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Dirichlet-Neumann alternating algorithm for an anisotropic quasi-linear problem in an unbounded domain with a concave angle

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Abstract. In this paper, based on the Kirchhoff transformation and the natural boundary reduction, a Dirichlet-Neumann (D-N) alternating algorithm is discussed for solving the anisotropic quasi-linear problem in an unbounded domain with a concave angle. By using the principle of the natural boundary reduction, the natural integral equation on the elliptical arc artificial boundary is obtained in this paper, and the convergence of the algorithm and analysis is proved. Meanwhile, the convergence rate for a typical domain is given in detail. Finally, some numerical examples are verified to show the feasibility of the method.

§1 Introduction

In many fields of scientific and physical engineering, such as continuum thermodynamics, fluid mechanics or magnetostatics, it is necessary to deal with some linear or nonlinear partial differential equation problems in the unbounded domain. When solving a linear or nonlinear partial differential equation problem in the unbounded domain, we find that the domain decomposition method is one of the most efficient techniques. The domain decomposition method is changed from the Coupling of NBEM and FEM method. Based on the natural boundary reduction[2-6], these methods are used to solve some unbounded domain problems by introducing an artificial boundary. So far, these techniques have been used to solve many linear problems [9-12], and generalized to solve quasi-linear problems [8,13,14,16]. In this paper, based on the Kirchhoff transformation [13, 14, 17], we consider a D-N alternating algorithm, which is a non-overlapping domain decomposition method, for an exterior anisotropic quasi-linear problem in an unbounded domain with a concave Angle.

Suppose Ω is an unbounded and simple connected domain with sufficiently smooth boundary $\partial\Omega = \Gamma \bigcup \Gamma_1 \bigcup \Gamma_2$, like Figure 1. The boundary Γ is a smooth curve, and

$$\Omega = \left\{(r, \theta) \mid r > R, 0 < \theta < \omega\right\},$$

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$$\Gamma=\{(r,\theta)\mid r=R,0<\theta<\omega\},$$
$$\Gamma_1=\{(r,\theta)\mid r>R,\theta=0\},$$
$$\Gamma_2=\{(r,\theta)\mid r>R,0=\theta\},$$

where ω is a concave angle, and $0<\omega\leq 2\pi$. We consider the following quasi-linear problem

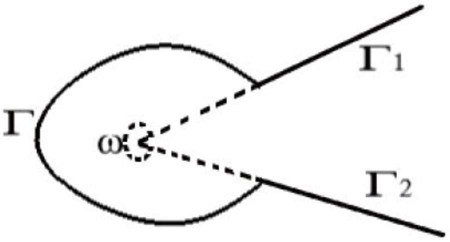

Figure 1. the illustration of domain.

$$\begin{cases}-\left(\frac{\partial}{\partial x}\left(\alpha a\left(x,u\right)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\beta a\left(x,u\right)\frac{\partial u}{\partial y}\right)\right)=f, & \text{in } \Omega,\\ \frac{\partial u}{\partial n}=0, & \text{on } \Gamma_1\cup\Gamma_2,\\ u=0, & \text{on } \Gamma,\\ u\left(x\right)\text{is unbounded}, & \text{as } |x|\longrightarrow\infty,\end{cases}\tag{1}$$

where $\beta>\alpha>0$, and $a(x,u)$ are given functions with some properties. Problem (1) has numerous mechanical applications. For example, in the field of image-sound physics, where problem (1) can be regarded as a wave-type governing equation, a is the speed of sound in a fluid and u is the velocity potential; in the field of two-dimensional elasticity, problem (1) can be considered as a simple displacement governing equation, a is stress-strain relationship, u is strain tensor and α,β are elastic constant. In this paper, following [8, 9], we suppose that the given function satisfies

$$0<C_0\leq a(x,u)\leq C_1,\forall u\in R,\tag{2}$$

and for almost all $x\in\Omega$, with two constants $C_0,C_1\in R$

$$|a(x,u)-a(x,v)|\leq C_L\left|u-v\right|,\forall u,v\in R,\tag{3}$$

and for almost all $x\in\Omega$, with a constant $C_L>0$. We also assume that $\frac{\partial a}{\partial s},\frac{\partial^2 a}{\partial s^2}$ are continuous. In the following, we suppose that the given function $f\in L^2(\Omega)$ has compact support, i.e., there exists a constant $K>0$, such that

$$suppf\subset\Omega_K=\left\{x\in R^2\mid |x|\leq K\right\}.\tag{4}$$

We also assume that

$$a(x,u)\equiv a_0(u), when\,|x|\leq K or\,|x|\geq K.\tag{5}$$

The rest of the paper is organized as follows. In section 2, we introduce the artificial boundary of elliptic arc boundary Γ_0, and derive the exact quasi-linear elliptical arc artificial boundary condition. In section 3, we give the nonoverlapping domain decomposition method and derive the convergence of the algorithm. In section 4, we give some numerical examples to illustrate the efficiency and feasibility of this method.

§2 Natural boundary reduction

Now, we introduce an elliptical arc artificial boundary Γ_0 in R^2, Γ_0 enclosing Γ such that

$$\Gamma_0 = \{(\mu, \phi) \mid \mu = \mu_1 > K, 0 < \phi < \omega\} \tag{2.1}$$

with $dist(\Gamma_0, \Gamma) > 0$, like Figure 2.

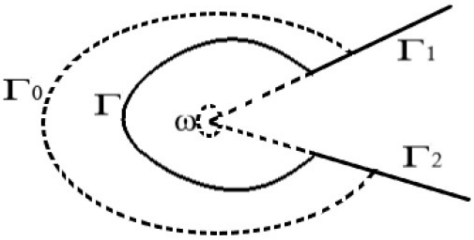

Figure 2. artificial boundary of domain.

