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Sweedler's duality of Hopf algebras in Yetter-Drinfeld's categories and an application

GU Yue[1]   WANG Shuan-hong[2,3,*]

Abstract. Let ${}^H_H\mathcal{YD}$ denote the Yetter-Drinfeld category over a Hopf algebra H with a bijective antipode. Let $(A, m, \mu, \Delta, \varepsilon, S)$ be a Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then we mainly study Sweedler's duality A^0 of A and show that $(A^0, m^{op}, \varepsilon^*, \Delta^{op}, \mu^*, S^*)$ is again a Hopf algebra in ${}^H_H\mathcal{YD}$ generalizing the main result in [8]. As an application of our result, we show that the space of binary-linearly recursive sequences is closed under the quantum convolution product of binary-linearly recursive sequences.

§1 Introduction

The concept of a Hopf algebra contains a symmetry between its algebraic structure and its coalgebraic structure and has many important applications (see [1], [6], [7] and [12]). A theory of linearly recursive sequences related Hopf algebras was first studied in 1980 by Peterson and Taft (see [9]) and later investigated in some papers (see [8], [14], [15], [16], [18] and [20]).

Throughout, $\Bbbk$ will denote a fixed field. All vector spaces will be over $\Bbbk$ and all homomorphisms will be $\Bbbk$-linear maps. Let C be a coalgebra with a coproduct $\Delta : C \longrightarrow C \otimes C$. We will adopt the Sweedler's "sigma" notation about Δ:

$$\Delta(c) = \sum c_1 \otimes c_2$$

for all $c \in C$ (cf. [12]).

In what follows, let (A, m_A, μ_A) be an associative algebra over $\Bbbk$. Then we have the coalgebra A^0 over $\Bbbk$ given by Sweedler in [12] as follows:

$$A^0 = \{ f \in A^* \mid \text{ Ker}f \text{ contains a cofinite ideal, } \}$$

where A^* is the linear dual space, and a cofinite ideal is an ideal J in A possessing that A/J is finite-dimensional. Furthermore, when A is a Hopf algebra, A^0 has a natural Hopf algebra structure which was described in [12, Section 6.2]. However, generally, A^{00} is not isomorphic to A, i.e, the Pontryagin duality does not hold (see [17]).

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*Corresponding author.

For $f \in A^*$ and $a, b \in A$, one sets $(a \vdash f)(b) = f(ba)$ (similarly, $(f \dashv a)(b) = f(ab)$). Then one gets that A^* is a A-A-bimodule. The following states are equivalent: for any $f \in A^*$

(1) $f \in A^0 = (m_A^*)^{-1}(A^* \otimes A^*)$;

(2) $A \vdash f$ is a finite-dimensional vector space;

(3) $dim(f \dashv A) < \infty$;

(4) $A \vdash f \dashv A$ is finite-dimensional.

We denote $\mathbf{LS}_{\Bbbk}$ the category of linear spaces and linear maps. Then $(\mathbf{LS}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ is a symmetric category. We can describe the Sweedler's duality in the following

Theorem A. A^0 forms a braided Hopf algebra in $\mathbf{LS}_{\Bbbk}$.

Given a Hopf algebra H with coproduct Δ, counit ϵ, and antipode S. Then one writes ${}_H\mathcal{M}$ the category of left H-modules. Similarly for ${}^H\mathcal{M}$ the category of left H-comodules.

For $V \in {}^H\mathcal{M}$, we write $\rho(v) = \sum v_{(-1)} \otimes v_0$ for all $v \in V$. A left-left Yetter-Drinfeld module M over H is an object both in ${}_H\mathcal{M}$ and ${}^H\mathcal{M}$ such that

$$\rho(h \cdot m) = \sum h_1 m_{(-1)} S h_3 \otimes h_2 \cdot m_0 \tag{1}$$

holds, for $m \in M$ and $h \in H$.

We have the category ${}^H_H\mathcal{YD}$ of left-left Yetter-Drinfeld modules. The category ${}^H_H\mathcal{YD}$ has the braiding given by

$$\tau_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto \sum m_{(-1)} \cdot n \otimes m_0. \tag{2}$$

We note that the category ${}^H_H\mathcal{YD}$ can provide a solution of quantum Yang-Baxter equation (cf. [6]). Some related further studies can be found in [4], [11] and [19].

In this paper, we will replace the category $\mathbf{LS}_{\Bbbk}$ in Theorem A by the category ${}^H_H\mathcal{YD}$, and then show the following main result.

Theorem B. Let A be a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then A^0 is a braided Hopf algebra in ${}^H_H\mathcal{YD}$.

This paper consists of four sections. In Section 2, we mainly show **Theorem B**, generalizing Ng and Taft's main result in [8].

In Section 3 and Section 4, we show that the space of binary-linearly recursive sequences is closed under the quantum convolution product of binary-linearly recursive sequences (see Theorem 4.1 and Theorem 4.5), generalizing the main result in [8] for the linear case.

In Section 5, one studies the structural properties of the space of binary linearly recursive sequences (see Theorem 5.4).

§2 The proof of Theorem B

In this section, H denotes a Hopf algebra with a bijective antipode S.

Given a bialgebra $(B, m_B, \mu_B, \Delta_B, \varepsilon_B)$ in category ${}^H_H\mathcal{YD}$. Recall from [1] that B^* is a left H-module with

$$(h \cdot f)(b) = f(S(h) \cdot b)$$

with $h, b \in H$ and $f \in B^*$.

Similarly for the coaction of H on B^* given by

$$\rho(f)(b) = S^{-1}(b_{(-1)}) \otimes f(b_0),$$

with $h \in H, b \in B$ and $f \in B^*$.

Given two algebras A and B in ${}^H_H\mathcal{YD}$. One has the algebra $A \underline{\otimes} B$ with the multiplication

$$(x \underline{\otimes} y)(a \underline{\otimes} b) = x(y_{(-1)} \cdot a) \underline{\otimes} y_0 b$$

for $x, a \in A$ and $y, b \in B$.

The following two lemmas are easy results.

Lemma 2.1. With the above notation,

(1) The $\dashv$ is a left H-linear;

(2) The $\vdash$ is a left H^{cop}-linear.

Lemma 2.2. B^0 is an H-submodule of B^*.

Proposition 2.3. B^0 is a subalgebra of B^*.

