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Approximate solutions and nonlinear scalarizations for set optimization with a co-radiant set

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Abstract. In normed linear spaces, by applying the Minkowski difference, the concepts of efficient solutions, weakly efficient solutions, proper efficient solutions and Henig efficient solutions are introduced for set optimization with respect to co-radiant sets, and the relationships among them is discussed. Optimality conditions for minimal solutions and strictly minimal solutions of scalar set optimization are established by using the generalized oriented distance functions, respectively. The relationship between the solutions of set optimization problem and the solutions of scalar problem is studied. Several examples are given to explain our result. Properties of Gerstewitz's functions with respect to co-radiant sets are discussed, which are used to establish sufficient conditions for weakly efficient solutions of set optimization.

§1 Introduction

Set-valued optimization is a generalization of vector optimization. In recent years, set-valued optimization problems have emerged as a vibrant and developing branch of applied mathematics due to its applications in several areas, for instance, mathematical finance, fuzzy optimization, game theory, control theory and many others, for details, see [1–6]. Vector criterion and set criterion are two well-known criteria to solve set-valued optimization problems. Vector criterion depends only on single element of the image set (see [2, 7]); whereas the set criterion depends on every element of the image set as it involves direct comparisons of image sets (see [9–11, 23, 28–34]).

The common approach is to transform it into a family of scalar optimization problems, which is called the scalarization technique. The scalarization techniques mainly include linear scalarization and nonlinear scalarization. The scalarization is one of the easiest techniques and

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the scalar problem provides more effective approaches of computing solutions in comparison to set optimization problems. And its core idea is to characterize set-valued optimization problems with numerical optimization problems. Several scalarization techniques for set optimization problems are available in the literature. Most of them are based on Gerstewitz's function, oriented distance funtion or their extensions. Hernández and Rodríguez-Marín [23] obtained scalar representations of set-valued optimization problems without any convexity assumption, they characterized this type of solutions by means of nonlinear scalarization, and the scalarizing function is a generalization of the Gerstewitz's nonconvex separation function. Also, they gave two existence theorems for set-valued optimization problems. Xu and Li [12] introduced a new nonlinear scalarization function, which is a generalization of the oriented distance function and established scalarizations for minimal and weak minimal solutions of set optimization, respectively.

In the middle of the 1980s, Loridan [14] introduced the approximate efficient solutions for vector optimization problems. Later, many scholars [15, 17] paid more attention to propose the concepts of approximate solutions of vector optimization problems. Notice that these approximate efficient solutions can be expressed uniformly. Chicco [16] introduced a new type of solution based on upper comprehensive sets in vector optimization, which is a generalization of weak and strong Pareto points. In recent years, many scholars use the co-radiant set to investigate various different solutions of vector optimization problems. A co-radiant set is more general than a cone and is also a main tool in the study of approximate solutions.

Motived by [17, 19–21], we introduce several kinds of approximate solutions for set optimization, and discuss the relationships among them. This work is devoted to investigating the properties of the special scalarizing functions with respect to co-radiant sets. By applying generalized oriented distance functions and Gerstewitz's functions with respect to co-radiant sets, we establish the optimality conditions for approximate solutions.

§2 Preliminaries

Let X and Y be real normed linear spaces. Denote by $\mathcal{P}(Y)$ and $\mathcal{B}(Y)$ the families of nonempty subsets and nonempty bounded subsets of Y, respectively. For a nonempty subset Q of Y, the cone generated by Q is defined as $\mathrm{cone}Q := \underset{\alpha\geq 0}{\cup}\alpha Q$. The set Q is said to be a cone if $Q = \mathrm{cone}Q$. Let Y^* be the topological dual space of Y. The dual cone Q^+ and strictly dual cone Q^{+i} of Q are, respectively, defined as

$$Q^+ := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0, \forall y \in Q\},$$

$$Q^{+i} := \{y^* \in Y^* \mid \langle y^*, y\rangle > 0, \forall y \in Q\backslash \{0\}\}.$$

We say that a set Q is proper if Q is nonempty and $Q \neq Y$. The set Q is pointed if $Q \cap (-Q) \subset \{0\}$, and it is co-radiant if $\forall \alpha \geq 1, \alpha Q \subset Q$.

For a co-radiant $C \subset Y$ and $\epsilon > 0$, the sets $C(\epsilon)$ and $C(0)$ are defined as $C(\epsilon) = \epsilon C$, and $C(0) = \cup_{\epsilon>0} C(\epsilon)$. For $\epsilon \geq 0$, it is said that Q is $C(\epsilon)$-closed if $Q + C(\epsilon)$ is a closed set; $C(\epsilon)$-bounded if for each neighborhood U of zero in Y there is some positive number t such that $Q \subset tU + C(\epsilon)$; $C(\epsilon)$-compact if any cover of Q of the form $\{U_\alpha + C(\epsilon) : U_\alpha \text{ are open}\}$

admits a finite subcover; and $C(\epsilon)$-proper if $Q + C(\epsilon) \neq Y$.

Let $P,\ Q \in \mathcal{P}(Y)$. The Minkowski difference [19] is defined as

$$P \dot{-} Q := \{y \in Y : y + Q \subset P\} = \bigcap_{q \in Q} (P - q).$$

Proposition 2.1 *Let C be a co-radiant set. If $0 < \epsilon_1 < \epsilon_2$, then $C(\epsilon_2) \subset C(\epsilon_1)$.*

Proof If $\epsilon_1 = 0$, it holds evidently. If $\epsilon_1 > 0$, it is proved in [20].

Proposition 2.2 *Let $C \subset Y$ be a proper convex co-radiant set, and* $\mathrm{int}C \neq \emptyset$. *Then for any $\epsilon \geq 0$, the following statements hold.*

(i) For any $\epsilon_1, \epsilon_2 > 0, C(\epsilon_1) + C(\epsilon_2) \subset C(\epsilon_1)$;

(ii) For any $\beta_1, \beta_2 > 0$, $\beta_1 \mathrm{cl}C(\epsilon) + \beta_2 \mathrm{int}C(\epsilon) \subset (\beta_1 + \beta_2)\mathrm{int}C(\epsilon)$;

(iii) $\mathrm{cl}C(\epsilon) + \mathrm{int}C(\epsilon) \subset \mathrm{int}C(\epsilon)$;

(iv) If $\emptyset \neq A \subset Y$, then $A + \mathrm{int}C(\epsilon) = \mathrm{int}(A + C(\epsilon))$.

Proof (i) The proposition has been proved in [20].

(ii) For any $c_1 \in \mathrm{cl}C(\epsilon), c_2 \in \mathrm{int}C(\epsilon)$, we have

$$\beta_1 c_1 + \beta_2 c_2 = (\beta_1 + \beta_2)\left(\frac{\beta_1}{\beta_1 + \beta_2} c_1 + \frac{\beta_2}{\beta_1 + \beta_2} c_2 \right) \in (\beta_1 + \beta_2)\mathrm{int}C(\epsilon),$$

which implies the proposition holds.

(iii) By (ii), we get $\mathrm{cl}C(\epsilon) + \mathrm{int}C(\epsilon) \subset 2\mathrm{int}C(\epsilon)$. Since C is co-radiant, one obtains $2\mathrm{int}C(\epsilon) \subset \mathrm{int}C(\epsilon)$. Consequently, $\mathrm{cl}C(\epsilon) + \mathrm{int}C(\epsilon) \subset \mathrm{int}C(\epsilon)$.

(iv) Since $A + \mathrm{int}C(\epsilon) \subset A + C(\epsilon)$ and $A + \mathrm{int}C(\epsilon)$ is an open set, we have $A + \mathrm{int}C(\epsilon) \subset \mathrm{int}(A + C(\epsilon))$.

Conversely, let $y \in \mathrm{int}(A + C(\epsilon))$, then there exists a neighborhood U of zero such that $y - U \subset A + C(\epsilon)$. Taking $k_0 \in \mathrm{int}C(\epsilon)$, from the absorption of U, there exists $\lambda_0 \in (0, +\infty)$ such that $\lambda_0 k_0 \in U$. Let $z = y - \lambda_0 k_0$. Then $z \in A + C(\epsilon)$ and $y = z + \lambda_0 k_0$. Hence,

$$y \in A + C(\epsilon) + \lambda_0 k_0 \subset A + C(\epsilon) + \lambda_0 \mathrm{int}C(\epsilon). \tag{2.1}$$

From Proposition 2.2(ii), we have $C(\epsilon) + \lambda_0 \mathrm{int}C(\epsilon) \subset (1 + \lambda_0)\mathrm{int}C(\epsilon)$. Since C is a co-radiant set, we get $(1 + \lambda_0)\mathrm{int}C(\epsilon) \subset \mathrm{int}C(\epsilon)$. From (2.1), we have $y \in A + \mathrm{int}C(\epsilon)$. Therefore, $\mathrm{int}(A + C(\epsilon)) \subset A + \mathrm{int}C(\epsilon)$. $\square$

Remark 2.1 If C is not a convex co-radiant set, then Proposition 2.2(iv) dose not hold. The following example illustrates it.

Example 2.1 Let $A = \mathbb{R}^2_+,\ C = \left\{(x, y) \mid\ x^2 + y^2 \leq 1\right\} \cup \{(-2, 0)\}$ and $\epsilon = 1$. Then C is not a convex co-radiant set, and it is easy to verify that $\mathrm{int}(A + C(\epsilon)) \not\subset A + \mathrm{int}C(\epsilon)$.

Lemma 2.1 [19] *If $A \in \mathcal{B}(Y)$, then $A \dot{-} A = \{0\}$.*

Definition 2.1 [18] Let $A \in \mathcal{P}(Y)$. The nonlinear function $\mathcal{D}_A : \mathcal{P}(Y) \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$\mathcal{D}_A(B) := \inf_{b \in B} \varDelta_A(b), \forall B \in \mathcal{P}(Y),$$

where $\varDelta_A(b) := d_A(b) - d_{Y \setminus A}(b)$, $d_A(b) := \inf_{a \in A} \|b - a\|$.

Proposition 2.3 [21] *For $A, B \in \mathcal{P}(Y)$, the following assertions hold.*

(i) If B is compact, then $\mathcal{D}_A(B) > -\infty$ and $\mathcal{D}_A(B) = \varDelta_A(\bar{b})$, for some $\bar{b} \in B$;

(ii) $\mathcal{D}_A(B) \geq 0$, if and only if, $\mathrm{int}A \cap B = \emptyset$;

(iii) If $\mathrm{cl}A \cap B \neq \emptyset$, *then $\mathcal{D}_A(B) \leq 0$;*

(iv) If $\mathcal{D}_A(B) \le 0$ *and* B *is compact, then* $\mathrm{cl}A \cap B \neq \emptyset$.

Proposition 2.4 *Suppose that* $C \subset Y$ *is a proper convex co-radiant set and* $0 \in C$, *then* C *is a cone.*

Proof For $c \in C$, $\lambda \ge 0$, we consider the following two cases. Case 1. If $\lambda \in [0,1)$, since C is a convex set, then $\lambda c = \lambda c + (1-\lambda)0 \in C$. Case 2. If $\lambda \ge 1$, since C is a co-radiant, then $\lambda c \in C$. Concluding these two cases, we obtain C is a cone. □

Lemma 2.2 [27] *Let* C *be a proper co-radiant subset of* Y. *Then the following assertions hold.*

$$\mathcal{D}_C(\lambda A) \le \lambda \mathcal{D}_C(A), \forall A \in \mathcal{P}(Y), \lambda \ge 1;$$
$$\mathcal{D}_C(\lambda A) \ge \lambda \mathcal{D}_C(A), \forall A \in \mathcal{P}(Y), \lambda \in (0,1].$$

Lemma 2.3 [27] *If* C *is a proper convex co-radiant subset of* Y, *then* $\mathcal{D}_C(A+B) \le \mathcal{D}_C(A)$ $+\mathcal{D}_C(B)$, *for any* $A, B \in \mathcal{P}(Y)$.