Γ_0 divides Ω into two parts, a bounded domain Ω_1 and an unbounded domain Ω_2, the problem (1) can be rewritten in the coupled form

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x,u\right)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta a\left(x,u\right)\frac{\partial u}{\partial y}\right)\right) = f, & \text{in } \Omega_1, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{11} \cup \Gamma_{21}, \\ u = 0, & \text{on } \Gamma. \end{cases} \tag{2.2}$$

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x,u\right)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta a\left(x,u\right)\frac{\partial u}{\partial y}\right)\right) = 0, & \text{in } \Omega_2, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{12} \cup \Gamma_{22}, \\ u\left(x\right) \text{is unbounded}, & \text{as } |x| \longrightarrow \infty, \end{cases} \tag{2.3}$$

where $u(x)$ and $\alpha a_0(u)n_x\frac{\partial u}{\partial x} + \beta a_0(u)n_y\frac{\partial u}{\partial y}$ are continuous on Γ_0, $\Gamma_{11} = \Omega_1 \cap \Gamma_1, \Gamma_{21} = \Omega_2 \cap \Gamma_1, \Gamma_{12} = \Omega_1 \cap \Gamma_2, \Gamma_{22} = \Omega_2 \cap \Gamma_2$, and $n = (n_x, n_y)$ is the unit exterior normal vector on Γ_0. Particularly, $a(x,u) \equiv a$ is independent of x and u when $|x| \geq K$, and the problem (2.3) is simplified to the linear exterior elliptic problem. We have the nonoverlapping domain decomposition algorithm.

Step 1: Choose an initial value $\lambda^0 \in H^{\frac{1}{2}}\left(\Gamma_0\right)$, and let $k = 0$.

Step 2: Solve a Dirichlet boundary value problem in the exterior domain Ω_2,

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x,u_2^k\right)\frac{\partial u_2^k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta a\left(x,u_2^k\right)\frac{\partial u_2^k}{\partial y}\right)\right) = 0, & \text{in } \Omega_2, \\ u_2^k = \lambda^k, & \text{on } \Gamma_0, \\ \frac{\partial u_2^k}{\partial n_1} = 0, & \text{on } \Gamma_{21} \cup \Gamma_{22}, \\ u_2^k\left(x\right) = o(1), & \text{as } |x| \longrightarrow \infty. \end{cases} \tag{2.4}$$

Step 3: Solve a mixed boundary value problem in the interior domain Ω_1,

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x,u_1^k\right)\frac{\partial u_1^k}{\partial x}\right)+\frac{\partial}{\partial y}\left(\beta a\left(x,u_1^k\right)\frac{\partial u_1^k}{\partial y}\right)\right)=f, & \text{in } \Omega_1, \\ \alpha a_0(u)n_x\frac{\partial u_1^k}{\partial x}+\beta a_0(u)n_y\frac{\partial u_1^k}{\partial y}=-(\alpha a_0(u)n_x\frac{\partial u_2^k}{\partial x}+\beta a_0(u)n_y\frac{\partial u_2^k}{\partial y}), & \text{on } \Gamma_0, \\ \frac{\partial u_1^k}{\partial n_2}=0, & \text{on } \Gamma_{12}\cup\Gamma_{22}, \\ u_1^k\left(x\right)=0, & \text{on } \Gamma. \end{cases} \tag{2.5}$$

Step 4: Update the boundary valve $0<\theta_k<1$,

$$\lambda^{k+1}=\theta_k u_1^k+(1-\theta_k)\lambda^k \quad \text{on } \Gamma_0. \tag{2.6}$$

Step 5: Put $k=k+1$, turn to Step 2.

The next, we introduce the so-called Kirchhoff transformation [14]:

$$w(x)=\int_0^{u(x)} a_0(\xi)d\xi, \quad x\in\Omega_2, \tag{2.7}$$

which gives

$$\nabla w=a_0(u)\nabla u, \tag{2.8}$$

$$\left(\alpha\frac{\partial w}{\partial x},\beta\frac{\partial w}{\partial y}\right)=\left(\alpha a_0(u)n_x\frac{\partial u}{\partial x},\beta a_0(u)n_y\frac{\partial u}{\partial y}\right). \tag{2.9}$$

From the problem (2.7), we have that w satisfies

$$w^k=\int_0^{\lambda^k} a_0(\xi)d\xi, \quad \text{on } \Gamma_0, \tag{2.10}$$

and

$$\begin{cases} -\left(\alpha\frac{\partial^2 w}{\partial x^2}+\beta\frac{\partial^2 w}{\partial y^2}\right)=0, & \text{in } \Omega_2, \\ \frac{\partial u}{\partial n}=0, & \text{on } \Gamma_{12}\cup\Gamma_{22}, \\ u\left(x\right) \text{ is unbounded}, & \text{as } |x|\longrightarrow\infty. \end{cases} \tag{2.11}$$

The next, we need to obtain the corresponding results for quasi-linear problem in Ω_2, by virtue of the Poisson integral formula and natural integral equation for the linear problem. For this purpose, we should discuss the relationship between elliptic coordinates (μ,ϕ) and Cartesian coordinates (x,y) first. The relationship can be expressed as below

$$\begin{cases} x=f_0\cosh\mu\cos\phi, \\ y=f_0\sinh\mu\sin\phi, \end{cases} \tag{2.12}$$

where

$$f_0=\sqrt{a^2-b^2}, a=f_0\cosh\mu_1, b=f_0\sinh\mu_1,$$
$$\mu_1=\ln\frac{a+b}{\sqrt{a^2-b^2}},$$
$$\bar{\Omega}=\left\{(\mu,\phi)\mid\mu>\mu_0,0<\phi<\omega\right\},\bar{\Gamma}=\left\{(\mu,\phi)\mid\mu=\mu_0,0<\phi<\omega\right\},$$
$$\bar{\Gamma_1}=\left\{(\mu,\phi)\mid\mu>\mu_0,\phi=0\right\},\bar{\Gamma_2}=\left\{(\mu,\phi)\mid\mu>\mu_0,\phi=\omega\right\},$$

Theorem 1. *The transformation (2.13) possesses the following property: The Jacobi determinant of equation (2.13) is*

$$J(\mu,\phi)=\begin{vmatrix} \frac{\partial\xi}{\partial\mu} & \frac{\partial\xi}{\partial\phi} \\ \frac{\partial\eta}{\partial\mu} & \frac{\partial\eta}{\partial\phi} \end{vmatrix}=f_0^2\left(\sinh^2\mu\cos^2\phi+\cosh^2\mu\sin^2\phi\right). \tag{2.13}$$

(1) For $u \in C^2(R^2)$, we have

$$\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \phi^2} = J \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{2.14}$$