Proof. Let $f, g \in B^*$. For $a, b \in B$, we compute

$$\begin{aligned}((fg) \dashv a)(b) &= (fg)(ab) = (f \otimes g)\Delta(ab) \\ &= f(a_1(a_{2(-1)} \cdot b_1)g(a_{20}b_2) \\ &= f(a_{2(-1)2} \cdot [(S^{-1}(a_{2(-1)1}) \cdot a_1)b_1])g(a_{20}b_2) \\ &= (S^{-1}(a_{2(-1)2}) \cdot f)[(S^{-1}(a_{2(-1)1}) \cdot a_1)b_1]g(a_{20}b_2) \\ &= [(S^{-1}(a_{2(-1)2}) \cdot f) \dashv (S^{-1}(a_{2(-1)1}) \cdot a_1)](b_1)(g \dashv a_{20})(b_2) \\ &= \Delta^*[(S^{-1}(a_{2(-1)}) \cdot (f \dashv a_1) \otimes g \dashv a_{20})](b),\end{aligned}$$

where we apply that B is an H-module algebra to the third equation above and use Lemma 2.1 to the final equation.

Therefore, we have

$$\begin{aligned}(fg) \dashv B &\subseteq \Delta^*[H \cdot (f \dashv B) \otimes g \dashv B] \\ &\subseteq \Delta^*[(H \cdot f) \dashv B) \otimes g \dashv B].\end{aligned}$$

By $f \in B^0$ and Lemma 2.2, one obtains $H \cdot f \in B^0$. Applying $f, g \in B^0$ and the right-hand side of the above containment is finite-dimensional, so $fg \in B^0$. Finally, it is straight to show that $\varepsilon_B^*(1) \in B^0$.

This concludes the proof. ■

Lemma 2.4. $i \circ \tau : (B^*)^{op} \underline{\otimes} (B^*)^{op} \longrightarrow (B \underline{\otimes} B)^{*op}$ is a homomorphism as algebra in ${}^H_H\mathcal{YD}$

Proof. It follows the definition of the coaction of H on B^*. ■

Then we can get Theorem B as follows.

Theorem 2.5. Assume that $(B, m_B, \mu_B, \Delta_B, \varepsilon_B, S_B)$ is a Hopf algebra in ${}^H_H\mathcal{YD}$. Then $(B^0, (m_{B^0})^{op}, \varepsilon_B^*, (\Delta_{B^0})^{op}, \mu_B^*, S_B^*)$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$.

Proof. Following [8, Theorem 3.4], we have to finish checking the following steps:

(1) B^0 is an H-subcomodule of B^*.

(2) Observe that $(m_{B^0})^{op} = \Delta_B^* \circ i \circ \tau : B^0 \otimes B^0 \longrightarrow B^*$. It is morphism in ${}^H_H\mathcal{YD}$. Obviously, $\varepsilon_B^* : \Bbbk \longrightarrow B^0$ is. Thus, $(B^0, (m_{B^0})^{op}, \varepsilon_B^*)$ is an algebra in the category ${}_H\mathcal{M}$.

(3) Note that $(\Delta_{B^0})^{op}$ is the composite map $B^0 \xrightarrow{m_B^*} i(B^0 \otimes B^0) \xrightarrow{(i \circ \tau)^{-1}} B^0 \otimes B^0$. This is morphism in ${}^H_H\mathcal{YD}$. $\mu_B^* : B^0 \longrightarrow k$. So, $(B^0, (\Delta_{B^0})^{op}, \mu_B^*)$ is a coalgebra in the category ${}_H\mathcal{M}$.

(4) $(\Delta_{B^0})^{op} : (B^0)^{op} \longrightarrow (B^0)^{op} \underline{\otimes} (B^0)^{op}$ is an algebra map.

(5) $S_B^*(B^0) \subseteq B^0$.

(6) $(m_{B^0})^{op}(S_B^* \otimes id_{B^0})(\Delta_{B^0})^{op} = \varepsilon_B^* \mu_B^*$ and $(m_{B^0})^{op}(id_{B^0} \otimes S_B^*)(\Delta_{B^0})^{op} = \varepsilon_B^* \mu_B^*$.

These checks are omitted here. ■

A Hopf algebra H is said to be quasitriangular provided there exists an invertible element $R=\sum R^{(1)}\otimes R^{(2)}\in H\otimes H$ satisfying $(r:=R)$,

(QT1) $\sum R^{(1)}{}_1\otimes R^{(1)}{}_2\otimes R^{(2)}=\sum(R^{(1)}\otimes 1\otimes R^{(2)})(1\otimes r^{(1)}\otimes r^{(2)})$;

(QT2) $\sum R^{(1)}\otimes R^{(2)}{}_1\otimes R^{(2)}{}_2=\sum(R^{(1)}\otimes 1\otimes R^{(2)})(r^{(1)}\otimes r^{(2)}\otimes 1)$;

(QT3) $\sum\Delta^{cop}(h)=R\Delta(h)R^{-1}$, with $h\in H$.

Furthermore, when $R^{-1}=\sum R^{(2)}\otimes R^{(1)}$, (H,R) is said to be triangular.

Dually, a coquasitriangular Hopf algebra is a pair $(H,<|>)$, where H is a Hopf algebra over $\Bbbk$ and $<|>$: $B\otimes B\to\Bbbk$ is a $k-$linear form (braiding) which is convolution invertible in $Hom_{\Bbbk}(H\otimes H,\Bbbk)$ such that the following compatibility relations hold, for any $h,g,l\in H$

(BR1) $<h\mid gl>=<h_1\mid g><h_2\mid l>$;

(BR2) $<hg\mid l>=<g\mid l_1><h\mid l_2>$;

(BR3) $<h_1\mid g_1>g_2h_2=h_1g_1<h_2\mid g_2>$.

If $<h_1\mid g_1><g_2\mid h_2>=\epsilon(g)\epsilon(h)$, then $(H,<|>)$ is called a cotriangular Hopf algebra.

Example 2.6. (1) Let (H,R) be a quasitriangular Hopf algebra. Then the category ${}_H\mathcal{M}$ left H-modules is a braided monoidal subcategory of ${}_H^H\mathcal{YD}$ with the following braiding

$$\tau_{M,N}:M\otimes N\longrightarrow N\otimes M,\quad m\otimes n\mapsto R^{(2)}\cdot n\otimes R^{(1)}\cdot m \tag{3}$$

for all $M,N\in{}_H\mathcal{M}$, $m\in M$ and $n\in N$.

