds$$
$$\leq \int_0^t \|\nabla\mu(s)\|^2_{L^2(\Omega)}ds + \int_0^t \|\mathbf{u}(s)\|^2_{L^6(\Omega)}\|\varphi_1(s)\|^2_{L^3(\Omega)}ds + \int_0^t \|\mathbf{u}_2(s)\|^2_{L^6(\Omega)}\|\varphi(s)\|^2_{L^3(\Omega)}ds$$
$$\leq \frac{C_1}{3}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)}ds.$$

□

§3 An optimal control problem

In the following, we will prove the existence of an optimal control by the monotonicity arguments and compactness theorem.

Theorem 3.1. *Suppose that $(H_1)-(H_3)$ are fulfilled. Then the optimal control problem (**CP**) admits a solution.*

Proof. Let $\{\mathbf{v}_n\}_{n=1}^\infty \subset U$ be a minimizing sequence for (**CP**) and let $T(\mathbf{v}_n) = (\varphi_n, \mathbf{u}_n, \mu_n)$ for each $n \in N$. Owing to the inequalities (9), (10), (11) and the assumption H_2, we get

$$\{\mathbf{v}_n\}_{n=1}^\infty \textit{ is uniformly bounded in } L^2(0,T;\mathbf{H}),$$
$$\{\varphi_n\}_{n=1}^\infty \textit{ is uniformly bounded in } L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)),$$
$$\{\mathbf{u}_n\}_{n=1}^\infty \textit{ is uniformly bounded in } L^2(0,T;\mathbf{V}),$$
$$\{\mu_n\}_{n=1}^\infty \textit{ is uniformly bounded in } L^2(0,T;H^1(\Omega)),$$
$$\{\partial_t\varphi_n\}_{n=1}^\infty \textit{ is uniformly bounded in } L^2(0,T;(H^1(\Omega))^*).$$

Thus, there exist

$$\mathbf{v} \in L^2(0,T;\mathbf{H}),$$
$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)),$$
$$\mathbf{u} \in L^2(0,T;\mathbf{V}),$$
$$\mu_1 \in L^2(0,T;H^1(\Omega)),$$
$$\varphi_t \in L^2(0,T;(H^1(\Omega))^*),$$

and we can extract subsequences $\{\mathbf{v}_{n_j}\}_{j=1}^\infty, \{\varphi_{n_j}\}_{j=1}^\infty, \{\mathbf{u}_{n_j}\}_{j=1}^\infty, \{\mu_{n_j}\}_{j=1}^\infty, \{\partial_t\varphi_{n_j}\}_{j=1}^\infty$ of $\{\mathbf{v}_n\}_{n=1}^\infty$, $\{\varphi_n\}_{n=1}^\infty$, $\{\mathbf{u}_n\}_{n=1}^\infty$, $\{\mu_n\}_{n=1}^\infty$, $\{\partial_t\varphi_n\}_{n=1}^\infty$, such that

$$\mathbf{v}_{n_j} \rightharpoonup \mathbf{v} \textit{ weakly in } L^2(0,T;\mathbf{H}),$$
$$\varphi_{n_j} \rightharpoonup \varphi \textit{ weakly star in } L^\infty(0,T;H^1(\Omega)),$$
$$\varphi_{n_j} \rightharpoonup \varphi \textit{ weakly in } L^2(0,T;H^3(\Omega)),$$
$$\mathbf{u}_{n_j} \rightharpoonup \mathbf{u} \textit{ weakly in } L^2(0,T;\mathbf{V}),$$
$$\mu_{n_j} \rightharpoonup \mu_1 \textit{ weakly in } L^2(0,T;H^1(\Omega)),$$
$$\partial_t\varphi_{n_j} \rightharpoonup \partial_t\varphi \textit{ weakly in } L^2(0,T;(H^1(\Omega))^*).$$

Furthermore, we can obtain

$$\varphi_{n_j} \to \varphi \text{ strong in } C([0,T]\,; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$$

by using the Aubin-Lions compactness theorem.

For any $\psi \in L^2(\Omega)$, we know

$$\begin{aligned}\int_\Omega f(\varphi_{n_j})\psi dx &\le \left\|f(\varphi_{n_j})\right\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \\ &\le c(1+\left\|\varphi_{n_j}\right\|^3_{L^6(\Omega)})\|\psi\|_{L^2(\Omega)} \le c(R)\|\psi\|_{L^2(\Omega)},\end{aligned}$$

such that

$$\mu_1 = -\Delta\varphi + f(\varphi) = \mu.$$

Thanks to

$$\int_\Omega (\nabla\varphi_{n_j}\mu_{n_j} - \nabla\varphi\mu) w dx = -\int_\Omega \nabla\mu_{n_j}(\varphi_{n_j}-\varphi) w dx - \int_\Omega \varphi(\nabla\mu_{n_j} - \nabla\mu) w dx$$

for any $w \in \mathbf{V}$ and

$$\int_\Omega (\mathbf{u}_{n_j}\varphi_{n_j} - \mathbf{u}\varphi)\cdot\nabla\psi dx = \int_\Omega \mathbf{u}_{n_j}\cdot\nabla\psi(\varphi_{n_j}-\varphi)dx + \int_\Omega(\mathbf{u}_{n_j}-\mathbf{u})\cdot\nabla\psi\varphi dx$$

for any $\psi \in H^1(\Omega)$, we derive

$$\begin{aligned}&\int_\Omega (\mathbf{u}_{n_j}\varphi_{n_j})\cdot\nabla\psi dx \to \int_\Omega (\mathbf{u}\varphi)\cdot\nabla\psi dx, \quad \text{as } j \to +\infty, \\ &\int_\Omega w\cdot(\nabla\varphi_{n_j}\mu_{n_j})dx \to \int_\Omega w\cdot\nabla\varphi\mu dx, \quad \text{as } j\to+\infty.\end{aligned}$$

Therefore, the pair $(\mathbf{v},(\varphi,\mathbf{u},\mu))$ is admissible for (**CP**). It follows from the weak lower sequential semicontinuity of the cost functional that $\mathbf{v}\in U$ is an optimal control for (**CP**). □

Define the solution operator

$$S : \mathbf{v} \to (\varphi, \mathbf{u}, \mu).$$

From Theorem 2.1, we realise that S is a Lipschitz continous mapping from $L^2(0,T;\mathbf{H})$ onto $\Lambda = L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^3(\Omega))\times L^2(0,T;H^1(\Omega))\times L^2(0,T;H^1(\Omega))$.

Let $\overline{\mathbf{v}}\in U$ be a local minimizer for (**CP**) and let $S(\overline{\mathbf{v}}) = (\overline{\varphi},\overline{\mathbf{u}},\overline{\mu})$ be the associated state. We will establish a differentiability result at any point for the control-to-state operator. In the following, for any fixed $\mathbf{h}\in L^2(0,T;\mathbf{H})$, we will consider the well-posedness of weak solution for the following linearized system

$$\begin{cases} \frac{\partial\omega(x,t)}{\partial t} + \nabla\cdot(\overline{\mathbf{u}}\omega)(x,t) + \nabla\cdot(\zeta\overline{\varphi})(x,t) = \Delta\theta(x,t), & (x,t)\in\Omega_T, \\ \theta(x,t) = -\Delta\omega(x,t) + f'(\overline{\varphi})\omega(x,t), & (x,t)\in\Omega_T, \\ -\Delta\zeta(x,t)+\zeta(x,t) = -\nabla q(x,t) - \overline{\varphi}\nabla\theta(x,t) - \omega\nabla\overline{\mu}(x,t) + \mathbf{h}(x,t), & (x,t)\in\Omega_T, \\ \nabla\cdot\zeta(x,t) = 0, & (x,t)\in\Omega_T, \\ \zeta(x,t) = 0, \quad \frac{\partial\omega(x,t)}{\partial\mathbf{n}} = \frac{\partial\theta(x,t)}{\partial\mathbf{n}} = 0, & (x,t)\in\partial\Omega_T, \\ \omega(x,0) = 0, \quad x\in\Omega, \end{cases} \tag{15}$$

where

$$\overline{\mu} = -\Delta\overline{\varphi} + f(\overline{\varphi}).$$

Theorem 3.2. *Let* $\mathbf{h}\in L^2(0,T;\mathbf{H})$. *The system (15) has a unique weak solution* (ω,ζ,θ) *with*

$$\omega \in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^3(\Omega)),$$

$$\zeta \in L^2(0,T;\mathbf{V}),$$
$$\theta \in L^2(0,T;H^1(\Omega)),$$
$$\omega_t \in L^2(0,T;(H^1(\Omega))^*).$$

Moreover, there exists a constant $C_2 > 0$ depending only on R, T and the initial data of the system such that

$$\max_{t\in[0,T]} \|\omega(t)\|^2_{H^1(\Omega)} + \int_0^T \|\omega(t)\|^2_{H^3(\Omega)}dt + \int_0^T \|\theta(t)\|^2_{H^1(\Omega)}dt + \int_0^T \|\zeta(t)\|^2_{H^1(\Omega)}dt$$
$$\leq C_2 \int_0^T \|\mathbf{h}(t)\|^2_{L^2(\Omega)}dt.$$

Proof. First, we eastablish the existence of weak solution to the system (15) using the Faedo-Galerkin method [25].

Let $A = -P\Delta$ be the Stokes operator where P is the Leray-Helmholtz projector from $(L^2(\Omega))^3$ onto $\mathbf{H}$. We all know that for the eigenvalue problem $A\xi = \lambda\xi$, there exists a sequence of non-decreasing numbers $\{\lambda_n\}_{n=1}^{\infty}$ and a sequence of functions $\{\xi_n\}_{n=1}^{\infty}$, which are orthonormal and complete in $\mathbf{H}$ such that for every $k \geq 1$, we have

$$A\xi_k = \lambda_k\xi_k \text{ and } \lim_{k\to+\infty} \lambda_k = +\infty.$$

Let us introduce the operator N which is the inverse of the Laplacian operator $-\Delta$, where $-\Delta$ is endowed with Neumann boundary conditions that impose zero average over Ω. It is well-known that there exists a sequence of non-decreasing numbers $\{\kappa_n\}_{n=1}^{\infty}$ and a sequence of functions $\{\psi_n\}_{n=1}^{\infty}$, which are orthonormal and complete in $L^2(\Omega)$ such that $\kappa_1 = 0$ and $\psi_1 = 1$ as well as for every $k \geq 2$, we have

$$N\psi_k = \frac{1}{\kappa_k}\psi_k \text{ and } \lim_{k\to+\infty} \kappa_k = +\infty.$$

For any $n \geq 1$, we introduce two finite-dimensional spaces $W_n = span\{\psi_1, \psi_2 ..., \psi_n\}$ and $\mathbf{H}_n = span\{\xi_1, \xi_2 ..., \xi_n\}$. Let P_n be the orthogonal projector from $L^2(\Omega)$ to W_n and let Γ_n be the orthogonal projector from $\mathbf{H}$ to $\mathbf{H}_n$. We will look for appropriate coefficients $(\alpha_i(t), \beta_i(t), \theta_i(t))(i = 1, 2..n)$ so that

$$\omega_n(t) = \sum_{i=1}^{n} \alpha_i(t)\psi_i,$$
$$\zeta_n(t) = \sum_{i=1}^{n} \beta_i(t)\xi_i,$$
$$\theta_n(t) = \sum_{i=1}^{n} \gamma_i(t)\psi_i.$$

These approximate solutions satisfy the problem

$$\begin{cases} \left\langle \frac{\partial\omega_n}{\partial t}, \psi \right\rangle - \langle \overline{\mathbf{u}}\omega_n + \zeta_n\overline{\varphi}, \nabla\psi \rangle + (\nabla\theta_n, \nabla\psi) = 0, \\ \langle \theta_n, \varsigma \rangle = \langle \nabla\omega_n, \nabla\varsigma \rangle + \langle f'(\overline{\varphi})\omega_n, \varsigma \rangle, \\ \langle \nabla\zeta_n, \nabla\xi \rangle + \langle \zeta_n, \xi \rangle = -\langle \overline{\varphi}\nabla\theta_n + \omega_n\nabla\overline{\mu}, \xi \rangle + \langle \mathbf{h}(x,t), \xi \rangle, \\ \alpha_i(0) = \langle \omega(x,0), \psi_i \rangle = 0, i = 1, 2.., n, \end{cases} \tag{16}$$

for any $\xi \in \mathbf{H}_n$ and $\psi, \varsigma \in W_n$.

