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A robustness analysis of joint predictions under a general linear model

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Abstract. Robustness refers to the relative stability of statistical inference results with respect to assumption conditions of regression models, namely, when the model assumptions change slightly, wether the corresponding statistical inference changes only slightly. This paper considers robustness analysis of joint predictions under a general linear model. We first establish the definitions and analytical expressions of Best Linear Unbiased Predictions (BLUPs) and Best Linear Minimum Bias Predictions (BLMBPs), and discuss algebraic and statistical properties of the BLUPs and the BLMBPs, including the robustness of joint predictions with respect to model matrices and covariance matrices under a general linear model by implementing the block matrix representation method and the matrix rank method.

§1 Introduction

We start with an outline of notation and some preliminary results adopted in the sequel. We denote by $\mathbb{R}^{m\times n}$ the set of all $m\times n$ real matrices, by $\mathbf{A}'$ the transpose of $\mathbf{A}$, $r(\mathbf{A})$ the rank of $\mathbf{A}$, i.e., the maximum order of the invertible sub-matrix of $\mathbf{A}$; and by $\mathscr{R}(\mathbf{A}) = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x}\in\mathbb{R}^{n\times 1}\}$ the range of a matrix $\mathbf{A}\in\mathbb{R}^{m\times n}$; by $\mathbf{I}_m$ the identity matrix of order m. The notation $\mathbf{A}\succcurlyeq\mathbf{0}$ means that $\mathbf{A}$ is nonnegative definite (nnd). Two symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size are said to satisfy the inequality $\mathbf{A}\succcurlyeq\mathbf{B}$, in the Löwner partial ordering if $\mathbf{A}-\mathbf{B}$ is nnd. The Moore–Penrose generalized inverse of a matrix $\mathbf{M}$, denoted by $\mathbf{M}^{+}$, is defined to be the unique matrix $\mathbf{G}$ satisfying the four Penrose equations $\mathbf{MGM}=\mathbf{M}$, $\mathbf{GMG}=\mathbf{G}$, $(\mathbf{MG})'=\mathbf{MG}$, and $(\mathbf{GM})'=\mathbf{GM}$. In what follows, we denote by $\mathbf{P}_{\mathbf{A}}=\mathbf{AA}^{+}$, $\mathbf{A}^{\perp}=\mathbf{E}_{\mathbf{A}}=\mathbf{I}_m-\mathbf{AA}^{+}$, and $\mathbf{F}_{\mathbf{A}}=\mathbf{I}_n-\mathbf{A}^{+}\mathbf{A}$, the three orthogonal projectors induced from $\mathbf{A}$, respectively. Further information about the orthogonal projectors $\mathbf{P}_{\mathbf{A}}$, $\mathbf{E}_{\mathbf{A}}$, and $\mathbf{F}_{\mathbf{A}}$ with their applications in the linear statistical models can be found, e.g., in [13]. It is also well known that the Löwner partial ordering is a surprisingly strong and useful property between two real symmetric matrices of

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the same size. For more results about the Löwner partial ordering of symmetric matrices and their applications in statistical analysis (see e.g., [13]).

As we know in regression theory, linear statistical model is one of the most widely-used parametric models in statistical data analysis and inference, and of course is a prominent topic in the field of statistical researches. On the other hand, it is also a fundamental model for more advanced statistical methods like logistic regression, survival analysis, multilevel modeling, and structural equation modeling, etc. In recent years, statisticians have approached general linear models by means of various novel matrix analysis techniques, and have established some explicit procedure of deriving closed-form formulas for calculating the Best Linear Unbiased Predictions (BLUPs) and the Best Linear Unbiased Estimations (BLUEs) of all unknown parameters in the contexts of general linear models; see e.g., [2,3,5,6,8,16,18,19] under various linear model assumptions.

In this paper, we consider the following two general linear models

$$\mathscr{M}: \quad \mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}, \quad \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad \mathrm{D}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \tag{1.1}$$

$$\mathscr{N}: \quad \mathbf{y} = \mathbf{X}_0\boldsymbol{\alpha} + \boldsymbol{\varepsilon}_0, \quad \mathrm{E}(\boldsymbol{\varepsilon}_0) = \mathbf{0}, \quad \mathrm{D}(\boldsymbol{\varepsilon}_0) = \boldsymbol{\Sigma}_0, \tag{1.2}$$

where we assume that $\mathscr{M}$ is a correct model, and $\mathscr{N}$ is a misspecified form of $\mathscr{M}$, $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observed random vector, $\mathbf{X}, \mathbf{X}_0 \in \mathbb{R}^{n\times p}$ are two known matrices of arbitrary ranks, $\boldsymbol{\alpha} \in \mathbb{R}^{p\times 1}$ is a fixed but unknown vector, and $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_0 \in \mathbb{R}^{n\times 1}$ are two random error vectors. We don't further assume any probability distribution assumptions for the random vectors in the models.

Observe that there are one unknown fixed parameter vector and two error term vectors in (1.1) and (1.2). Thus it is necessary to propose and establish certain joint linear estimators and predictors of these unknown terms. In fact, much previous and recent work on the problems of joint estimations and predictions of all unknown parameters in a given model were proposed and studied in the statistical literature (see e.g., [4, 7, 8, 16-19]) among others. In view of these facts, we construct a general vector of parametric functions involving the fixed parameter and error terms as follows

$$\boldsymbol{\phi} = \mathbf{K}\boldsymbol{\alpha} + \mathbf{H}\boldsymbol{\varepsilon}, \tag{1.3}$$

where it is assumed that $\mathbf{K} \in \mathbb{R}^{s\times p}$ and $\mathbf{H} \in \mathbb{R}^{s\times n}$ are given matrices of arbitrary ranks. Note that (1.3) includes the two unknown parameter vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ in (1.1) as its spacial cases. Hence the combined form can help develop a unified estimation and prediction theory under the assumptions in (1.1) and (1.3). In this case, we have

$$\mathrm{E}(\boldsymbol{\phi}) = \mathbf{K}\boldsymbol{\alpha}, \quad \mathrm{Cov}\{\boldsymbol{\phi}, \mathbf{y}\} = \mathbf{H}\boldsymbol{\Sigma}, \quad \mathrm{Cov}(\boldsymbol{\phi}) = \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' \tag{1.4}$$

under (1.1). Also under the misspecified assumptions in (1.2), a general vector of parametric functions composed by the two unknown vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}_0$ in (1.2) is

$$\boldsymbol{\phi}_0 = \mathbf{K}_0\boldsymbol{\alpha} + \mathbf{H}_0\boldsymbol{\varepsilon}_0, \tag{1.5}$$

where it is assumed that $\mathbf{K}_0 \in \mathbb{R}^{s\times p}$ and $\mathbf{H}_0 \in \mathbb{R}^{s\times n}$.

We now present some background details of the work. Robustness is a basic concern in statistical inference of regression models, which means that statistical inference has relative stability with respect to the given model assumption conditions. The robustness of BLUEs on model matrices and covariance matrices can be studied separately or simultaneously. For instance, [11, 14, 22] researched robustness of BLUEs relative to covariance matrices separately for linear fixed-effects models; [10] researched robustness of BLUEs relative to model matrices; and [11] researched the robustness of BLUEs relative to model matrices and covariance matrices simultaneously. For linear random-effects models, [1, 21] approached the robustness of BLUEs relative to model matrices and covariance matrices can be studied separately and simultaneously.