Lemma 2.4 [24] *If* $C \subset Y$ *is a convex set with nonempty interior, then*

$$\mathop{\cup}_{\epsilon>0} \mathrm{int} C(\epsilon) = \mathrm{int}\left(\mathop{\cup}_{\epsilon>0} C(\epsilon)\right).$$

Let $\emptyset \neq S \subset X$, $F : X \to 2^Y$. We consider the following set optimization problem

$$\text{(SOP)} \ \ \min F(x) \quad \text{s.t.} \ \ x \in S.$$

In the remainder of the paper, we assume that $C \subset Y$ is a proper convex co-radiant set, $\mathrm{int}C \neq \emptyset$.

In the following, we introduce some concepts of solutions by using Minkowski difference.

Definition 2.2 Let $\epsilon \ge 0$ and $\bar{x} \in S$.

(i) We say that $\bar{x} \in S$ is an ϵ-efficient solution of (SOP) with respect to C if

$$\left(F(x) \dot{-} F(\bar{x})\right) \cap -C(\epsilon) \subset \{0\}, \ \forall x \in S.$$

(ii) We say that $\bar{x} \in S$ is a weakly ϵ-efficient solution of (SOP) with respect to C if

$$\left(F(x) \dot{-} F(\bar{x})\right) \cap -\mathrm{int}C(\epsilon) = \emptyset, \ \forall x \in S.$$

(iii) We say that $\bar{x} \in S$ is a proper ϵ-efficient solution of (SOP) with respect to C if

$$\mathrm{clcone}\left(\mathop{\cup}_{x \in S}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) \cap -C(0) \subset \{0\}.$$

(iv) We say that $\bar{x} \in S$ is a Henig ϵ-efficient solution of (SOP) with respect to C, if there exists a closed proper convex co-radiant set K with $\mathrm{int}K \neq \emptyset$ such that $C \backslash \{0\} \subset \mathrm{int}K$, and

$$\left(F(S) \dot{-} F(\bar{x})\right) \cap -K(\epsilon) \subset \{0\}.$$

We denote by ϵ-$AE(F,C)$, ϵ-$WAE(F,C)$, ϵ-$PBAE(F,C)$ and ϵ-$PHAE(F,C)$ the sets of ϵ-efficient solutions, weakly ϵ-efficient solutions, proper ϵ-efficient solutions and Henig ϵ-efficient solutions of (SOP), respectively.

Theorem 2.1 *If* $\epsilon \ge 0$, *then* ϵ-$PHAE(F,C) \subset \epsilon$-$AE(F,C)$.

Proof If $\bar{x} \in \epsilon$-$PHAE(F,C)$, there exists a closed proper convex co-radiant set K with $\mathrm{int}K \neq \emptyset$ such that $C \backslash \{0\} \subset \mathrm{int}K$ and $(F(S) \dot{-} F(\bar{x})) \cap -K(\epsilon) \subset \{0\}$. We need to show $\left(F(x) \dot{-} F(\bar{x})\right) \cap -C(\epsilon) \subset \{0\}, \forall x \in S$. Case 1. If $0 \in C$, by Proposition 2.4 we have C is a cone. Taking $c_0 \in C \backslash \{0\}$, since C is a cone and $C \backslash \{0\} \subset \mathrm{int}K$, we get $\frac{1}{n}c_0 \in \mathrm{int}K$, for any $n \in \mathbb{N}$, hence, $0 \in \mathrm{cl}(\mathrm{int}K) = \mathrm{cl}K = K$. Case 2. If $0 \notin C$, then $C = C \backslash \{0\} \subset \mathrm{int}K \subset K$. Summarizing above two cases, we have $C \subset K$, so, $C(\epsilon) \subset K(\epsilon)$. Since $F(x) \dot{-} F(\bar{x}) \subset F(S) \dot{-} F(\bar{x})$, $\forall x \in S$, we have $\left(F(x)\dot{-}F(\bar{x})\right)\cap -C(\epsilon)\subset\{0\}$, $\forall x\in S$. $\square$

Theorem 2.2 *If* $\epsilon_2>\epsilon_1\geq 0$, *then* ϵ_1-$PBAE(F,C)\subset\epsilon_2$-$PBAE(F,C)$.

Proof If $\bar{x}\in\epsilon_1$-$PBAE(F,C)$, then

$$\operatorname{clcone}\left(\bigcup_{x\in S}\left(F(x)\dot{-}F(\bar{x})\right)+C(\epsilon_1)\right)\cap -C(0)\subset\{0\}.$$

From Proposition 2.1, we get $C(\epsilon_2)\subset C(\epsilon_1)$. Hence,

$$\operatorname{clcone}\left(\bigcup_{x\in S}\left(F(x)\dot{-}F(\bar{x})\right)+C(\epsilon_2)\right)\cap -C(0)\subset\{0\},$$

therefore, $\bar{x}\in\epsilon_2$-$PBAE(F,C)$. $\square$

Theorem 2.3 *If* $C(0)\setminus\{0\}$ *is an open set, then* $\bigcap_{\epsilon>0}\epsilon$-$PBAE(F,C)=0$-$PBAE(F,C)$.

Proof From Theorem 2.2, we have 0-$PBAE(F,C)\subset\bigcap_{\epsilon>0}\epsilon$-$PBAE(F,C)$. Hence, we only need to prove $\bigcap_{\epsilon>0}\epsilon$-$PBAE(F,C)\subset 0$-$PBAE(F,C)$. Let $\bar{x}\in S$ and $\bar{x}\notin 0$-$PBAE(F,C)$. Then there exists $d\neq 0$ such that $d\in -C(0)$ and $d\in\operatorname{clcone}\left(\bigcup_{x\in S}\left(F(x)\dot{-}F(\bar{x})\right)+C(0)\right)$. Hence, there exist $\hat{\lambda}_n>0$, $\hat{x}_n\in S$, $\hat{y}_n\in F(\hat{x}_n)\dot{-}F(\bar{x})$, $\hat{\epsilon}_n>0$ and $\hat{c}_n\in C$, such that $d=\lim_{n\to+\infty}\hat{\lambda}_n(\hat{y}_n+\hat{\epsilon}_n\hat{c}_n)$. Since $C(0)\setminus\{0\}$ is an open set, then there exists $N\in\mathbb{N}$ such that $\hat{\lambda}_n(\hat{y}_n+\hat{\epsilon}_n\hat{c}_n)\in -C(0)\setminus\{0\}$, for any $n>N$. Therefore, $\operatorname{clcone}\left(\bigcup_{x\in S}\left(F(x)\dot{-}F(\bar{x})\right)+C(\hat{\epsilon}_n)\right)$ $\cap-C(0)\not\subset\{0\}$, which implies $\bar{x}\notin\hat{\epsilon}_n$-$PBAE(F,C)$. Hence, $\bar{x}\notin\bigcap_{\epsilon>0}\epsilon$-$PBAE(F,C)$. $\square$

§3 Nonlinear scalarizations in terms of generalized oriented distance functions

We consider the scalarization problem

$$(P_\phi):\quad \min\phi(x)\quad \text{s.t. } x\in S,$$

where $\emptyset\neq S\subset X$, $\phi:S\to\mathbb{R}$.

If $A,B\in\mathcal{P}(Y),\varphi\in Y^*$, then $\varphi(A)\geq\varphi(B)$ stands for $\varphi(a)\geq\varphi(b)$, for any $a\in A,b\in B$.

Definition 3.1 [17] Let $\epsilon\geq 0$, $\bar{x}\in S$.

(i) If

$$\phi(x)\geq\phi(\bar{x})-\epsilon,\forall x\in S,$$

then $\bar{x}$ is said to be an ϵ -minimal solution of (P_ϕ).

(ii) If

$$\phi(x)>\phi(\bar{x})-\epsilon,\forall x\in S\setminus\{\bar{x}\},$$

then $\bar{x}$ is said to be a strict ϵ -minimal solution of (P_ϕ).

We consider the following scalar optimization problem

$$\left(P_{C(\epsilon),Q}\right):\min_{x\in S}\mathcal{D}_{-C(\epsilon)}\left(F(x)\dot{-}Q\right),$$

where $\emptyset\neq S\subset X$, and $F(x),Q\in\mathcal{P}(Y)$.

Lemma 3.1 *If* $C\subset Y$ *is a proper convex co-radiant set, then* $0\notin\operatorname{int}C(\epsilon)$ *for any* $\epsilon\geq 0$.

Proof Suppose to the contrary that there exists $\hat{\epsilon}\geq 0$ such that $0\in\operatorname{int}C(\hat{\epsilon})$. If $\hat{\epsilon}>0$, then we have $\frac{0}{\hat{\epsilon}} \in \mathrm{int}C \subset C$. If $\hat{\epsilon} = 0$, from Lemma 2.4, we have $0 \in \mathrm{int}C\left(0\right) = \underset{\epsilon>0}{\cup}\mathrm{int}C\left(\epsilon\right)$, then there exists $\epsilon_1 > 0$ such that $0 \in \mathrm{int}C\left(\epsilon_1\right)$, hence, $\frac{0}{\epsilon_1} \in \mathrm{int}C \subset C$. Therefore, we get $0 \in \mathrm{int}C \subset C$, and from Proposition 2.4, we know C is a cone. Since $0 \in \mathrm{int}C$, we get $C = Y$, and this contradicts with that C is proper. $\square$

Lemma 3.2 *Let $\epsilon \geq 0$, $C \subset Y$ be a proper convex co-radiant set. Then $\Delta_{-C(\epsilon)}\left(0\right) = \epsilon d\left(0, C\right)$.*

Proof We first prove $\Delta_{-C(\epsilon)}\left(0\right) = \epsilon\Delta_{-C}\left(0\right)$. We consider the following two cases. Case 1. If $\epsilon = 0$, then $C\left(0\right) = \underset{\epsilon>0}{\cup}\epsilon C$, we get $\Delta_{-C(0)}\left(0\right) = d_{-C(0)}\left(0\right) - d_{Y\backslash -C(0)}\left(0\right) = 0 = 0\Delta_{-C}\left(0\right)$. Case 2. If $\epsilon > 0$, then we have $\Delta_{-C(\epsilon)}\left(0\right) = d_{-C(\epsilon)}\left(0\right) - d_{Y\backslash -C(\epsilon)}\left(0\right) = \epsilon\left(d_{-C}\left(0\right) - d_{Y\backslash -C}\left(0\right)\right) = \epsilon\Delta_{-C}\left(0\right)$. Therefore, $\forall\epsilon \geq 0$, we have $\Delta_{-C(\epsilon)}\left(0\right) = \epsilon\Delta_{-C}\left(0\right)$. Next, we need to show $\Delta_{-C}\left(0\right) = d\left(0, C\right)$. From Definition 2.1, we know that $\Delta_{-C}\left(0\right) = d\left(0, -C\right) - d\left(0, Y\backslash -C\right) = d\left(0, C\right) - d\left(0, Y\backslash -C\right)$. Hence, we only need to show $d\left(0, Y\backslash -C\right) = 0$. We consider the following two cases. Case 1. If $0 \in C$, by Proposition 2.4, we have C is a cone, hence, $d\left(0, Y\backslash -C\right) = 0$. Case 2. If $0 \notin C$, then $0 \notin -C$, hence, $d\left(0, Y\backslash -C\right) = 0$. Therefore, $\Delta_{-C}\left(0\right) = d\left(0, C\right)$, and so, $\Delta_{-C(\epsilon)}\left(0\right) = \epsilon\Delta_{-C}\left(0\right) = \epsilon d\left(0, C\right)$. $\square$