Let $\psi = 1$ in the first equation of (16)

$$\int_\Omega \omega_n(x,t)dx = \int_\Omega \omega_n(x,0)dx = 0.$$

Let $\psi = \theta_n,\ \varsigma = \frac{\partial \omega_n}{\partial t},\ \xi = \zeta_n$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla\omega_n\right\|^2_{L^2(\Omega)} + \left\|\nabla\theta_n\right\|^2_{L^2(\Omega)} + \left\|\zeta_n\right\|^2_{H^1(\Omega)} \\
=&\int_\Omega \overline{\mathbf{u}}\cdot\nabla\theta_n\omega_n dx - \int_\Omega \zeta_n\cdot\nabla\overline{\mu}\omega_n dx - \int_\Omega f'(\overline{\varphi})\omega_n\frac{\partial\omega_n}{\partial t}dx + \int_\Omega \mathbf{h}(x,t)\cdot\zeta_n dx \\
\le&\|\overline{\mathbf{u}}\|_{L^6(\Omega)}\|\nabla\theta_n\|_{L^2(\Omega)}\|\omega_n\|_{L^3(\Omega)} + \|\zeta_n\|_{L^6(\Omega)}\|\nabla\overline{\mu}\|_{L^2(\Omega)}\|\omega_n\|_{L^3(\Omega)} \\
&-\int_\Omega f'(\overline{\varphi})\omega_n\frac{\partial\omega_n}{\partial t}dx + \|\mathbf{h}(x,t)\|_{L^2(\Omega)}\|\zeta_n\|_{L^2(\Omega)},
\end{aligned} \tag{17}$$

which implies

$$\begin{aligned}
&\frac{d}{dt}\left\|\nabla\omega_n\right\|^2_{L^2(\Omega)} + \frac{3}{2}\left\|\nabla\theta_n\right\|^2_{L^2(\Omega)} + \left\|\zeta_n\right\|^2_{H^1(\Omega)} \\
\le& c(\|\overline{\mathbf{u}}\|^2_{H^1(\Omega)} + \|\nabla\overline{\mu}\|^2_{L^2(\Omega)})\left\|\nabla\omega_n\right\|^2_{L^2(\Omega)} + C\left\|\mathbf{h}\right\|^2_{L^2(\Omega)} \\
&-\frac{d}{dt}\int_\Omega f'(\overline{\varphi})|\omega_n|^2dx + \int_\Omega f''(\overline{\varphi})\overline{\varphi}_t|\omega_n|^2dx.
\end{aligned}$$

Integrating the above inequality over $[0,t]$, we derive

$$\begin{aligned}
&\left\|\nabla\omega_n(t)\right\|^2_{L^2(\Omega)} + \frac{3}{2}\int_0^t\left\|\nabla\theta_n(s)\right\|^2_{L^2(\Omega)}ds + \int_0^t\left\|\zeta_n(s)\right\|^2_{H^1(\Omega)}ds \\
\le& c\int_0^t\left(\|\overline{\mathbf{u}}(s)\|^2_{H^1(\Omega)} + \|\nabla\overline{\mu}(s)\|^2_{L^2(\Omega)}\right)\left\|\nabla\omega_n(s)\right\|^2_{L^2(\Omega)}ds - \frac{1}{2}\int_\Omega f'(\overline{\varphi}(t))|\omega_n(t)|^2dx \\
&+C\int_0^t\|\mathbf{h}(s)\|^2_{L^2(\Omega)}ds + \frac{1}{2}\int_0^t\int_\Omega f''(\overline{\varphi}(s))\overline{\varphi}_t|\omega_n|^2dxds.
\end{aligned}$$

With the help of (6), we can perform further calculations on the above equation

$$\begin{aligned}
&\left\|\nabla\omega_n(t)\right\|^2_{L^2(\Omega)} + \frac{3}{2}\int_0^t\left\|\nabla\theta_n(s)\right\|^2_{L^2(\Omega)}ds + \int_0^t\left\|\zeta_n(s)\right\|^2_{H^1(\Omega)}ds \\
\le& c\int_0^t\left(\|\overline{\mathbf{u}}(s)\|^2_{H^1(\Omega)} + \|\nabla\bar{\mu}(s)\|^2_{L^2(\Omega)}\right)\left\|\nabla\omega_n(s)\right\|^2_{L^2(\Omega)}ds + \frac{c}{2}\int_\Omega\left|1+|\overline{\varphi}(t)|^2\right||\omega_n(t)|^2dx \\
&+C\int_0^t\|\mathbf{h}(s)\|^2_{L^2(\Omega)}ds + \frac{1}{2}\int_0^t\int_\Omega(1+|\overline{\varphi}(s)|)\overline{\varphi}_t|\omega_n|^2dxds \\
\le& c\int_0^t\left(\|\overline{\mathbf{u}}(s)\|^2_{H^1(\Omega)} + \|\nabla\bar{\mu}(s)\|^2_{L^2(\Omega)}\right)\left\|\nabla\omega_n(s)\right\|^2_{L^2(\Omega)}ds \\
&+\frac{c}{2}(1+\|\overline{\varphi}(t)\|^2_{L^6(\Omega)})\|\omega_n(t)\|_{L^2(\Omega)}\|\omega_n(t)\|_{L^6(\Omega)} \\
&+\frac{1}{2}\int_0^t\|\bar{\varphi}_t\|_{(H^1(\Omega))^*}\|\bar{\varphi}(s)\|_{H^1(\Omega)}\left\|\omega_n\right\|^2_{L^6(\Omega)}ds + \frac{1}{2}\int_0^t\|\bar{\varphi}_t\|_{(H^1(\Omega))^*}\|\Delta\bar{\varphi}(s)\|_{L^2(\Omega)}\left\|\omega_n\right\|^2_{L^6(\Omega)}ds \\
&+\frac{1}{2}\int_0^t\|\bar{\varphi}_t\|_{(H^1(\Omega))^*}\|\bar{\varphi}(s)\|_{L^6(\Omega)}\|\omega_n(s)\|_{L^6(\Omega)}\|\nabla\omega(s)\|_{L^6(\Omega)}ds \\
&+C\int_0^t\|\mathbf{h}(s)\|^2_{L^2(\Omega)}ds.
\end{aligned} \tag{18}$$

Let $\psi = \lambda\omega_n$ and $\varsigma = -\lambda\Delta\omega_n$ in (16) for λ determined later.

$$
\begin{aligned}
&\frac{\lambda}{2}\frac{d}{dt}\|\omega_n\|^2_{L^2(\Omega)} + \lambda\|\Delta\omega_n\|^2_{L^2(\Omega)}\\
&= \lambda\int_\Omega f'(\overline{\varphi})\omega_n\Delta\omega_n dx - \lambda\int_\Omega(\zeta_n\cdot\nabla\overline{\varphi})\omega_n dx\\
&\le \lambda\|f'(\overline{\varphi})\|_{L^\infty(\Omega)}\|\omega_n\|_{L^2(\Omega)}\|\Delta\omega_n\|_{L^2(\Omega)} + \lambda\|\zeta_n\|_{L^6(\Omega)}\|\nabla\overline{\varphi}\|_{L^2(\Omega)}\|\omega_n\|_{L^3(\Omega)},
\end{aligned}
$$

which entails that

$$
\begin{aligned}
&\lambda\|\omega_n(t)\|^2_{L^2(\Omega)} + \lambda\int_0^t\|\Delta\omega_n(s)\|^2_{L^2(\Omega)}ds\\
&\le c\int_0^t\left(\lambda\|f'(\overline{\varphi}(s))\|^2_{L^\infty(\Omega)} + \lambda^2\|\nabla\overline{\varphi}(s)\|^2_{L^2(\Omega)}\right)\|\nabla\omega_n(s)\|^2_{L^2(\Omega)}ds + \frac{1}{2}\int_0^t\|\zeta_n(s)\|^2_{H^1(\Omega)}ds.
\end{aligned}
\tag{19}
$$

Hence, adding (18) with (19) we end up with

$$
\begin{aligned}
&\frac{1}{2}\|\nabla\omega_n(t)\|^2_{L^2(\Omega)} + (\lambda - c(1+\|\overline{\varphi}(t)\|^4_{L^6(\Omega)})\|\omega_n(t)\|^2_{L^2(\Omega)} + (\lambda-1)\int_0^t\|\Delta\omega_n(s)\|^2_{L^2(\Omega)}ds\\
&+\frac{3}{2}\int_0^t\|\nabla\theta_n\|^2_{L^2(\Omega)}ds + \frac{1}{2}\int_0^t\|\zeta_n(s)\|^2_{H^1(\Omega)}ds\\
\le{}& c\int_0^t(\|\overline{\mathbf{u}}(s)\|^2_{H^1(\Omega)} + \|\nabla\bar{\mu}(s)\|^2_{L^2(\Omega)} + 1 + \|\bar{\varphi}_t\|_{(H^1(\Omega))^*}\|\bar{\varphi}(s)\|_{L^6(\Omega)} + \|\bar{\varphi}_t\|^2_{(H^1(\Omega))^*}\|\bar{\varphi}(s)\|^2_{L^6}\\
&+\|\overline{\varphi}_t\|_{H^1(\Omega)^*}\|\Delta\overline{\varphi}(s)\|_{L^2(\Omega)} + \lambda\|f'(\bar{\varphi}(s))\|^2_{L^\infty(\Omega)} + \lambda^2\|\nabla\bar{\varphi}(s)\|^2_{L^2(\Omega)})\|\nabla\omega_n(s)\|^2_{L^2(\Omega)}ds\\
&+C\int_0^t\|\mathbf{h}(\mathbf{s})\|^2_{L^2(\Omega)}ds.
\end{aligned}
$$

Choosing $\lambda = 1 + c\max\limits_{t\in[0,T]}(1+\|\bar{\varphi}(t)\|^4_{L^6(\Omega)}) > 0$, we infer from the Gronwall inequality that

$$
\begin{aligned}
&\|\omega_n(t)\|^2_{H^1(\Omega)} + \int_0^t(\|\Delta\omega_n(s)\|^2_{L^2(\Omega)} + \|\nabla\theta_n(s)\|^2_{L^2(\Omega)} + \|\zeta_n(s)\|^2_{H^1(\Omega)})ds\\
&\le \frac{C_2}{2}\int_0^t\|\mathbf{h}(\mathbf{s})\|^2_{L^2(\Omega)}ds.
\end{aligned}
\tag{20}
$$