(2) For the domain Ω_2,

$$\frac{\partial u}{\partial v} = -\frac{1}{\sqrt{J}} \frac{\partial u}{\partial \mu}, \tag{2.15}$$

where v denotes the unit exterior normal vector on Γ_0,

$$v = -\frac{1}{\sqrt{J}} \left(f_0 \sinh \mu \cos \phi, f_0 \cosh \mu \sin \phi \right). \tag{2.16}$$

Now, we assume that $w(x)$ is the solution of the problem (2.11), and the value $w \mid_{|u|=u_1}$ is given

$$w \mid_{\Gamma_0} = w_0(\mu_1, \phi)$$

and the boundary can be described as this form

$$\Gamma_0 == \left\{ (x, y) \mid ax^2 + by^2 = 1 \right\}, \text{with } \beta b > \alpha a > 0.$$

We introduce $x = \sqrt{\alpha}\xi, y = \sqrt{\beta}\eta$, the boundary Γ_0 is changed to the elliptic arc boundary $\bar{\Gamma_0} = \left\{ (\xi, \eta) \mid \alpha a \xi^2 + \beta b \eta^2 = 1 \right\}$, the unit exterior normal vector on $\bar{\Gamma_0}$ is

$$\vec{v} = -\frac{1}{\sqrt{\alpha a \cos^2 \phi + \beta b \cos^2 \phi}} \left(\sqrt{\alpha a} \cos \phi, \sqrt{\beta b} \sin \phi \right). \tag{2.17}$$

The problem (2.11) is transformed into

$$\begin{cases} -\triangle w = -\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} \right) = 0, & \text{in } \bar{\Omega_2}, \\ \frac{\partial w}{\partial n} = 0, & \text{on } \bar{\Gamma_{12}} \cup \bar{\Gamma_{22}}, \\ w(x) = \circ(1), & \text{as } |x| \longrightarrow \infty. \end{cases} \tag{2.18}$$

Similar with (2.12), let

$$\begin{cases} \xi = f_0 \cosh \mu \cos \phi, \\ \eta = f_0 \sinh \mu \sin \phi, \end{cases}$$

we have

$$f_0 = \sqrt{\frac{b\beta - a\alpha}{a\alpha b\beta}} R, \mu_0 = \ln \frac{\sqrt{a\alpha} + \sqrt{b\beta}}{\sqrt{b\beta - a\alpha}},$$

$$\bar{\Omega_2} = \left\{ (\mu, \phi) \mid \mu > \mu_1, 0 < \phi < \omega \right\}, \bar{\Gamma_0} = \left\{ (\mu, \phi) \mid \mu = \mu_1, 0 < \phi < \omega \right\}$$

. Based on the natural reduction [3], there are the Poisson integral formulas[4]

$$w(\mu, \phi) = \frac{1}{\omega} \int_0^\omega w_0(\mu, \phi') d\phi' + \frac{2}{\omega} \sum_{n=1}^{+\infty} e^{\frac{n(\mu_1 - \mu)\pi}{\omega}} \int_0^\omega \cos \frac{n\pi(\phi - \phi')}{\omega} w_0(\mu, \phi') d\phi', \tag{2.19}$$

and the natural integral equation

$$\frac{\partial w}{\partial n} = \frac{2\pi}{\omega^2 \sqrt{J_0}} \sum_{n=1}^{+\infty} n \int_0^\omega w(\mu_1, \phi') \cos \frac{n\pi\phi'}{\omega} \cos \frac{n\pi\phi}{\omega} d\phi', \tag{2.20}$$

where$J_0 = f_0^2(\cosh^2 \mu_1 - \cos^2 \phi)$.

From (2.8), we obtain

$$\frac{\partial w}{\partial n} = a_0(u) \frac{\partial u}{\partial n}. \tag{2.21}$$

Combining (2.19), (2.20) and (2.21), we obtain the exact artificial boundary condition of u

on Γ_0,

$$\left|\alpha a_0(u)n_x\frac{\partial u}{\partial x}+\beta a_0(u)n_y\frac{\partial u}{\partial y}\right|_{\mu=\mu_1} \\ =-\frac{\pi}{\omega^2 R}\sqrt{\frac{\alpha a\beta b}{a\cos^2\phi+b\sin^2\phi}}\int_0^{\omega}\sum_{n=1}^{\infty}n\cos n(\phi-\phi')\left(\int_0^{u(\mu_1,\phi)}a_0(y)dy\right)d\phi \\ \triangleq \kappa_1(u(\mu_1,\phi)). \tag{2.22}$$

§3 Variational problem and finite element approximation

3.1 The equivalent variational problems

Now,we will use $W^{m,p}$ denoting the standard Sobolev spaces. $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$denoting the corresponding norms and semi-norms. In particular, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega}$ $=\|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,\Omega}=|\cdot|_{m,2,\Omega}$.

Let us introduce the space

$$V=\left\{v\in H^1(\Omega_1)\mid v\,|_{\Gamma}=0\right\}, \tag{3.1}$$

and the corresponding norms

$$\|v\|_{0,\Omega_1}=\sqrt{\int_{\Omega_1}|v|^2\,dx},\|v\|_{1,\Omega_1}=\sqrt{\int_{\Omega_1}(|v|^2+\left|\nabla v\right|^2)dx}.$$

The boundary value problem (2.22) is equivalent to the following variational problem

$$\begin{cases}\text{Find } u\in V,\text{such that}\\ A(u;u,v)+B(u;u,v)=F(v)\quad v\in V,\end{cases} \tag{3.2}$$

with

$$A(w;u,v)=\int_{\Omega_1}a(x,w)\left(\alpha\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\beta\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dx, \tag{3.3}$$

$$B(w;u,v)=\sum_{n=1}^{\infty}\frac{\sqrt{\alpha\beta}}{n\pi}\int_0^{\omega}\int_0^{\omega}a_0(w(\mu_1,\phi')\frac{\partial u(\mu_1,\phi')}{\partial\phi'}\frac{\partial v(\mu_1,\phi)}{\partial\phi}\cos n(\phi'-\phi)d\phi'd\phi, \tag{3.4}$$

$$F(v)=\int_{\Omega_1}f(x)v(x)dx. \tag{3.5}$$

Lemma 1. *[13] There exists $C_0',C_1'>0$, such that*

$$|B(w;u,v)|\le C_0'\|u\|_{1,\Omega_1}\|v\|_{1,\Omega_1},B(w;u,v)\ge C_1'\|u\|_{1,\Omega_1}^2,\quad u,v,w\in V.$$