Let A and B be algebras in ${}_H\mathcal{M}$. Then we have the braided product $A\#B$ (called braided algebra) of A and B given via

$$(a\#x)(b\#y)=a(R^{(2)}\cdot b)\#(R^{(1)}\cdot x)y \tag{4}$$

for any $a,b\in A$ and $x,y\in B$.

(2) Let $(H,<|>)$ be a coquasitriangular Hopf algebra. Then the category ${}^H\mathcal{M}$ left H-comodules is a braided monoidal subcategory of ${}_H^H\mathcal{YD}$ with the following braiding

$$\tau_{M,N}:M\otimes N\longrightarrow N\otimes M,\quad m\otimes n\mapsto<m_{(-1)}\mid n_{(-1)}>n_0\otimes m_0 \tag{5}$$

for all $M,N\in{}_H\mathcal{M}$, $m\in M$ and $n\in N$.

Let A and B be algebras in ${}_H\mathcal{M}$. Then we have the braided product $A\#B$ (called braided algebra) of A and B given via

$$(a\#x)(b\#y)=<x_{(-1)}\mid b_{(-1)}>ab_0\#x_0y \tag{6}$$

for any $a,b\in A$ and $x,y\in B$.

As a straightforward result of Theorem 1.5, we have

Corollary 2.7. ([8, Theorem 3.4]) Given a quasitriangular Hopf algebra (H,R) with a bijective antipode S. If $(A,m_A,\mu_A,\Delta_A,\varepsilon_A,S_A)$ is a braided Hopf algebra in ${}_H\mathcal{M}$, then $(A^0,(m_{A^0})^{op},\varepsilon_A^*,(\Delta_{A^0})^{op},\mu_A^*,S_A^*)$ is also a braided Hopf algebra in ${}_H\mathcal{M}$.

Theorem 2.8. Given a coquasitriangular Hopf algebra $(H,|)$ with a bijective antipode S. When $(A,m_A,\mu_A,\Delta_A,\varepsilon_A,S_A)$ is a braided Hopf algebra ${}^H\mathcal{M}$, $(A^0,(m_{A^0})^{op},\varepsilon_A^*,(\Delta_{A^0})^{op},\mu_A^*,S_A^*)$ is also a braided Hopf algebra in ${}^H\mathcal{M}$.

Finally, as an application, let $A=\Bbbk[x,y]$ be the bialgebra with x group-like element and with y $(x,1)$-primitive. Consider a cyclic group $G=\langle g\rangle$ of order n. One has a Hopf algebra $H=\Bbbk[G]$ with h group-like for any $h\in G$. Moreover, H is quasitriangular with $R=(1/n)\sum_{i,j=0}^{n-1}p^{-ij}(g^i\otimes g^j)$, where p is a primitive nth root of unity in $\Bbbk$ (see [6] or [8]). Thus ${}_H\mathcal{M}$ is a braided monoidal subcategory of ${}_H^H\mathcal{YD}$. We can study the Sweedler's duality $A^0 = \Bbbk[x, y]^0$ in Section 3.

§3 Binary-linearly recursive sequences

Let $A = \Bbbk[x]$ be the polynomial algebra in one variable x. It has a bialgebra structure given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$. On one hand, we can identify an element f in the dual space A^* with the sequences $(f_n)_{n\geq 0} = (f_0, f_1, f_2, \cdots)$, where $f_n = f(x^n)$ for $n \geq 0$. On the other hand, A has a dual coalgebra $A^0 = \{f \in A^* \mid f(J) = 0$ for some cofinite ideal J of A, i.e., A/J is finite dimensional$\}$. Since a cofinite ideal J of $A = \Bbbk[x]$ is just a nonzero ideal generated by a monic polynomial $h(x) = x^r - h_1 x^{r-1} - \cdots - h_r$, the condition $f(J) = 0$ means that $f_n = h_1 f_{n-1} + \cdots + h_r f_{n-r}$ for all $n > r$. This meas that f is linearly recursive, satisfying the recursive relation $h(x)$. Thus the space of linearly recursive sequences has a Hopf algebra structure. See the references [3],[9],[13],[14] and [15] for details of this structure.

Let $q \neq 0$ in $\Bbbk$. In 1997, Ng and Taft [8] showed that the space of linearly recursive sequences is closed under the quantum convolution product $(f_n) *_q (g_n) = (h_n)$, where $h_n = \sum_{i=0}^n \begin{pmatrix} n \\ i \end{pmatrix}_q f_i g_{n-i}$ when q is a root of unity.

We now consider the bialgebra $A = \Bbbk[x, y]$ with x group-like element and with y a $(x, 1)$-primitive. $A = \Bbbk[x] \otimes \Bbbk[y]$ as an algebra, and thus $A^0 = \Bbbk[x]^0 \otimes \Bbbk[y]^0$ as a coalgebra. We identify each f in A^* as a binary-sequence $(f_{i,j})$ for $i, j > 0$, where $f_{i,j} = f(x^i y^j)$. A row of such a binary-sequence is a sequence $\{f_{i,p} \mid p \geq 0\}$ for a fixed $i \geq 0$, which we say is parallel to the y-axis, or a sequence $\{f_{p,j} \mid p \geq 0\}$ for a fixed $j \geq 0$, which we say is parallel to the x-axis.

Let f be in A^0, $f(J) = 0$ for a cofinite ideal J of A. For each i, j, the powers of x (y) spans a finite-dimensional space in A/J, so there is a minimal monic $h_i(x)$ $(h_j(x))$ in $\Bbbk[x]$ such that each row of f parallel to the y (x)-axis satisfies $h_i(x)$ $(h_j(x))$. Thus J contains the cofinite elementary ideal Γ generated by $h_i(x)h_j(y)$.

Recall from [8] that given a $q \neq 0$ in $\Bbbk$ and an integer $n > 0$, one has the Gaussian polynomial for $0 \leq i \leq n$

$$\begin{pmatrix} n \\ i \end{pmatrix}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!},$$

where $(n)_q = 1 + q + \cdots + q^{n-1}$ and the q-factorial of n is given by $(0)!_q = 1$ and

$$(n)!_q = (1)_q(2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}$$

if $n > 0$. It is a polynomial in q with coefficients in $\mathbb{Z}$. If $q = 1$, then $\begin{pmatrix} n \\ i \end{pmatrix}$ is the ordinary binomial coefficient.