Due to

$$
\left\|\frac{\partial\omega_n}{\partial t}\right\|_{(H^1(\Omega))^*} \le \|\overline{\mathbf{u}}\|_{L^6(\Omega)}\|\omega_n\|_{L^3(\Omega)} + \|\zeta_n\|_{L^6(\Omega)}\|\overline{\varphi}\|_{L^3(\Omega)} + \|\nabla\theta_n\|_{L^2(\Omega)}
$$

and

$$
\begin{aligned}
&\|\nabla\Delta\omega_n\|_{L^2(\Omega)}\\
&\le \|\nabla(f(\bar{\varphi})\omega_n)\|_{L^2(\Omega)} + \|\nabla\theta_n\|_{L^2(\Omega)}\\
&\le \|f'(\bar{\varphi})\|_{L^3(\Omega)}\|\nabla\omega_n\|_{L^6(\Omega)} + \|f''(\bar{\varphi})\|_{L^6(\Omega)}\|\nabla\bar{\varphi}\|_{L^6(\Omega)}\|\omega_n\|_{L^6(\Omega)} + \|\nabla\theta_n\|_{L^2(\Omega)}\\
&\le c(1+\|\overline{\varphi}\|^2_{L^6(\Omega)})\|\nabla\omega_n\|_{L^6(\Omega)} + c(1+\|\bar{\varphi}\|_{L^6(\Omega)})\|\nabla\bar{\varphi}\|_{L^6(\Omega)}\|\omega_n\|_{L^6(\Omega)} + \|\nabla\theta_n\|_{L^2(\Omega)},
\end{aligned}
$$

we derive

$$
\begin{aligned}
&\{\omega_n\}_{n=1}^\infty \ \textit{is uniformly bounded in } L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^3(\Omega)),\\
&\{\zeta_n\}_{n=1}^\infty \ \textit{is uniformly bounded in } L^2(0,T;\mathbf{V}),\\
&\{\theta_n\}_{n=1}^\infty \ \textit{is uniformly bounded in } L^2(0,T;H^1(\Omega)),
\end{aligned}
$$

$$\left\{\frac{\partial \omega_n}{\partial t}\right\}_{n=1}^{\infty} \textit{ is uniformly bounded in } L^2(0,T;(H^1(\Omega))^*).$$

Therefore, there exist

$$\begin{aligned} &\omega \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)),\\ &\zeta \in L^2(0,T;\mathbf{V}),\\ &\theta \in L^2(0,T;H^1(\Omega)),\\ &\omega_t \in L^2(0,T;H^1(\Omega)^*) \end{aligned}$$

and we can extract subsequences $\{\omega_{n_j}\}_{j=1}^{\infty}$, $\{\zeta_{n_j}\}_{j=1}^{\infty}$, $\{\theta_{n_j}\}_{j=1}^{\infty}$, $\left\{\frac{\partial \omega_{n_j}}{\partial t}\right\}_{j=1}^{\infty}$ of $\{\omega_n\}_{n=1}^{\infty}$, $\{\zeta_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$, $\{\frac{\partial \omega_n}{\partial t}\}_{n=1}^{\infty}$, such that

$$\begin{aligned} \omega_{n_j} &\rightharpoonup \omega \textit{ weakly star in } L^\infty(0,T;H^1(\Omega)),\\ \omega_{n_j} &\rightharpoonup \omega \textit{ weakly in } L^2(0,T;H^3(\Omega)),\\ \zeta_{n_j} &\rightharpoonup \zeta \textit{ weakly in } L^2(0,T;\mathbf{V}),\\ \theta_{n_j} &\rightharpoonup \theta \textit{ weakly in } L^2(0,T;H^1(\Omega)),\\ \frac{\partial \omega_{n_j}}{\partial t} &\rightharpoonup \omega_t \textit{ weakly in } L^2(0,T;(H^1(\Omega))^*). \end{aligned}$$

Passing to the limit, we obtain the existence of a weak solution for system (15).

Finally, we can prove the uniqueness of weak solution which is in common with [7]. We omit the proof here. □

Based on the results about the well-posedness of weak solution of (15), we can get the following: Theorem 3.3 and Theorem 3.4.

Theorem 3.3. *Suppose that $(H_1)-(H_3)$ are satisfied. Then the control-to-state mapping S is Fréchet differentiable for any $\bar{\mathbf{v}} \in U$. Furthermore, for any $\mathbf{h} \in L^2(0,T;\mathbf{H})$, the Fréchet derivative $DS(\bar{\mathbf{v}})\mathbf{h}$ is defined as follows: for any $\mathbf{h} \in L^2(0,T;\mathbf{H})$, we have $DS(\bar{\mathbf{v}})\mathbf{h} = (\omega,\zeta,\theta)$, where (ω,ζ,θ) is the unique weak solution for the linearized system (18).*

Proof. Let any $\bar{\mathbf{v}} \in U$ be fixed and let $S(\bar{\mathbf{v}}) = (\overline{\varphi},\overline{\mathbf{u}},\overline{\mu})$ be the associated solution for the state system (1). Since U is an open subset of $L^2(0,T;\mathbf{H})$, there exists some $\alpha > 0$ such that for any $\mathbf{h} \in L^2(0,T;\mathbf{H})$ with $\|\mathbf{h}\|_{L^2(0,T;\mathbf{H})} \le \alpha$, we have $\bar{\mathbf{v}} + \mathbf{h} \in U$. Let $(\varphi^h,\mathbf{u}^h,\mu^h)$ be the unique weak solution for the state system (1) with $\bar{\mathbf{v}} + \mathbf{h}$ and let (ω,ζ,θ) be the unique weak solution for the linearized system (15) with $\mathbf{h}$.

Denote by

$$\phi(t) = \varphi^h(t) - \overline{\varphi}(t) - \omega(t) = \varphi(t) - \omega(t),$$

$$\eta(t) = \mathbf{u}^h(t) - \overline{\mathbf{u}}(t) - \zeta(t) = \mathbf{u}(t) - \zeta(t),$$

and

$$\rho(t) = \mu^h(t) - \overline{\mu}(t) - \theta(t) = \mu(t) - \theta(t)$$

for any $t \ge 0$.

Then (ϕ,η,ρ) satisfies the following equations

$$\begin{cases} \phi(x,t)_t = -\mathbf{u}\cdot\nabla\varphi(x,t) - \eta\cdot\nabla\overline{\varphi}(x,t) - \bar{\mathbf{u}}\cdot\nabla\phi(x,t) + \Delta\rho(x,t), \quad (x,t)\in\Omega_T, \\ \rho(x,t) = -\Delta\phi(x,t) + f(\varphi^h)(x,t) - f(\overline{\varphi})(x,t) - f'(\overline{\varphi})\omega(x,t), \quad (x,t)\in\Omega_T, \\ -\Delta\eta(x,t) + \eta(x,t) + \nabla p(x,t) = -\varphi\nabla\mu(x,t) - \phi\nabla\overline{\mu}(x,t) - \overline{\varphi}\nabla\rho(x,t), \quad (x,t)\in\Omega_T, \\ \nabla\cdot\eta(x,t) = 0, \quad (x,t)\in\Omega_T, \\ \eta(x,t) = 0, \frac{\partial\phi(x,t)}{\partial\mathbf{n}} = \frac{\partial\rho(x,t)}{\partial\mathbf{n}} = 0, \quad (x,t)\in\partial\Omega_T, \\ \phi(x,0) = 0, \quad x\in\Omega. \end{cases} \tag{21}$$

Taking the inner product of the first equation and the third equation of (21) with $-\Delta\phi$ and η, respectively, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\phi\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^1(\Omega)} \\ =& -\int_\Omega(\Delta\rho - \mathbf{u}\cdot\nabla\varphi - \eta\cdot\nabla\overline{\varphi} - \overline{\mathbf{u}}\cdot\nabla\phi)\Delta\phi dx + \int_\Omega(-\varphi\nabla\mu - \phi\nabla\overline{\mu} - \overline{\varphi}\nabla\rho)\cdot\eta dx \\ =& -\int_\Omega(\Delta\rho - \mathbf{u}\cdot\nabla\varphi)\Delta\phi dx + \int_\Omega\overline{\mathbf{u}}\cdot\nabla\phi\Delta\phi dx + \int_\Omega(\eta\cdot\nabla\varphi\mu + \eta\cdot\nabla\phi\overline{\mu})dx \\ &+ \int_\Omega\eta\cdot\nabla\overline{\varphi}(f(\varphi^h) - f(\overline{\varphi}) - f'(\overline{\varphi})\omega)dx \\ =& -\int_\Omega(\Delta\rho - \mathbf{u}\cdot\nabla\varphi)\Delta\phi dx + \int_\Omega\eta\cdot\nabla\varphi\mu dx \\ &+ \int_\Omega\overline{\mathbf{u}}\cdot\nabla\phi\Delta\phi + \eta\cdot\nabla\phi\overline{\mu} + \eta\cdot\nabla\overline{\varphi}(f(\varphi^h) - f(\bar{\varphi}) - f'(\bar{\varphi})\omega)dx. \end{aligned} \tag{22}$$

In the following, we will divide the right-hand side of equation (22) into three parts of computation.

In the first part, we find

$$\begin{aligned} &-\int_\Omega(\Delta\rho - \mathbf{u}\cdot\nabla\varphi)\Delta\phi dx \\ =&\int_\Omega\nabla(-\Delta\phi + f(\varphi^h) - f(\overline{\varphi}) - f'(\overline{\varphi})\omega)\nabla\Delta\phi - \mathbf{u}\cdot\varphi\nabla\Delta\phi dx \\ \le& -\|\nabla\Delta\phi\|^2_{L^2(\Omega)} + \|\mathbf{u}\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\nabla\Delta\phi\|_{L^2(\Omega)} \\ &+ \|\nabla\Delta\phi\|_{L^2(\Omega)}\left\|\nabla(f(\varphi^h) - f(\overline{\varphi}) - f'(\overline{\varphi})\omega)\right\|_{L^2(\Omega)} \\ \le& -\|\nabla\Delta\phi\|^2_{L^2(\Omega)} + c\|\mathbf{u}\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\Delta\phi\|_{L^2(\Omega)} \\ &+ \|\nabla\Delta\phi\|_{L^2(\Omega)}\left\|\nabla(f(\varphi^h) - f(\overline{\varphi}) - f'(\overline{\varphi})\omega)\right\|_{L^2(\Omega)}, \end{aligned} \tag{23}$$

where we will use the Mean Value Theorem to handle the last term on the right-hand side of (23) and demonstrate later.

Then, we can calculate the second part as follows

$$\begin{aligned} &\int_\Omega\eta\cdot\nabla\varphi\mu dx \\ \le&\|\eta\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)} + \|\eta\|_{L^3(\Omega)}\|\varphi\|_{L^6(\Omega)}\left\|\nabla(f(\varphi^h) - f(\overline{\varphi}))\right\|_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&\le c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)} \\
&\quad + c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\left(\left\|f'(\varphi^h)\nabla\varphi + (f'(\varphi^h) - f'(\bar\varphi))\nabla\bar\varphi\right\|_{L^2(\Omega)}\right) \\
&\le c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)} + c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}(\left\|f'(\varphi^h)\right\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&\quad + c\|\varphi\|_{L^6(\Omega)}(1 + \left\|\varphi^h\right\|_{L^6(\Omega)} + \|\bar\varphi\|_{L^6(\Omega)})\|\nabla\bar\varphi\|_{L^6(\Omega)}) \\
&\le c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)} \\
&\quad + c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\left\|f'(\varphi^h)\right\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \\
&\quad + c\|\eta\|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}(1 + \left\|\nabla\varphi^h\right\|_{L^2(\Omega)} + \|\nabla\bar\varphi\|_{L^2(\Omega)})\|\Delta\bar\varphi\|_{L^2(\Omega)}.
\end{aligned} \tag{24}
$$

In the last part, we obtain

$$
\begin{aligned}
&\int_\Omega \bar{\mathbf{u}}\cdot\nabla\phi\Delta\phi + \eta\cdot\nabla\phi\bar\mu + \eta\cdot\nabla\overline{\varphi}(f(\varphi^h) - f(\bar\varphi) - f'(\bar\varphi)\omega)dx \\
\le&\|\bar{\mathbf{u}}\|_{L^3(\Omega)}\|\phi\|_{L^6(\Omega)}\|\nabla\Delta\phi\|_{L^2(\Omega)} + \|\eta\|_{L^3(\Omega)}\|\phi\|_{L^6(\Omega)}\|\nabla\overline{\mu}\|_{L^2(\Omega)} \\
&+ \|\eta\|_{L^3(\Omega)}\|\overline{\varphi}\|_{L^6(\Omega)}\left\|\nabla(f(\varphi^h) - f(\bar\varphi) - f'(\bar\varphi)\omega)\right\|_{L^2(\Omega)} \\
\le& c\|\bar{\mathbf{u}}\|_{H^1(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}\|\nabla\Delta\phi\|_{L^2(\Omega)} + c\|\eta\|_{H^1(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}\|\overline{\mu}\|_{H^1(\Omega)} \\
&+ c\|\eta\|_{H^1(\Omega)}\|\nabla\overline{\varphi}\|_{L^2(\Omega)}\left\|\nabla(f(\varphi^h) - f(\bar\varphi) - f'(\bar\varphi)\omega)\right\|_{L^2(\Omega)}.
\end{aligned} \tag{25}
$$