Because linear models include unknown parameter vectors and error vectors, it is necessary to consider prediction and estimation problems of joint functions composed of parameter vectors and error vectors. As some novel contributions in this respect, we present in this paper a review of some recent developments concerning predictions and estimations of all unknown parameter vectors in the contexts of (1.1)–(1.4). The coverage of this paper includes solving the following three problems:

(I) deriving analytic expressions and algebraic and statistical properties of Linear Unbiased Predictions (LUPs) and BLUPs of unknown parameter function in the context of (1.1);

(II) deriving the analytic expressions and algebraic and statistical properties of Linear Minimum Bias Predictions (LMBPs) and Best Linear Unbiased Predictions (BLUPs) and Best Linear Minimum Bias Predictions (BLMBPs) of unknown parameter functions in the context of (1.1);

(III) establishing relationships between BLUPs and BLMBPs in the contexts of (1.1) and (1.2).

§2 Some preliminary results

In this section, we review some relevant methods and techniques that can be conveniently used in the analysis of linear regression models. Recall that block matrix and the rank of matrix are two basic conceptual objects in linear algebra and matrix theory. On the other hand, they have been taken as two useful analysis tools for dealing with various basic and advanced problems in theoretical and computational mathematics because they provide us the capacity of constructing and analyzing various simple and complicated matrix expressions and equalities in a clear and concise way.

Lemma 2.1. ([9]) *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$r[\,\mathbf{A},\, \mathbf{B}\,] = r(\mathbf{A}) + r(\mathbf{E_A B}) = r(\mathbf{B}) + r(\mathbf{E_B A}), \tag{2.1}$$

$$r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = r(\mathbf{A}) + r(\mathbf{C F_A}) = r(\mathbf{C}) + r(\mathbf{A F_C}), \tag{2.2}$$

$$r\begin{bmatrix}\mathbf{AA}' & \mathbf{B}\\ \mathbf{B}' & \mathbf{0}\end{bmatrix} = r[\,\mathbf{A},\, \mathbf{B}\,] + r(\mathbf{B}). \tag{2.3}$$

In addition, the following results hold.

(a) $r[\,\mathbf{A},\, \mathbf{B}\,] = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A}) \Leftrightarrow \mathbf{AA}^{+}\mathbf{B} = \mathbf{B} \Leftrightarrow \mathbf{E_A B} = \mathbf{0}$.

(b) $r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{C}') \subseteq \mathscr{R}(\mathbf{A}') \Leftrightarrow \mathbf{C}\mathbf{A}^{+}\mathbf{A} = \mathbf{C} \Leftrightarrow \mathbf{C}\mathbf{F}_{\mathbf{A}} = \mathbf{0}.$

Lemma 2.2. ([20]) *Let* $\mathbf{X} \in \mathbb{R}^{n\times p}, \mathbf{X}_0 \in \mathbb{R}^{n\times q}$, *and let* $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}, \mathbf{\Sigma}_0 \in \mathbb{R}^{n\times n}$ *be two nnd matrices. Then*

$$r\begin{bmatrix}\mathbf{X} & \mathbf{X}_0 & \mathbf{\Sigma}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{bmatrix} = r[\mathbf{X}, \mathbf{X}_0, \mathbf{\Sigma}] + r(\mathbf{X}), \quad r\begin{bmatrix}\mathbf{X} & \mathbf{X}_0 & \mathbf{\Sigma}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0'\end{bmatrix} = r[\mathbf{X}, \mathbf{X}_0, \mathbf{\Sigma}_0] + r(\mathbf{X}_0), \tag{2.4}$$

$$r\begin{bmatrix}\mathbf{X} & \mathbf{X}_0 & \mathbf{\Sigma} & \mathbf{\Sigma}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0'\end{bmatrix} = r[\mathbf{X}, \mathbf{X}_0, \mathbf{\Sigma}, \mathbf{\Sigma}_0] + r(\mathbf{X}) + r(\mathbf{X}_0). \tag{2.5}$$

The following lemma is well known in matrix theory and its applications.

Lemma 2.3. ([12]) *The linear matrix equation* $\mathbf{A}\mathbf{X} = \mathbf{B}$ *is consistent if and only if* $r[\mathbf{A}, \mathbf{B}] = r(\mathbf{A})$, *or equivalently,* $\mathbf{A}\mathbf{A}^{+}\mathbf{B} = \mathbf{B}$. *In this case, the general solution of the equation can be written in the following parametric form* $\mathbf{X} = \mathbf{A}^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{U}$, *where* $\mathbf{U}$ *is an arbitrary matrix.*

It is well known that statistical theory and optimization theory have been closely linked in many ways. In particular, statisticians search different kinds of best estimations and predictions of unknown parameters in parametric regression models by means of various mathematical optimization tools and techniques. In order to obtain analytical expressions of BLUPs/BLUEs under linear regression models, we shall utilize the following remarkable analytical result concerning analytical solutions of a constrained quadratic matrix-valued function optimization problem.

Lemma 2.4. ([17]) *Let* $\mathbf{A} \in \mathbb{R}^{p\times q}$, $\mathbf{B} \in \mathbb{R}^{n\times q}$, $\mathbf{C} \in \mathbb{R}^{p\times m}$, *and* $\mathbf{D} \in \mathbb{R}^{n\times m}$ *be given,* $\mathbf{M} \in \mathbb{R}^{m\times m}$ *be nnd matrix, and assume that the matrix equation* $\mathbf{G}\mathbf{A} = \mathbf{B}$ *is solvable for the variable matrix* $\mathbf{G} \in \mathbb{R}^{n\times p}$. *Then the following constrained quadratic matrix-valued function optimization problem in the Löwner partial ordering*

$$f(\mathbf{G}) = (\mathbf{G}\mathbf{C} + \mathbf{D})\mathbf{M}(\mathbf{G}\mathbf{C} + \mathbf{D})' = \min \quad s.t. \quad \mathbf{G}\mathbf{A} = \mathbf{B} \tag{2.6}$$

is always solvable for $\mathbf{G}$. *In this case, the variable matrix satisfying* (2.6) *is determined by the matrix equation*

$$\mathbf{G}_0\left[\mathbf{A},\, \mathbf{C}\mathbf{M}\mathbf{C}'\mathbf{A}^{\perp}\right] = \left[\mathbf{B},\, -\mathbf{D}\mathbf{M}\mathbf{C}'\mathbf{A}^{\perp}\right]. \tag{2.7}$$

Correspondingly, the analytical expressions of $\mathbf{G}_0$ *and* $f(\mathbf{G}_0)$ *as well as an additive decomposition of* $f(\mathbf{G})$ *are given by*

$$\mathbf{G}_0 = \underset{\mathbf{G}\mathbf{A}=\mathbf{B}}{\operatorname{argmin}} f(\mathbf{G}) = \left[\mathbf{B},\, -\mathbf{D}\mathbf{M}\mathbf{C}'\mathbf{A}^{\perp}\right]\left[\mathbf{A},\, \mathbf{C}\mathbf{M}\mathbf{C}'\mathbf{A}^{\perp}\right]^{+} + \mathbf{U}\left[\mathbf{A},\, \mathbf{C}\mathbf{M}\mathbf{C}'\right]^{\perp}, \tag{2.8}$$

$$f(\mathbf{G}_0) = \min_{\mathbf{G}\mathbf{A}=\mathbf{B}} f(\mathbf{G}) = \mathbf{W}\mathbf{M}\mathbf{W}' - \mathbf{W}\mathbf{M}\mathbf{C}'\left(\mathbf{A}^{\perp}\mathbf{C}\mathbf{M}\mathbf{C}'\mathbf{A}^{\perp}\right)^{+}\mathbf{C}\mathbf{M}\mathbf{W}', \tag{2.9}$$

$$f(\mathbf{G}) = f(\mathbf{G}_0) + (\mathbf{G}\mathbf{C} + \mathbf{D})\,\mathbf{M}\mathbf{C}'\left(\mathbf{A}^{\perp}\mathbf{C}\mathbf{M}\mathbf{C}'\mathbf{A}^{\perp}\right)^{+}\mathbf{C}\mathbf{M}\,(\mathbf{G}\mathbf{C} + \mathbf{D})', \tag{2.10}$$

where $\mathbf{W} = \mathbf{B}\mathbf{A}^{+}\mathbf{C} + \mathbf{D}$ *and* $\mathbf{U} \in \mathbb{R}^{n\times p}$ *is arbitrary.*

In the following, we describe the predictability and estimability of unknown parameters under (1.1). In our analysis, we employ the conventional statistical techniques in the mainstream of regression theory. We begin with the classic conceptional definition of consistency of (1.1), which was first introduced and studied in [14,15].