Theorem 3.1 *Suppose that $\bar{x} \in S$, $F\left(\bar{x}\right) \in \mathcal{B}\left(Y\right)$, $\epsilon \geq 0$, and $\beta = d\left(0, C\right)$. Then $\bar{x} \in \epsilon$-$WAE\left(F, C\right)$ if and only if $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$.*

Proof If $\bar{x} \in \epsilon$-$WAE\left(F, C\right)$, then

$$\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -\mathrm{int}C\left(\epsilon\right) = \emptyset, \ \forall x \in S. \tag{3.1}$$

Suppose to the contrary that $\bar{x}$ is not an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$, then there exists $\hat{x} \in S$ such that

$$\mathcal{D}_{-C(\epsilon)}\left(F\left(\hat{x}\right) \dot{-} F\left(\bar{x}\right)\right) < \mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right) \dot{-} F\left(\bar{x}\right)\right) - \epsilon\beta. \tag{3.2}$$

From Lemma 2.1, we get $F\left(\bar{x}\right) \dot{-} F\left(\bar{x}\right) = \{0\}$. Hence, $\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right) \dot{-} F\left(\bar{x}\right)\right) - \epsilon\beta = \Delta_{-C(\epsilon)}\left(0\right) - \epsilon\beta$. From (3.2) and Lemma 3.2, we have $\mathcal{D}_{-C(\epsilon)}\left(F\left(\hat{x}\right) \dot{-} F\left(\bar{x}\right)\right) < \epsilon d\left(0, C\right) - \epsilon\beta = 0$. By Proposition 2.3(ii), we get $\left(F\left(\hat{x}\right) \dot{-} F\left(\bar{x}\right)\right) \cap \mathrm{int}\left(-C\left(\epsilon\right)\right) \neq \emptyset$, which leads to a contradiction to (3.1), therefore, $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$.

Conversely, if $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$, then

$$\mathcal{D}_{-C(\epsilon)}\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \geq \mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right) \dot{-} F\left(\bar{x}\right)\right) - \epsilon\beta = 0, \ \forall x \in S. \tag{3.3}$$

From Proposition 2.3(ii), we get $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -C\left(\epsilon\right) = \emptyset$, $\forall x \in S$. Therefore, $\bar{x} \in \epsilon$-$WAE\left(F, C\right)$. $\square$

In the following, we give an example to illustrate that the necessity of the Theorem 3.1 may not hold, if $Q \neq F\left(\bar{x}\right)$ in $\left(P_{C(\epsilon),Q}\right)$.

Example 3.1 Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ + \{(1, 1)\}$, $S = \mathbb{R}^2_+$, $\epsilon = 1$, $\bar{x} = (0, 2)$, $F : S \to 2^S$ and $F(x) = \{x, \bar{x}\}$ for all $x \in S$. If we consider $Q = \{(-1, -1)\}$, then it is clear that $\bar{x} \in WAE\left(F, C, \epsilon\right)$. But for $\hat{x} = (0, 0)$, we have $\mathcal{D}_{-C(\epsilon)}\left(F\left(\hat{x}\right) \dot{-} Q\right) = \sqrt{8} < \sqrt{20} - \sqrt{2} = \mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right) \dot{-} Q\right) - \epsilon\beta$. This implies $\bar{x}$ is not an $\epsilon\beta$-solution of $\left(P_{C(\epsilon),Q}\right)$.

Theorem 3.2 0-$WAE\left(F, C\right) = \underset{\epsilon>0}{\cap}\epsilon$-$WAE\left(F, C\right)$.

Proof We first prove that 0-$WAE\left(F, C\right) \subset \underset{\epsilon>0}{\cap}\epsilon$-$WAE\left(F, C\right)$. If we take $\bar{x} \in$ 0-$WAE\left(F, C\right)$, then we have $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap \left(-\mathrm{int}C\left(0\right)\right) = \emptyset$, for any $x \in S$. Hence, $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -\mathrm{int}\left(\underset{\epsilon>0}{\cup}C\left(\epsilon\right)\right) = \emptyset$, which implies that $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -\mathrm{int}C\left(\epsilon\right) = \emptyset$, for any $\epsilon > 0$, $x \in S$.

Therefore, $\bar{x} \in \bigcap\limits_{\epsilon>0} \epsilon\text{-}WAE\left(F,C\right)$.

Next, we only need to prove that $\bigcap\limits_{\epsilon>0} \epsilon\text{-}WAE\left(F,C\right) \subset 0\text{-}WAE\left(F,C\right)$. Let $\bar{x} \in \bigcap\limits_{\epsilon>0} \epsilon\text{-}WAE\left(F,C\right)$, then for any $\epsilon>0$ and $x \in S$, $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -\mathrm{int}C\left(\epsilon\right) = \emptyset$. Hence, $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap \left(\bigcup\limits_{\epsilon>0} -\mathrm{int}C\left(\epsilon\right)\right) = \emptyset$, from Lemma 2.4, we get $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -\mathrm{int}\left(\bigcup\limits_{\epsilon>0} C\left(\epsilon\right)\right) = \emptyset$. Therefore, $\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \cap -\mathrm{int}C\left(0\right) = \emptyset$, which implies $\bar{x} \in 0\text{-}WAE\left(F,C\right)$. □

Theorem 3.3 *If* $\epsilon \geq 0$, *then* $\epsilon\text{-}PBAE\left(F,C\right) \subset \epsilon\text{-}WAE\left(F,C\right)$.

Proof Let $\bar{x} \in S$ and $\bar{x} \notin \epsilon\text{-}WAE\left(F,C\right)$. Then there exist $d \in \mathrm{int}C\left(\epsilon\right)$ and $x_1 \in S$, such that $-d \in F\left(x_1\right) \dot{-} F\left(\bar{x}\right)$. Since $d \in \mathrm{int}C\left(\epsilon\right)$, it follows that there exists $\beta > 0$ such that $\left(1-\beta\right)d \in C\left(\epsilon\right)$. Hence, we have $-d = \frac{1}{\beta}\left(-d + \left(1-\beta\right)d\right) \in \mathrm{clcone}\left(F\left(x_1\right) \dot{-} F\left(\bar{x}\right) + C\left(\epsilon\right)\right) \subset \mathrm{clcone}\left(\bigcup\limits_{x\in S}\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) + C\left(\epsilon\right)\right)$. From $-d \in -\mathrm{int}C\left(\epsilon\right) \subset -C\left(0\right)$ and Lemma 3.1, we have $0 \neq -d \in \mathrm{clcone}\left(\bigcup\limits_{x\in S}\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) + C\left(\epsilon\right)\right) \cap -C\left(0\right)$. Therefore, $\bar{x} \notin \epsilon\text{-}PBAE\left(F,C\right)$. □

We consider the following scalar problem $\left(P_{C(\epsilon)}\right)$

$$\min_{x\in S} \mathcal{D}_{-C(\epsilon)}\left(F\left(x\right)\right),$$

where $\emptyset \neq S \subset X$.

Theorem 3.4 *Let* $\beta = d\left(0,C\right)$ *and* $\epsilon \geq 0$. *If* $\bar{x}$ *is an optimal solution of* $\left(P_{C(\epsilon)}\right)$, *then* $\bar{x}$ *is an* $\epsilon\beta$*-minimal solution of* $\left(P_{C(\epsilon),F(\bar{x})}\right)$. *Hence,* $\bar{x} \in \epsilon\text{-}WAE\left(F,C\right)$.

Proof If $\bar{x}$ is an optimal solution of $\left(P_{C(\epsilon)}\right)$, then $\mathcal{D}_{-C(\epsilon)}\left(F\left(x\right)\right) \geq \mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right)$, $\forall x \in S$. From Lemma 2.3, we get

$$\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right) \leq \mathcal{D}_{-C(\epsilon)}\left(F\left(x\right) + \{0\}\right) \leq \mathcal{D}_{-C(\epsilon)}\left(F\left(x\right)\right) + \mathcal{D}_{-C(\epsilon)}\left(\{0\}\right), \forall x \in S. \tag{3.4}$$

Since $\mathcal{D}_{-C(\epsilon)}\left(\{0\}\right) = \varDelta_{-C(\epsilon)}\left(0\right)$, it follows from Lemma 3.2 that $\varDelta_{-C(\epsilon)}\left(0\right) = \epsilon\beta$, hence, $\mathcal{D}_{-C(\epsilon)}\left(\{0\}\right) = \epsilon\beta$. From (3.4), we get $\mathcal{D}_{-C(\epsilon)}\left(F\left(x\right)\right) \geq \mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right) - \epsilon\beta$, $\forall x \in S$, therefore, $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$. From Theorem 3.1, we have $\bar{x} \in \epsilon\text{-}WAE\left(F,C\right)$. □

The following example illustrates that the inverse of Theorem 3.4 is incorrect.

Example 3.2 Let $S = \mathbb{R}^2_+$, $C = \left\{\left(x_1,x_2\right) \mid x_1 \geq 0, x_2 \geq 0, x_2 \geq 1 - x_1\right\}$, $\bar{x} = \left(0,2\right)$. Consider the set-valued map $F : S \to 2^S$ defined by $F\left(x\right) = \{x, \bar{x}\}$ for all $x \in S$. In the following, we illustrate that for any $\epsilon \geq 0$, $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$. In fact, for any $x \in S$, we have $\mathcal{D}_{-C(\epsilon)}\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \geq 0$. From Lemma 3.2, we get $\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right) \dot{-} F\left(\bar{x}\right)\right) - \epsilon d\left(0,C\right) = \mathcal{D}_{-C(\epsilon)}\left(0\right) - \epsilon d\left(0,C\right) = 0$. Hence, $\mathcal{D}_{-C(\epsilon)}\left(F\left(x\right) \dot{-} F\left(\bar{x}\right)\right) \geq \mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right) \dot{-} F\left(\bar{x}\right)\right) - \epsilon\beta$, namely, $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(\epsilon),F(\bar{x})}\right)$. In the following, we show that $\bar{x}$ is not an optimal solution of $\left(P_{C(\epsilon)}\right)$. In fact, if $0 \leq \epsilon \leq 2$, then $\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right) = \sqrt{\epsilon^2+4}$, and $\mathcal{D}_{-C(\epsilon)}\left(F\left(0,0\right)\right) = \min\left\{\mathcal{D}_{-C(\epsilon)}\left(\left(0,2\right)\right), \mathcal{D}_{-C(\epsilon)}\left(\left(0,0\right)\right)\right\} = \min\left\{\sqrt{\epsilon^2+4}, \frac{\sqrt{2}}{2}\epsilon\right\} = \frac{\sqrt{2}}{2}\epsilon$. Hence, $\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right) = \sqrt{\epsilon^2+4} > \frac{\sqrt{2}}{2}\epsilon = \mathcal{D}_{-C(\epsilon)}\left(F\left(0,0\right)\right)$. If $\epsilon > 2$, then $\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right) = \sqrt{2} + \frac{\sqrt{2}}{2}\epsilon$, hence, $\mathcal{D}_{-C(\epsilon)}\left(F\left(0,0\right)\right) = \min\left\{\sqrt{2} + \frac{\sqrt{2}}{2}\epsilon, \frac{\sqrt{2}}{2}\epsilon\right\} = \frac{\sqrt{2}}{2}\epsilon$. Therefore, $\mathcal{D}_{-C(\epsilon)}\left(F\left(\bar{x}\right)\right) = \sqrt{2} + \frac{\sqrt{2}}{2}\epsilon > \frac{\sqrt{2}}{2}\epsilon = \mathcal{D}_{-C(\epsilon)}\left(F\left(0,0\right)\right)$.