We will focus particularly on the last term of (25). In the first, we get

$$
\begin{aligned}
&f(\varphi^h) - f(\overline{\varphi}) - f'(\overline{\varphi})\omega \\
&= (\varphi^h - \overline{\varphi})\int_0^1 \left[f'(\tau\varphi^h + (1-\tau)\overline{\varphi}) - f'(\overline{\varphi})\right]d\tau + f'(\overline{\varphi})(\varphi^h - \overline{\varphi} - \omega) \\
&= \varphi\int_0^1 \left[f'(\tau\varphi^h + (1-\tau)\bar\varphi) - f'(\bar\varphi)\right]d\tau + f'(\bar\varphi)\phi.
\end{aligned} \tag{26}
$$

Then, taking the derivative with respect to the spatial variable in equation (26), we have

$$
\begin{aligned}
&\nabla\left(f(\varphi^h) - f(\bar\varphi) - f'(\bar\varphi)\omega\right) \\
=&\nabla\varphi\int_0^1 \left[f'(\tau\varphi^h + (1-\tau)\bar\varphi) - f'(\bar\varphi)\right] + f'(\bar\varphi)\nabla\phi + f''(\bar\varphi)\nabla\bar\varphi\phi \\
&+ \varphi\int_0^1 \left[f''(\tau\varphi^h + (1-\tau)\overline{\varphi})(\tau\nabla\varphi^h + (1-\tau)\nabla\overline{\varphi}) - f''(\overline{\varphi})\nabla\overline{\varphi}\right]d\tau \\
=&\nabla\varphi\int_0^1\int_0^1 f''(s(\tau\varphi^h + (1-\tau)\overline{\varphi}) + (1-s)\overline{\varphi})(\tau\varphi^h + (1-\tau)\overline{\varphi} - \overline{\varphi})dsd\tau \\
&+ \varphi\int_0^1 [f''(\tau\varphi^h + (1-\tau)\overline{\varphi})\tau\nabla\varphi \\
&+ \nabla\overline{\varphi}\int_0^1 f'''(s(\tau\varphi^h + (1-\tau)\overline{\varphi}) + (1-s)\overline{\varphi})(\tau\varphi^h + (1-\tau)\overline{\varphi} - \overline{\varphi})ds]d\tau \\
&+ f'(\bar\varphi)\nabla\phi + f''(\bar\varphi)\nabla\bar\varphi\phi \\
=&A_h\varphi\nabla\varphi + B_h\varphi^2\nabla\bar\varphi + f'(\bar\varphi)\nabla\phi + f''(\bar\varphi)\nabla\bar\varphi\phi.
\end{aligned} \tag{27}
$$

Take the L^2-norm on equation(27)

$$\begin{aligned}
&\left\|\nabla\left(f(\varphi^h)-f(\bar{\varphi})-f'(\bar{\varphi})\omega\right)\right\|_{L^2(\Omega)}\\
\le&\|A_h\|_{L^\infty(\Omega)}\|\varphi\|_{L^3(\Omega)}\|\nabla\varphi\|_{L^6(\Omega)}+\|B_h\|_{L^\infty(\Omega)}\|\varphi\|^2_{L^6(\Omega)}\|\nabla\bar{\varphi}\|_{L^6(\Omega)}\\
&+\|f'(\bar{\varphi})\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}+\|f''(\bar{\varphi})\|_{L^\infty(\Omega)}\|\nabla\bar{\varphi}\|_{L^3(\Omega)}\|\phi\|_{L^6(\Omega)}\\
\le&c\|A_h\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)}+c\|B_h\|_{L^\infty(\Omega)}\|\nabla\varphi\|^2_{L^2(\Omega)}\|\Delta\bar{\varphi}\|_{L^2(\Omega)}\\
&+\|f'(\bar{\varphi})\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}+c\|f''(\bar{\varphi})\|_{L^\infty(\Omega)}\|\Delta\bar{\varphi}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)},
\end{aligned}\tag{28}$$

where

$$A_h=\int_0^1\tau\int_0^1 f''(s\tau\varphi^h+(1-s\tau)\overline{\varphi})dsd\tau+\int_0^1\tau f''(\tau\varphi^h+(1-\tau)\overline{\varphi})d\tau,\tag{29}$$

and

$$B_h=\int_0^1\tau\int_0^1 f'''(s\tau\varphi^h+(1-s\tau)\overline{\varphi})dsd\tau.\tag{30}$$

In view of theorem 2.1, we have

$$\|A_h\|_{L^\infty}+\|B_h\|_{L^\infty}\le c\tag{31}$$

with $\|\mathbf{h}\|_{L^2(0,T;\mathbf{H})}\le\alpha$.

From (22) to (31), we derive

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\phi\|^2_{L^2(\Omega)}+\|\eta\|^2_{H^1(\Omega)}+\|\nabla\Delta\phi\|^2_{L^2(\Omega)}\\
&\le\alpha_1(t)\|\nabla\phi\|^2_{L^2(\Omega)}+\alpha_2(t),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(t)&=c(\|f'(\overline{\varphi})\|^2_{L^\infty(\Omega)}+\|f''(\overline{\varphi})\|^2_{L^\infty(\Omega)}\|\Delta\overline{\varphi}\|^2_{L^2(\Omega)}+\|\overline{\mathbf{u}}\|^2_{H^1(\Omega)}+\|\overline{\mu}\|^2_{H^1(\Omega)}),\\
&\le c((1+\|\varphi\|^2_{L^\infty(\Omega)})^2+(1+\|\varphi\|_{L^\infty(\Omega)})^2\|\Delta\overline{\varphi}\|^2_{L^2(\Omega)}+\|\overline{\mathbf{u}}\|^2_{H^1(\Omega)}+\|\overline{\mu}\|^2_{H^1(\Omega)})\\
&\le c((1+\|\Delta\varphi\|^2_{L^2(\Omega)})+(1+\|\Delta\varphi\|_{L^2(\Omega)})\|\Delta\overline{\varphi}\|^2_{L^2(\Omega)}+\|\overline{\mathbf{u}}\|^2_{H^1(\Omega)}+\|\overline{\mu}\|^2_{H^1(\Omega)})
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2(t)=&c(\|\nabla\varphi\|^2_{L^2(\Omega)}\|\mathbf{u}\|^2_{H^1(\Omega)}+\|A_h\|^2_{L^\infty(\Omega)}\|\nabla\varphi\|^2_{L^2(\Omega)}\|\Delta\varphi\|^2_{L^2(\Omega)}\\
&+\|B_h\|^2_{L^\infty(\Omega)}\|\nabla\varphi\|^4_{L^2(\Omega)}\|\Delta\overline{\varphi}\|^2_{L^2(\Omega)}\\
&+\|\nabla\varphi\|^2_{L^2(\Omega)}\|\nabla\Delta\varphi\|^2_{L^2(\Omega)}+\|\nabla\varphi\|^4_{L^2(\Omega)}\left\|f'(\varphi^h)\right\|^2_{L^\infty(\Omega)}\\
&+\|\nabla\varphi\|^4_{L^2(\Omega)}(1+\left\|\nabla\varphi^h\right\|^2_{L^2(\Omega)}+\|\nabla\overline{\varphi}\|^2_{L^2(\Omega)})\|\Delta\overline{\varphi}\|^2_{L^2(\Omega)}).
\end{aligned}$$

Thanks to the Gronwall inequality, we get

$$\begin{aligned}
&\max_{t\in[0,T]}\|\nabla\phi(t)\|^2_{L^2(\Omega)}+\int_0^T(\|\eta(t)\|^2_{H^1(\Omega)}+\|\nabla\Delta\phi(t)\|^2_{L^2(\Omega)})dt\\
&\le(\int_0^T\alpha_2(t)dt)\exp^{\int_0^T\alpha_1(t)dt},
\end{aligned}$$

where the integrability of $\alpha_1(t)$ and $\alpha_2(t)$ is attributed to above estimates and Theorem 2.1. By virtue of (11), Theorem 2.1, Theorem 3.2 and

$$\begin{aligned}
&\|\phi_t\|_{(H^1(\Omega))^*}\\
\le&\|\mathbf{u}\|_{L^6(\Omega)}\|\varphi\|_{L^3(\Omega)}+\|\eta\|_{L^6(\Omega)}\|\overline{\varphi}\|_{L^3(\Omega)}+\|\overline{\mathbf{u}}\|_{L^6(\Omega)}\|\phi\|_{L^3(\Omega)}+\|\nabla\Delta\phi\|_{L^2(\Omega)}\\
&+\left\|\nabla(f(\varphi^h)-f(\overline{\varphi})-f'(\overline{\varphi})\omega)\right\|_{L^2(\Omega)},
\end{aligned}$$

we deduce

$$\max_{t\in(0,T)} \|\nabla\phi(t)\|^2_{L^2(\Omega)} + \int_0^T (\|\eta(t)\|^2_{H^1(\Omega)} + \|\nabla\Delta\phi(t)\|^2_{L^2(\Omega)} + \|\phi_t(t)\|^2_{(H^1(\Omega))^*})dt \\ \le \lambda(t, R, \|\mathbf{h}\|_{L^2(0,T;\mathbf{H})})\, \|\mathbf{h}\|^2_{L^2(0,T;\mathbf{H})}, \tag{32}$$

where $\lambda(t, R, \|\mathbf{h}\|_{L^2(0,T;L^2(\Omega))})$ can be computed by above estimates and it satisfies

$$\lim_{\|\mathbf{h}\|_{L^2(0,T;L^2(\Omega))}\to 0^+} \lambda(t, R, \|\mathbf{h}\|_{L^2(0,T;\mathbf{H})}) = 0.$$

□

Theorem 3.4. *Suppose that $(H_1) - (H_3)$ are fulfilled. Then the mapping DS is Lipschitz continuous on U in the following sense: there is constant $C_3 > 0$ such that for any $\mathbf{v}, \overline{\mathbf{v}} \in U$ and any $\mathbf{h} \in L^2(0,T;\mathbf{H})$ we have*

$$\|DS(\mathbf{v})\mathbf{h} - DS(\overline{\mathbf{v}})\mathbf{h}\|_\Lambda \le C_3\|\mathbf{v} - \overline{\mathbf{v}}\|_{L^2(0,T;\mathbf{H})}\|\mathbf{h}\|_{L^2(0,T;\mathbf{H})},$$

where $\Lambda = (L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega))) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$.