Definition 2.5. Model (1.1) is said to be consistent if and only if the vector inclusion $\mathbf{y} \in \mathscr{R}[\,\mathbf{X},\, \mathbf{\Sigma}\,]$ holds with probability 1.

Definition 2.6. Let $\boldsymbol{\phi}$ be as given in (1.3). The linear statistic $\mathbf{Ly}$, where $\mathbf{L} \in \mathbb{R}^{s\times n}$, is said to have the same expectation with $\boldsymbol{\phi}$, or $\boldsymbol{\phi}$ is said to be *predictable* by $\mathbf{y}$ in (1.1), if and only if $\mathrm{E}\,(\mathbf{Ly} - \boldsymbol{\phi}) = \mathbf{0}$ holds.

Definition 2.7. Let $\boldsymbol{\phi}$ be as given in (1.3) and assume that $\boldsymbol{\phi}$ is predictable under (1.1). If there exists a matrix $\mathbf{L}$ such that

$$\mathrm{Cov}\,(\mathbf{Ly} - \boldsymbol{\phi}) = \min \quad \text{s.t.} \quad \mathrm{E}(\mathbf{Ly} - \boldsymbol{\phi}) = \mathbf{0} \tag{2.11}$$

holds in the Löwner partial ordering, then the linear statistic $\mathbf{Ly}$ is defined to be the BLUP of $\boldsymbol{\phi}$ in (1.3), and is denoted by

$$\mathbf{Ly} = \mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi}) = \mathrm{BLUP}_{\mathscr{M}}\,(\mathbf{K}\boldsymbol{\alpha} + \mathbf{H}\boldsymbol{\varepsilon})\,. \tag{2.12}$$

In particular, if $\boldsymbol{\phi} = \mathbf{K}\boldsymbol{\alpha}$, then the linear statistic $\mathbf{Ly}$ satisfying (2.11) is the BLUE of $\mathbf{K}\boldsymbol{\alpha}$ under (1.1), and is denoted by

$$\mathbf{Ly} = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\alpha}). \tag{2.13}$$

Based on the above definitions, we have the following result on the predictability of $\boldsymbol{\phi}$ in (1.3).

Lemma 2.8. *The vector $\boldsymbol{\phi}$ in* (1.3) *is predictable by $\mathbf{y}$ in* (1.1), *i.e.*, $\mathrm{E}\,(\mathbf{Ly} - \boldsymbol{\phi}) = \mathbf{0}$ *holds for some $\mathbf{L}$, if and only if*

$$\mathbf{LX} = \mathbf{K} \Leftrightarrow \mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}'). \tag{2.14}$$

Below, we present some known results on the matrix equation and explicit formulas of the BLUPs of $\boldsymbol{\phi}$ in (1.3) under $\mathscr{M}$, as well as some fundamental properties of the BLUPs.

Lemma 2.9. ([17]) *Assume that $\boldsymbol{\phi}$ in* (1.3) *is predictable by $\mathbf{y}$ in* (1.1), *namely,* (2.14) *holds. Then*

$$\mathrm{Cov}\,(\mathbf{L}_0\mathbf{y} - \boldsymbol{\phi}) = \min \quad s.t. \quad \mathrm{E}\,(\mathbf{L}_0\mathbf{y} - \boldsymbol{\phi}) = \mathbf{0} \Leftrightarrow \mathbf{L}_0[\,\mathbf{X},\, \mathrm{Cov}(\mathbf{y})\mathbf{X}^{\perp}\,] = [\,\mathbf{K},\, \mathrm{Cov}\{\boldsymbol{\phi},\, \mathbf{y}\}\mathbf{X}^{\perp}\,]. \tag{2.15}$$

The matrix equation in (2.15) *is consistent, i.e.*,

$$[\,\mathbf{K},\, \mathrm{Cov}\{\boldsymbol{\phi},\, \mathbf{y}\}\mathbf{X}^{\perp}\,][\,\mathbf{X},\, \mathrm{Cov}(\mathbf{y})\mathbf{X}^{\perp}\,]^{+}[\,\mathbf{X},\, \mathrm{Cov}(\mathbf{y})\mathbf{X}^{\perp}\,] = [\,\mathbf{K},\, \mathrm{Cov}\{\boldsymbol{\phi},\, \mathbf{y}\}\mathbf{X}^{\perp}\,] \tag{2.16}$$

holds under (2.14), *and the general solution of the matrix equation and the corresponding* $\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *can be written as*

$$\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi}) = \mathbf{L}_0\mathbf{y}, \tag{2.17}$$

where the general solution $\mathbf{L}_0$ of the matrix equation in (2.15) *is*

$$\begin{aligned}\mathbf{L}_0 &= [\,\mathbf{K},\, \mathrm{Cov}\{\boldsymbol{\phi},\, \mathbf{y}\}\mathbf{X}^{\perp}\,][\,\mathbf{X},\, \mathrm{Cov}(\mathbf{y})\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\, \mathrm{Cov}(\mathbf{y})\mathbf{X}^{\perp}\,]^{\perp}\\ &= [\,\mathbf{K},\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,][\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,]^{\perp}.\end{aligned} \tag{2.18}$$

In addition, the following results hold.

(a) ([p. 123, 13]) $r[\mathbf{X}, \mathbf{\Sigma X}^{\perp}] = r[\mathbf{X}, \mathbf{\Sigma}]$, $\mathscr{R}[\mathbf{X}, \mathbf{\Sigma X}^{\perp}] = \mathscr{R}[\mathbf{X}, \mathbf{\Sigma}]$, *and* $\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{\Sigma X}^{\perp}) = \{\mathbf{0}\}$.

(b) $\mathbf{L}_0$ *is unique if and only if* $r[\mathbf{X}, \mathbf{\Sigma}] = n$.

(c) $\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is unique if and only if* (1.1) *is consistent.*

(d) *The covariance matrix of* $\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is*

$$\mathrm{Cov}(\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})) = ([\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+})\mathbf{\Sigma}([\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+})'; \tag{2.19}$$

the covariance matrix between $\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *and* $\boldsymbol{\phi}$ *is*

$$\mathrm{Cov}\{\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi}), \boldsymbol{\phi}\} = [\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+}\mathbf{\Sigma H}'; \tag{2.20}$$

the difference of the covariance matrices of $\boldsymbol{\phi}$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is*

$$\begin{aligned}&\mathrm{Cov}(\boldsymbol{\phi}) - \mathrm{Cov}(\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi}))\\ &= \mathbf{H\Sigma H}' - ([\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+})\mathbf{\Sigma}([\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+})';\end{aligned} \tag{2.21}$$

the covariance matrix of the difference of $\boldsymbol{\phi}$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is*

$$\mathrm{Cov}(\boldsymbol{\phi} - \mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})) = ([\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+} - \mathbf{H})\mathbf{\Sigma}([\mathbf{K}, \mathbf{H\Sigma X}^{\perp}][\mathbf{X}, \mathbf{\Sigma X}^{\perp}]^{+} - \mathbf{H})'. \tag{2.22}$$

Correspondingly, we have the following results regarding the matrix equation and explicit formulas of the BLUPs of $\boldsymbol{\phi}_0$ in (1.5) under $\mathscr{N}$.