Let x and y be variables subject to the quantum plane relation $yx = qxy$. Then for any $n > 0$ we have

$$(x+y)^n = \sum_{0 \leq i \leq n} \begin{pmatrix} n \\ i \end{pmatrix} x^i y^{n-i}. \tag{7}$$

Let $q = 1$. Then we have that $\begin{pmatrix} n \\ k \end{pmatrix}$ is the ordinary binomial coefficient.

Proposition 3.1. ([8,Lemma 5.2]) Let q be a primitive nth root of 1. For integers $a \geq b \geq 0$, write $a = a'n + r, b = b'n + s$ for $0 \leq r, s < n$. Then

$$\left(\begin{array}{c} a \\ b \end{array}\right)_q = \left(\begin{array}{c} a' \\ b' \end{array}\right)\left(\begin{array}{c} r \\ s \end{array}\right)_q,$$

where $\left(\begin{array}{c} r \\ s \end{array}\right)_q = 0$ if $r < s$.

We will let $A = \Bbbk_q[x, y]$ with $yx = qxy$, here $0 \neq q \in \Bbbk$. Then $f \in A^*$ is regarded as the binary sequences $(f_{m,n})_{m,n\geq 0} = (f_{0,0}, f_{0,1}, \cdots, f_{0,n},\ f_{1,0}, \cdots, f_{1,n}, \cdots,\ f_{m,0}, \cdots, f_{m,n}, \cdots)$ where

$$f_{m,n} = f(x^m y^n) = q^{-mn} f(y^n x^m) = q^{-mn} f_{n,m}$$

for all $m, n \geq 0$. We call them the q-binary sequences. A has a bialgebra structure with a group-like element x and with a $(x, 1)$-primitive element y, i. e., we have a comultiplication Δ given by the following

$$\Delta(x) = x \otimes x, \quad \Delta(y) = x \otimes y + y \otimes 1,$$
$$\varepsilon(x) = 1, \quad \varepsilon(y) = 0,$$

requiring Δ and ε to be algebra homomorphisms from A to $A \otimes A$ and A to $\Bbbk$, respectively. Thus one has

$$\Delta(x^m y^n) = \sum_{0\leq k\leq n} \left(\begin{array}{c} n \\ k \end{array}\right)_q x^{m+k} y^{n-k} \otimes x^m y^k \tag{8}$$

for any $m, n \geq 0$. Therefore, the quantum convolution product on A^* is given by $f_{m,n} *_q g_{m,n} = h_{m,n}$, where $h_{m,n} = \sum_{0\leq k\leq n} \left(\begin{array}{c} n \\ k \end{array}\right)_q f_{m+k,n-k} \otimes g_{m,k}$ for $m, n \geq 0$.

By a cofinite ideal J of $A = \Bbbk_q[x, y]$ we mean a nonzero ideal generated by a monic binary polynomial

$$\begin{aligned} h(x, y) =& x^r y^s - h_{1,0} x^{r-1} y^s - \cdots - h_{r,0} y^s \\ & - h_{0,1} x^r y^{s-1} - h_{1,1} x^{r-1} y^{s-1} - \cdots - h_{r,1} y^{s-1} \\ & - h_{0,2} x^r y^{s-2} - h_{1,2} x^{r-1} y^{s-2} - \cdots - h_{r,2} y^{s-2} \\ & \cdots \quad \cdots \quad \cdots \quad \cdots \\ & - h_{0,s} x^r - h_{1,s} x^{r-1} - \cdots - h_{r,s}. \end{aligned}$$

By the condition $f(J) = 0$, we have the following cases.

Case 1: If $f(x^{m-r} h(x, y) y^{n-s}) = 0$, then we have a binary linearly recursive sequence $f = (f_{m,n})_{m\geq r,n\geq s}$ satisfying the recursive relation $h(x, y)$, where

$$\begin{aligned} f_{m,n} =& h_{1,0} f_{m-1,n} + h_{2,0} f_{m-2,n} + \cdots + h_{r,0} f_{m-r,n} \\ & + h_{0,1} f_{m,n-1} + h_{1,1} f_{m-1,n-1} + \cdots + h_{r,1} f_{m-r,n-1} \\ & + h_{0,2} f_{m,n-2} + h_{1,2} f_{m-1,n-2} + \cdots + h_{r,2} f_{m-r,n-2} \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + h_{0,s} f_{m,n-s} + h_{1,s} f_{m-1,n-s} + \cdots + h_{r,s} f_{m-r,n-s}. \end{aligned}$$

Case 2: If $f(x^{m-r} y^{n-s} h(x, y)) = 0$, then we have a parameterized binary linearly recursive sequence $f = (f_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h(x,y)$, where

$$\begin{aligned} f_{m,n} =& q^{-(n-s)}h_{1,0}f_{m-1,n} + q^{-2(n-s)}h_{2,0}f_{m-2,n} + \cdots + q^{-r(n-s)}h_{r,0}f_{m-r,n} \\ &+ h_{0,1}f_{m,n-1} + q^{-(n-s)}h_{1,1}f_{m-1,n-1} + q^{-2(n-s)}h_{2,1}f_{m-2,n-1} \\ &+ \cdots + q^{-r(n-s)}h_{r,1}f_{m-r,n-1} \\ &+ h_{0,2}f_{m,n-2} + q^{-(n-s)}h_{1,2}f_{m-1,n-2} + q^{-2(n-s)}h_{2,2}f_{m-2,n-2} \\ &+ \cdots + q^{-r(n-s)}h_{r,2}f_{m-r,n-2} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &+ h_{0,s}f_{m,n-s} + q^{-(n-s)}h_{1,s}f_{m-1,n-s} + q^{-2(n-s)}h_{2,s}f_{m-2,n-2} \\ &+ \cdots + q^{-r(n-s)}h_{r,s}f_{m-r,n-s}. \end{aligned}$$