Proof. Let $\overline{\mathbf{v}} + \mathbf{k} \in U$ for any fixed $\overline{\mathbf{v}} \in U$ and $\mathbf{k} \in L^2(0,T;\mathbf{H})$. Denote by

$$S(\overline{\mathbf{v}} + \mathbf{k}) = (\varphi^k, \mathbf{u}^k, \mu^k), S(\overline{\mathbf{v}}) = (\overline{\varphi}, \overline{\mathbf{u}}, \overline{\mu}),$$

and

$$(\varphi, \mathbf{u}, \mu) = (\varphi^k - \overline{\varphi}, \mathbf{u}^k - \overline{\mathbf{u}}, \mu^k - \overline{\mu}) = S(\overline{\mathbf{v}} + \mathbf{k}) - S(\overline{\mathbf{v}}).$$

Let any $\mathbf{h} \in L^2(0,T;\mathbf{H})$ be fixed. Denote by

$$(\omega^k, \zeta^k, \theta^k) = DS(\overline{\mathbf{v}} + \mathbf{k})\mathbf{h}, (\overline{\omega}, \overline{\zeta}, \overline{\theta}) = DS(\overline{\mathbf{v}})\mathbf{h},$$

and

$$(\omega, \zeta, \theta) = (\omega^k - \overline{\omega}, \zeta^k - \overline{\zeta}, \theta^k - \overline{\theta}) = DS(\overline{\mathbf{v}} + \mathbf{k})\mathbf{h} - DS(\overline{\mathbf{v}})\mathbf{h}.$$

Then (ω, ζ, θ) satisfies the following equations

$$\begin{cases} \frac{\partial\omega(x,t)}{\partial t} + \nabla\cdot(\overline{\mathbf{u}}\omega + \mathbf{u}\omega^k)(x,t) + \nabla\cdot(\zeta\overline{\varphi} + \zeta^k\varphi)(x,t) = \Delta\theta(x,t), (x,t) \in \Omega_T, \\ \theta(x,t) = -\Delta\omega(x,t) + f'(\overline{\varphi})\omega(x,t) + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k(x,t), (x,t) \in \Omega_T, \\ -\Delta\zeta(x,t) + \zeta(x,t) = -\nabla p(x,t) - \overline{\varphi}\nabla\theta(x,t) - \varphi\nabla\theta^k(x,t) - \omega\nabla\overline{\mu}(x,t) \\ \qquad -\omega^k\nabla\mu(x,t), (x,t) \in \Omega_T, \\ \nabla\cdot\zeta(x,t) = 0, (x,t) \in \Omega_T, \\ \zeta(x,t) = 0, \quad \frac{\partial\omega(x,t)}{\partial\mathbf{n}} = \frac{\partial\theta(x,t)}{\partial\mathbf{n}} = 0, (x,t) \in \partial\Omega_T, \\ \omega(x,0) = 0, x \in \Omega. \end{cases} \tag{33}$$

Taking the inner product of the first equation and the third equation of (33) with $-\Delta\omega$ and ζ, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|^2_{L^2(\Omega)} + \|\nabla\Delta\omega\|^2_{L^2(\Omega)} + \|\zeta\|^2_{H^1(\Omega)} \\ = -\int_\Omega \nabla\Delta\omega\cdot(\overline{\mathbf{u}}\omega + \mathbf{u}\omega^k + \zeta^k\varphi)dx - \int_\Omega (\zeta\overline{\varphi} - \nabla\Delta\omega)\cdot\nabla(f'(\bar{\varphi})\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k)dx \\ - \int_\Omega \zeta\cdot(\varphi\nabla\theta^k + \omega\nabla\overline{\mu} + \omega^k\nabla\mu)dx,$$

such that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|^2_{L^2(\Omega)} + \|\nabla\Delta\omega\|^2_{L^2(\Omega)} + \|\zeta\|^2_{H^1(\Omega)}$$

$$\begin{aligned}
\leq & \|\nabla\Delta\omega\|_{L^2(\Omega)}(\|\overline{\mathbf{u}}\|_{L^6(\Omega)}\|\omega\|_{L^3(\Omega)} + \|\mathbf{u}\|_{L^6(\Omega)}\|\omega^k\|_{L^3(\Omega)} + \|\zeta^k\|_{L^6(\Omega)}\|\varphi\|_{L^3(\Omega)}) \\
& + (\|\zeta\|_{L^6(\Omega)}\|\overline{\varphi}\|_{L^3(\Omega)} + \|\nabla\Delta\omega\|_{L^2(\Omega)})\|\nabla(f'(\bar{\varphi})\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k)\|_{L^2(\Omega)} \\
& + \|\zeta\|_{L^6(\Omega)}(\|\varphi\|_{L^3(\Omega)}\|\nabla\theta^k\|_{L^2(\Omega)} + \|\omega\|_{L^3(\Omega)}\|\nabla\overline{\mu}\|_{L^2(\Omega)} + \|\omega^k\|_{L^3(\Omega)}\|\nabla\mu\|_{L^2(\Omega)}) \\
\leq & c\|\nabla\Delta\omega\|_{L^2(\Omega)}(\|\overline{\mathbf{u}}\|_{H^1(\Omega)}\|\nabla\omega\|_{L^2(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)}\|\nabla\omega^k\|_{L^2(\Omega)} + \|\zeta^k\|_{H^1}\|\nabla\varphi\|_{L^2(\Omega)}) \\
& + (c\|\zeta\|_{H^1(\Omega)}\|\nabla\overline{\varphi}\|_{L^2(\Omega)} + \|\nabla\Delta\omega\|_{L^2(\Omega)})\|\nabla(f'(\bar{\varphi})\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k)\|_{L^2(\Omega)} \\
& + c\|\zeta\|_{H^1(\Omega)}(\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\theta^k\|_{L^2(\Omega)} + \|\nabla\omega\|_{L^2(\Omega)}\|\nabla\overline{\mu}\|_{L^2(\Omega)} + \|\nabla\omega^k\|_{L^2(\Omega)}\|\nabla\mu\|_{L^2(\Omega)}).
\end{aligned}$$

We can transform it into the following

$$\begin{aligned}
& \frac{d}{dt}\|\nabla\omega\|^2_{L^2(\Omega)} + \|\nabla\Delta\omega\|^2_{L^2(\Omega)} + \|\zeta\|^2_{H^1(\Omega)} \\
\leq & c\|\nabla\Delta\omega\|_{L^2(\Omega)}(\|\overline{\mathbf{u}}\|_{H^1(\Omega)}\|\nabla\omega\|_{L^2(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)}\|\nabla\omega^k\|_{L^2(\Omega)} + \|\zeta^k\|_{H^1}\|\nabla\varphi\|_{L^2(\Omega)}) \\
& + (c\|\zeta\|_{H^1(\Omega)}\|\nabla\overline{\varphi}\|_{L^2(\Omega)} + \|\nabla\Delta\omega\|_{L^2(\Omega)})\|\nabla(f'(\bar{\varphi})\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k)\|_{L^2(\Omega)} \\
& + c\|\zeta\|_{H^1(\Omega)}(\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\theta^k\|_{L^2(\Omega)} + \|\nabla\omega\|_{L^2(\Omega)}\|\nabla\overline{\mu}\|_{L^2(\Omega)} + \|\nabla\omega^k\|_{L^2(\Omega)}\|\nabla\mu\|_{L^2(\Omega)}) \\
\leq & c(\|\overline{\mathbf{u}}\|^2_{H^1(\Omega)}\|\nabla\omega\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{H^1(\Omega)}\|\nabla\omega^k\|^2_{L^2(\Omega)} + \|\zeta^k\|^2_{H^1(\Omega)}\|\nabla\varphi\|^2_{L^2(\Omega)}) \\
& + c(\|\nabla\overline{\varphi}\|^2_{L^2(\Omega)} + 1)\|\nabla(f'(\bar{\varphi})\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k)\|^2_{L^2(\Omega)} \\
& + c(\|\nabla\varphi\|^2_{L^2(\Omega)}\|\nabla\theta^k\|^2_{L^2(\Omega)} + \|\nabla\omega\|^2_{L^2(\Omega)}\|\nabla\overline{\mu}\|^2_{L^2(\Omega)} + \|\nabla\omega^k\|^2_{L^2(\Omega)}\|\nabla\mu\|^2_{L^2(\Omega)}).
\end{aligned} \tag{34}$$

As for the last term of the penultimate row in (34), we find

$$\begin{aligned}
& \|\nabla(f'(\bar{\varphi})\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\omega^k)\|^2_{L^2(\Omega)} \\
= & \|f'(\bar{\varphi})\nabla\omega + f''(\bar{\varphi})\nabla\overline{\varphi}\omega + (f'(\varphi^k) - f'(\bar{\varphi}))\nabla\omega^k + f''(\overline{\varphi})\nabla\varphi\omega^k + (f''(\varphi^k) - f''(\overline{\varphi}))\nabla\varphi^k\omega^k\|^2_{L^2(\Omega)} \\
= & \|f'(\bar{\varphi})\nabla\omega\|^2_{L^2(\Omega)} + \|f''(\bar{\varphi})\nabla\bar{\varphi}\omega\|^2_{L^2(\Omega)} + \|f'(\varphi^k) - f'(\overline{\varphi}))\nabla\omega^k\|^2_{L^2(\Omega)} \\
& + \|f''(\bar{\varphi})\nabla\varphi\omega^k\|^2_{L^2(\Omega)} + \left\|\omega^k\varphi\nabla\varphi^k\int_0^1 f'''(\tau\varphi^k + (1-\tau)\bar{\varphi})d\tau\right\|^2_{L^2(\Omega)} \\
\leq & \|f'(\bar{\varphi})\|^2_{L^\infty(\Omega)}\|\nabla\omega\|^2_{L^2(\Omega)} + \|f''(\bar{\varphi})\|^2_{L^\infty(\Omega)}\|\nabla\overline{\varphi}\|^2_{L^6(\Omega)}\|\omega\|^2_{L^3(\Omega)} \\
& + c\|\nabla\varphi\|^2_{L^2(\Omega)}(1 + \|\nabla\varphi^k\|^2_{L^2(\Omega)} + \|\nabla\overline{\varphi}\|^2_{L^2(\Omega)})\|\nabla\omega^k\|^2_{L^6(\Omega)} \\
& + c\|\nabla\varphi\|^2_{L^2(\Omega)}\|\Delta\varphi^k\|^2_{L^2(\Omega)}\|\nabla\omega^k\|^2_{L^2(\Omega)}\left\|\int_0^1 f'''(\tau\varphi^k + (1-\tau)\bar{\varphi})d\tau\right\|^2_{L^\infty(\Omega)} \\
& + \|f''(\bar{\varphi})\|^2_{L^\infty(\Omega)}\|\nabla\varphi\|^2_{L^2(\Omega)}\|\omega^k\|^2_{L^\infty(\Omega)}.
\end{aligned}$$

We infer from Theorem 2.1 and Theorem 3.2 that

$$\begin{aligned}
& \max_{t\in[0,T]}\|\nabla\omega(t)\|^2_{L^2(\Omega)} + \int_0^T\|\nabla\Delta\omega(t)\|^2_{L^2(\Omega)}dt + \int_0^T\|\zeta(t)\|^2_{H^1(\Omega)}dt \\
\leq & \frac{C_3}{2}\|\mathbf{k}\|^2_{L^2(0,T;\mathbf{H})}\|\mathbf{h}\|^2_{L^2(0,T;\mathbf{H})},
\end{aligned} \tag{35}$$

where $C_3 > 0$ depends on $R,\ T$ and $\|\varphi_0\|_{H^1(\Omega)}$.