Lemma 3.3 $C\left(0\right) \cap -C\left(0\right) \subset \{0\}$.

Proof Suppose to the contrary that there exists $\hat{c} \in C\left(0\right) \setminus \{0\}$ such that $\hat{c} \in -C\left(0\right)$, then

$-\hat{c} \in C(0) \setminus \{0\}$. Since C is a convex co-radiant set, we have $0 = \hat{c} - \hat{c} \in C(0) \setminus \{0\}$, this leads a contradiction.

Theorem 3.5 *Let* $\beta = \mathcal{D}_{-C(0)}(C)$, $\epsilon \geq 0$ *and* $F(\bar{x}) \in \mathcal{B}(Y)$. *If* $\bar{x} \in \epsilon$-$PBAE(F, C)$, *then* $\bar{x}$ *is an* $\epsilon\beta$*-minimal solution of* $\left(P_{C(0),F(\bar{x})}\right)$.

Proof From $\bar{x} \in \epsilon$-$PBAE(F, C)$, we get $\operatorname{clcone}\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) \cap -C(0) \subset \{0\}$, which implies that $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) \cap (-C(0) \setminus \{0\}) = \emptyset$. In the following, we prove that for any $x \in S$, $y \in F(x) \dot{-} F(\bar{x})$, $c \in C$, $y + \epsilon c \notin -C(0) \setminus \{0\}$. In fact, if $\epsilon > 0$, it is evident. If $\epsilon = 0$, by Lemma 3.3 and $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) \cap (-C(0) \setminus \{0\}) = \emptyset$, we obtain $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right)\right) \cap (-C(0) \setminus \{0\}) = \emptyset$. Hence, $y \notin -C(0) \setminus \{0\}$. Since $0 \in \partial(\operatorname{cl}(C(0)))$, we get $d_{Y\setminus(-C(0))}(y + \epsilon c) = d_{Y\setminus(-C(0)\setminus\{0\})}(y + \epsilon c) = 0$.

Therefore, for any $x \in S$, $y \in F(x) \dot{-} F(\bar{x})$, $c \in C$, we have $\Delta_{-C(0)}(y + \epsilon c) = d_{-C(0)}(y + \epsilon c)$ $-d_{Y\setminus -C(0)}(y + \epsilon c) = d_{-C(0)}(y + \epsilon c) \geq 0$, so $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x}) + \epsilon c\right) \geq 0$. Since $C(0)$ is a proper convex co-radiant set, from Lemma 2.3, we have

$$\begin{aligned}\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) &\geq \mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x}) + \epsilon c\right) - \mathcal{D}_{-C(0)}(\epsilon c) \\ &\geq -\mathcal{D}_{-C(0)}(\epsilon c).\end{aligned} \tag{3.5}$$

Hence, for any $x \in S$, we have $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) \geq -\mathcal{D}_{-C(0)}(C(\epsilon))$. In the following, we calculate $\mathcal{D}_{-C(0)}(C(\epsilon))$. If $\epsilon > 0$, then

$$\begin{aligned}\mathcal{D}_{-C(0)}(C(\epsilon)) &= \inf_{c_1\in C(\epsilon)} \Delta_{-C(0)}(c_1) = \inf_{c_1\in C(\epsilon)} \left(\, d_{-C(0)}(c_1) - d_{Y\setminus -C(0)}(c_1)\, \right) \\ &= \inf_{c_1\in C(\epsilon)} d_{-C(0)}(c_1) = \epsilon \inf_{c\in C} d_{-C(0)}(c) = \epsilon \mathcal{D}_{-C(0)}(C) = \epsilon\beta,\end{aligned}$$

therefore, we have

$$\mathcal{D}_{-C(0)}(C(\epsilon)) = \epsilon\beta. \tag{3.6}$$

If $\epsilon = 0$, then

$$\mathcal{D}_{-C(0)}(C(0)) = \inf_{c\in C(0)} \Delta_{-C(0)}(c) = \inf_{c\in C(0)} \left(d_{-C(0)}(c) - d_{Y\setminus -C(0)}(c)\right) = \inf_{c\in C(0)} d_{-C(0)}(c) = 0,$$

therefore, we have

$$\mathcal{D}_{-C(0)}(C(0)) = 0. \tag{3.7}$$

From Lemma 3.2, we get $\mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) = \mathcal{D}_{-C(0)}(0) = \Delta_{-C(0)}(0) = 0$. Therefore, by (3.5)-(3.7), we have $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) \geq \mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta$. So, $\bar{x}$ is an $\epsilon\beta$-minimal solution of $\left(P_{C(0),F(\bar{x})}\right)$. □

The following example illustrates that the inverse of Theorem 3.5 is incorrect.

Example 3.3 Let $X = Y = \mathbb{R}^2$, $C = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_2 \geq 1 - x_1\right\}$, $S = \mathbb{R}^2_+$, $\epsilon = 1$ and $\bar{x} = (0, 2)$. Consider the set-valued map $F : S \to 2^S$ defined by $F(x) = \{x, \bar{x}\}$ for any $x \in S$. In the following, we illustrate that $\bar{x}$ is an $\epsilon\beta$-minimal solution of scalar optimization problem $\left(P_{C(0),F(\bar{x})}\right)$. For any $x \in S$, we have $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) = \mathcal{D}_{-C(0)}\{(0, 0), (x_1, x_2 - 2)\} \geq 0$, where $x = (x_1, x_2)$. Since $\mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta = \mathcal{D}_{-C(0)}\{(0, 0)\} - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$, therefore, we get $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) \geq \mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-}\right.$

$F(\bar{x})\,) - \epsilon\beta$ for any $x \in S$. Hence, $\bar{x}$ is an $\epsilon\beta$-minimal solution of scalar optimization problem $\left(P_{C(0),F(\bar{x})}\right)$. On the other hand, we have clcone $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) \cap -C(0) = \{(x_1, x_2) \mid x_1 = 0, x_2 < 0\} \not\subset \{0\}$. Therefore, $\bar{x} \notin \epsilon$-$PBAE(F, C)$.

Theorem 3.6 *Let* $C(0) \setminus \{0\}$ *be an open set,* $\beta = \inf\limits_{c\in C} d_{\partial(C(0))}(c)$, $\epsilon \geq 0$ *and* $F(\bar{x}) \in \mathcal{B}(Y)$. *If* $\bar{x}$ *is a strict* $\epsilon\beta$*-minimal solution of* $\left(P_{C(0),F(\bar{x})}\right)$, *then* $\bar{x} \in \epsilon$*-*$PBAE(F, C)$.

Proof Suppose to the contrary that $\bar{x} \notin \epsilon$-$PBAE(F, C)$, then there exists $d \neq 0$ such that $d \in -C(0)$ and $d \in$ clcone $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right)$. Hence, there exist $\lambda_n > 0$, $\hat{x}_n \in S$, $\hat{y}_n \in F(\hat{x}_n) \dot{-} F(\bar{x})$ and $\hat{c}_n \in C(\epsilon)$, such that $d = \lim\limits_{n\to+\infty} \lambda_n(\hat{y}_n + \hat{c}_n) \in -C(0) \setminus \{0\}$. Since $C(0) \setminus \{0\}$ is an open set, there exists $N \in \mathbb{N}$ such that $\lambda_n(\hat{y}_n + \hat{c}_n) \in -C(0) \setminus \{0\}$, for any $n > N$. Hence, $\hat{y}_n + \hat{c}_n \in -C(0)$. By Lemma 2.3, we get $\mathcal{D}_{-C(0)}(\{\hat{y}_n\}) \leq \mathcal{D}_{-C(0)}(\{\hat{y}_n + \hat{c}_n\}) + \mathcal{D}_{-C(0)}(\{-\hat{c}_n\})$. Hence, $\varDelta_{-C(0)}(\hat{y}_n) \leq \varDelta_{-C(0)}(\hat{y}_n + \hat{c}_n) + \varDelta_{-C(0)}(-\hat{c}_n) \leq -d_{Y\setminus -C(0)}(-\hat{c}_n) = -d_{\partial(-C(0))}(-\hat{c}_n) = -d_{\partial(C(0))}(\hat{c}_n) \leq -\inf\limits_{c_1\in C(\epsilon)} d_{\partial(C(0))}(c_1)$. Therefore, if $\epsilon > 0$, $\varDelta_{-C(0)}(\hat{y}_n) \leq -\inf\limits_{c_1\in C(\epsilon)} d_{\partial(C(0))}(c_1) = -\epsilon \inf\limits_{c\in C} d_{\partial(C(0))}(c) = -\epsilon\beta$, we have

$$\varDelta_{-C(0)}(\hat{y}_n) \leq -\epsilon\beta. \tag{3.8}$$

If $\epsilon = 0$, then we have $\varDelta_{-C(0)}(\hat{y}_n) \leq -\inf\limits_{c_1\in C(0)} d_{\partial(C(0))}(c_1) = 0$, therefore, we have

$$\varDelta_{-C(0)}(\hat{y}_n) \leq 0. \tag{3.9}$$

From (3.8)-(3.9), we have $\forall\epsilon \geq 0$, $\varDelta_{-C(0)}(\hat{y}_n) \leq -\epsilon\beta$, hence,

$$\mathcal{D}_{-C(0)}\left(F(\hat{x}_n) \dot{-} F(\bar{x})\right) \leq \varDelta_{-C(0)}(\hat{y}_n) \leq -\epsilon\beta. \tag{3.10}$$

On the other hand, since $\bar{x}$ is a strict $\epsilon\beta$-minimal solution of $\left(P_{C(0),F(\bar{x})}\right)$, it follows from Lemma 3.2 that $\mathcal{D}_{-C(0)}\left(F(\hat{x}_n) \dot{-} F(\bar{x})\right) > \mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta = -\epsilon\beta$. This contradicts with (3.10), therefore, $\bar{x} \in \epsilon$-$PBAE(F, C)$. □

Remark 3.1 In Example 3.3, we observe that $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) > \mathcal{D}_{-C(0)}\left(\,F(\bar{x}) \dot{-} F(\bar{x})\,\right) - \epsilon\beta$, $\forall x \in S\setminus\{\bar{x}\}$. Hence, $\bar{x}$ is a strict $\epsilon\beta$-minimal solution of scalar optimization problem $\left(P_{C(0),F(\bar{x})}\right)$, but $\bar{x} \notin \epsilon$-$PBAE(F, C)$. It is clear that $C(0) \setminus \{0\}$ is not an open set. Hence, if $C(0) \setminus \{0\}$ is not an open set, then the conclusion of Theorem 3.6 may not hold.

The following example illustrates that the inverse of Theorem 3.6 is incorrect.