Case 3: If $f(h(x,y)x^{m-r}y^{n-s}) = 0$, then we have a parameterized binary linearly recursive sequence $f = (f_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h(x,y)$, where

$$\begin{aligned} f_{m,n} =& h_{1,0}f_{m-1,n} + h_{2,0}f_{m-2,n} + \cdots + h_{r,0}f_{m-r,n} \\ &+ q^{-(m-r)}h_{0,1}f_{m,n-1} + q^{-(m-r)}h_{1,1}f_{m-1,n-1} + q^{-(m-r)}h_{2,1}f_{m-2,n-1} \\ &+ \cdots + q^{-(m-r)}h_{r,1}f_{m-r,n-1} \\ &+ q^{-2(m-r)}h_{0,2}f_{m,n-2} + q^{-2(m-r)}h_{1,2}f_{m-1,n-2} + q^{-2(m-r)}h_{2,2}f_{m-2,n-2} \\ &+ \cdots + q^{-2(m-r)}h_{r,2}f_{m-r,n-2} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &+ q^{-s(m-r)}h_{0,s}f_{m,n-s} + q^{-s(m-r)}h_{1,s}f_{m-1,n-s} + \cdots + q^{-s(m-r)}h_{r,s}f_{m-r,n-s}. \end{aligned}$$

Remarks: (1) Set $h'_{i,j} = q^{-i(n-s)}h_{i,j}$ in Case 1. We then have a new monic binary polynomial $h'(x,y)$ as follows:

$$\begin{aligned} h'(x,y) =& x^r y^s - q^{n-s}h'_{1,0}x^{r-1}y^s - \cdots - q^{r(n-s)}h'_{r,0}y^s \\ &- h'_{0,1}x^r y^{s-1} - q^{n-s}h'_{1,1}x^{r-1}y^{s-1} - \cdots - q^{r(n-s)}h'_{r,1}y^{s-1} \\ &- h'_{0,2}x^r y^{s-2} - q^{n-s}h'_{1,2}x^{r-1}y^{s-2} - \cdots - q^{r(n-s)}h'_{r,2}y^{s-2} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \\ &- h'_{0,s}x^r - q^{n-s}h'_{1,s}x^{r-1} - \cdots - q^{r(n-s)}h'_{r,s}. \end{aligned}$$

In this situation, we get a new binary linearly recursive sequence $f' = (f'_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h'(x,y)$, where

$$\begin{aligned} f'_{m,n} =& h'_{1,0}f_{m-1,n} + h'_{2,0}f_{m-2,n} + \cdots + h'_{r,0}f_{m-r,n} \\ &+ h'_{0,1}f_{m,n-1} + h'_{1,1}f_{m-1,n-1} + \cdots + h'_{r,1}f_{m-r,n-1} \\ &+ h'_{0,2}f_{m,n-2} + h'_{1,2}f_{m-1,n-2} + \cdots + h'_{r,2}f_{m-r,n-2} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &+ h'_{0,s}f_{m,n-s} + h'_{1,s}f_{m-1,n-s} + \cdots + h'_{r,s}f_{m-r,n-s}, \end{aligned}$$

which satisfies the relation $h'(x,y)$.

(2) Similarly, set $h''_{i,j} = q^{-j(m-r)}h_{i,j}$ in Case 2. We then have a new monic binary polynomial $h''(x,y)$, hence we can obtain a new binary linearly recursive sequence $f'' = (f''_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h''(x,y)$,

Example 3.2. Let $h(x,y)=(x-a)(y-b)$ for any $a,b\in k$. Then we have a binary linearly recursive sequence

$$f_{m,n}=bf_{m,n-1}+af_{m-1,n}-abf_{m-1,n-1}$$

satisfying $h(x,y)$, for all $m\geq 1$ and $n\geq 1$. If take $f_{0,0}=1$, $f_{0,i}=0$ and $f_{i,0}=0$ for all $i\in\mathbb{N}$, then one gets a binary linearly recursive sequence $(1,0,0,\cdots,\ 0,-ab,-ab^2,-ab^3,\cdots,\ 0,-a^2b,-a^2b^2,-a^2b^3,\cdots)$.

Remark: In the case of $q=1$, the Case 1, the Case 2 and the Case 3 are the same.

§4 Quantum convolution for $q=1$

Let $q=1$. Then we have $f_{m,n}=f_{n,m}$ for any $m,n\geq 0$ and $\left(\begin{array}{c}n\\k\end{array}\right)$ is the ordinary binomial coefficient. In this section we will study the binary linearly recursive sequence in Case 1 satisfying the recursive relation $h(x,y)$ as in Section 2.

In what follows, by a method similar to that in [8] we can consider $A=\mathbb{k}[x,y]$ in ${}_H\mathcal{M}$ (see the end of Section 1) via

$$g^i\cdot(x^ky^l)=p^{i(k+l)}x^ky^l,\quad \text{for all } x^k,y^l\in A,\ g^i\in H. \tag{9}$$

It is not hard to verify that A is an algebra in ${}_H\mathcal{M}$. By (2) and (5), we have

$$\begin{aligned}
&\tau_{A,A}(x^ky^l\otimes x^sy^t)\\
=&\frac{1}{n}\sum_{i,j=0}^{n-1}p^{-ij}(g^j\cdot(x^sy^t)\otimes g^i\cdot(x^ky^l))\\
=&\frac{1}{n}\sum_{i,j=0}^{n-1}p^{-ij}p^{j(s+t)}p^{i(k+l)}(x^sy^t\otimes x^ky^l)\\
=&\frac{1}{n}\sum_{i,j=0}^{n-1}p^{-ij}p^{jr}p^{iu}(x^sy^t\otimes x^ky^l)\\
=&\frac{1}{n}\sum_{i=0}^{n-1}p^{iu}(\sum_{j=0}^{n-1}p^{j(r-i)})(x^sy^t\otimes x^ky^l),
\end{aligned}$$

where we write $s+t=an+r, k+l=bn+u$ for some $0\leq r,u<n$.