Furthermore, with the help of (33) we calculate the following two terms

$$\begin{aligned}&\left\|\frac{\partial\omega}{\partial t}\right\|_{(H^1(\Omega))^*}\\ \leq&\|\overline{\mathbf{u}}\|_{L^6(\Omega)}\|\omega\|_{L^3(\Omega)}+\|\mathbf{u}\|_{L^6(\Omega)}\left\|\omega^k\right\|_{L^3(\Omega)}+\|\zeta\|_{L^6(\Omega)}\|\overline{\varphi}\|_{L^3(\Omega)}\\ &+\left\|\zeta^k\right\|_{L^6(\Omega)}\|\varphi\|_{L^3(\Omega)}+\|\nabla\theta\|_{L^2(\Omega)},\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega\theta dx\right|&=\left|\int_\Omega f(\overline{\varphi})\omega+(f(\varphi^k)-f(\overline{\varphi}))\omega^k dx\right|\\ &\leq\|f(\overline{\varphi})\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega)}+c\|\varphi\|_{L^3(\Omega)}(1+\left\|\varphi^k\right\|_{L^6(\Omega)}+\|\overline{\varphi}\|_{L^6(\Omega)})\left\|\omega^k\right\|_{L^2(\Omega)}\\ &\leq c(1+\|\overline{\varphi}\|^2_{L^4(\Omega)})\|\omega\|_{L^2(\Omega)}+c\|\nabla\varphi\|_{L^2(\Omega)}(1+\left\|\nabla\varphi^k\right\|_{L^2(\Omega)}+\|\nabla\overline{\varphi}\|_{L^2(\Omega)})\left\|\nabla\omega^k\right\|_{L^2(\Omega)},\end{aligned}$$

such that we derive

$$\begin{aligned}&\int_0^T\|\omega_t(t)\|^2_{(H^1(\Omega))^*}dt+\int_0^T\|\theta(t)\|^2_{H^1(\Omega)}dt\\ \leq&\frac{C_3}{2}\|\mathbf{k}\|^2_{L^2(0,T;\mathbf{H})}\|\mathbf{h}\|^2_{L^2(0,T;\mathbf{H})}.\end{aligned}$$

□

We will derive the variational inequality that the optimal control satisfies. Indeed, it follows from the quadratic form of J that the reduced cost functional $\mathrm{J}(\mathbf{v})=J(S(\mathbf{v}),\mathbf{v})$ is Fréchet differentiable at every $\overline{\mathbf{v}}\in U$ with the *Fréchet* derivative given by

$$\mathrm{DJ}(\overline{\mathbf{v}})=D_{(\overline{\varphi},\overline{\mathbf{u}},\overline{\mu})}J(S(\overline{\mathbf{v}}),\overline{\mathbf{v}})\circ DS(\overline{\mathbf{v}})+D_{\overline{\mathbf{v}}}J(S(\overline{\mathbf{v}}),\overline{\mathbf{v}}).$$

Recalling that U_{ad} is a closed and convex subset of $L^2(0,T;\mathbf{L}^2(\Omega))$, we know that for any minimizer $\overline{\mathbf{v}}\in U_{ad}$,

$$D\mathrm{J}(\overline{\mathbf{v}})(\mathbf{v}-\overline{\mathbf{v}})\geq 0,\quad\forall\mathbf{v}\in U_{ad}.$$

Hence, we derive the following result.

Theorem 3.5. *Suppose that $(H_1)-(H_3)$ are fulfilled. If $\overline{\mathbf{v}}\in U_{ad}$ is an optimal control for the control problem (CP) with associated state $S(\overline{\mathbf{v}})=(\overline{\varphi},\overline{\mathbf{u}},\overline{\mu})$, then for any $\mathbf{v}\in U_{ad}$ and any $t\in[0,T]$, we have*

$$\begin{aligned}&\lambda_0\int_0^T\int_\Omega(\bar{\varphi}(x,t)-\varphi_{\Omega_T}(x,t))\omega(x,t)dxdt+\lambda_1\int_\Omega(\bar{\varphi}(x,T)-\varphi_\Omega(x))\omega(x,T)dx\\ &+\lambda_2\int_0^T\int_\Omega(\overline{\mathbf{u}}(x,t)-\mathbf{u}_{\Omega_T}(x,t))\cdot\zeta(x,t)dxdt+\lambda_3\int_0^T\int_\Omega\overline{\mathbf{v}}(x,t)\cdot(\mathbf{v}(x,t)-\overline{\mathbf{v}}(x,t))dxdt\geq 0,\end{aligned}\tag{36}$$

where $(\omega,\zeta,\theta)=DS(\overline{\mathbf{v}})(\mathbf{v}-\overline{\mathbf{v}})$ is the unique weak solution for the linearized system (18) with $\mathbf{h}=\mathbf{v}-\overline{\mathbf{v}}$.

In order to establish the necessary first-order optimality conditions for (**CP**), we need to eliminate ζ and ω from (36). To this end, we will prove the well-posedness of weak solution of the adjoint state system and simplify the inequality (36). To this aim, we need to give the following result about the regularity of weak solution for the state system.

Theorem 3.6. *Assume that $\varphi_0\in H^2(\Omega)$ in system (1). Then for any $\mathbf{v}\in U$, there exists a constant $C_4>0$ depending only on R, T and the initial data of the system such that the* *associated solution* $(\varphi, \mathbf{u}, \mu)$ *for the state system satisfies*

$$\max_{t\in[0,T]} \|\varphi(t)\|_{H^2(\Omega)} + \|\varphi\|_{L^2(0,T;H^4(\Omega))} \le C_4.$$

Proof. Multiplying the first equation of (1) by $\Delta^2\varphi$ and integrating the resulting equation, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta\varphi\|^2_{L^2(\Omega)} + \left\|\Delta^2\varphi\right\|^2_{L^2(\Omega)} \\
&= -\int_\Omega \mathbf{u}\cdot\nabla\varphi\Delta^2\varphi dx + \int_\Omega \Delta f(\varphi)\Delta^2\varphi dx \\
&\le \|\mathbf{u}\|_{L^6(\Omega)}\|\nabla\varphi\|_{L^3(\Omega)}\left\|\Delta^2\varphi\right\|_{L^2(\Omega)} + \|\Delta f(\varphi)\|_{L^2(\Omega)}\left\|\Delta^2\varphi\right\|_{L^2(\Omega)} \\
&\le c(\|\mathbf{u}\|^2_{H^1(\Omega)}\|\Delta\varphi\|^2_{L^2(\Omega)} + \|\Delta f(\varphi)\|^2_{L^2(\Omega)}) + \frac{1}{2}\left\|\Delta^2\varphi\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{37}$$

By using $Ladyzhenskaya's$ inequality and $Young$ inequality, we observe that

$$\begin{aligned}
&\|\Delta f(\varphi)\|^2_{L^2(\Omega)} \\
&= \left\|f''(\varphi)(\nabla\varphi)^2 + f'(\varphi)\Delta\varphi\right\|^2_{L^2(\Omega)} \\
&\le c\|f''(\varphi)\|^2_{L^\infty(\Omega)}\|\nabla\varphi\|^4_{L^4(\Omega)} + c\|f'(\varphi)\|^2_{L^\infty(\Omega)}\|\Delta\varphi\|^2_{L^2(\Omega)} \\
&\le c\|f''(\varphi)\|^2_{L^\infty(\Omega)}\|\Delta\varphi\|^3_{L^2(\Omega)} + c\|f'(\varphi)\|^2_{L^\infty(\Omega)}\|\Delta\varphi\|^2_{L^2(\Omega)}.
\end{aligned} \tag{38}$$

With the help of the assumptions on f, we find

$$\|f''(\varphi)\|^2_{L^\infty(\Omega)} \le c(1 + \|\varphi\|_{L^\infty(\Omega)})^2 \le c(1 + \|\Delta\varphi\|_{L^2(\Omega)}), \tag{39}$$

$$\|f'(\varphi)\|^2_{L^\infty(\Omega)} \le c(1 + \|\varphi\|^2_{L^\infty(\Omega)})^2 \le c(1 + \|\Delta\varphi\|^2_{L^2(\Omega)}). \tag{40}$$

Exploiting the inequalities (11), (37), (38), (39) and (40), we derive

$$\frac{d}{dt}\|\Delta\varphi\|^2_{L^2(\Omega)} + \left\|\Delta^2\varphi\right\|^2_{L^2(\Omega)} \le c(1 + \|\mathbf{u}\|^2_{H^1(\Omega)} + \|\Delta\varphi\|^2_{L^2(\Omega)})\|\Delta\varphi\|^2_{L^2(\Omega)},$$

which yields

$$\max_{t\in[0,T]}\|\Delta\varphi\|^2_{L^2(\Omega)} + \int_0^T \left\|\Delta^2\varphi(t)\right\|^2_{L^2(\Omega)}dt \le C_4. \tag{41}$$

□

Theorem 3.7. *Assume* $(H_1)-(H_3)$ *and* $\overline{\varphi}_0 \in H^2(\Omega)$ *hold. If* $\overline{\mathbf{v}} \in U_{ad}$ *is an optimal control for the control problem* (**CP**) *with the associated state* $S(\overline{\mathbf{v}}) = (\overline{\varphi}, \overline{\mathbf{u}}, \overline{\mu})$, *then the following adjoint state system*

$$\begin{cases}
-\frac{\partial\rho(x,t)}{\partial t} - \nabla\cdot(\overline{\mathbf{u}}\rho)(x,t) + \nabla\cdot(\vartheta\overline{\mu})(x,t) - \Delta\xi(x,t) + f'(\overline{\varphi})\xi(x,t) = \lambda_0(\bar{\varphi} - \varphi_{\Omega_T})(x,t), \\
(x,t)\in\Omega_T, \\
\xi(x,t) = -\Delta\rho(x,t) - \nabla\cdot(\vartheta\overline{\varphi})(x,t), \quad (x,t)\in\Omega_T, \\
-\Delta\vartheta(x,t) + \vartheta(x,t) = -\nabla q(x,t) + \overline{\varphi}\nabla\rho(x,t) + \lambda_2(\overline{\mathbf{u}}(x,t) - \mathbf{u}_{\Omega_T}(x,t)), \quad (x,t)\in\Omega_T, \\
\nabla\cdot\vartheta(x,t) = 0, (x,t)\in\Omega_T, \\
\vartheta(x,t) = 0, \quad \frac{\partial\rho(x,t)}{\partial\mathbf{n}} = \frac{\partial\xi(x,t)}{\partial\mathbf{n}} = 0, \quad (x,t)\in\partial\Omega_T, \\
\rho(x,T) = \lambda_1(\overline{\varphi}(x,T) - \varphi_{\Omega_T}(x)), \quad x\in\Omega,
\end{cases} \tag{42}$$

has a unique solution $(\rho, \vartheta, \xi) \in \mathrm{H}$ *and for any* $\mathbf{v} \in U_{ad}$, *we have*

$$\int_0^T \int_\Omega (\vartheta(x,t) + \lambda_1\overline{\mathbf{v}}(x,t))\cdot(\mathbf{v}(x,t) - \overline{\mathbf{v}}(x,t))dxdt \ge 0, \tag{43}$$

where $\mathrm{H} = \Big(L^\infty\left(0,T;L^2(\Omega)\right)\cap H^1\Big(0,T;\left(H^2(\Omega)\right)^*\Big)\cap L^2\left(0,T;H^2(\Omega)\right)\Big)\times L^2\left(0,T;H^1(\Omega)\right)\times L^2\left(0,T;L^2(\Omega)\right)$.

Proof. As mentioned in the proof of Theorem 3.2, let A be the Stokes operator [26] and let N be the inverse of the Laplacian operator $-\Delta$ endowed with Neumann boundary condition imposing zero average over Ω. Furthermore, let $\{\omega_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ be the eigenfunctions of the operator A and N, respectively. For any $n\geq 1$, we introduce two finite-dimensional spaces $W_n = span\{\psi_1,\psi_2...,\psi_n\}$ and $\mathbf{H}_n = span\{\omega_1,\omega_2...,\omega_n\}$. Let P_n be the orthogonal projector from $L^2(\Omega)$ onto W_n and let Γ_n be the orthogonal projector from $\mathbf{H}$ onto $\mathbf{H}_n$.