Lemma 2.10. *Assume that* $\boldsymbol{\phi}_0$ *in* (1.5). *Then*

$$\mathrm{Cov}(\mathbf{G}_0\mathbf{y} - \boldsymbol{\phi}_0) = \min \ \ s.t. \ \mathrm{E}(\mathbf{G}_0\mathbf{y} - \boldsymbol{\phi}_0) = \mathbf{0} \Leftrightarrow \mathbf{G}_0[\mathbf{X}_0, \mathrm{Cov}(\mathbf{y})\mathbf{X}_0^{\perp}] = [\mathbf{K}_0, \mathrm{Cov}\{\boldsymbol{\phi}_0, \mathbf{y}\}\mathbf{X}_0^{\perp}]. \tag{2.23}$$

The matrix equation in (2.23) *is consistent, i.e.,*

$$[\mathbf{K}_0, \mathrm{Cov}\{\boldsymbol{\phi}_0, \mathbf{y}\}\mathbf{X}_0^{\perp}][\mathbf{X}_0, \mathrm{Cov}(\mathbf{y})\mathbf{X}_0^{\perp}]^{+}[\mathbf{X}_0, \mathrm{Cov}(\mathbf{y})\mathbf{X}_0^{\perp}] = [\mathbf{K}_0, \mathrm{Cov}\{\boldsymbol{\phi}_0, \mathbf{y}\}\mathbf{X}_0^{\perp}], \tag{2.24}$$

and the general solution of the matrix equation and the corresponding $\mathrm{BLUP}_{\mathscr{N}}(\boldsymbol{\phi}_0)$ *can be written as*

$$\mathrm{BLUP}_{\mathscr{N}}(\boldsymbol{\phi}_0) = \mathbf{G}_0\mathbf{y}, \tag{2.25}$$

where the general solution $\mathbf{G}_0$ *of the matrix equation in* (2.23) *is*

$$\begin{aligned}\mathbf{G}_0 &= [\mathbf{K}_0, \mathrm{Cov}\{\boldsymbol{\phi}_0, \mathbf{y}\}\mathbf{X}_0^{\perp}][\mathbf{X}_0, \mathrm{Cov}(\mathbf{y})\mathbf{X}_0^{\perp}]^{+} + \mathbf{U}_0[\mathbf{X}_0, \mathrm{Cov}(\mathbf{y})\mathbf{X}_0^{\perp}]^{\perp}\\ &= [\mathbf{K}_0, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}][\mathbf{X}_0, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}]^{+} + \mathbf{U}_0[\mathbf{X}_0, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}]^{\perp}.\end{aligned} \tag{2.26}$$

Lemma 2.11. ([17]) *Let* $\boldsymbol{\phi}$ *be as given in* (1.3). *Then the following results hold.*

(a) *If* $\boldsymbol{\phi}$ *is predictable by* $\mathbf{y}$ *in* (1.1), *then* $\mathbf{P}\boldsymbol{\phi}$ *is predictable by* $\mathbf{y}$ *in* (1.1) *as well for any matrix* $\mathbf{P} \in \mathbb{R}^{t\times s}$, *and* $\mathrm{BLUP}_{\mathscr{M}}(\mathbf{P}\boldsymbol{\phi}) = \mathbf{P}\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})$ *holds.*

(b) *If* $\boldsymbol{\phi}$ *in* (1.3) *is predictable by* $\mathbf{y}$ *in* (1.1), *then* $\mathbf{K}\boldsymbol{\alpha}$ *is estimable by* $\mathbf{y}$ *in* (1.1) *as well, and the* BLUP *of* $\boldsymbol{\phi}$ *can be decomposed as the sum*

$$\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi}) = \mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\alpha}) + \mathrm{BLUP}_{\mathscr{M}}(\mathbf{H}\boldsymbol{\varepsilon}), \tag{2.27}$$

and the following formulas for covariance matrices hold:

$$\mathrm{Cov}\left\{\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\alpha}),\, \mathrm{BLUP}_{\mathscr{M}}(\mathbf{H}\boldsymbol{\varepsilon})\right\} = \mathbf{0}, \tag{2.28}$$

$$\mathrm{Cov}(\mathrm{BLUP}_{\mathscr{M}}(\boldsymbol{\phi})) = \mathrm{Cov}(\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\boldsymbol{\alpha})) + \mathrm{Cov}(\mathrm{BLUP}_{\mathscr{M}}(\mathbf{H}\boldsymbol{\varepsilon})). \tag{2.29}$$

Definition 2.12. Let $\boldsymbol{\phi}$ be as given in (1.3), and $\mathbf{Ly}$ be any linear predictor for $\boldsymbol{\phi}$. Then the mean square error matrix (MSEM) is defined to be

$$\mathrm{MSEM} = \mathrm{E}((\mathbf{Ly}-\boldsymbol{\phi})(\mathbf{Ly}-\boldsymbol{\phi})'). \tag{2.30}$$

It is easy to verify that

$$\begin{aligned}\mathrm{MSEM} &= \mathrm{E}((\mathbf{Ly}-\boldsymbol{\phi})(\mathbf{Ly}-\boldsymbol{\phi})')\\ &= \mathrm{E}((\mathbf{LX}\boldsymbol{\alpha}+\mathbf{L}\boldsymbol{\varepsilon}-\mathbf{K}\boldsymbol{\alpha}-\mathbf{H}\boldsymbol{\varepsilon})(\mathbf{LX}\boldsymbol{\alpha}+\mathbf{L}\boldsymbol{\varepsilon}-\mathbf{K}\boldsymbol{\alpha}-\mathbf{H}\boldsymbol{\varepsilon})')\\ &= \mathrm{E}((\mathbf{L}\boldsymbol{\varepsilon}-\mathbf{H}\boldsymbol{\varepsilon})(\mathbf{L}\boldsymbol{\varepsilon}-\mathbf{H}\boldsymbol{\varepsilon})') + (\mathbf{LX}-\mathbf{K})\boldsymbol{\alpha}\boldsymbol{\alpha}'(\mathbf{LX}-\mathbf{K})'\\ &= (\mathbf{L}-\mathbf{H})\boldsymbol{\Sigma}(\mathbf{L}-\mathbf{H})' + (\mathbf{LX}-\mathbf{K})\boldsymbol{\alpha}\boldsymbol{\alpha}'(\mathbf{LX}-\mathbf{K})'\\ &= \mathrm{Cov}(\mathbf{Ly}-\boldsymbol{\phi}) + (\mathbf{LX}-\mathbf{K})\boldsymbol{\alpha}\boldsymbol{\alpha}'(\mathbf{LX}-\mathbf{K})'.\end{aligned}$$

Also recall that

$$\|\mathbf{LX}-\mathbf{K}\|_F^2 = \mathrm{tr}((\mathbf{K}-\mathbf{LX})(\mathbf{K}-\mathbf{LX})') = \mathrm{tr}((\mathbf{K}-\mathbf{LX})'(\mathbf{K}-\mathbf{LX})), \tag{2.31}$$

where $\|\mathbf{A}\|_F = \sqrt{\mathrm{trace}(\mathbf{A}'\mathbf{A})}$ is defined to the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$.

Definition 2.13. Let $\boldsymbol{\phi}$ be as given in (1.3), and $\mathbf{Ly}$ be any linear predictor for $\boldsymbol{\phi}$. If

$$\mathbf{L}_0 = \mathrm{argmin}\, \|\mathbf{LX}-\mathbf{K}\|^2 = \mathrm{argmin}\, \mathrm{tr}((\mathbf{K}-\mathbf{LX})'(\mathbf{K}-\mathbf{LX})),$$

then $\mathbf{L}_0\mathbf{y}$ is said to be LMBP of $\boldsymbol{\phi}$ under $\mathscr{M}$. In this case, if $\mathbf{L}_0$ satisfies

$$\mathrm{Cov}(\mathbf{Ly}-\boldsymbol{\phi}) \succcurlyeq \mathrm{Cov}(\mathbf{L}_0\mathbf{y}-\boldsymbol{\phi}) \tag{2.32}$$

holds in the Löwner partial ordering for $\boldsymbol{\phi}$, then the $\mathbf{L}_0\mathbf{y}$ is said to be BLMBP of $\boldsymbol{\phi}$ under $\mathscr{M}$.