If $i\neq r$, then $\sum_{j=0}^{n-1}p^{j(r-i)}=0$. Therefore, $\tau_{A,A}(x^ky^l\otimes x^sy^t)=p^{(s+t)(k+l)}(x^sy^t\otimes x^ky^l)$. Thus we have a braided algebra $A\#A$ with

$$(x^{k_1}y^{l_1}\#x^{s_1}y^{t_1})(x^{k_2}y^{l_2}\#x^{s_2}y^{t_2})=p^{(s_1+t_1)(k_2+l_2)}(x^{k_1+k_2}y^{l_1+l_2}\#x^{s_1+s_2}y^{t_1+t_2}) \tag{10}$$

for all $x^{k_1}y^{l_1}\#x^{s_1}y^{t_1},x^{k_2}y^{l_2}\#x^{s_2}y^{t_2}\in A\#A$.

Note that $q=1$, so $xy=yx$. It follows from (6) that $(x\#y)(y\#1)=p(xy\#y)=p(y\#1)(x\#y)$. If we regard A as a bialgebra in the category ${}_H\mathcal{M}$, then by Majid's bosonization [3], this requires Δ to be an algebra morphism in ${}_H\mathcal{M}$. Note that $(x\#y)^l=p^{\frac{1}{2}l(l+1)}(x^l\#y^l)$, and so, by (1) and (6) we have

$$\Delta(x^my^l)=\sum_{0\leq k\leq l}\left(\begin{array}{c}l\\k\end{array}\right)_p p^{\frac{1}{2}[(l-k)(l-k+1)+m(m+2l+1)]}x^{m+l-k}y^k\#x^my^{l-k} \tag{11}$$

for $m, l \geq 0$.

Notice that x^m is not a group-like element since $\Delta(x^m) = p^{\frac{1}{2}m(m+1)}(x^m\#y^m)$, and y^n is $(x^n, 1)$-primitive, i.e., $\Delta(y^n) = x^n\#y^n + y^n\#1$ since $\begin{pmatrix} n \\ k \end{pmatrix}_p = 0$ for $1 \leq k \leq n-1$. The counit ε of A is given as usual by $\varepsilon(x^m) = 1, \varepsilon(y^k) = \delta_{0,k}$. It is easy to check that Δ and ε are morphisms in ${}_H\mathcal{M}$. But, A is not a Hopf algebra in ${}_H\mathcal{M}$ unless $x^2 = 1$ with $S(x) = x$ and $S(y) = -xy$.

In what follows, Theorem 3.1 shows that the space A^0 of binary linearly recursive sequences is a bialgebra in ${}_H\mathcal{M}$. The quantum convolution product in A^* is given $(f_{m,l}) *_p (g_{m,l}) = (h_{m,l})$, where

$$h_{m,l} = \sum_{0\leq k\leq l} \begin{pmatrix} l \\ k \end{pmatrix}_p p^{\frac{1}{2}[(l-k)(l-k+1)+m(m+2l+1)]} f_{m+l-k,k} g_{m,l-k}. \tag{12}$$

Thus, we have

Theorem 4.1. Let p be a root of unity in k. Then the binary linearly recursive sequences is closed under the quantum convolution product $(f_{m,l}) *_p (g_{m,l}) = (h_{m,l})$ where $h_{m,l}$ is given by (12).

Remarks: 1) If $x = 1$ in $A = k[x, y]$, then by (7) we have

$$\Delta(y^l) = \sum_{0\leq k\leq l} \begin{pmatrix} l \\ k \end{pmatrix}_p y^k\#y^{l-k}$$

for all $l \geq 0$. Hence, we can get the result in [8, Theorem 4.1].

2) If $y = 1$ in $A = k[x, y]$, then by (7) one has

$$\Delta(x^m) = p^{\frac{1}{2}m(m+1)}(x^m\#y^m).$$

In this case, we have

$$h_m = p^{\frac{1}{2}m(m+1)} f_{m+l-k} g_m. \tag{13}$$

Notice that the algebraic structure of linearly recursive sequences under Hadamard product was described in [3]. We call the product given by (9) a quantum Hadamard product. Then we have

Corollary 4.2. The linearly recursive sequences are closed under the quantum Hadamard product $(f_m) *_p (g_m) = (h_m)$, where h_m is given by (13).

Next, we give a direct proof of Theorem 4.1.

First, we have

Lemma 4.3. Let $f = (f_{m,l})$ be the binary linearly recursive sequence over k satisfying the relation $h(x, y)$. Let $\alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_u$ be the roots of $h(x, y)$ in $\bar{k}$, the algebraic closure of k. Then, for any $i, j \geq 0, s, t \geq 0$, the subsequence $f^{(i,j)(s,t)}$ given by

$$\begin{aligned} f^{(i,j)(s,t)} =&(f_{i,s}, f_{i,s+t}, f_{i,s+2t}, \cdots \\ &f_{i+j,s}, f_{i+j,s+t}, f_{i+j,s+2t}, \cdots \\ &f_{i+2j,s}, f_{i+2j,s+t}, f_{i+2j,s+2t}, \cdots \\ &\cdots \quad \cdots \quad \cdots \quad \cdots) \end{aligned}$$

is a binary linearly recursive sequence over $\Bbbk$ satisfying the relation $h_{j,t}(x, y) = (x-\alpha_1^j)\cdots(x-\alpha_k^j)(y-\beta_1^t)\cdots(y-\beta_u^t)$.

Proof. It suffices to show $h_{j,t}(x,y) \rightharpoonup f^{(i,j)(s,t)} = 0$ in order to verify that $f^{(i,j)(s,t)}$ satisfies $h_{j,t}(x,y)$, where $\rightharpoonup$ is defined by $(a \rightharpoonup f)(b) = f(ba)$ for $a, b \in A$, and $f \in A^*$, for A is a commutative algebra.