We will look for appropriate coefficients $(\alpha_i(t),\beta_i(t),\gamma_i(t))(i=1,2..n)$ so that

$$\rho_n(t)=\sum_{i=1}^n\alpha_i(t)\psi_i,$$
$$\vartheta_n(t)=\sum_{i=1}^n\beta_i(t)\omega_i,$$
$$\xi_n(t)=\sum_{i=1}^n\gamma_i(t)\psi_i.$$

These approximate solutions satisfy the system

$$\begin{cases}-\left\langle\frac{\partial\rho_n}{\partial t},\psi\right\rangle+(\overline{\mathbf{u}}\rho_n-\vartheta_n\overline{\mu},\nabla\psi)+(\nabla\xi_n,\nabla\psi)+\langle f'(\bar\varphi)\xi_n-\lambda_0(\overline{\varphi}(x,t)-\varphi_{\Omega_T}),\psi\rangle=0,\\(\xi_n,\sigma)=(\nabla\rho_n,\nabla\sigma)+(\vartheta_n\overline{\varphi},\nabla\sigma),\\(\nabla\vartheta_n,\nabla\phi)+(\vartheta_n,\phi)=\langle\overline{\varphi}\nabla\rho_n+\lambda_2(\overline{\mathbf{u}}-\mathbf{u}_{\Omega_T}),\phi\rangle,\\(\rho_n(\cdot,T),\psi_i)=(\lambda_3(\overline{\varphi}(\cdot,T)-\varphi_\Omega),\psi_i),\ i=1,2...,n,\end{cases}\tag{44}$$

for any $\phi\in\mathbf{H}_n$ and $\psi,\sigma\in W_n$.

Let $\sigma=1$ in the second equation of (44)

$$\int_\Omega\xi_n(x,t)dx=0.$$

Let $\psi=\rho_n$, $\sigma=P_n(-\Delta\rho_n+f'(\bar\varphi)\rho_n)$ and $\phi=\vartheta_n$ in system (44)

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|\rho_n\|^2_{L^2(\Omega)}+\|\vartheta_n\|^2_{H^1(\Omega)}+\|\Delta\rho_n\|^2_{L^2(\Omega)}\\
=&\int_\Omega(\vartheta_n\cdot\nabla\rho_n)(\overline{\varphi}+\overline{\mu})dx+\lambda_2\int_\Omega(\overline{\mathbf{u}}(x,t)-\mathbf{u}_{\Omega_T})\cdot\vartheta_ndx-\int_\Omega\vartheta_n\cdot\nabla\overline{\varphi}\Delta\rho_ndx\\
&+\int_\Omega f'(\bar\varphi)\rho_n\Delta\rho_ndx-\int_\Omega f(\overline{\varphi})(\vartheta_n\cdot\nabla\rho_n)dx+\lambda_0\int_\Omega(\overline{\varphi}(x,t)-\varphi_{\Omega_T})\rho_ndx\\
=&\int_\Omega(\vartheta_n\cdot\nabla\rho_n)(\overline{\varphi}-\Delta\overline{\varphi})dx+\lambda_2\int_\Omega(\overline{\mathbf{u}}(x,t)-\mathbf{u}_{\Omega_T})\cdot\vartheta_ndx-\int_\Omega\vartheta_n\cdot\nabla\overline{\varphi}\Delta\rho_ndx\\
&+\int_\Omega f'(\bar\varphi)\rho_n\Delta\rho_ndx+\lambda_0\int_\Omega(\overline{\varphi}(x,t)-\varphi_{\Omega_T})\rho_ndx\\
\leq&\|\vartheta_n\|_{L^3(\Omega)}\|\rho_n\|_{L^2(\Omega)}\|\nabla\overline{\varphi}-\nabla\Delta\overline{\varphi}\|_{L^6(\Omega)}+\|\nabla\vartheta_n\|_{L^2(\Omega)}\|\rho_n\|_{L^2(\Omega)}\|\overline{\varphi}-\Delta\overline{\varphi}\|_{L^\infty(\Omega)}\\
&+\|\nabla\vartheta_n\|_{L^2(\Omega)}\|\nabla\bar\varphi\|_{L^6(\Omega)}\|\nabla\rho_n\|_{L^3(\Omega)}+\|\vartheta_n\|_{L^6(\Omega)}\|\Delta\overline{\varphi}\|_{L^2(\Omega)}\|\nabla\rho_n\|_{L^3(\Omega)}\\
&+\|f'(\overline{\varphi})\|_{L^\infty(\Omega)}\|\rho_n\|_{L^2(\Omega)}\|\Delta\rho_n\|_{L^2(\Omega)}+\lambda_0\|\overline{\varphi}(x,t)-\varphi_{\Omega_T}\|_{L^2(\Omega)}\|\rho_n\|_{L^2(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_2 \|\overline{\mathbf{u}}(x,t) - \mathbf{u}_{\Omega_T}\|_{L^2(\Omega)} \|\vartheta_n\|_{L^2(\Omega)} \\
\leq & c\|\vartheta_n\|_{H^1(\Omega)} \|\rho_n\|_{L^2(\Omega)} (\|\Delta\overline{\varphi}\|_{L^2(\Omega)} + \left\|\Delta^2\overline{\varphi}\right\|_{L^2(\Omega)}) \\
&+ c\|\vartheta_n\|_{H^1(\Omega)} \|\rho_n\|_{L^2(\Omega)} \|\nabla\overline{\varphi} - \nabla\Delta\overline{\varphi}\|^{\frac{1}{2}}_{L^2(\Omega)} \left\|\Delta\overline{\varphi} - \Delta^2\overline{\varphi}\right\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&+ \lambda_2 \|\overline{\mathbf{u}}(x,t) - \mathbf{u}_{\Omega_T}\|_{L^2(\Omega)} \|\vartheta_n\|_{L^2(\Omega)} + c\|\vartheta_n\|_{H^1(\Omega)} \|\bar{\varphi}\|_{H^2(\Omega)} \|\nabla\rho_n\|_{L^3(\Omega)} \\
&+ c\|\rho_n\|_{L^2(\Omega)} \|\Delta\rho_n\|_{L^2(\Omega)} (1 + \|\Delta\overline{\varphi}\|_{L^2(\Omega)}) + \lambda_0 \|\overline{\varphi}(x,t) - \varphi_{\Omega_T}\|_{L^2(\Omega)} \|\rho_n\|_{L^2(\Omega)}.
\end{aligned}
$$

After simplification, we obtain

$$
\begin{aligned}
&- \frac{d}{dt} \|\rho_n\|^2_{L^2(\Omega)} + \|\vartheta_n\|^2_{H^1(\Omega)} + \|\Delta\rho_n\|^2_{L^2(\Omega)} \\
\leq & c(1 + \|\Delta\bar{\varphi}\|^2_{L^2(\Omega)} + \left\|\Delta^2\bar{\varphi}\right\|^2_{L^2(\Omega)} + \|\nabla\Delta\bar{\varphi}\|^2_{L^2(\Omega)} + \|\Delta\bar{\varphi}\|^4_{L^2(\Omega)} + \|\Delta\bar{\varphi}\|^8_{L^2(\Omega)}) \|\rho_n\|^2_{L^2(\Omega)} \\
&+ c \|\overline{\mathbf{u}}(x,t) - \mathbf{u}_{\Omega_T}\|^2_{L^2(\Omega)} + c \|\overline{\varphi}(x,t) - \varphi_{\Omega_T}\|^2_{L^2(\Omega)},
\end{aligned} \tag{45}
$$

where we have used Gagliardo-Nirenberg interpolation inequality and Young's inequality. Integrating the above inequality over $[t, T]$, we infer from the Gronwall inequality and Theorem 4.6 that

$$
\begin{aligned}
&\max_{s\in[t,T]} \|\rho_n(s)\|^2_{L^2(\Omega)} + \int_0^T \|\vartheta_n(s)\|^2_{H^1(\Omega)}\, ds + \int_0^T \|\Delta\rho_n(s)\|^2_{L^2(\Omega)} ds \\
&\leq c(R).
\end{aligned} \tag{46}
$$

Furthermore, we find

$$
\begin{aligned}
\left\|\frac{\partial\rho_n}{\partial t}\right\|_{(H^2(\Omega))^*} \leq & \|\overline{\mathbf{u}}\|_{L^3(\Omega)} \|\rho_n\|_{L^2(\Omega)} + \|\vartheta_n\|_{L^3(\Omega)} \|\overline{\mu}\|_{L^2(\Omega)} + \|\xi_n\|_{L^2(\Omega)} \\
&+ \|f(\overline{\varphi})\|_{L^3(\Omega)} \|\xi_n\|_{L^2(\Omega)} + \lambda_0 \|\overline{\varphi}(x,t) - \varphi_{\Omega_T}\|_{L^2(\Omega)} \\
\leq & c\|\overline{\mathbf{u}}\|_{H^1(\Omega)} \|\rho_n\|_{L^2(\Omega)} + c\|\vartheta_n\|_{H^1(\Omega)} + \|\xi_n\|_{L^2(\Omega)} + c(1 + \|\overline{\varphi}\|^2_{L^6(\Omega)}) \|\xi_n\|_{L^2(\Omega)} \\
&+ \lambda_0 \|\overline{\varphi}(x,t) - \varphi_{\Omega_T}\|_{L^2(\Omega)},
\end{aligned}
$$

and

$$
\|\xi_n\|_{L^2(\Omega)} \leq \|\Delta\rho_n\|_{L^2(\Omega)} + \|\vartheta\|_{L^3(\Omega)} \|\nabla\overline{\varphi}\|_{L^6(\Omega)}.
$$

Therefore, we obtain

$$
\begin{aligned}
&\{\rho_n\}_{n=1}^{\infty} \ \textit{is uniformly bounded in}\ L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\
&\{\vartheta_n\}_{n=1}^{\infty} \ \textit{is uniformly bounded in}\ L^2(0,T;\mathbf{V}), \\
&\{\xi_n\}_{n=1}^{\infty} \ \textit{is uniformly bounded in}\ L^2(0,T;L^2(\Omega)), \\
&\{(\rho_n)_t\}_{n=1}^{\infty} \ \textit{is uniformly bounded in}\ L^2(0,T;(H^2(\Omega))^*),
\end{aligned}
$$

which entail that there exist

$$
\begin{aligned}
&\rho \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\
&\vartheta \in L^2(0,T;\mathbf{V}), \\
&\xi \in L^2(0,T;L^2(\Omega)), \\
&\rho_t \in L^2(0,T;(H^2(\Omega))^*)
\end{aligned}
$$

and with the help of the Kakutani theorem and the Banach-Alaoglu-Bourbaki theorem we can extract subsequences $\{\rho_{n_j}\}_{j=1}^{\infty}$, $\{\vartheta_{n_j}\}_{j=1}^{\infty}$, $\{\xi_{n_j}\}_{j=1}^{\infty}$, $\left\{\frac{\partial\rho_{n_j}}{\partial t}\right\}_{j=1}^{\infty}$ of $\{\rho_n\}_{n=1}^{\infty}$, $\{\vartheta_n\}_{n=1}^{\infty}$,

$\{\xi_n\}_{n=1}^{\infty}$, $\left\{\frac{\partial\rho_n}{\partial t}\right\}_{n=1}^{\infty}$ such that they satisfy

$$\begin{aligned}
\rho_{n_j} &\rightharpoonup \rho \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)),\\
\rho_{n_j} &\rightharpoonup \rho \textit{ weakly in } L^2(0,T;H^2(\Omega)),\\
\vartheta_{n_j} &\rightharpoonup \vartheta \textit{ weakly in } L^2(0,T;\mathbf{V}),\\
\xi_{n_j} &\rightharpoonup \xi \textit{ weakly in } L^2(0,T;L^2(\Omega)),\\
\frac{\partial\rho_{n_j}}{\partial t} &\rightharpoonup \frac{\partial\rho}{\partial t} \textit{ weakly in } L^2(0,T;(H^2(\Omega))^*).
\end{aligned}$$

Passing to the limit, we obtain the existence of weak solution for system (42).