In practice, we need to truncate the series in (2.22) for some nonnegative integer N, that is,

$$\left|\alpha a_0(u)n_x\frac{\partial u}{\partial x}+\beta a_0(u)n_y\frac{\partial u}{\partial y}\right|_{u=\mu_1}=\kappa_1^N(u(\mu_1,\phi)), \tag{3.6}$$

with

$$\kappa_1^N(u(\mu_1,\phi))=-\frac{\pi}{\omega^2R}\sqrt{\frac{\alpha a\beta b}{a\cos^2\phi+b\sin^{\phi}}}\int_0^{\omega}\sum_{n=1}^{N}n\cos n(\phi-\phi')\left(\int_0^{u(\mu_1,\phi)}a_0(y)dy\right)d\phi. \tag{3.7}$$

Then, we consider the following approximate problem

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x,u^N\right)\frac{\partial u^N}{\partial x}\right)+\frac{\partial}{\partial y}\left(\beta a\left(x,u^N\right)\frac{\partial u^N}{\partial y}\right)\right)=0, & \text{in } \Omega_1,\\ \frac{\partial u^N}{\partial n}=0, & \text{on } \Gamma_{11}\cup\Gamma_{21},\\ u^N=0, & \text{on } \Gamma.\\ \alpha a_0(u^N)n_x\frac{\partial u^N}{\partial x}+\beta a_0(u^N)n_y\frac{\partial u^N}{\partial y}=\kappa_1^N(u^N(\mu_1,\phi)), & \text{on } \Gamma_0. \end{cases} \tag{3.8}$$

The problem (3.8) is equivalent to the following variational problem

$$\begin{cases} \text{Find } u^N\in V, \text{such that}\\ A(u^N;u^N,v)+B_N(u^N;u^N,v)=F(v) \quad v\in V, \end{cases} \tag{3.9}$$

where

$$B_N(w;u,v)=\sum_{n=1}^{N}\frac{\sqrt{\alpha\beta}}{n\pi}\int_0^\omega\int_0^\omega a_0(w(\mu_1,\phi'))\frac{\partial u(\mu_1,\phi')}{\partial\phi'}\frac{\partial v(\mu_1,\phi)}{\partial\phi}\cos n(\phi'-\phi)d\phi'd\phi. \tag{3.10}$$

Similar with Lemma 1, we have

Lemma 2. *[13] There exists $C_0'', C_1''>0$, such that*

$$|B_N(w;u,v)|\le C_0''\left\|u\right\|_{1,\Omega_1}\left\|v\right\|_{1,\Omega_1}, B_N(w;u,v)\ge C_1''\left\|u\right\|_{1,\Omega_1}^2, \quad u,v,w\in V.$$

3.2 Finite element approximation

Suppose τ_h is a regular and quasi-uniform triangulation on Ω_1, such that

$$\Omega_1=\bigcup_{K\in\tau_h}K, \tag{3.11}$$

where K is a (curved) triangle, and h denotes the maximal side of the triangles. Let

$$V_h=\left\{v_h\in V\mid v\mid_k \text{ is a linear polynomial,} \forall K\in\tau_h\right\}. \tag{3.12}$$

The approximate problem of (3.9) can be written as

$$\begin{cases} \text{Find } u_h^N\in V_h(\Omega_1), \text{such that}\\ A(u_h^N;u_h^N,v_h)+B_N(u_h^N;u_h^N,v_h)=F(v_h) \quad v_h\in V_h(\Omega_1). \end{cases} \tag{3.13}$$

We divide the arc Γ_0 into M parts, and take a finite element subdivision in Ω_1, which their nodes on Γ_0 are coincident. Let θ_i represents the split point,$i=1,2,\dots,M-1$, where $\theta_0\equiv 0<\theta_1<\theta_2<\dots<\theta_{N-1}<\omega\equiv\theta_M$, and let $h_i=\theta_i-\theta_{i-1}, i=1,2,\dots,M-1$. Take $L_i(\theta)$ as the following piecewise linear interpolation function

$$L_0(\theta)=\begin{cases} \frac{\theta_1-\theta}{h_1}, & \theta_0\le\theta\le\theta_1,\\ 0, & \text{else;} \end{cases}$$

$$L_i(\theta)=\begin{cases} \frac{\theta-\theta_{i-1}}{h_i}, & \theta_{i-1}\le\theta\le\theta_i,\\ \frac{\theta_{i+1}-\theta}{h_{i+1}}, & \theta_i\le\theta\le\theta_{i+1}, 1\le i\le M+1,\\ 0, & \text{else;} \end{cases}$$

$$L_0(\theta)=\begin{cases} \frac{\theta-\theta_{N-1}}{h_i}, & \theta_{N-1}\le\theta\le\theta_M,\\ 0, & \text{else.} \end{cases}$$

Therefore, we know $V_h = \{L_i(\theta)\}_{i=1}^{M}$, and get

$$u_h = \sum_{i=1}^{M} U_i L_i(\theta), v_h = \sum_{i=1}^{M} V_i L_i(\theta). \tag{3.14}$$

Coming (3.13), we can get the finite element equation

$$QU = b,$$

where, $Q = Q_1 + Q_2$. The matrix $Q_1 = (q_{ij})_{NN}$ is the stiffness matrix obtained from the finite element in Ω_1, while $Q_2 = \left(q_{ij}^2\right)_{M\times M}$ follows from the natural boundary element method on Γ_0, and see reference [1] for the calculation formula Q_1. In this paper, we analyze the use of linear interpolation in the case of uniform boundary Γ_0 partition, and we know $h = h_i = \frac{\omega}{M}, \theta_i = i \cdot h = \frac{i\omega}{M}$, and get

$$q_{i,j} = q_{j,i} = \frac{32M^2}{\pi^3} \sum_{n=1}^{+\infty} \frac{1}{n^3} \sin^4(\frac{n\pi}{2M}) \cos(\frac{n(i-j)\pi}{M}), i, j = 1, 2, \ldots, M. \tag{3.15}$$

From the discrete problem (3.13), we can get the following the discrete form of nonoverlapping domain decomposition algorithm:

Step 1: Choose an initial value $\lambda_h^0 \in H^{\frac{1}{2}}(\Gamma_0)$, and let $k = 0$.