Example 3.4 Let $X = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ + \{(1, 1)\}$, $S = \mathbb{R}^2_+ + \{(-1, -1)\}$, $\epsilon = 1$ and $\bar{x} = (0, 0)$. Consider the set-valued map $F : S \to 2^S$ defined by $F(x) = \{x, \bar{x}\}$. It is clear that $\beta = 1$ and $C(0) \setminus \{0\}$ is an open set. In the following, we illustrate that $\bar{x} \in \epsilon$-$PBAE(F, C)$. We have that clcone $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) = \mathbb{R}^2_+$, by calculation. Hence, clcone $\left(\underset{x\in S}{\cup}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right) \cap -C(0) = \emptyset \subset \{0\}$. On the other hand, for $\hat{x} = (-1, -1) \in S\setminus\{\bar{x}\}$, we have $\mathcal{D}_{-C(0)}\left(F(\hat{x}) \dot{-} F(\bar{x})\right) = \mathcal{D}_{-C(0)}\{(0, 0), (-1, -1)\} = -1$. Since $\mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta = \mathcal{D}_{-C(0)}\{(0, 0)\} - 1 = -1$, it follows that $\mathcal{D}_{-C(0)}\left(F(\hat{x}) \dot{-} F(\bar{x})\right) = -1 = \mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta$. Hence, $\bar{x}$ is not a strict $\epsilon\beta$-minimal solution of scalar optimization problem $\left(P_{C(0),F(\bar{x})}\right)$.

Theorem 3.7 *Let* $\beta = -\inf\limits_{c\in C} d_{\partial(-C(0))}(c)$, $\epsilon \geq 0$ *and* $F(\bar{x}) \in \mathcal{B}(Y)$. *If* $\bar{x}$ *is a strict* $\epsilon\beta$*-minimal* *solution of* $\left(P_{C(0),F(\bar{x})}\right)$, *then* $\bar{x} \in \epsilon\text{-}AE\,(F,C)$.

Proof Suppose to the contrary that $\bar{x} \notin \epsilon\text{-}AE\,(F,C)$, then there exist $\hat{x} \in S$, $0 \neq \hat{y} \in F(\hat{x}) \dot{-} F(\bar{x})$ such that $\hat{y} \in -C(\epsilon)$. From $F(\bar{x}) \dot{-} F(\bar{x}) = \{0\}$, we have $\hat{x} \neq \bar{x}$. Hence, we have $\hat{y} - \epsilon c \in -C(\epsilon) - C(\epsilon) \subset -C(0)$, for any $c \in C$. Hence,

$$\mathcal{D}_{-C(0)}\left(\{\hat{y} - \epsilon c\}\right) = \varDelta_{-C(0)}(\hat{y} - \epsilon c) \leq 0, \ \forall c \in C. \tag{3.11}$$

On the other hand, since $\bar{x}$ is a strict $\epsilon\beta$-minimal solution of $\left(P_{C(0),F(\bar{x})}\right)$, it follows from Lemma 3.2 that $\mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) > \mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta = \mathcal{D}_{-C(0)}(\{0\}) - \epsilon\beta = \varDelta_{-C(0)}(0) - \epsilon\beta = -\epsilon\beta$, $\forall x \in S \backslash \{\bar{x}\}$. For $c \in C$, $\varDelta_{-C(0)}(c) = d_{-C(0)}(c) - d_{Y \backslash -C(0)}(c) = d_{-C(0)}(c) = d_{\partial(-C(0))}(c)$. Hence, $-\beta = \inf\limits_{c \in C} d_{\partial(-C(0))}(c) = \inf\limits_{c \in C} \varDelta_{-C(0)}(c) = \mathcal{D}_{-C(0)}(C)$. By using Lemmas 2.3 and 3.2, we know $\mathcal{D}_{-C(0)}\left(\, F(x) \dot{-} F(\bar{x}) - C(\epsilon)\, \right) \geq \mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) - \mathcal{D}_{-C(0)}(C(\epsilon)) = \mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) - \epsilon \mathcal{D}_{-C(0)}(C) = \mathcal{D}_{-C(0)}\left(F(x) \dot{-} F(\bar{x})\right) + \epsilon\beta > 0$. This contradicts with (3.11), hence, $\bar{x} \in \epsilon\text{-}AE\,(F,C)$. $\square$

The following example illustrates that the inverse of Theorem 3.7 is incorrect.

Example 3.5 Let $X = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ + \{(1,1)\}$, $S = \mathbb{R}^2_+ \cup \{(-1,0)\}$, $\epsilon = 1$ and $\bar{x} = (0,0)$. Consider the set-valued map $F : S \to 2^S$ defined by $F(x) = \{x, \bar{x}\}$. A direct calculation gives $\beta = \sqrt{2}$. It is clear that $\left(F(x) \dot{-} F(\bar{x})\right) \cap -C = \emptyset \subset \{0\}$, $\forall x \in S$, therefore, $\bar{x} \in \epsilon\text{-}AE\,(F,C)$. For $\hat{x} = (-1,0) \in S \backslash \{\bar{x}\}$, we have $\mathcal{D}_{-C(0)}\left(F(\hat{x}) \dot{-} F(\bar{x})\right) = \mathcal{D}_{-C(0)}\{(0,0),(-1,0)\} = 0$. Hence, we get $\mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta = \mathcal{D}_{-C(0)}\{(0,0)\} - \left(-\sqrt{2}\right) = \sqrt{2}$, which implies that $\mathcal{D}_{-C(0)}\left(F(\hat{x}) \dot{-} F(\bar{x})\right) = 0 < \sqrt{2} = \mathcal{D}_{-C(0)}\left(F(\bar{x}) \dot{-} F(\bar{x})\right) - \epsilon\beta$. Therefore, $\bar{x}$ is not a strict $\epsilon\beta$-minimal solution of scalar optimization problem $\left(P_{C(0),F(\bar{x})}\right)$.

Lemma 3.4 *If* $\epsilon \geq 0$ *and* $c_1 \in C$, *then* $\epsilon c_1 + C(0) \subset C(\epsilon)$.

Proof Let $y \in \epsilon c_1 + C(0)$, then there exist $c_2 \in C$, $\lambda > 0$, such that $y = \epsilon c_1 + \lambda c_2$. Hence, $y = (\epsilon + \lambda)\left(\frac{\epsilon}{\epsilon+\lambda} c_1 + \frac{\lambda}{\epsilon+\lambda} c_2\right)$, since C is convex, we have $y \in C(\epsilon + \lambda)$. From Proposition 2.1, we get $y \in C(\epsilon + \lambda) \subset C(\epsilon)$, so, $y \in C(\epsilon)$. $\square$

Theorem 3.8 *Let* $\epsilon \geq 0$ *and* $C \backslash \{0\}$ *be an open set. If* $F(x) \in \mathcal{B}(Y)$ *for any* $x \in S$, *then* $\epsilon\text{-}PBAE\,(F,C) = \epsilon\text{-}AE\,(F,C)$.

Proof We first prove that $\epsilon\text{-}AE\,(F,C) \subset \epsilon\text{-}PBAE\,(F,C)$. Let $\bar{x} \in S$ and $\bar{x} \notin \epsilon\text{-}PBAE\,(F,C)$. Then there exists $d \neq 0$ such that $d \in -C(0)$ and $d \in \operatorname{clcone}\left(\bigcup\limits_{x \in S}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon)\right)$. Hence, if $\epsilon = 0$, then there exist $\lambda_n > 0$, $\epsilon_n > 0$, $\hat{x}_n \in S$, $\hat{c}_n \in C$ and $\hat{y}_n \in F(\hat{x}_n) \dot{-} F(\bar{x})$, such that $d = \lim\limits_{n \to +\infty} \lambda_n(\hat{y}_n + \epsilon_n \hat{c}_n) \in -C(0) \setminus \{0\}$. Since $C \backslash \{0\}$ is open, we know that $C(0) \setminus \{0\}$ is also open. Hence, there exists $N \in \mathbb{N}$ such that

$$\lambda_n(\hat{y}_n + \epsilon_n \hat{c}_n) \in -C(0) \setminus \{0\}, \tag{3.12}$$

where $n > N$. Hence, $\hat{y}_n + \epsilon_n \hat{c}_n \in -C(0) \setminus \{0\}$. We claim that $\hat{y}_n \neq 0$, $\forall n > N$. If there exists $n_1 > N$ such that $\hat{y}_{n_1} = 0$. From (3.12) we get $\hat{c}_{n_1} \in -C(0) \setminus \{0\}$, which contradicts the fact that $C(0) \cap (-C(0) \setminus \{0\}) = \emptyset$. Hence, from Lemma 3.4 we have $0 \neq \hat{y}_n \in -\epsilon_n \hat{c}_n - C(0) \subset -C(\epsilon_n) \subset -C(0)$, which implies that $\bar{x} \notin 0\text{-}AE\,(F,C)$. If $\epsilon > 0$, then there exist $\lambda_n > 0$, $\hat{x}_n \in S$, $\hat{c}_n \in C$ and $\hat{y}_n \in F(\hat{x}_n) \dot{-} F(\bar{x})$, such that $d = \lim\limits_{n \to +\infty} \lambda_n(\hat{y}_n + \epsilon \hat{c}_n) \in -C(0) \setminus \{0\}$. From the above similar proof, we get $0 \neq \hat{y}_n \in -\epsilon \hat{c}_n - C(0) \subset -C(\epsilon)$, which implies that $\bar{x} \notin \epsilon\text{-}AE\,(F,C)$. Therefore, $\epsilon\text{-}AE\,(F,C) \subset \epsilon\text{-}PBAE\,(F,C)$.

Next, since $C\backslash\{0\}$ is an open set, it follows that ϵ-$AE(F,C) = \epsilon$-$WAE(F,C)$. From Theorem 3.3, we have ϵ-$PBAE(F,C) = \epsilon$-$AE(F,C)$. $\square$

Remark 3.2 If the condition that $C\backslash\{0\}$ is an open set reduces to that $C(0)\backslash\{0\}$ is an open set, then we still have ϵ-$AE(F,C) \subset \epsilon$-$PBAE(F,C)$ for any $\epsilon \geq 0$. But the proposition ϵ-$PBAE(F,C) \subset \epsilon$-$AE(F,C)$ *for any* $\epsilon \geq 0$ is not correct.

The following example illustrates that there exists $\epsilon_0 \geq 0$ such that ϵ_0-$PBAE(F,C) \not\subset \epsilon_0$-$AE(F,C)$, although $C\backslash\{0\}$ is an open set.

Example 3.6 Let $X = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ + \{(2,2)\}$, $S = \left(\mathbb{R}^2_+ + \{(-1,-1)\}\right) \cup \{(-2,-2)\}$, $\epsilon_0 = 1$ and $\bar{x} = (0,0)$. Consider the set-valued map $F : S \to 2^S$ defined by $F(x) = \{x, \bar{x}\}$. It is clear that $C(0)\backslash\{(0,0)\}$ is an open set. We have that $\operatorname{clcone}\left(\bigcup_{x\in S}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon_0)\right) = \mathbb{R}^2_+$. Hence, $\operatorname{clcone}\left(\bigcup_{x\in S}\left(F(x) \dot{-} F(\bar{x})\right) + C(\epsilon_0)\right) \cap -C(0) = \emptyset \subset \{(0,0)\}$, which implies that $\bar{x} \in \epsilon_0$-$PBAE(F,C)$. On the other hand, taking $x_1 = (-2,-2)$, we have $\left(F(x_1) \dot{-} F(\bar{x})\right) \cap -C(1) = \{(-2,-2)\} \not\subset \{(0,0)\}$, hence, $\bar{x} \notin \epsilon_0$-$AE(F,C)$.