Note that $f^{(i,j)(s,t)}(x^v y^w) = f_{i+vj,s+wt} = f(x^{i+vj}y^{s+wt})$ for $v, w \geq 0$. So, for all $g(x,y) \in k[x,y]$, $f^{(i,j)(s,t)}(g(x,y)) = f(x^i y^s g(x^j y^t))$. Thus, we have

$$\begin{aligned} & h_{j,t}(x,y) \rightharpoonup f^{(i,j)(s,t)}(x^v y^w) \\ =& f^{(i,j)(s,t)}(x^v y^w h_{j,t}(x,y)) \\ =& f(x^i y^s x^{vj} y^{wt} h_{j,t}(x^j, y^t)) \\ =& f(x^{i+vj} y^{s+wt} h_{j,t}(x^j, y^t)). \end{aligned}$$

For $h_{j,t}(x^j, y^t) = I(x,y)h(x,y)$ for some $I(x,y) \in k[x,y]$, and $h(x,y) \rightharpoonup f = 0$, one has $f(x^{i+vj}y^{s+wt}h_{j,t}(x^j,y^t)) = (h(x,y) \rightharpoonup f)(x^{i+vj}y^{s+wt}I(x,y)) = 0$, concluding the proof. ■

Example 4.4. In Example 2.2, we have a subsequence

$$\begin{aligned} f^{(1,2)(3,4)} =& (f_{1,3}, f_{1,7}, f_{1,11}, \cdots \\ & f_{3,3}, f_{3,7}, f_{3,11}, \cdots \\ & f_{5,3}, f_{5,7}, f_{5,11}, \cdots) \end{aligned}$$

satisfying $h_{2,4}(x,y) = (x - a^2)(y - b^4)$. Explicitly, one has a binary linearly recursive sequence $g_{m,n} = b^4 g_{m,n-1} + a^2 g_{m-1,n} - a^2 b^4 g_{m-1,n-1}$ satisfying $h_{2,4}(x,y)$, and $(g_{m,n})_{m\geq 1, n\geq 1}$ is the same as $f^{(1,2)(3,4)}$. For example, $g_{0,0} = f_{1,3} = -ab^3$, $g_{0,1} = f_{1,7} = -ab^7$, $g_{1,0} = f_{3,3} = -a^3b^3$, and it follows from the formula $g_{m,n}$ that

$$\begin{aligned} g_{1,1} &= b^4 g_{1,0} + a^2 g_{0,1} - a^2 b^4 g_{0,0} \\ &= b^4(-a^3b^3) + a^2(-ab^7) - a^2b^4(-ab^3) \\ &= -a^3b^7 - a^3b^7 + a^3b^7 = -a^3b^7, \end{aligned}$$

which equals to $f_{3,7}$.

Theorem 4.5. Let p be a primitive nth root of unity. Let $f = (f_{m,l})$ and $g = (g_{m,l})$ be binary linearly recursive sequences, satisfying $h(x,y) = \prod_{i,j}(x - \alpha_i)^{m_i}(y - \beta_j)^{n_j}$ and $J(x,y) = \prod_{k,r}(x - \gamma_k)^{s_k}(y - \lambda_r)^{t_r}$, respectively. Then the quantum product $(f_{m,l}) *_p (g_{m,l}) = (h_{m,l})$ where $h_{m,l}$ is given by (8), is a binary linearly recursive sequence, satisfying $W(x^n, y^n)$; here $W(x,y) = \prod_{i,k;j,r}[x - (\alpha_i^n + \gamma_k^n)^{m_i+s_k-1}][y - (\beta_j^n + \lambda_r^n)^{n_j+t_r-1}]$.

Proof. For $0 \leq a, b \leq n$ and $c, d \geq 0$, we have

$$\begin{aligned} & h^{(a,n)(b,n)}(x^c y^d) \\ =& h_{a+cn,b+dn} \\ =& \sum_{u=0}^{b+dn} \begin{pmatrix} b+dn \\ u \end{pmatrix}_p p^{\frac{1}{2}[(b+dn-u)(b+dn-u+1)+(a+cn)(a+2b+(c+2d)n+1)]} \\ & f_{a+b+(c+d)n-u,u} g_{a+cn,b+dn-u} \quad \text{(by (8))} \\ =& \sum_{s=0}^{b}\sum_{j=0}^{d} \begin{pmatrix} b+dn \\ u \end{pmatrix}_p p^{\frac{1}{2}\{[(b-s)+(d-j)n][(b-s)+(d-j)n+1]+(a+cn)[a+2b+(c+2d)n+1]\}} \\ & f_{a+b-s+(c+d-j)n,s+jn} g_{a+cn,b-s+(d-j)n} \end{aligned}$$

$$=\sum_{s=0}^{b}\sum_{j=0}^{d}\begin{pmatrix} d \\ j \end{pmatrix}\begin{pmatrix} b \\ s \end{pmatrix}_p p^{\frac{1}{2}\{[(b-s)+(d-j)n][(b-s)+(d-j)n+1]+(a+cn)[a+2b+(c+2d)n+1]\}}$$

$$f_{a+b-s+(c+d-j)n,s+jn}g_{a+cn,b-s+(d-j)n} \quad \text{(by Proposition 3.1)}$$

$$=\sum_{s=0}^{b}\begin{pmatrix} b \\ s \end{pmatrix}_p p^{\frac{1}{2}[(b-s)(b-s+1)+a(a+2b+1)]}$$

$$(\sum_{j=0}^{d}\begin{pmatrix} d \\ j \end{pmatrix}(f^{(a+b-s,n)(s,n)})_{(c+d-j,j)}(g^{(a,n)(b-s,n)})_{(c,d-j)})$$

$$=\sum_{s=0}^{b}\begin{pmatrix} b \\ s \end{pmatrix}_p p^{\frac{1}{2}[(b-s)(b-s+1)+a(a+2b+1)]}$$

$$(\sum_{i=0}^{d}\begin{pmatrix} d \\ i \end{pmatrix}(f^{(a+b-s,n)(s,n)})_{(c+i,d-i)}(g^{(a,n)(b-s,n)})_{(c,i)}).$$

Therefore,

$$h^{(a,n)(b,n)}=\sum_{s=0}^{b}\begin{pmatrix} b \\ s \end{pmatrix}_p p^{\frac{1}{2}[(b-s)(b-s+1)+a(a+2b+1)}f^{(a+b-s,n)(s,n)}*g^{(a,n)(b-s,n)},$$

where $*$ is the usual convolution product. It follows from Lemma 4.3 that $f^{(a+b-s,n)(s,n)}$ and $g^{(a,n)(b-s,n)}$ are binary linearly recursive sequences, satisfying $h_{n,n}(x,y)=\prod_{i,j}(x-\alpha_i^n)^{m_i}(y-\beta_j^n)^{n_j}$ and $J_{n,n}(x,y)=\prod_{k,r}(x-\gamma_k^n)^{s_k}(y-\lambda_r^n)^{t_r}$, respectively. Hence Theorem 4.5 holds. In fact, h is the interlacing of the sequences $h^{(0,n)(0,n)}$, $h^{(0,n)(1,n)}$, $\cdot$, $h^{(0,n)(n-1,n)}$, $\cdots$, $h^{(1,n)(0,n)}$, $\cdots$, $h^{(1,n)(n-1,n)}$, $\cdots$, $h^{(n-1,n)(0,n)}$, $\cdots$, $h^{(n-1,n)(n-1,n)}$.