Step 2: Solve a Dirichlet boundary value problem in the exterior domain Ω_2,

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x, u_{2h}^k\right) \frac{\partial u_{2h}^k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta a\left(x, u_{2h}^k\right) \frac{\partial u_{2h}^k}{\partial y}\right)\right) = 0, & \text{in } \Omega_2, \\ u_{2h}^k = \lambda^k, & \text{on } \Gamma_0, \\ \frac{\partial u_{2h}^k}{\partial n_1} = 0, & \text{on } \Gamma_{21} \cup \Gamma_{22}, \\ u_{2h}^k(x) = o(1), & \text{as } |x| \longrightarrow \infty. \end{cases} \tag{3.16}$$

Step 3: Solve a mixed boundary value problem in the interior domain Ω_1,

$$\begin{cases} -\left(\frac{\partial}{\partial x}\left(\alpha a\left(x, u_{2h}^k\right) \frac{\partial u_{1h}^k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta a\left(x, u_{2h}^k\right) \frac{\partial u_{1h}^k}{\partial y}\right)\right) = f, & \text{in } \Omega_1, \\ \alpha a_0(u) n_x \frac{\partial u_{1h}^k}{\partial x} + \beta a_0(u) n_y \frac{\partial u_{1h}^k}{\partial y} = -(\alpha a_0(u) n_x \frac{\partial u_{2h}^k}{\partial x} + \beta a_0(u) n_y \frac{\partial u_{2h}^k}{\partial y}), & \text{on } \Gamma_0, \\ \frac{\partial u_{1h}^k}{\partial n_2} = 0, & \text{on } \Gamma_{12} \cup \Gamma_{22}, \\ u_{1h}^k(x) = 0, & \text{on } \Gamma. \end{cases} \tag{3.17}$$

Step 4: Update the boundary valve $0 < \theta_k < 1$,

$$\lambda_h^{k+1} = \theta_k u_{1h}^k + (1 - \theta_k)\lambda_h^k \quad \text{on } \Gamma_0. \tag{3.18}$$

Step 5: Put $k = k + 1$, turn to Step 2.

Therefore, we can get a system of algebraic equations for our quasi-linear problem with the following form

$$\begin{pmatrix} Q_{11} + Q_2 & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{3.19}$$

where U_1, U_2 are vectors, whose components are function values at nodes on Γ_0 and at interior nodes of Ω_1 respectively. The matrix $Q_1 = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$ is the stiffness matrix obtained from the finite element in Ω_1, while Q_2 follows from the natural boundary element method on Γ_0.

The problem (3.19) can also be rewritten as follows:

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} b_1 - Q_2 U_1 \\ b_2 \end{pmatrix}, \tag{3.20}$$

then, we have the iterative algorithm

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} U_{1K} \\ U_{2K} \end{pmatrix} = \begin{pmatrix} b_1 - Q_2 \Lambda_K \\ b_2 \end{pmatrix}, \tag{3.21}$$

with

$$\Lambda_{k+1} = \theta_k U_{1k} + (1-\theta_k), k = 0, 1, 2, \ldots \tag{3.22}$$

By condition (1.2), we obtain that Q_1 is a positive definite matrix, so the Q_{22}^{-1} exists. We assume that $S_h = S_h^{(1)} + Q_2$ is the discrete analogue of the Steklov-Poincare operator on Γ_0, with

$$\begin{gathered} S_h^{(1)} = Q_{11} - Q_{12}^T Q_{22}^{-1} Q_{21}, \\ B = b_1 - Q_{12} Q_{22}^{-1} b_2. \end{gathered}$$

We have the preconditioned Richardson iteration

$$S_h^1(\Lambda_{k+1} - \Lambda_k) = \theta_k (B - S_h \Lambda_k). \tag{3.23}$$

Theorem 2. *[14] The D-N alternating algorithm (2.4)-(2.6) is equivalent to the preconditioned Richardson iteration (3.23).*

Theorem 3. *[14] If* $0 < \min \theta_k < \max \theta_k < 1$, *the discrete nonoverlapping alternating method (3.16)-(3.18) is convergent, and both the convergence rate and the condition number of* $[S_h^{(1)}]^{-1} S_h$ *are independent of the mesh size* h.

§4 Convergence analysis of the method

Theorem 4. *If* $0 < \theta_k < 1$, *then the D-N alternating algorithm (2.4)-(2.6) is convergent.*

Proof: We assume that the exact solution to problem (1.1) is u, and let $\lambda = u \mid_{\Gamma_0}, u_k = u \mid_{\Omega_k}, k = 1, 2$. According to (2.4)-(2.6), we let $e_i^k = u_i - u_i^k, i = 1, 2$ and $u^k = e_1^n \mid_{\Gamma_0} = \lambda - \lambda^k$. We suppose $\Gamma = \{(\mu, \phi) \mid \mu = \mu_0, 0 < \phi < \omega\}, \mu_1 > \mu_0$, and combing (2.4) and (2.5), we get

$$\begin{cases} -\triangle e_2^k = 0, & \text{in } \Omega_2, \\ e_2^k = \mu^k, & \text{on } \Gamma_0, \\ \frac{\partial e_2^k}{\partial n_1} = 0, & \text{on } \Gamma_{12} \cup \Gamma_{22}, \\ w^k(x) = \circ(1), & \text{as } |x| \longrightarrow \infty, \end{cases} \tag{4.1}$$

$$\begin{cases} -\triangle e_1^k = 0, & \text{in } \Omega_1, \\ \frac{\partial e_1^k}{\partial n_1} = -\frac{\partial e_2^k}{\partial n_2}, & \text{on } \Gamma_1, \\ \frac{\partial e_1^k}{\partial n_1} = 0, & \text{on } \Gamma_{12} \cup \Gamma_{22}, \\ w^k(x) = 0, & \text{on } \Gamma, \end{cases} \tag{4.2}$$

$$\mu^{k+1} = \theta_n e_1^k \mid_{\Gamma_0} + (1-\theta_n)\lambda^k, \tag{4.3}$$