§4 Nonlinear scalarizations in terms of generalized Gerstewitz's functions

We assume that $C \subset Y$ is a proper convex co-radiant set and $\operatorname{int}C \neq \emptyset$. For any $\epsilon \geq 0$, denote by $\mathcal{P}_{C(\epsilon)}(Y)$ the family of all $C(\epsilon)$-proper nonempty subsets of Y.

For any $\epsilon \geq 0$, it is known that the proper convex co-radiant set $C(\epsilon)$ induces the following ordering relations in Y. For $x, y \in Y$

$$x \leq_\epsilon y \;\Leftrightarrow\; y - x \in C(\epsilon),$$

$$x <_\epsilon y \;\Leftrightarrow\; y - x \in \operatorname{int}C(\epsilon).$$

We denote by $\leq^l_\epsilon$ the following order relations with respect to a co-radiant set, for $A, B \in \mathcal{P}(Y)$,

$$A \leq^l_\epsilon B \;\Leftrightarrow\; B \subset A + C(\epsilon),$$

$$A <^l_\epsilon B \;\Leftrightarrow\; B \subset A + \operatorname{int}C(\epsilon).$$

Definition 4.1 A function $\Phi : \mathcal{P}_{C(\epsilon)}(Y) \to \mathbb{R} \cup \{-\infty\}$ is said to be

(i) $\leq^l_\epsilon$-increasing (strictly $<^l_\epsilon$-increasing) on $\mathcal{P}_{C(\epsilon)}(Y)$, if for $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$,

$$A \leq^l_\epsilon B \left(A <^l_\epsilon B\right) \Longrightarrow \Phi(A) \leq \Phi(B)\,(\Phi(A) < \Phi(B));$$

(ii) $\leq^l_\epsilon$-decreasing (strictly $<^l_\epsilon$-decreasing) on $\mathcal{P}_{C(\epsilon)}(Y)$, if for $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$,

$$A \leq^l_\epsilon B \left(A <^l_\epsilon B\right) \Longrightarrow \Phi(B) \leq \Phi(A)\,(\Phi(B) < \Phi(A)).$$

In what follows, we consider a generalized Gerstewitz's function with respect to a co-radiant set, not necessarily a cone. Let $e \in \operatorname{int}C(\epsilon)$, for a fixed $a \in Y$, the generalized Gerstewitz's function $h^e_\epsilon(a,\cdot) : Y \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$h^e_\epsilon(a,y) \;:=\; \inf\{t \in \mathbb{R} : y + te \in a + C(\epsilon)\},\ \forall y \in Y.$$

The following lemma gives two important properties of the generalized Gerstewitz's function with respect to a co-radiant set.

Lemma 4.1 [26] *For a fixed* $a \in Y$, *the following statements hold.*

(i) $h_\epsilon^e(a,y) < r \;\Leftrightarrow\; y + re \in a + \text{int}C(\epsilon)$;

(ii) $h_\epsilon^e(a,y) \le r \;\Leftrightarrow\; y + re \in a + \text{cl}C(\epsilon)$.

The extension of Gerstewitz's function has been considered in [23] by replacing the point a in Y by a set A in $\mathcal{P}(Y)$. In the following we recall the extension of Gerstewitz's function with respect to a co-radiant set. For a set $A \in \mathcal{P}(Y)$, the extension of Gerstewitz's function $h_\epsilon^e(A,\cdot) : Y \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$h_\epsilon^e(A,y) := \inf\{t \in \mathbb{R} : y + te \in A + C(\epsilon)\}, \forall y \in Y.$$

Remark 4.1 The extension of Gerstewitz's function with respect to a co-radiant set can also be expressed as

$$h_\epsilon^e(A,y) = \inf\{h_\epsilon^e(a,y) : a \in A\}.$$

The following lemma gives two important properties of the extension of Gerstewitz's function with respect to a co-radiant set.

Lemma 4.2 [26] *Let* $A \in \mathcal{P}_{C(\epsilon)}(Y)$, $r \in \mathbb{R}$. *For a fixed* $y \in Y$, *then we have* $h_\epsilon^e(A,y) < r \;\Leftrightarrow\; y + re \in A + \text{int}C(\epsilon)$.

Remark 4.2 From Proposition 2.2(iv), if $A \in \mathcal{P}_{C(\epsilon)}(Y)$, $r \in \mathbb{R}$ and $y \in Y$, then we have $h_\epsilon^e(A,y) < r \;\Leftrightarrow\; y + re \in \text{int}(A + C(\epsilon))$.

Lemma 4.3 *Let* $A \in \mathcal{P}_{C(\epsilon)}(Y)$ *be a compact set,* $r \in \mathbb{R}$. *Then* $h_\epsilon^e(A,y) \le r \;\Leftrightarrow\; y + re \in \text{cl}(A + C(\epsilon)), \forall y \in Y$.

Proof If $h_\epsilon^e(A,y) < r$, from Lemma 4.2 we have $y + re \in A + \text{int}C(\epsilon)$, hence, we get $y + re \in \text{cl}(A + C(\epsilon))$. If $h_\epsilon^e(A,y) = r$, then for any $n \in \mathbb{N}$, there exists $r_n \in \left[r, r + \frac{1}{n}\right)$ such that $y + r_n e \in A + C(\epsilon)$. Taking $n \to +\infty$, we have $y + re \in \text{cl}(A + C(\epsilon))$.

Conversely, since $y + re \in \text{cl}(A + C(\epsilon))$, there exist $a_n \in A, c_n \in C(\epsilon)$ such that $y + re = \lim\limits_{n \to +\infty}(a_n + c_n)$. Since A is compact, there exists a subsequence $a_{n_k} \to a \in A$, hence, $y + re = a + \lim\limits_{k \to +\infty} c_{n_k} \in a + \text{cl}C(\epsilon)$. Since $C(\epsilon)$ is a convex co-radiant set, we have $y + re + \frac{1}{n}e - a \in \text{cl}C(\epsilon) + \frac{1}{n}\text{int}C(\epsilon) \subset \text{int}C(\epsilon)$. Hence, $y + re + \frac{1}{n}e \in A + \text{int}C(\epsilon)$, from Lemma 4.2 we have $h_\epsilon^e(a,y) < r + \frac{1}{n}$. Therefore, we get $h_\epsilon^e(a,y) \le r$, as $n \to +\infty$. From $h_\epsilon^e(A,y) \le h_\epsilon^e(a,y)$, we have $h_\epsilon^e(A,y) \le r$. □

Corollary 4.1 *Let* $A \in \mathcal{P}_{C(\epsilon)}(Y)$ *be a compact set,* $r \in \mathbb{R}$. *Then* $h_\epsilon^e(A,y) \le r \;\Leftrightarrow\; y + re \in A + \text{cl}C(\epsilon)$.

Proof Since A is a compact set, we have $\text{cl}(A + C(\epsilon)) = A + \text{cl}C(\epsilon)$. □

Corollary 4.2 *Let* $A \in \mathcal{P}_{C(\epsilon)}(Y)$ *be a* $C(\epsilon)$*-closed set,* $r \in \mathbb{R}$. *Then* $h_\epsilon^e(A,y) \le r \;\Leftrightarrow\; y + re \in \text{cl}(A + C(\epsilon))$.

Proof The Proof of sufficiency is similar to that of Lemma 4.3, and the necessity can be gained by definition. □

Remark 4.3 (i) Since $A + C(\epsilon) \subset A + \text{cl}C(\epsilon) \subset \text{cl}(A + C(\epsilon))$ and A is a $C(\epsilon)$-closed set, we have $A + C(\epsilon) = A + \text{cl}C(\epsilon) = \text{cl}(A + C(\epsilon))$. Therefore, if $A \in \mathcal{P}_{C(\epsilon)}(Y)$ is a $C(\epsilon)$-closed set and $r \in \mathbb{R}$, then $h_\epsilon^e(A,y) \le r \;\Leftrightarrow\; y + re \in A + \text{cl}C(\epsilon)$.

(ii) From Remark 4.2 and Corollary 4.2, it follows that if $A \in \mathcal{P}_{C(\epsilon)}(Y)$ is a $C(\epsilon)$-closed set and $r \in \mathbb{R}$, then $h_\epsilon^e(A,y) = r \;\Leftrightarrow\; y + re \in \partial(A + C(\epsilon))$, $\forall y \in Y$.

Lemma 4.4 *If* $\alpha > 0,\ \lambda_2 \ge \lambda_1 > 0,\ e \in \text{int}C(\epsilon)$, *then* $\lambda_2 e + \frac{\lambda_2}{\alpha}\text{int}C(\epsilon) \subset \lambda_1 e + \frac{\lambda_1}{\alpha}\text{int}C(\epsilon)$.

Proof From Proposition 2.1 we have $\lambda_2 e - \lambda_1 e + \frac{\lambda_2}{\alpha}\mathrm{int}C(\epsilon) \subset (\lambda_2 - \lambda_1)\,\mathrm{int}C(\epsilon) + \frac{\lambda_1}{\alpha}\mathrm{int}C(\epsilon)$. From $\lambda_2 \geq \lambda_1 > 0$ and Proposition 2.2(ii), we have $\lambda_2 e - \lambda_1 e + \frac{\lambda_2}{\alpha}\mathrm{int}C(\epsilon) \subset \left(\lambda_2 - \lambda_1 + \frac{\lambda_1}{\alpha}\right)\mathrm{int}C(\epsilon) \subset \frac{\lambda_1}{\alpha}\mathrm{int}C(\epsilon)$. □

The following proposition gives the monotonicity of $h_\epsilon^e(\cdot, y)$.

Proposition 4.1 [23] *Let $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$, $y \in Y$. Then the following statements hold.*

(i) If $A \leq_\epsilon^l B$, then $h_\epsilon^e(A, y) \leq h_\epsilon^e(B, y)$;

(ii) If $A <_\epsilon^l B$ and B is a $C(\epsilon)$-closed set, then $h_\epsilon^e(A, y) < h_\epsilon^e(B, y)$.

The following proposition gives the monotonicity of $h_\epsilon^e(A, \cdot)$.

Proposition 4.2 *Let $A \in \mathcal{P}_{C(\epsilon)}(Y)$. Then the following statements hold.*

(i) If $x \leq_\epsilon y$, then $h_\epsilon^e(A, y) \leq h_\epsilon^e(A, x)$;

(ii) If $x <_\epsilon y$ and A is a compact set, then $h_\epsilon^e(A, y) < h_\epsilon^e(A, x)$.

Proof (i) Since $x \leq_\epsilon y$, we have $y \in x + C(\epsilon)$. If $x + re \in A + C(\epsilon)$, according to Proposition 2.2(i), we get $y + re \in x + re + C(\epsilon) \subset A + C(\epsilon) + C(\epsilon) \subset A + C(\epsilon)$. Therefore, $\{t \in \mathbb{R} : x + re \in A + C(\epsilon)\} \subset \{t \in \mathbb{R} : y + re \in A + C(\epsilon)\}$, which implies

$$\inf\{t \in \mathbb{R} : y + re \in A + C(\epsilon)\} \leq \inf\{t \in \mathbb{R} : x + re \in A + C(\epsilon)\}.$$

Hence, $h_\epsilon^e(A, y) \leq h_\epsilon^e(A, x)$.