This finishes the proof. ■

§5 Structural properties of binary linearly recursive sequence space

Let p be a primitive nth root of 1, $\mathcal{BL}$ the space of binary linearly recursive sequences. Denote the quantum convolution product on $\mathcal{BL}$ by $*_p$. Define a map $\Phi:\mathcal{BL}\longrightarrow\mathcal{BL}$ by $\Phi(f)=f^{(0,n)(0,n)}$, where $f^{(0,n)(0,n)}=(f_{0.0},f_{0,n},f_{0,2n},\cdots,f_{n,0},f_{n,n},f_{n,2n},\cdots,f_{2n,0},f_{2n,n},f_{2n,2n},\cdots)$. Obviously, Φ is surjective.

Let $f,g\in\mathcal{BL}$, $h=f*_p g$. By the proof of Theorem 4.5, we get $h^{(0,n)(0,n)}=f^{(0,n)(0,n)}*g^{(0,n)(0,n)}$. Hence $\Phi(f*_p g)=\Phi(h)=h^{(0,n)(0,n)}=f^{(0,n)(0,n)}*g^{(0,n)(0,n)}=\Phi(f)*\Phi(g)$. Therefore, we have

Lemma 5.1. Φ is a surjective algebra map of $(\mathcal{BL},*_p)$ onto $(\mathcal{BL},*)$.

Let $Z^{(i,j)}$ be the sequence with 1 in the (i,j)th position (coordinate) and zero elsewhere. Then, by (8) one has

Lemma 5.2. (i) $Z^{(i,j)}*_p Z^{(s,t)}=\begin{pmatrix} t+j \\ j \end{pmatrix}_p p^{\frac{1}{2}[t(t+1)+s(i+t+2j+1)]}Z^{(s,t+j)}$,

(ii) $Z^{(0,j)}*_p Z^{(0,t)}=\begin{pmatrix} t+j \\ j \end{pmatrix}_p p^{\frac{1}{2}t(t+1)}Z^{(0,t+j)}$,

(iii) $Z^{(i,0)}*_p Z^{(s,0)}=p^{\frac{1}{2}s(s+1)}Z^{(s,0)}$,

(iv) $Z^{(i,0)} *_p Z^{(0,t)} = p^{\frac{1}{2}t(t+1)} Z^{(0,t)}$,

(v) $Z^{(0,j)} *_p Z^{(s,0)} = p^{\frac{1}{2}s(s+2j+1)} Z^{(s,j)}$.

Lemma 5.3. The kernel of Φ is a nilpotent ideal of $(\mathcal{BL}, *_p)$.

Proof. By the definition of Φ, an element f in $Ker\,\Phi$ is of the form $\sum_{i,j\not\equiv 0(mod\ n)} \alpha_{i,j} Z^{(i,j)}$ (an infinite sum), so an element in $(Ker\Phi)^n$ is of the form

$$\sum_{i_1,\cdots,i_n;j_1,\cdots,j_n\not\equiv 0(mod\ n)} \alpha_{i_1,\cdots,i_n;j_1,\cdots,j_n} Z^{(i_1,j_1)} *_p \cdots *_p Z^{(i_n,j_n)}.$$

It suffices to verify that $Z^{(i_1,j_1)} *_p \cdots *_p Z^{(i_n,j_n)} = 0$ if $i_1,\cdots,i_n;j_1,\cdots,j_n \not\equiv 0(mod\ n)$. Set $i_u = a_u n + r_u, j_v = b_v n + s_v$ with $0 < r_u, s_v < n$ for $u, v = 1,\cdots,n$. There are $u', v' \le n-1$ such that $r_1+\cdots+r_{u'} < n, s_1+\cdots+s_{v'} < n$, but $r_1+\cdots+r_{u'}+r_{u'+1} \ge n$, and $s_1+\cdots+s_{v'}+s_{v'+1} \ge n$. So, by Lemma 5.2(i), for some β in k, we have

$$\begin{aligned} &Z^{(i_1,j_1)} *_p \cdots *_p Z^{(i_{u'+1},j_{v'+1})} \\ =&\beta Z^{(i_{u'},j_1+\cdots+j_{v'})} *_p Z^{(i_{u'+1},j_{v'+1})} \\ =&\alpha \begin{pmatrix} j_1+\cdots+j_{v'}+j_{v'+1} \\ j_{v'+1} \end{pmatrix}_p Z^{(i_{u'+1},j_1+\cdots+j_{v'}+j_{v'+1})}, \end{aligned}$$

where $\alpha = \beta p^{\frac{1}{2}[j_{v'+1}(j_{v'+1}+1)+i_{u'+1}(i_{u'}+2j_1+\cdots+2j_{v'}+j_{v'+1}+1)]}$. Note that $j_1+\cdots+j_{v'}+j_{v'+1})$ $= n(b_1+\cdots+b_{v'+1}) + (s_1+\cdots+s_{v'+1}-n) + n = n(b_1+\cdots+b_{v'+1}+1) + (s_1+\cdots+s_{v'+1}$ $-n)$, here $s_1+\cdots+s_{v'+1}-n < n$. By Proposition 3.1 and an argument similar to [8, Lemma 5.5], we have $\begin{pmatrix} j_1+\cdots+j_{v'}+j_{v'+1} \\ j_{v'+1} \end{pmatrix}_p = 0$ and so $(Ker\Phi)^n = 0$.

This concludes the proof. ■

Theorem 5.4. $f \in \mathcal{BL}$ is $*_p$-invertible if and only if $\Phi(f)$ is $*$-invertible.

Proof. Similar to [8, Theorem 5.6]. It follows from Lemma 5.1 and Lemma 5.3 that Theorem 5.4 holds. ■

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Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing 211816, China.
Email: 202110006988@njtech.edu.cn

[2]School of Mathematics, Southeast University, Nanjing 210096, China.

[3]Shing-Tung Yau Center of Southeast University, Southeast University, Nanjing 210096, China.
Email: shuanhwang@seu.edu.cn