and let

$$\mu^k = \sum_{n=-\infty}^{+\infty} a_n e^{in\phi} \in H^{\frac{1}{2}}(\Gamma_0), a_n = \frac{1}{\omega}\int_0^{\omega} \mu_0 e^{-in\phi} d\phi, \tag{4.4}$$

by the natural integral equation (3.7), we know

$$\frac{\partial e_1^k}{\partial n} = -\kappa_1(\mu^k) = -\frac{1}{\sqrt{J}}\sum_{n=-\infty}^{+\infty} |n|\, b_n e^{in\phi}, b_n = \frac{1}{\omega}\int_0^{\omega} \lambda^n e^{-in\phi} d\phi, \tag{4.5}$$

and we get

$$\begin{cases} -\bigtriangleup e_1^k = 0, & \text{in } \Omega_1, \\ e_1^k = 0, & \text{on } \Gamma, \\ \frac{\partial e_1^k}{\partial n} = \frac{1}{\sqrt{J}}\sum_{n=-\infty}^{+\infty} |n|\, b_n e^{in\phi}, & \text{on } \Gamma_0. \end{cases} \tag{4.6}$$

In order to get the exact solution of the problem (4.6), we introduce the following Lemma 3.

Lemma 3. *[8] If $u \in H^1(\Omega_1)$ is the solution of (4.7),*

$$\begin{cases} -\bigtriangleup u = 0, & \textit{in } \tilde{\Omega_1}, \\ u = u_0, & \textit{on } \tilde{\Gamma}, \\ \frac{\partial u}{\partial n} = u_n & \textit{on } \tilde{\Gamma_0}, \end{cases} \tag{4.7}$$

where $\tilde{\Omega_1}$ is the circular ring domain between $\tilde{\Gamma}$ and $\tilde{\Gamma_0}$, $R_0 > R, R_0, R$ are the radii of $\tilde{\Gamma_0}, \tilde{\Gamma}$,

$$u_0 = \sum_{n=-\infty}^{+\infty} c_n e^{in\varphi} \in H^{\frac{1}{2}}(\tilde{\Gamma}), u_n = \sum_{n=-\infty}^{+\infty} |n|\, d_n e^{in\varphi} + d_0 \in H^{\frac{1}{2}}(\tilde{\Gamma_0}), \tag{4.8}$$

and there exists a unique

$$u(r,\varphi) = \sum_{n=-\infty}^{+\infty} \left(\frac{c_n R^{|n|}(r^{|n|} + R_0^{2|n|} r^{-|n|}) + d_n R_0^{|n|+1}(r^{|n|} - R^{|n|+1} r^{-|n|})}{R^{2|n|} + R_0^{2|n|}} \right) e^{in\varphi} + c_0 + R_1 b_0 \ln \frac{r}{R}. \tag{4.9}$$

Similar with Lemma 3, we can get

$$e_1^k = -\sum_{n=-\infty}^{+\infty} \left(\frac{c_n(e^{|n|(\mu-\mu_1)} + e^{|n|(\mu_1-\mu)}) + d_n(e^{|n|(\mu-\mu_0)} + e^{|n|(\mu_0-\mu)})}{(e^{|n|(\mu_1-\mu)} + e^{|n|(\mu_0-\mu)})} \right) e^{in\phi}, \tag{4.10}$$

and

$$k_1(e_1^k) = -\frac{1}{\sqrt{J}}\sum_{n=-\infty}^{+\infty} |n| \left(\frac{c_n(e^{|n|(\mu-\mu_1)} + e^{|n|(\mu_1-\mu)}) + d_n(e^{|n|(\mu-\mu_0)} + e^{|n|(\mu_0-\mu)})}{(e^{|n|(\mu_1-\mu)} + e^{|n|(\mu_0-\mu)})} \right) e^{in\phi}. \tag{4.11}$$

Let

$$H_n(\mu) = \frac{e^{|n|(\mu-\mu_0)} - e^{|n|(\mu_0-\mu)}}{e^{|n|(\mu_1-\mu_0)} + e^{|n|(\mu_0-\mu_1)}}, \tag{4.12}$$

therefore, we have

$$e_1^k = -\sum_{n=-\infty}^{+\infty} a_n H_n(\mu_1) e^{in\phi}, \kappa_1(e_1^k) = -\frac{1}{\sqrt{J}}\sum_{n=-\infty}^{+\infty} |n| a_n H_n(\mu_1) e^{in\phi}, \tag{4.13}$$

then

$$\begin{aligned}\frac{\partial e_1^{k+1}}{\partial n} &= -\kappa_1(\lambda - \lambda^{k+1}) \\ &= \kappa_1(\theta_k u_1^k + (1-\theta_k)\lambda^k - \lambda) \\ &= -\theta_k \kappa_1(e_1^k) - (1-\theta_k)\kappa_1(e_2^k) \\ &= \frac{1}{\sqrt{J}} \sum_{n=-\infty}^{+\infty} |n| a_n(\theta_k H_n(\mu_1) - 1 + \theta_k) e^{in\phi}. \qquad (4.14)\end{aligned}$$

Let

$$E^m = \frac{\pi}{\omega} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\sqrt{1+n^2}} |a_n|^2, \qquad (4.15)$$

$$\begin{aligned}E^{m+1} &= \frac{\pi}{\omega} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\sqrt{1+n^2}} |a_n|^2 (\theta_k H_n(\mu_1) - 1 + \theta_k)^2 \\ &= (1-\theta_k)^2 E^m + \frac{\pi}{\omega} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\sqrt{1+n^2}} |a_n|^2 \theta_k H_n(\mu_1)[\theta_k(H_n(\mu_1)+2) - 2]. \qquad (4.16)\end{aligned}$$

Assuming that $\delta_1 = \inf \frac{2}{2+H_n(\mu_1)}$, we have $\frac{2}{3} < \delta_1 < 1$. If $0 < \delta_k < \delta_1, k = 0, 1, 2, \ldots$, then

$$E^{m+1} < (1-\theta_k)^2 E^m, \qquad (4.17)$$

and by the trace theorem, we have

$$\left\| e_1^k \right\|_{1,\Omega_1}^2 \leq C E^m \longrightarrow 0, m \longrightarrow \infty. \qquad (4.18)$$

From (4.16), we have

$$\begin{aligned}E^{m+1} &= \frac{\pi}{\omega} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\sqrt{1+n^2}} |a_n|^2 (\theta_k H_n(\mu_1) - 1 + \theta_k)^2 \\ &= (1-2\theta_k)^2 E^m + \frac{\pi}{\omega} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\sqrt{1+n^2}} |a_n|^2 \theta_k H_n'(\mu_1)[\theta_k(H_n'(\mu_1) - 2) + 1], \qquad (4.19)\end{aligned}$$

where

$$H_n'(\mu_1) = \frac{1 - H_n(\mu_1)}{2}. \qquad (4.20)$$

We suppose that

$$\sigma = \sup_{n \in Z^+} \frac{1}{2 - H_n(\mu_1)}, \qquad (4.21)$$

then, we get $\sigma = \frac{2}{3}$.