(ii) Let $h_\epsilon^e(A, x) = r$. Hence, we have $h_\epsilon^e(A, x) \leq r$. From Lemma 4.3, we get $x + re \in \mathrm{cl}(A + C(\epsilon))$. Since A is a compact set, it follows that there exists $a_1 \in A$ such that $x + re - a_1 \in \mathrm{cl}C(\epsilon)$. From $x <_\epsilon y$, we have $y \in x + \mathrm{int}C(\epsilon)$. Hence, $y + re - a_1 \in x + re - a_1 + \mathrm{int}C(\epsilon) \subset \mathrm{cl}C(\epsilon) + \mathrm{int}C(\epsilon) \subset \mathrm{int}C(\epsilon)$, which implies $y + re \in a_1 + \mathrm{int}C(\epsilon) \subset A + \mathrm{int}C(\epsilon)$. According to Lemma 4.2, we have $h_\epsilon^e(A, y) < r$, therefore, $h_\epsilon^e(A, y) < h_\epsilon^e(A, x)$. □

In the following we introduce the generalized Gerstewitz's function with respect to a co-radiant set $H_\epsilon^e(\cdot,\cdot) : \mathcal{P}_{C(\epsilon)}(Y) \times \mathcal{P}_{C(\epsilon)}(Y) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$H_\epsilon^e(A, B) := \sup_{b \in B} h_\epsilon^e(A, b).$$

The following lemma gives two important properties of the generalized Gerstewitz's function with respect to a proper convex co-radiant set.

Lemma 4.5 *Let $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$, $r \in \mathbb{R}$. Then the following propositions hold.*

(i) If A is a compact set, then $H_\epsilon^e(A, B) \leq r \;\Leftrightarrow\; B + re \subset \mathrm{cl}(A + C(\epsilon))$;

(ii) $H_\epsilon^e(A, B) < r \;\Rightarrow\; B + re \subset A + \mathrm{int}C(\epsilon)$.

Proof (i) Since $H_\epsilon^e(A, B) \leq r$, we have $h_\epsilon^e(A, b) \leq r$, for any $b \in B$. According to Lemma 4.3, we have $b + re \in \mathrm{cl}(A + C(\epsilon))$, for any $b \in B$. Therefore, $B + re \subset \mathrm{cl}(A + C(\epsilon))$. Similarly, we can show the converse inclusion also holds.

(ii) Since $H_\epsilon^e(A, B) < r$, we have $h_\epsilon^e(A, b) < r$, for any $b \in B$. According to Lemma 4.2, we have $b + re \in A + \mathrm{int}C(\epsilon)$, for any $b \in B$. Therefore, $B + re \subset A + \mathrm{int}C(\epsilon)$. □

Corollary 4.3 *Let $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$, $r \in \mathbb{R}$, A be a compact set. Then $H_\epsilon^e(A, B) \leq r \;\Leftrightarrow\; B + re \subset A + \mathrm{cl}C(\epsilon)$.*

Proof The proof follows directly from Corollary 4.1. □

Corollary 4.4 *Let $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$, $r \in \mathbb{R}$, A be a compact set. Then $H_\epsilon^e(A, B) \leq r \;\Leftrightarrow\; B + re \subset \mathrm{cl}(A + C(\epsilon))$.*

Proof The proof follows directly from Corollary 4.2. □

Lemma 4.6 *Let* $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$ *and* A *be a* $C(\epsilon)$*-closed set. If* $H_\epsilon^e(A,B) < +\infty$*, then* $H_\epsilon^e(A,B) = \min\{t \in \mathbb{R} : B + te \subset A + C(\epsilon)\}$*.*

Proof By Corollary 4.4, we have $H_\epsilon^e(A,B) \le r \ \Leftrightarrow\ B + re \subset A + C(\epsilon)$. Hence,

$$\{t \in \mathbb{R} \mid B + te \subset A + C(\epsilon)\} = \{t \in \mathbb{R} \mid H_\epsilon^e(A,B) \le t\} = [H_\epsilon^e(A,B), +\infty),$$

which implies that $H_\epsilon^e(A,B) = \min\{t \in \mathbb{R} : B + te \subset A + C(\epsilon)\}$. $\square$

Remark 4.4 If $C(\epsilon)$ is a cone, then Lemma 4.6 reduces to [23, Proposition 3.2].

Lemma 4.7 *Let* $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$ *and* B *be a* $C(\epsilon)$*-compact set. If* $H_\epsilon^e(A,B) < +\infty$*, then* $H_\epsilon^e(A,B) = \max\limits_{b \in B}\{h_\epsilon^e(A,b)\}$*.*

Proof Let $H_\epsilon^e(A,B) = \sup\limits_{b \in B} h_\epsilon^e(A,b) = m$. Suppose to the contrary that $h_\epsilon^e(A,b) < m$, for any $b \in B$. From Lemma 4.2 we get $b + me \in A + \mathrm{int}C(\epsilon)$. Hence, there exists $\lambda_b > 0$ such that $b + me - \lambda_b e \in A + \mathrm{int}C(\epsilon)$. So, we have $b + me - \frac{\lambda_b}{2}e \in \frac{\lambda_b}{2}e + A + \mathrm{int}C(\epsilon) \subset \frac{\lambda_b}{2}\mathrm{int}C(\epsilon) + A + C(\epsilon)$. Therefore, we get $B \subset \bigcup\limits_{b \in B}\left(-\left(m - \frac{\lambda_b}{2}\right)e + \frac{\lambda_b}{2}\mathrm{int}C(\epsilon) + A\right) + C(\epsilon)$. Since B is a $C(\epsilon)$-compact set and $-\left(m - \frac{\lambda_b}{2}\right)e + \frac{\lambda_b}{2}\mathrm{int}C(\epsilon) + A$ is an open set, there are a finite number $\{\lambda_{b_1}, \ldots, \lambda_{b_r}\}$ satisfying $B \subset \bigcup\limits_{i=1}^{r}\left(-\left(m - \frac{\lambda_{b_i}}{2}\right)e + \frac{\lambda_{b_i}}{2}\mathrm{int}C(\epsilon) + A\right) + C(\epsilon)$. Without loss of generality, we assume that $\min\{\lambda_{b_i} : i = 1, 2, \ldots, r\} = \lambda_{b_1}$. From Lemma 4.4 we have $\frac{\lambda_{b_i}}{2}e + \frac{\lambda_{b_i}}{2}\mathrm{int}C(\epsilon) \subset \frac{\lambda_{b_1}}{2}e + \frac{\lambda_{b_1}}{2}\mathrm{int}C(\epsilon)$, hence,

$$B \subset -\left(m - \frac{\lambda_{b_1}}{2}\right)e + \frac{\lambda_{b_1}}{2}\mathrm{int}C(\epsilon) + A + C(\epsilon). \tag{4.1}$$

Let $m_0 = m - \frac{\lambda_{b_1}}{2}$, from (4.1) and Proposition 2.2(ii), we have $B + m_0 e \subset A + \frac{\lambda_{b_1}}{2}\mathrm{int}C(\epsilon) + C(\epsilon) \subset A + \mathrm{int}C(\epsilon) \subset A + C(\epsilon)$. Hence, $\forall b \in B$, $b + m_0 e \in A + C(\epsilon)$, so $h_\epsilon^e(A,b) \le m_0$, $\forall b \in B$. Therefore, we have $H_\epsilon^e(A,B) \le m_0$. Since $m_0 < m$, this contradicts with $H_\epsilon^e(A,B) = m$, therefore, $H_\epsilon^e(A,B) = \max\limits_{b \in B}\{h_\epsilon^e(A,b)\}$. $\square$

Remark 4.5 If $C(\epsilon)$ is a cone, then Lemma 4.7 reduces to [23, Proposition 3.4].

Lemma 4.8 *Suppose that* $B \in \mathcal{P}_{C(\epsilon)}(Y)$*, then* B *is* $C(\epsilon)$*-bounded if and only if* $H_\epsilon^e(C(\epsilon), B) < +\infty$*.*

Proof If B is $C(\epsilon)$-bounded, then for the neighborhood of zero $U = -e + \mathrm{int}C(\epsilon)$, there exists $\alpha > 0$ such that $B \subset \alpha U + C(\epsilon)$, hence, we have $B \subset -\alpha e + \alpha\mathrm{int}C(\epsilon) + C(\epsilon)$. From Proposition 2.2(ii), we have $\alpha\mathrm{int}C(\epsilon) + C(\epsilon) \subset (\alpha + 1)\mathrm{int}C(\epsilon)$. Since $C(\epsilon)$ is a co-radiant set, we have $(\alpha + 1)\mathrm{int}C(\epsilon) \subset C(\epsilon)$. Therefore, we get $B + \alpha e \subset C(\epsilon)$, by definition we have $H_\epsilon^e(C(\epsilon), B) \le \alpha < +\infty$.

Conversely, suppose that $H_\epsilon^e(C(\epsilon), B) < +\infty$, then there exists $r \in \mathbb{R}$ such that $H_\epsilon^e(C(\epsilon), B) < r$. From Lemma 4.5(ii), we have

$$B + re \subset C(\epsilon) + \mathrm{int}C(\epsilon). \tag{4.2}$$

Let U_0 be any neighborhood of zero. Then there exists $\lambda > 0$ such that $-re \in \lambda U_0$. From (4.2) we have $B \subset -re + \mathrm{int}C(\epsilon) \subset \lambda U_0 + C(\epsilon)$, which implies that B is $C(\epsilon)$-bounded. $\square$

Remark 4.6 If $C(\epsilon)$ is a cone, then Lemma 4.8 reduces to Lemma 3.5 of [23].

Lemma 4.9 *Let* $A, B \in \mathcal{P}_{C(\epsilon)}(Y)$*,* A *be a* $C(\epsilon)$*-bounded set. Then* B *is* $C(\epsilon)$*-bounded if and only if* $H_\epsilon^e(A,B) < +\infty$*.*

Proof For any $a \in A$, $b \in B$, by definition we can get $h_\epsilon^e(a,b) = h_\epsilon^e(0, b - a)$. From Proposition 2.2(i) we have $h_{\epsilon}^{e}(0,b-a)\le h_{\epsilon}^{e}(C(\epsilon),b-a)$, hence, $h_{\epsilon}^{e}(a,b)\le h_{\epsilon}^{e}(C(\epsilon),b-a)$. From Remark 4.1 we have $h_{\epsilon}^{e}(A,b)\le h_{\epsilon}^{e}(C(\epsilon),b-a)$. Picking $a_0\in A$, we obtain $h_{\epsilon}^{e}(A,b)\le h_{\epsilon}^{e}(C(\epsilon),b-a_0)$. Therefore, $H_{\epsilon}^{e}(A,B)\le H_{\epsilon}^{e}(C(\epsilon),B-a_0)$. From Lemma 4.8 we have

$$H_{\epsilon}^{e}(C(\epsilon),B-a_0)<+\infty.$$

Hence, $H_{\epsilon}^{e}(A,B)<+\infty$.