In the following, we will prove the uniqueness of solution for the system (42). Let $(\rho_1,\vartheta_1,\xi_1,q_1)$, $(\rho_2,\vartheta_2,\xi_2,q_2)$ be two weak solutions for the system (42) and let $\rho=\rho_1-\rho_2$, $\vartheta=\vartheta_1-\vartheta_2$, $\xi=\xi_1-\xi_2$ and $q=q_1-q_2$, then (ρ,ϑ,ξ,q) satisfies the following equations

$$\begin{cases}
-\frac{\partial\rho(x,t)}{\partial t}-\nabla\cdot(\overline{\mathbf{u}}\rho)(x,t)+\nabla\cdot(\vartheta\overline{\mu})(x,t)-\Delta\xi(x,t)+f'(\overline{\varphi})\xi(x,t)=0, & (x,t)\in\Omega_T,\\
\xi(x,t)=-\Delta\rho(x,t)-\nabla\cdot(\vartheta\overline{\varphi})(x,t), & (x,t)\in\Omega_T,\\
-\Delta\vartheta(x,t)+\vartheta(x,t)=-\nabla q(x,t)+\overline{\varphi}\nabla\rho(x,t), & (x,t)\in\Omega_T,\\
\nabla\cdot\vartheta(x,t)=0, & (x,t)\in\Omega_T,\\
\vartheta(x,t)=0,\quad \frac{\partial\rho(x,t)}{\partial\mathbf{n}}=\frac{\partial\xi(x,t)}{\partial\mathbf{n}}=0, & (x,t)\in\partial\Omega_T,\\
\rho(x,T)=0, & x\in\Omega.
\end{cases}$$

Repeating the proof of (46) with $\lambda_0=\lambda_1=\lambda_2=0$, we get$\rho_1=\rho_2$, $\vartheta_1=\vartheta_2$, $\xi_1=\xi_2$, and we can derive the result.

Multiplying the first equation, the second equation and the third equation of (15) with ρ, ϑ and ξ, respectively, and integrating over Ω and $[0,T]$, for $\mathbf{h}=\mathbf{v}-\overline{\mathbf{v}}$ we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega\frac{\partial\omega}{\partial t}\rho dxdt+\int_0^T\int_\Omega\nabla\cdot(\overline{\mathbf{u}}\omega)\rho dxdt+\int_0^T\int_\Omega\nabla\cdot(\zeta\overline{\varphi})\rho dxdt+\int_0^T\int_\Omega\nabla\theta\cdot\nabla\rho dxdt\\
&\int_0^T\int_\Omega\nabla\omega\cdot\nabla\xi dxdt+\int_0^T\int_\Omega f'(\overline{\varphi})\omega\xi dxdt-\int_0^T\int_\Omega\theta\xi dxdt+\int_0^T\int_\Omega\nabla\zeta\cdot\nabla\vartheta dxdt\\
&+\int_0^T\int_\Omega\zeta\cdot\vartheta dxdt+\int_0^T\int_\Omega\overline{\varphi}(\vartheta\cdot\nabla\theta)dxdt+\int_0^T\omega(\vartheta\cdot\nabla\overline{\mu})dxdt\\
&=\int_0^T\int_\Omega(\mathbf{v}-\overline{\mathbf{v}})\cdot\vartheta dxdt,
\end{aligned}\tag{47}$$

where (ρ,ϑ,ξ) is the weak solution of (42). Taking the inner product of the first equation, the second equation and the third equation of (42) with ω, θ and ζ, we get

$$\begin{aligned}
&-\int_0^T\int_\Omega\frac{\partial\rho}{\partial t}\omega dxdt-\int_0^T\int_\Omega\nabla\cdot(\overline{\mathbf{u}}\rho)\omega dxdt+\int_0^T\int_\Omega\nabla\cdot(\vartheta\overline{\mu})\omega dxdt+\int_0^T\int_\Omega\nabla\xi\cdot\nabla\omega dxdt\\
&+\int_0^T\int_\Omega f'(\overline{\varphi})\xi\omega dxdt+\int_0^T\int_\Omega\nabla\rho\cdot\nabla\theta dxdt-\int_0^T\int_\Omega\nabla\cdot(\vartheta\overline{\varphi})\theta dxdt-\int_0^T\int_\Omega\xi\theta dxdt\\
&+\int_0^T\int_\Omega\nabla\vartheta\cdot\nabla\zeta dxdt+\int_0^T\int_\Omega\vartheta\cdot\zeta dxdt\\
=&\int_0^T\int_\Omega(\overline{\varphi}\zeta)\cdot\nabla\rho dxdt+\lambda_0\int_0^T\int_\Omega(\overline{\varphi}(x,t)-\varphi_{\Omega_T})\omega dxdt+\lambda_2\int_0^T\int_\Omega(\overline{\mathbf{u}}(x,t)
\end{aligned}$$

$$- \mathbf{u}(x,t) \cdot \zeta dxdt, \tag{48}$$

where (ω, ζ, θ) is the weak solution of (15) with $\mathbf{h} = \mathbf{v} - \overline{\mathbf{v}}$. we infer from equation (47) and equation (48) that

$$\begin{aligned}
&\lambda_0 \int_{\Omega_T} (\bar{\varphi}(x,t) - \varphi_{\Omega_T})\omega dxdt \\
&+ \lambda_1 \int_{\Omega} (\bar{\varphi}(x,T) - \varphi_{\Omega})\omega(x,T)dx + \lambda_2 \int_{\Omega_T} (\overline{\mathbf{u}}(x,t) - \mathbf{u}(x,t)) \cdot \zeta dxdt \\
&= \int_0^T \int_{\Omega} (\mathbf{v} - \overline{\mathbf{v}}) \cdot \vartheta dxdt.
\end{aligned} \tag{49}$$

Therefore, from Theorem 4.5 and equation (49), we deduce

$$\int_0^T \int_{\Omega} (\mathbf{v}(x,t) - \overline{\mathbf{v}}(x,t)) \cdot (\vartheta + \lambda_3 \overline{\mathbf{v}}(x,t)) dxdt \geq 0. \tag{50}$$

□

Remark 3.1. *The system (1), the adjoint system (42) and the variational inequality (50) form together the first-order necessary optimality conditions. Moreover, since U_{ad} is a nonempty, convex and closed subset of $L^2(0,T;\mathbf{H})$, then from (50) we derive the following conclusions:*

(1) In the case $\lambda_3 > 0$, the optimal control $\overline{\mathbf{v}}$ is equivalent to the following condition

$$\overline{\mathbf{v}}(x,t) = \mathrm{P}(-\frac{\vartheta}{\lambda_3}),$$

where P is the orthogonal projector in $L^2(0,T;\mathbf{H})$ onto U_{ad}. Thanks to projection property, we obtain

$$\overline{\mathbf{v}}_i(x,t) = \max\left\{(\mathbf{v}_i)_{\min}, \min\left\{-\lambda_3^{-1}\vartheta_i, (\mathbf{v}_i)\right\}_{\max}\right\}, i = 1,2,3, \ for\ a.\ e.\ (x,t) \in \Omega_T,$$

where $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(2) In the case $\lambda_3 = 0$, we derive that

$$\overline{\mathbf{v}}_i(x,t) = \begin{cases} (\mathbf{v}_i)_{\min}, & if\ \ \vartheta_i > 0, \\ (\mathbf{v}_i)_{\max}, & if\ \ \vartheta_i < 0, \end{cases}$$

for $i = 1,2,3$.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] P Knopf, A Signori. *Existence of weak solutions to multiphase Cahn-Hilliard-Darcy and Cahn-Hilliard-Brinkman models for stratified tumor growth with chemotaxis and general source terms*, Commun Partial Differ Equ, 2022, 47(2): 233-278.

[2] S Bosia, M Conti, M Grasselli. *On the Cahn-Hilliard-Brinkman system*, Commun Math Sci, 2015, 13(6): 1541-1567.

[3] F Li, C Zhong, B You. *Finite-dimensional global attractor of the Cahn-Hilliard-Brinkman system*, J Math Anal Appl, 2016, 434(1): 599-616.

[4] F Li, B You. *Well-Posedness and global attractor of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions*, Dyn Partial Differ Equ, 2016, 13(1): 75-90.

[5] F Li, B You. *On the dimension of global attractor of the Cahn-Hilliard-Brinkman system with dynamic boundary conditions*, Discrete Contin Dyn Syst Ser B, 2021, 26(12): 6387-6403.

[6] Z Pu, X Xiao. *The global attractors of the Cahn-Hilliard-Brinkman system*, Acta Math Sci Ser A, 2022, 42(4): 1027-1040.

[7] F Li, B You. *Optimal distributed control of the Cahn-Hilliard-Brinkman system with regular potential*, Nonlinear Anal, 2019, 182: 226-247.

[8] C Collins, J Shen, S M Wise. *An efficient, energy stable scheme for the Cahn-Hilliard-Brinkman system*, Commun Comput Phys, 2013, 13(4): 929-957.

[9] M Ebenbeck, H Garcke. *Analysis of a Cahn-Hilliard-Brinkman model for tumour growth with chemotaxis*, J Differ Equ, 2019, 266(9): 5998-6036.

[10] S Frigeri, M Grasselli, P Krejci. *Strong solutions for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes system*, J Differ Equ, 2013, 255(9): 2587-2614.

[11] M Ebenbeck, H Garcke. *On a Cahn-Hilliard-Brinkman model for tumour growth and its singular limits*, SIAM J Math Anal, 2019, 51(3): 1868-1912.

[12] M Ebenbeck, P Knopf. *Optimal medication for tumours modelled by a Cahn-Hilliard-Brinkman equation*, Calc Var Partial Differ Equations, 2019, 58: 131.

[13] M Ebenbeck, P Knopf. *Optimal control theory and advanced optimality conditions for a diffuse interface model of tumour growth*, ESAIM Control Optim Calc Var, 2020, 26: 71.

[14] M Grasselli, F D Porta. *On the nonlocal Cahn-Hilliard-Brinkman and Cahn-Hilliard-Hele-Shaw system*, Commun Pure Appl Anal, 2016, 16(2): 299-317.

[15] N M P Lakshmi, D Sheetal. *Nonlocal Cahn-Hilliard-Brinkman system with regular potential: regularity and optimal control*, J Dyn Control Syst, 2021, 27(2): 221-246.

[16] P Colli, M H Farshbaf-Shaker, G Gilardi, et al. *Optimal boundary control of a viscous Cahn-Hilliard system with dynamic boundary conditions and double obstacle potentials*, SIAM J Control Optim, 2015, 53(4): 2696-2721.

[17] P Colli, G Gilardi, J Sprekels. *A boundary control problem for the pure Cahn-Hilliard equation with dynamic boundary conditions*, Adv Nonlinear Anal, 2015, 4(4): 311-325.

[18] A V Fursikov, M D Gunzburger, L S Hou. *Optimal boundary control for the evolutionary Navier-Stokes system: three-dimensional case*, SIAM J Control Optim, 2005, 43(6): 2191-2232.

[19] H Garcke, K F Lam, E Rocca. *Optimal control of treatment time in a diffuse interface model of tumor growth*, Appl Math Optim, 2018, 78(3): 495-544.

[20] M Hintermuller, T Keil, D Wegner. *Optimal control of a semidiscrete Cahn-Hilliard-Navier-Stokes system with nonmatched fluid densities*, SIAM J Control Optim, 2017, 55(3): 1954-1989.

[21] M Hintermuller, D Wegner. *Optimal control of a semidiscrete Cahn-Hilliard-Navier-Stokes system*, SIAM J Control Optim, 2014, 52(1): 747-772.

[22] X Zhao, C Liu. *Optimal control of the convective Cahn-Hilliard equation*, Appl Anal, 2013, 92(5): 1028-1045.

[23] R Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland Amsterdam, 1984.

[24] S Frigeri, E Rocca, J Sprekels. *Optimal distributed control of a nonlocal Cahn-Hilliard/ Navier-Stokes system in two dimensions*, SIAM J Control Optim, 2016, 54(1): 221-250.

[25] R Temam. *Infinite-Dimensional Systems in Mechanics and Physics*, New York: Springer-Verlag, 1997.

[26] H Sohr. *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*, Basel: Birkhäuser Basel, 2001.

School of Mathematics, Sichuan Normal University, Chengdu 610066, China.
Email: 407044728@qq.com