For $\delta_2 < \theta_k < 1, k = 0, 1, 2, \ldots$, the convergence result can be obtained. Therefore, for $0 < \theta_k < 1$, the D-N alternating algorithm (2.4)-(2.6) is convergent.

§5 Numerical examples

We will give some numerical examples using the method developed above to test its effectiveness. In the following, we choose the finite element space as given in (3.1). For simplicity, we let $\triangle\mu = \frac{\mu_1-\mu_0}{m}$, $\triangle\phi = \frac{\omega}{M}$, and let $e(k) = \left\|u - u^k_{h,N}\right\|_{L^\infty(\Omega_1)}$, $e_h(k) = \left\|u^k_{h,N} - u^{k-1}_{h,N}\right\|_{L^\infty(\Omega_1)}$, $q_h(k)$ $= \frac{e_h(k-1)}{e_h(k)}$ to simulate the convergence rate.

Example 1. *We assume*

$$\begin{aligned} \Omega &= \{(\mu,\phi) \mid \mu > \mu_0, 0 < \phi < \omega\}, \\ \Gamma &= \{(\mu,\phi) \mid \mu = \mu_0, 0 < \phi < \omega\}, \\ \Omega &= \{(\mu,\phi) \mid \mu > \mu_0, \phi = 0\}, \\ \Gamma &= \{(\mu,\phi) \mid \mu > \mu_0, \phi = \omega\}. \end{aligned}$$

with $\omega = \frac{7}{4}\pi, \mu_0 = 0.8, f_0 = 1.6$. *(According to reference [8-10], we make* $\varepsilon = 0.5$ *for easy analysis.) We introduce an elliptical boundary*

$$\Gamma_0 = \{(\mu,\phi) \mid \mu = \mu_1, 0 < \phi < \omega\}, \mu_1 = 1.5.$$

We take our numerical results for problem (1) with

$$a(x,u) = \begin{cases} 9 - r^2 + \frac{1}{1+u^2}, & 0.8 \le r \le 1.5, \\ \frac{1}{1+u^2}, & r > 1.5, \end{cases}$$

$$r = \sqrt{x^2+y^2}, u = \tan(\frac{y}{r^2}),$$

$$f = -\frac{1}{r^4\cos(\frac{y}{r^2})^3}(2(x^2\sin(\frac{y}{r^2}) - x^2\cos(\frac{y}{r^2})y + y^2\sin(\frac{y}{r^2}) - y^3\cos(\frac{y}{r^2}) - 9\sin(\frac{y}{r^2}))).$$

The numerical results are given in Table 1, Table 2 and Figure 3.

Table 1. The relationship between meshes and convergence rate ($N = 100, \theta_k = 0.9, \varepsilon = 0.5$).

mesh	Iteration number						
	k	0	1	2	3	4	5
h	e(k)	7.42E-01	5.34E-01	3.80E-01	3.25E-01	2.30E-01	2.29E-01
	$e_h(k)$	-	3.58E-02	1.56E-02	8.57E-03	5.35E-03	3.63E-03
	$q_h(k)$	-	-	2.2915	1.8212	1.6019	1.4751
h/2	e(k)	7.24E-01	4.59E-01	3.22E-01	3.15E-01	2.24E-01	1.83E-01
	$e_h(k)$	-	2.53E-02	1.09E-02	5.93E-03	3.67E-03	2.48E-03
	$q_h(k)$	-	-	2.3229	1.8378	1.6129	1.4833
h/4	e(k)	7.11E-01	4.26E-01	2.95E-01	2.87E-01	2.19E-01	1.81E-01
	$e_h(k)$	-	1.44E-02	6.45E-03	3.59E-03	2.26E-03	1.54E-03
	$q_h(k)$	-	-	2.2272	1.7959	1.5882	1.4663

Table 2. The relationship between ε and convergence rate $(N = 100, \theta_k = 0.9, m = 4, M = 17)$.

ε	Iteration number						
	k	0	1	2	3	4	5
0.9	e(k)	7.42E-01	5.34E-01	3.80E-01	3.25E-01	2.30E-01	2.29E-01
	$e_h(k)$	-	2.44E-02	1.05E-02	5.72E-03	3.55E-03	2.40E-03
	$q_h(k)$	-	-	2.2915	1.8212	1.6019	1.4751
0.75	e(k)	7.47E-01	5.00E-01	3.62E-01	3.51E-01	2.50E-01	2.07E-01
	$e_h(k)$	-	2.47E-02	1.06E-02	5.80E-03	3.60E-03	2.43E-03
	$q_h(k)$	-	-	2.3216	1.8365	1.6119	1.4824
0.5	e(k)	7.24E-01	4.59E-01	3.22E-01	3.15E-01	2.24E-01	1.83E-01
	$e_h(k)$	-	2.53E-02	1.09E-02	5.93E-03	3.67E-03	2.48E-03
	$q_h(k)$	-	-	2.3229	1.8378	1.6129	1.4833
0.3	e(k)	7.27E-01	4.64E-01	3.37E-01	3.28E-01	2.30E-01	1.95E-01
	$e_h(k)$	-	2.56E-02	1.10E-02	6.00E-03	3.72E-03	2.51E-03
	$q_h(k)$	-	-	2.3217	1.838	1.6135	1.4838
0.1	e(k)	7.64E-01	5.61E-01	4.30E-01	3.43E-01	2.63E-01	2.59E-01
	$e_h(k)$	-	2.51E-02	1.10E-02	5.99E-03	3.72E-03	2.51E-03
	$q_h(k)$	-	-	2.2801	1.837	1.6124	1.4833