§3 Main Results

Concerning the expressions of the LMBPs and the BLMBPs for $\boldsymbol{\phi}$ under $\mathscr{M}$ and their properties, we have the following results.

Theorem 3.1. *Let* $\boldsymbol{\phi}$ *be as given in* (1.3), *and* $\mathbf{Ly}$ *be any linear predictor of* $\boldsymbol{\phi}$. *Then the following results hold.*

(a) $\mathbf{Ly}$ *is an LMBP for* $\boldsymbol{\phi}$ *under* $\mathscr{M}$ *if and only if* $\mathbf{L}$ *satisfies* $\mathbf{LXX}' = \mathbf{KX}'$.

(b) $\mathbf{Ly}$ *is a BLMBP for* $\boldsymbol{\phi}$ *under*$\mathscr{M}$ *if and only if* $\mathbf{L}$ *satisfies the following equality*

$$\mathbf{L}[\mathbf{XX}',\, \boldsymbol{\Sigma}\mathbf{X}^{\perp}] = [\mathbf{KX}',\, \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}]. \tag{3.1}$$

The matrix equation in (3.1) *is always consistent, and the general solution of the matrix equation and the corresponding* $\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})$ *under model* (1.1) *can be expressed as*

$$\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi}) = \mathbf{Ly} = \left([\mathbf{KX}',\, \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}][\mathbf{XX}',\boldsymbol{\Sigma}\mathbf{X}^{\perp}]^{+} + \mathbf{U}[\mathbf{XX}',\, \boldsymbol{\Sigma}\mathbf{X}^{\perp}]^{\perp}\right)\mathbf{y}. \tag{3.2}$$

In addition, the following results hold.

(c) $r[\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}] = r[\mathbf{X},\, \mathbf{\Sigma}]$, $\mathscr{R}[\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}] = \mathscr{R}[\mathbf{X},\, \mathbf{\Sigma}]$, *and* $\mathscr{R}(\mathbf{X}\mathbf{X}') \cap \mathscr{R}(\mathbf{\Sigma}\mathbf{X}^{\perp}) = \mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{\Sigma}\mathbf{X}^{\perp}) = \{\mathbf{0}\}$.

(d) $\mathbf{L}$ *is unique if and only if* $r[\mathbf{X},\, \mathbf{\Sigma}] = n$.

(e) $\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is unique if and only if* (1.1) *is consistent.*

(f) *The covariance matrix of* $\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is*

$$\mathrm{Cov}(\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})) = ([\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}]^{+})\mathbf{\Sigma}([\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}]^{+})';$$

the covariance matrix between $\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})$ *and* $\boldsymbol{\phi}$ *is*

$$\mathrm{Cov}\{\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi}),\, \boldsymbol{\phi}\} = [\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}]^{+}\mathbf{\Sigma}\mathbf{H}';$$

the difference of the covariance matrices of $\boldsymbol{\phi}$ *and* $\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is*

$$\begin{aligned}&\mathrm{Cov}(\boldsymbol{\phi}) - \mathrm{Cov}(\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi}))\\ &= \mathbf{H}\mathbf{\Sigma}\mathbf{H}' - ([\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}]^{+})\mathbf{\Sigma}([\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}]^{+})';\end{aligned}$$

the covariance matrix of the difference of $\boldsymbol{\phi}$ *and* $\mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi})$ *is*

$$\begin{aligned}&\mathrm{Cov}(\boldsymbol{\phi} - \mathrm{BLMBP}_{\mathscr{M}}(\boldsymbol{\phi}))\\ &= ([\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\mathbf{\Sigma}\mathbf{X}^{\perp}]^{+} - \mathbf{H})\mathbf{\Sigma}([\mathbf{K}\mathbf{X},\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}][\mathbf{X}\mathbf{X}',\mathbf{\Sigma}\mathbf{X}^{\perp}]^{+} - \mathbf{H})'.\end{aligned}$$

Proof. Note that

$$\begin{aligned}&\mathrm{tr}((\mathbf{K} - \mathbf{L}\mathbf{X})'(\mathbf{K} - \mathbf{L}\mathbf{X})) = \mathrm{tr}((\mathbf{K} - \mathbf{L}\mathbf{X})(\mathbf{K} - \mathbf{L}\mathbf{X})')\\ =&\mathrm{tr}((\mathbf{K} - \mathbf{K}\mathbf{X}^{+}\mathbf{X} - \mathbf{L}\mathbf{X} + \mathbf{K}\mathbf{X}^{+}\mathbf{X})(\mathbf{K} - \mathbf{K}\mathbf{X}^{+}\mathbf{X} - \mathbf{L}\mathbf{X} + \mathbf{K}\mathbf{X}^{+}\mathbf{X})')\\ =&\mathrm{tr}((\mathbf{K} - \mathbf{K}\mathbf{X}^{+}\mathbf{X} - (\mathbf{L}\mathbf{X} - \mathbf{K}\mathbf{X}^{+}\mathbf{X}))(\mathbf{K} - \mathbf{K}\mathbf{X}^{+}\mathbf{X} - (\mathbf{L}\mathbf{X} - \mathbf{K}\mathbf{X}^{+}\mathbf{X}))')\\ =&\mathrm{tr}((\mathbf{K}\mathbf{F}_{\mathbf{X}} - (\mathbf{L}\mathbf{X} - \mathbf{K}\mathbf{X}^{+}\mathbf{X}))(\mathbf{K}\mathbf{F}_{\mathbf{X}} - (\mathbf{L}\mathbf{X} - \mathbf{K}\mathbf{X}^{+}\mathbf{X}))')\\ =&\mathrm{tr}(\mathbf{K}\mathbf{F}_{\mathbf{X}}\mathbf{K}') + \mathrm{tr}((\mathbf{L}\mathbf{X} - \mathbf{K}\mathbf{X}^{+}\mathbf{X})(\mathbf{L}\mathbf{X} - \mathbf{K}\mathbf{X}^{+}\mathbf{X})').\end{aligned} \tag{3.3}$$

By Definition 2.12, $\mathbf{L}\mathbf{y}$ is LMBP of $\boldsymbol{\phi}$ under $\mathscr{M}$ if and only if $\mathbf{L}\mathbf{X} = \mathbf{K}\mathbf{X}^{+}\mathbf{X}$, or equivalently, $\mathbf{L}\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}'$, as required for (a).

Also note that

$$\mathrm{Cov}(\mathbf{L}\mathbf{y} - \boldsymbol{\phi}) = \mathrm{Cov}((\mathbf{L} - \mathbf{H})\boldsymbol{\varepsilon}) = (\mathbf{L} - \mathbf{H})\mathbf{\Sigma}(\mathbf{L} - \mathbf{H})'. \tag{3.4}$$

Under (3.4), we see from Lemma 2.3 that Result (b) is equivalent to finding a solution $\mathbf{L}_0$ of the consistent matrix equation $\mathbf{L}\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}'$ in (a) such that

$$\mathrm{Cov}(\mathbf{L}\mathbf{y} - \boldsymbol{\phi}) \succcurlyeq \mathrm{Cov}(\mathbf{L}_0\mathbf{y} - \boldsymbol{\phi}) \quad \text{for all} \quad \mathbf{L}\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}' \tag{3.5}$$

holds in the Löwner partial ordering. Furthermore from Lemma 2.4, there always exists a solution $\mathbf{L}_0$ of $\mathbf{L}\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}'$ such that (3.5) holds, and the $\mathbf{L}_0$ is determined by the matrix equation $\mathbf{L}_0[\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}] = [\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}]$, establishing the matrix equation in (b). The proofs of Results (c)–(f) are similar to these of (a)–(d) in Lemma 2.9, and thus are omitted. □

Corollary 3.2. *Let* $\boldsymbol{\phi}_0$ *be as given in* (1.5), *and* $\mathbf{L}\mathbf{y}$ *be any linear predictor of* $\boldsymbol{\phi}_0$. *Then the following results hold.*

(a) $\mathbf{G}_0\mathbf{y}$ *is LMBP of* $\boldsymbol{\phi}_0$ *under* $\mathscr{N}$ *if and only if* $\mathbf{G}_0$ *satisfies* $\mathbf{G}_0\mathbf{X}_0\mathbf{X}_0' = \mathbf{K}_0\mathbf{X}_0'$.