Conversely, suppose $H_{\epsilon}^{e}(A,B)<r$ with $r\in\mathbb{R}$. From Lemma 4.5(ii), we have

$$B+re\subset A+\mathrm{int}C(\epsilon). \tag{4.3}$$

Since A is a $C(\epsilon)$-bounded set, we can know $-re+A$ is also $C(\epsilon)$-bounded. Hence, for any neighborhood of zero U, there exists $\lambda_0>0$ such that $-re+A\subset\lambda_0U+C(\epsilon)$. From (4.3) we have $B\subset -re+A+C(\epsilon)\subset\lambda_0U+C(\epsilon)+C(\epsilon)\subset\lambda_0U+C(\epsilon)$, which implies that B is $C(\epsilon)$-bounded. □

Remark 4.7 If $C(\epsilon)$ is a cone, then Lemma 4.9 reduces to Theorem 3.6 of [23].

The following proposition gives the monotonicity of $H_{\epsilon}^{e}(A,\cdot)$ and $H_{\epsilon}^{e}(\cdot,A)$, respectively.

Proposition 4.3 *Let $A\in\mathcal{P}_{C(\epsilon)}(Y)$. Then the following statements hold.*

(i) $H_{\epsilon}^{e}(A,\cdot)$ is decreasing on $\mathcal{P}_{C(\epsilon)}(Y)$ with respect to $\le_{\epsilon}^{l}$;

(ii) $H_{\epsilon}^{e}(\cdot,A)$ is increasing on $\mathcal{P}_{C(\epsilon)}(Y)$ with respect to $\le_{\epsilon}^{l}$.

Proof (i) Let $B,D\in\mathcal{P}_{C(\epsilon)}(Y)$ and $B\le_{\epsilon}^{l}D$. Hence, for any $d\in D$, there exists $\hat{b}\in B$ such that $\hat{b}\le_{\epsilon}d$. From Proposition 4.2(i), we have $h_{\epsilon}^{e}(A,d)\le h_{\epsilon}^{e}\left(A,\hat{b}\right)$. Therefore, $H_{\epsilon}^{e}(A,D)\le H_{\epsilon}^{e}(A,B)$.

(ii) By using Proposition 4.1(i), similar to the proof of Proposition 4.3(i), we can show Proposition 4.3(ii) holds. □

Remark 4.8 If $C(\epsilon)$ is a cone, then Proposition 4.3(i) and (ii) reduces to [23, Theorem 3.8(ii) and (v)], respectively.

The following proposition shows that the strict monotonicity of $H_{\epsilon}^{e}(A,\cdot)$ and $H_{\epsilon}^{e}(\cdot,A)$, respectively.

Proposition 4.4 *The following statements hold.*

(i) If $A\in\mathcal{P}_{C(\epsilon)}(Y)$ is a compact set, then $H_{\epsilon}^{e}(A,\cdot)$ is strictly decreasing on the family of all $C(\epsilon)$-compact sets with respect to $<_{\epsilon}^{l}$;

(ii) If $A\in\mathcal{P}_{C(\epsilon)}(Y)$ is a $C(\epsilon)$-compact set, then $H_{\epsilon}^{e}(\cdot,A)$ is strictly increasing on the family of all $C(\epsilon)$-bounded sets with respect to $<_{\epsilon}^{l}$.

Proof (i) Let $B,D\in\mathcal{P}_{C(\epsilon)}(Y)$ and $B<_{\epsilon}^{l}D$. Since A is a compact set, we have A is a $C(\epsilon)$-compact set. Hence, by Lemmas 4.7 and 4.9, there exists $\hat{d}\in D$ such that

$$H_{\epsilon}^{e}(A,D)=h_{\epsilon}^{e}\left(A,\hat{d}\right). \tag{4.4}$$

Since $B<_{\epsilon}^{l}D$, there exists $\hat{b}\in B$ such that $\hat{b}<_{\epsilon}\hat{d}$. From Proposition 4.2(ii) we have $h_{\epsilon}^{e}\left(A,\hat{d}\right)<h_{\epsilon}^{e}\left(A,\hat{b}\right)$, hence, by (4.4) we conclude $H_{\epsilon}^{e}(A,D)<H_{\epsilon}^{e}(A,B)$.

(ii) By using Proposition 4.1(ii), similar to the proof of Proposition 4.4(i), we can show Proposition 4.4(ii) holds. □

Remark 4.9 If $C(\epsilon)$ is a cone, then Proposition 4.4 reduces to Theorem 3.9 of [23].

The following theorems give sufficient conditions for the weakly ϵ-efficient solution of (SOP) with respect to C.

Theorem 4.1 *Let* $\bar{x} \in S$, $F(\bar{x})$ *be a* $C(\epsilon)$*-closed set, and* $F(s) \in \mathcal{P}_{C(\epsilon)}(Y)$ *for any* $s \in S$. *If there exists* $y_0 \in Y$ *such that* $h_\epsilon^e(F(\bar{x}), y_0) = \min\{h_\epsilon^e(F(x), y_0) : x \in S\}$, *then* $\bar{x} \in \epsilon$-$WAE(F, C)$.

Proof Suppose to the contrary that $\bar{x} \notin \epsilon$-$WAE(F, C)$, there exist $d \in \mathrm{int}C(\epsilon)$ and $x_1 \in S$, such that $-d \in F(x_1) \overset{\cdot}{-} F(\bar{x})$. Hence,

$$F(\bar{x}) \subset F(x_1) + d. \tag{4.5}$$

Since $h_\epsilon^e(F(\bar{x}), y_0) = \min\{h_\epsilon^e(F(x), y_0) : x \in S\}$, we have $h_\epsilon^e(F(\bar{x}), y_0) \leq h_\epsilon^e(F(x), y_0)$, for any $x \in S$. From Corollary 4.2, we get $y_0 + h_\epsilon^e(F(x), y_0)e \in F(\bar{x}) + C(\epsilon)$. By (4.5) and Proposition 2.2(iii), we have $y_0 + h_\epsilon^e(F(x), y_0)e \in d + F(x_1) + C(\epsilon) \subset \mathrm{int}C(\epsilon) + F(x_1) + C(\epsilon) \subset F(x_1) + \mathrm{int}C(\epsilon)$. Therefore, $h_\epsilon^e(F(x_1), y_0) < h_\epsilon^e(F(x), y_0)$, $\forall x \in S$, this leads a contradiction. Hence, $\bar{x} \in \epsilon$-$WAE(F, C)$. □

Theorem 4.2 *Let* $F(x) \in \mathcal{P}_{C(\epsilon)}(Y)$ *and* $F(x)$ *be a compact set for any* $x \in S$. *If there exists* $\bar{x} \in S$ *such that either of the following holds:*

(a) $H_\epsilon^e(F(x), F(\bar{x})) \geq H_\epsilon^e(F(x), F(x))$, $\forall x \in S$,

(b) $H_\epsilon^e(F(\bar{x}), F(x)) \leq H_\epsilon^e(F(\bar{x}), F(\bar{x}))$, $\forall x \in S$,

then $\bar{x} \in \epsilon$-$WAE(F, C)$.

Proof Suppose to the contrary that $\bar{x} \notin \epsilon$-$WAE(F, C)$, there exists $d \in \mathrm{int}C(\epsilon)$ and $x_1 \in S$, such that $-d \in F(x_1) \overset{\cdot}{-} F(\bar{x})$, hence, $F(\bar{x}) \subset F(x_1) + d \subset F(x_1) + \mathrm{int}C(\epsilon)$. Therefore,

$$F(x_1) <_\epsilon^l F(\bar{x}). \tag{4.6}$$

If (a) holds, then by (4.6) and Proposition 4.4(i), we have

$$H_\epsilon^e(F(x_1), F(\bar{x})) < H_\epsilon^e(F(x_1), F(x_1)),$$

which is a contradiction. Similarly, we can prove the result when condition (b) holds. □

The following example illustrates that the converse of Theorem 4.2 may not hold.

Example 4.1 Let $Y = \mathbb{R}^2$, $C = \mathbb{R}_+^2$, $S = \{0, 1\}$, $\epsilon = 1$ and $e = (1, 1)$. Consider the set-valued map $F : S \to 2^Y$ defined by

$$F(x) = \begin{cases} [0,1] \times [0,1], x = 0, \\ [2,4] \times [2,4], x = 2. \end{cases}$$

It is clear that $F(x) \in \mathcal{P}_{C(\epsilon)}(Y)$, $F(x)$ is a compact set, for any $x \in S$. Let $\bar{x} = 2$. Since $F(0) \overset{\cdot}{-} F(2) = \emptyset$ and $F(2) \overset{\cdot}{-} F(2) = \{(0,0)\}$, hence, $\left(F(x) \overset{\cdot}{-} F(\bar{x})\right) \cap -\mathrm{int}C(\epsilon) = \emptyset$, $\forall x \in S$. Therefore, $\bar{x} \in \epsilon$-$WAE(F, C)$. For any $x \in S$, we have

$$H_\epsilon^e(F(x), F(x)) = \sup_{y \in F(x)} h_\epsilon^e(F(x), y) = \sup_{y \in F(x)} \inf\{t \in \mathbb{R} : y + te \in F(x) + C\} = 0.$$

If $\hat{x} = 0$, then we have

$$H_\epsilon^e(F(\hat{x}), F(\bar{x})) = \sup_{y \in F(\bar{x})} h_\epsilon^e(F(\hat{x}), y) = \sup_{y \in F(\bar{x})} \inf\{t \in \mathbb{R} : y + te \in F(\hat{x}) + C\} = -2 < 0.$$

Therefore, (a) of Theorem 4.2 does not hold.

If $\tilde{x} = 0$, then we have

$$H_\epsilon^e(F(\bar{x}), F(\tilde{x})) = \sup_{y \in F(\tilde{x})} h_\epsilon^e(F(\bar{x}), y) = \sup_{y \in F(\tilde{x})} \inf\{t \in \mathbb{R} : y + te \in F(\bar{x}) + C\} = 2 < 0.$$

Therefore, (b) of Theorem 4.2 does not hold.

§5 Conclusion

In this paper, based on the Minkowski difference, we extend the vector optimization problem in [17] to the set optimization problem. And we introduce the concepts of ϵ-efficient solutions, weakly ϵ-efficient solutions, proper ϵ-efficient solutions and Henig ϵ-efficient solutions of set optimization with respect to a co-radiant set, and obtain the relationships among these solutions, such as, $0\text{-}WAE\,(F,C)=\bigcap\limits_{\epsilon>0}\epsilon\text{-}WAE\,(F,C)$, $0\text{-}PBAE\,(F,C)=\bigcap\limits_{\epsilon>0}\epsilon\text{-}PBWAE\,(F,C)$. In this paper, Lemma 2.4 generalizes Proposition 2.3 (ii) in [24]. Theorem 3.1 shows that the weakly ϵ-efficient solution of set optimization problem is equivalent to the $\epsilon\beta$-minimal solution of scalar optimization problem $\left(P_{C(\epsilon),F(\bar{x})}\right)$. The relationship between the optimal solutions of scalar optimization problem $\left(P_{C(\epsilon)}\right)$ and the $\epsilon\beta$-minimal solutions of scalar optimization problem $\left(P_{C(\epsilon),F(\bar{x})}\right)$ is discussed. And the relationships among ϵ-efficient solutions, proper ϵ-efficient solutions for set optimization with respect to co-radiant sets, the $\epsilon\beta$-minimal solutions and strict $\epsilon\beta$-minimal solutions of scalar optimization problem $\left(P_{C(\epsilon),F(\bar{x})}\right)$ are discussed. We investigate the generalized Gerstewitz's function with respect to co-radiant sets, and establish the sufficient optimality conditions and equivalent optimality condition for weakly ϵ-efficient solutions of set optimization.

Declarations

Conflict of interest The authors declare no conflict of interest.

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