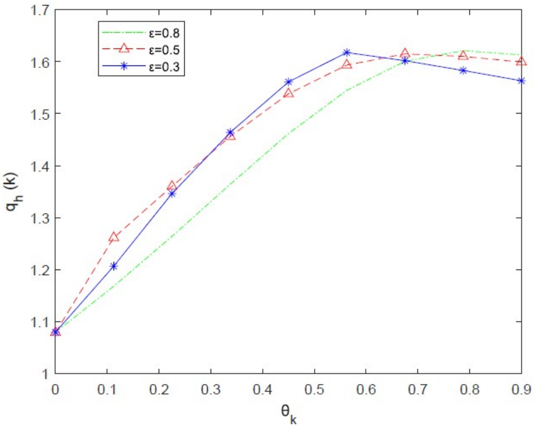

Figure 3. The relationship between θ_k and convergence rate $(N = 100, m = 4, M = 17), k = 4$.

Example 2. *We assume*

$$\Omega = \{(\mu, \phi) \mid \mu > \mu_0, 0 < \phi < \omega\}, \Gamma = \{(\mu, \phi) \mid \mu = \mu_0, 0 < \phi < \omega\},$$

$$\Omega = \{(\mu, \phi) \mid \mu > \mu_0, \phi = 0\}, \Gamma = \{(\mu, \phi) \mid \mu > \mu_0, \phi = \omega\}.$$

with $\omega = \frac{3}{2}\pi, \mu_0 = 0.8, f_0 = 2$. *(According to reference [8-10], we make* $\varepsilon = 0.5$ *for easy analysis.) We introduce an elliptical boundary*

$$\Gamma_0 = \{(\mu, \phi) \mid \mu = \mu_1, 0 < \phi < \omega\}, \mu_1 = 1.5.$$

We take our numerical results for problem (1), with

$$a(x,u)=\begin{cases}1+r^2+\frac{1}{1+u^2}, & 0.8\le r\le 1.5,\\ \frac{1}{1+u^2}, & r>1.5,\end{cases}$$

$$r=\sqrt{x^2+y^2}, u=\tan(\frac{y}{r^2}),$$

$$f=-\frac{1}{r^4\cos(\frac{y}{r^2})^3}(2(-x^2\cos(\frac{y}{r^2})+x^2\sin(\frac{y}{r^2})+\sin(\frac{y}{r^2})-y^3\cos(\frac{y}{r^2})+y\sin(\frac{y}{r^2}))).$$

The numerical results are given in Table 3, Table 4 and Figure 4.

Table 3. The relationship between meshes and convergence rate ($N=100,\theta_k=0.8,\varepsilon=0.6$).

mesh	Iteration number						
	k	0	1	2	3	4	5
h	e(k)	9.21E-01	8.65E-01	8.44E-01	7.80E-01	7.27E-01	6.84E-01
	$e_h(k)$	-	4.04E-02	1.92E-02	1.06E-02	6.59E-03	4.44E-03
	$q_h(k)$	-	-	2.0987	1.8088	1.6137	1.4855
h/2	e(k)	9.16E-01	8.39E-01	7.96E-01	7.66E-01	6.88E-01	6.05E-01
	$e_h(k)$	-	2.25E-02	1.08E-02	5.96E-03	3.70E-03	2.49E-03
	$q_h(k)$	-	-	2.0903	1.8041	1.6111	1.484
h/4	e(k)	8.71E-01	8.81E-01	7.88E-01	7.36E-01	6.36E-01	5.90E-01
	$e_h(k)$	-	8.42E-03	4.55E-03	2.72E-03	1.77E-03	1.23E-03
	$q_h(k)$	-	-	1.8498	1.6727	1.5386	1.441

Table 4. The relationship between ε and convergence rate ($N=100,\theta_k=0.8,m=4,M=17$).

ε	Iteration number						
	k	0	1	2	3	4	5
0.95	e(k)	9.16E-01	8.34E-01	7.92E-01	7.62E-01	6.86E-01	5.95E-01
	$e_h(k)$	-	2.18E-02	1.04E-02	5.80E-03	3.61E-03	2.43E-03
	$q_h(k)$	-	-	2.0831	1.7999	1.6087	1.4826
0.8	e(k)	9.16E-01	8.39E-01	7.96E-01	7.66E-01	6.88E-01	6.05E-01
	$e_h(k)$	-	2.25E-02	1.08E-02	5.96E-03	3.70E-03	2.49E-03
	$q_h(k)$	-	-	2.3216	1.8365	1.6119	1.4824
0.6	e(k)	9.15E-01	8.37E-01	7.91E-01	7.58E-01	6.78E-01	6.46E-01
	$e_h(k)$	-	1.99E-02	9.65E-03	5.41E-03	3.38E-03	2.29E-03
	$q_h(k)$	-	-	2.057	1.7848	1.6002	1.4774
0.4	e(k)	9.14E-01	8.35E-01	7.85E-01	7.52E-01	6.70E-01	6.23E-01
	$e_h(k)$	-	1.70E-02	8.51E-03	4.86E-03	3.08E-03	2.10E-03
	$q_h(k)$	-	-	1.9945	1.75	1.5809	1.4662
0.2	e(k)	9.10E-01	8.30E-01	7.73E-01	7.41E-01	6.60E-01	5.94E-01
	$e_h(k)$	-	1.23E-02	6.80E-03	4.10E-03	2.67E-03	1.86E-03
	$q_h(k)$	-	-	1.8134	1.6581	1.5335	1.4398

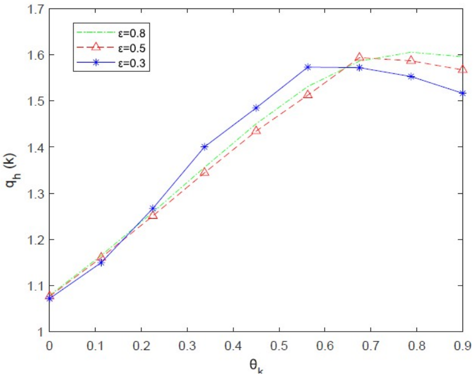

Figure 4. The relationship between θ_k and convergence rate $(N = 100, m = 4, M = 17), k = 5$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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