(b) $\mathbf{G}_0\mathbf{y}$ *is BLMBP of* $\boldsymbol{\phi}_0$ *under* $\mathscr{N}$ *if and only if* $\mathbf{G}_0$ *satisfies the following equality*

$$\mathbf{G}_0[\mathbf{X}_0\mathbf{X}_0',\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}] = [\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}]. \tag{3.6}$$

Eq. (3.6) *is always consistent, and the general solution of the matrix equation and the corresponding* $\mathrm{BLMBP}_{\mathscr{N}}(\boldsymbol{\phi}_0)$ *under* (1.2) *can be expressed as*

$$\mathrm{BLMBP}_{\mathscr{N}}(\boldsymbol{\phi}_0) = \mathbf{G}_0\mathbf{y} = ([\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}][\mathbf{X}_0\mathbf{X}_0',\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}]^{+} + \mathbf{U}_0[\mathbf{X}_0\mathbf{X}_0',\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}]^{\perp})\mathbf{y}. \tag{3.7}$$

In the section, we address the problem on robustness of BLUP and BLMBP of unknown parameters in relation to model matrix and covariance matrix.

Theorem 3.3. *Let* $\boldsymbol{\phi}$, $\boldsymbol{\phi}_0$, *and* $\mathrm{BLUP}_{\mathscr{N}}(\boldsymbol{\phi}_0)$ *be as given in* (1.3), (1.5) *and* (2.25), *and let*

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{XX}' & \mathbf{X}_0 & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} \mathbf{XX}' & \mathbf{\Sigma} & \mathbf{X}_0 & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix},$$

$$\mathbf{N}_1 = [\,\mathbf{KX}',\, \mathbf{K}_0,\, \mathbf{H}\mathbf{\Sigma}_0\,], \quad \mathbf{N}_2 = [\,\mathbf{KX}',\, \mathbf{H\Sigma},\, \mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\,].$$

Then the following results hold.

(a) *The following two statements are equivalent*:

(i) *All the BLUPs of* $\boldsymbol{\phi}_0$ *under* (1.2) *are LMBPs of* $\boldsymbol{\phi}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{M}_1') \supseteq \mathscr{R}(\mathbf{N}_1')$ *and* $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$.

(b) *The following two statements are equivalent*:

(i) *All the BLUPs of* $\boldsymbol{\phi}_0$ *under* (1.2) *are BLMBPs of* $\boldsymbol{\phi}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\, \mathbf{\Sigma}\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$.

Proof. By Theorem 3.1(a), $\mathbf{G}_0$ in (2.26) satisfies the following equation

$$\mathbf{G}_0\mathbf{XX}' = \mathbf{KX}'. \tag{3.8}$$

Substituting the expression of $\mathbf{G}_0$ in (2.26) into (3.8), we obtain

$$\begin{aligned} &\big([\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+} + \mathbf{U}_0[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\big)\,\mathbf{XX}' = \mathbf{KX}' \\ \Leftrightarrow\ & \mathbf{U}_0[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{XX}' = \mathbf{KX}' - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{XX}' \end{aligned} \tag{3.9}$$

holds for all $\mathbf{U}_0$, so (3.9) is further equivalent to the following conditions

$$\begin{bmatrix} \mathbf{KX}' - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{XX}' \\ [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{XX}' \end{bmatrix} = \mathbf{0}. \tag{3.10}$$

Applying (2.1), (2.2), and simplifying by elementary block matrix operations (EBMOs), we obtain

$$r\begin{bmatrix} \mathbf{KX}' - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{XX}' \\ [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{XX}' \end{bmatrix}$$

$$= r\begin{bmatrix} \mathbf{K}\mathbf{X}' - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\mathbf{X}' & \mathbf{0} \\ \mathbf{X}\mathbf{X}' & [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]$$

$$= r\begin{bmatrix} \mathbf{K}\mathbf{X}' & [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \\ \mathbf{X}\mathbf{X}' & [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$$

$$= r\begin{bmatrix} \mathbf{X}\mathbf{X}' & \mathbf{X}_0 & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \\ \mathbf{K}\mathbf{X}' & \mathbf{K}_0 & \mathbf{H}_0\mathbf{\Sigma}_0 \end{bmatrix} - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]. \tag{3.11}$$

Setting (3.11) equal to zero, we see that (3.10) is equivalent to

$$r\begin{bmatrix} \mathbf{M}_1 \\ \mathbf{N}_1 \end{bmatrix} = r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}_0).$$

Also note from (2.2) and (2.4) that

$$r(\mathbf{M}_1) = r[\mathbf{X}, \mathbf{X}_0, \mathbf{\Sigma}_0] + r(\mathbf{X}_0) \geqslant r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}_0), \tag{3.12}$$

$$r\begin{bmatrix} \mathbf{M}_1 \\ \mathbf{N}_1 \end{bmatrix} \geqslant r(\mathbf{M}_1) \geqslant r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}_0). \tag{3.13}$$

Combining (3.12) and (3.13) yields $r\begin{bmatrix} \mathbf{M}_1 \\ \mathbf{N}_1 \end{bmatrix} = r(\mathbf{M}_1) = r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}_0)$, or equivalently, $\mathscr{R}(\mathbf{M}_1') \supseteq \mathscr{R}(\mathbf{N}_1')$ and $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$ by Lemma 2.1 (a) and (b), thus establishing the equivalence of (i) and (ii) in (a).

From Theorem 3.1(b), $\mathbf{G}_0$ in (2.26) satisfies the matrix equation in (3.1)

$$\mathbf{G}_0[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] = [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,]. \tag{3.14}$$

Substituting the expression of $\mathbf{G}_0$ in (2.26) into (3.14) yields

$$\begin{aligned} &\left([\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+} + \mathbf{U}_0[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\right)[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] = [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] \\ &\Leftrightarrow \mathbf{U}_0[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] \\ &= [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] \end{aligned} \tag{3.15}$$

holds for all $\mathbf{U}_0$, so (3.15) is further equivalent to the following conditions

$$\begin{bmatrix} [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] \\ [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] \end{bmatrix} = \mathbf{0}. \tag{3.16}$$

Applying (2.1), (2.2), and simplifying by EBMOs, we obtain

$$\begin{aligned} &r\begin{bmatrix} [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] \\ [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] \end{bmatrix} \\ &= r\begin{bmatrix} [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] - [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] & \mathbf{0} \\ [\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \end{bmatrix} \\ &\quad - r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \\ &= r\begin{bmatrix} [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] & [\,\mathbf{K}_0,\, \mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \\ [\,\mathbf{X}\mathbf{X}',\, \mathbf{\Sigma}\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,] \end{bmatrix} - r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] \end{aligned}$$

$$= r\begin{bmatrix} \mathbf{XX}' & \mathbf{\Sigma} & \mathbf{X}_0 & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \\ \mathbf{KX}' & \mathbf{H\Sigma} & \mathbf{K}_0 & \mathbf{H}_0\mathbf{\Sigma}_0 \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]. \tag{3.17}$$

Setting (3.17) equal to zero, we see that (3.16) is equivalent to

$$r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} = r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}) + r(\mathbf{X}_0).$$

Also note from (2.2) and (2.5) that

$$r(\mathbf{M}_2) = r[\,\mathbf{X},\, \mathbf{\Sigma},\, \mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}) + r(\mathbf{X}_0) \geqslant r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}) + r(\mathbf{X}_0), \tag{3.18}$$

$$r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} \geqslant r(\mathbf{M}_2) \geqslant r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}) + r(\mathbf{X}_0). \tag{3.19}$$

Combining (3.18) and (3.19) yields $r\begin{bmatrix} \mathbf{M}_2 \\ \mathbf{N}_2 \end{bmatrix} = r(\mathbf{M}_2) = r[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,] + r(\mathbf{X}) + r(\mathbf{X}_0)$, or equivalently, $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ and $\mathscr{R}[\,\mathbf{X},\, \mathbf{\Sigma}\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$ by Lemma 2.1 (a) and (b), thus establishing the equivalence of (i) and (ii) in (b). □

Corollary 3.4. *Let* $\mathbf{H} = \mathbf{0}$ *and* $\mathbf{H}_0 = \mathbf{0}$ *in* (1.3) *and* (1.5), *and let*

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{XX}' & \mathbf{X}_0 & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} \mathbf{XX}' & \mathbf{\Sigma} & \mathbf{X}_0 & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix},$$

$$\mathbf{N}_1 = [\,\mathbf{KX}',\, \mathbf{K}_0, \mathbf{0}\,], \quad \mathbf{N}_2 = [\,\mathbf{KX}',\, \mathbf{0},\, \mathbf{K}_0,\, \mathbf{0}\,].$$

Then the following results hold.

(a) *The following two statements are equivalent*:

(i) *All the BLUEs of* $\mathbf{K}_0\boldsymbol{\alpha}$ *under model* (1.2) *are linear minimum bias estimators (LMBEs) of* $\mathbf{K}\boldsymbol{\alpha}$ *under model* (1.1).

(ii) $\mathscr{R}(\mathbf{M}_1') \supseteq \mathscr{R}(\mathbf{N}_1')$ *and* $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$.

(b) *The following two statements are equivalent*:

(i) *All the BLUEs of* $\mathbf{K}_0\boldsymbol{\alpha}$ *under* (1.2) *are best linear minimum bias estimators (BLMBEs) of* $\mathbf{K}\boldsymbol{\alpha}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{M}_2') \supseteq \mathscr{R}(\mathbf{N}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\, \mathbf{\Sigma}\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \mathbf{\Sigma}_0\,]$.

Theorem 3.5. *Let* $\boldsymbol{\phi}$, $\boldsymbol{\phi}_0$, *and* $\mathrm{BLMBP}_{\mathscr{N}}(\boldsymbol{\phi}_0)$ *be as given in* (1.3), (1.5), *and* (3.7), *and let*

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{XX}' & \mathbf{X}_0\mathbf{X}_0' & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} \mathbf{XX}' & \mathbf{\Sigma} & \mathbf{X}_0\mathbf{X}_0' & \mathbf{\Sigma}_0 \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0' \end{bmatrix},$$

$$\mathbf{Q}_1 = [\,\mathbf{KX}',\, \mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\mathbf{\Sigma}_0\,], \quad \mathbf{Q}_2 = [\,\mathbf{KX}',\, \mathbf{H\Sigma},\, \mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\mathbf{\Sigma}_0\,].$$

Then the following results hold.

(a) *The following two statements are equivalent*:

(i) *All the BLMBPs of* $\boldsymbol{\phi}_0$ *under model* (1.2) *are LMBPs of* $\boldsymbol{\phi}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{P}_1') \supseteq \mathscr{R}(\mathbf{Q}_1')$ *and* $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,]$.

(b) *The following two statements are equivalent*:

(i) *All the BLMBPs of* $\boldsymbol{\phi}_0$ *under* (1.2) *are BLMBPs of* $\boldsymbol{\phi}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{P}_2') \supseteq \mathscr{R}(\mathbf{Q}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\, \boldsymbol{\Sigma}\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,]$.

Proof. By Theorem 3.1(a), $\mathbf{G}_0$ in (3.7) satisfies the following equation

$$\mathbf{G}_0\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}'. \tag{3.20}$$

Substituting the expression of $\mathbf{G}_0$ in (3.7) into (3.20), we obtain

$$\begin{aligned}&\big([\mathbf{K}_0\mathbf{X}_0', \mathbf{H}_0\boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}][\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+} + \mathbf{U}_0[\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\big)\,\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}'\\ &\Leftrightarrow \mathbf{U}_0[\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X}\mathbf{X}' = \mathbf{K}\mathbf{X}' - [\,\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\mathbf{X}'\end{aligned} \tag{3.21}$$

holds for all $\mathbf{U}_0$, so (3.21) is further equivalent to the following conditions

$$\begin{bmatrix}\mathbf{K}\mathbf{X}' - [\,\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\mathbf{X}'\\ [\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X}\mathbf{X}'\end{bmatrix} = \mathbf{0}. \tag{3.22}$$

Applying (2.1), (2.2), and simplifying by EBMOs, we obtain

$$\begin{aligned}&r\begin{bmatrix}\mathbf{K}\mathbf{X}' - [\,\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\mathbf{X}'\\ [\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}\mathbf{X}\mathbf{X}'\end{bmatrix}\\ &= r\begin{bmatrix}\mathbf{K}\mathbf{X}' - [\,\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}\mathbf{X}\mathbf{X}' & \mathbf{0}\\ \mathbf{X}\mathbf{X}' & [\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]\end{bmatrix} - r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]\\ &= r\begin{bmatrix}\mathbf{K}\mathbf{X}' & [\,\mathbf{K}_0\mathbf{X}_0',\, \mathbf{H}_0\boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]\\ \mathbf{X}\mathbf{X}' & [\,\mathbf{X}_0\mathbf{X}_0',\, \boldsymbol{\Sigma}_0\mathbf{X}_0^{\perp}\,]\end{bmatrix} - r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,]\\ &= r\begin{bmatrix}\mathbf{X}\mathbf{X}' & \mathbf{X}_0\mathbf{X}_0' & \boldsymbol{\Sigma}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0'\\ \mathbf{K}\mathbf{X}' & \mathbf{K}_0\mathbf{X}_0' & \mathbf{H}_0\boldsymbol{\Sigma}_0\end{bmatrix} - r(\mathbf{X}_0) - r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,].\end{aligned} \tag{3.23}$$

Setting (3.23) equal to zero, we see that (3.22) is equivalent to

$$r\begin{bmatrix}\mathbf{P}_1\\ \mathbf{Q}_1\end{bmatrix} = r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,] + r(\mathbf{X}_0).$$

Also note from (2.2) and (2.4) that

$$r(\mathbf{P}_1) = r[\mathbf{X}, \mathbf{X}_0, \boldsymbol{\Sigma}_0] + r(\mathbf{X}_0) \geqslant r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,] + r(\mathbf{X}_0), \tag{3.24}$$

$$r\begin{bmatrix}\mathbf{P}_1\\ \mathbf{Q}_1\end{bmatrix} \geqslant r(\mathbf{P}_1) \geqslant r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,] + r(\mathbf{X}_0). \tag{3.25}$$

Combining (3.24) and (3.25) yields $r\begin{bmatrix}\mathbf{P}_1\\ \mathbf{Q}_1\end{bmatrix} = r(\mathbf{P}_1) = r[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,] + r(\mathbf{X}_0)$, or equivalently, $\mathscr{R}(\mathbf{P}_1') \supseteq \mathscr{R}(\mathbf{Q}_1')$ and $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,]$ by Lemma 2.1 (a) and (b), thus establishing the equivalence of (i) and (ii) in (a).

From Theorem 3.1(b), $\mathbf{G}_0$ in (3.7) satisfies the matrix equation in (3.1)

$$\mathbf{G}_0[\,\mathbf{X}\mathbf{X}',\, \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,] = [\,\mathbf{K}\mathbf{X}',\, \mathbf{H}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]. \tag{3.26}$$

Substituting the expression of $\mathbf{G}_0$ in (3.7) into (3.26), we obtain

$$\begin{aligned}&([\,\mathbf{K}_0\mathbf{X}_0',\,\mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}+\mathbf{U}_0[\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp})[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,]=[\,\mathbf{K}\mathbf{X}',\,\mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,]\\ &\Leftrightarrow \mathbf{U}_0[\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,]=[\,\mathbf{K}\mathbf{X}',\,\mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,]\\ &\qquad\qquad -[\,\mathbf{K}_0\mathbf{X}_0',\,\mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,] \quad (3.27)\end{aligned}$$

holds for all $\mathbf{U}_0$, so (3.27) is further equivalent to the following condition

$$\begin{bmatrix}[\,\mathbf{K}\mathbf{X}',\,\mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,]-[\,\mathbf{K}_0\mathbf{X}_0',\,\mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,]\\ [\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,]\end{bmatrix}=\mathbf{0}. \tag{3.28}$$

Applying (2.1), (2.2), and simplifying by EBMOs, we obtain

$$\begin{aligned}&r\begin{bmatrix}[\,\mathbf{K}\mathbf{X}',\,\mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,]-[\,\mathbf{K}_0\mathbf{X}_0',\,\mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,]\\ [\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{\perp}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,]\end{bmatrix}\\ &=r\begin{bmatrix}[\,\mathbf{K}\mathbf{X}',\,\mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,]-[\,\mathbf{K}_0\mathbf{X}_0',\,\mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,][\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]^{+}[\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,] & \mathbf{0}\\ [\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]\end{bmatrix}\\ &\quad -r[\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]\\ &=r\begin{bmatrix}[\,\mathbf{K}\mathbf{X}',\,\mathbf{H}\mathbf{\Sigma}\mathbf{X}^{\perp}\,] & [\,\mathbf{K}_0\mathbf{X}_0',\,\mathbf{H}_0\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]\\ [\,\mathbf{X}\mathbf{X}',\,\mathbf{\Sigma}\mathbf{X}^{\perp}\,] & [\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\mathbf{X}_0^{\perp}\,]\end{bmatrix}-r[\,\mathbf{X}_0\mathbf{X}_0',\,\mathbf{\Sigma}_0\,]\\ &=r\begin{bmatrix}\mathbf{X}\mathbf{X}' & \mathbf{\Sigma} & \mathbf{X}_0\mathbf{X}_0' & \mathbf{\Sigma}_0\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0'\\ \mathbf{K}\mathbf{X}' & \mathbf{H}\mathbf{\Sigma} & \mathbf{K}_0\mathbf{X}_0' & \mathbf{H}_0\mathbf{\Sigma}_0\end{bmatrix}-r(\mathbf{X})-r(\mathbf{X}_0)-r[\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]. \qquad (3.29)\end{aligned}$$

Setting (3.29) equal to zero, we see that (3.28) is equivalent to

$$r\begin{bmatrix}\mathbf{P}_2\\ \mathbf{Q}_2\end{bmatrix}=r[\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]+r(\mathbf{X})+r(\mathbf{X}_0).$$

Also note from (2.2) and (2.5) that

$$r(\mathbf{P}_2)=r[\,\mathbf{X},\,\mathbf{\Sigma},\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]+r(\mathbf{X})+r(\mathbf{X}_0)\geqslant r[\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]+r(\mathbf{X})+r(\mathbf{X}_0), \tag{3.30}$$

$$r\begin{bmatrix}\mathbf{P}_2\\ \mathbf{Q}_2\end{bmatrix}\geqslant r(\mathbf{P}_2)\geqslant r[\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]+r(\mathbf{X})+r(\mathbf{X}_0). \tag{3.31}$$

Combining (3.30) and (3.31) yields $r\begin{bmatrix}\mathbf{P}_2\\ \mathbf{Q}_2\end{bmatrix}=r(\mathbf{P}_2)=r[\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]+r(\mathbf{X})+r(\mathbf{X}_0)$, or equivalently, $\mathscr{R}(\mathbf{P}_2')\supseteq\mathscr{R}(\mathbf{Q}_2')$ and $\mathscr{R}[\,\mathbf{X},\,\mathbf{\Sigma}\,]\subseteq\mathscr{R}[\,\mathbf{X}_0,\,\mathbf{\Sigma}_0\,]$ by Lemma 2.1(a) and (b), thus establishing the equivalence of (i) and (ii) in (b). □

If the given matrices are taken as some special forms in (1.3) and (1.5), we can obtain the following corollary from Theorem 3.5.

Corollary 3.6. *Let* $\mathbf{H}=\mathbf{0},\mathbf{H}_0=\mathbf{0}$ *in* (1.3) *and* (1.5), *and define*

$$\mathbf{P}_1=\begin{bmatrix}\mathbf{X}\mathbf{X}' & \mathbf{X}_0\mathbf{X}_0' & \mathbf{\Sigma}_0\\ \mathbf{0} & \mathbf{0} & \mathbf{X}_0'\end{bmatrix},\quad \mathbf{P}_2=\begin{bmatrix}\mathbf{X}\mathbf{X}' & \mathbf{\Sigma} & \mathbf{X}_0\mathbf{X}_0' & \mathbf{\Sigma}_0\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_0'\end{bmatrix},$$

$$\mathbf{Q}_1 = [\,\mathbf{K}\mathbf{X}',\, \mathbf{K}_0\mathbf{X}_0', \mathbf{0}\,], \quad \mathbf{Q}_2 = [\,\mathbf{K}\mathbf{X}',\, \mathbf{0},\, \mathbf{K}_0\mathbf{X}_0',\, \mathbf{0}\,].$$

Then the following results hold.

(a) *The following statements are equivalent*:

(i) *All the BLMBEs of* $\mathbf{K}_0\boldsymbol{\alpha}$ *under* (1.2) *are LMBEs of* $\mathbf{K}\boldsymbol{\alpha}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{P}_1') \supseteq \mathscr{R}(\mathbf{Q}_1')$ *and* $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,]$.

(b) *The following statements are equivalent*:

(i) *All the BLMBEs of* $\mathbf{K}_0\boldsymbol{\alpha}$ *under* (1.2) *are BLMBEs of* $\mathbf{K}\boldsymbol{\alpha}$ *under* (1.1).

(ii) $\mathscr{R}(\mathbf{P}_2') \supseteq \mathscr{R}(\mathbf{Q}_2')$ *and* $\mathscr{R}[\,\mathbf{X},\, \boldsymbol{\Sigma}\,] \subseteq \mathscr{R}[\,\mathbf{X}_0,\, \boldsymbol{\Sigma}_0\,]$.

§4 Conclusion

We offered a robustness analysis of joint prediction on the model matrix and the covariance matrix under a general linear model using the classic MSEM criterion from a new perspective and analytical solutions of a specified quadratic matrix value optimization problem and some precise matrix analysis tools, discussed some algebraic and statistical properties of BLMBPs, and established an additive decomposition equality under a general linear model. It is believed that this work provides some valuable overview of the most important ideas and results in solving biased predition/estimation problems under linear statistical models. We hope that the study can bring deep understandings of the MSEM criteria and the corresponding BLMBPs under general model assumptions, and also this work can be extended to the cases of solving various biased predition/estimation problems in the contexts of linear random-effects models, linear mixed-effects models, and multivariate linear models with and without restrictions.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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