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Optimality analysis for nonsmooth vector equilibrium problems with constraints via generalized subdifferentials

Tran Van Su[1,*] Dinh Dieu Hang[2]

Abstract. This paper is devoted to the study of KKT-type optimality via the generalized subdifferentials of a nonsmooth vector equilibrium problem with set, inequality and equality constraints (CVEP, for brevity) and its applications. First, we provide the notion of the generalized subdifferentials associated to the contingent epiderivatives (also called the Aubin-Frankowska's generalized subdifferentials) and the Clarke's generalized subgradients. Additionally, we provide some regularity conditions (RC1) and (RC3-s) for any index $s \in I = \{1, \ldots, p\}$. Some KKT-type necessary optimality conditions for the efficient solution types of problem CVEP under some suitable regularity conditions are derived. Besides, some strong KKT-type necessary optimality conditions become sufficient optimality conditions under some suitable assumptions on the pseudoconvexity, quasiconvexity and quasilinearly of objective and constraint functions. Finally, an application of such result to the vector optimization problem with constraints (CVOP) and the vector variational inequality problem with constraints (CVVI) is presented. Some illustrative examples are also provided for our findings.

§1 Introduction

In the last several decades, primal and dual optimality conditions for the efficient solution types of nonsmooth vector equilibrium problems have been paid much attention from researchers due to a wide range of applications in different areas of engineering and economics; see ([3,4,7,8,9,15,16,19,21,29,34,35,42,44,49] and the references therein). There are a lot of literature dealing with weak/strict efficiency conditions in nonsmooth vector equilibrium problems using the tool of convex analysis, variational analysis and nonsmooth analysis; see ([5,12,13,14,15,16,17,18,22,23,24,25,26,30,31,33,36,37,38,39,40,41,43,45,46,47,48,50,51,52] and the references therein). As we all know, the concept of generalized gradient for locally Lipschitz functions and constraint qualifications for programs with Lipschitz conditions has been an important subject of study, e.g., in Clarke [6], Constantin [10], Jiménez and Novo [12,13], Jourani

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*Corresponding author.

[15,16], Nobakhtian [21] and Luu [30,31,32]. However, the Clarke subdifferential only was applied for locally Lipschitz programs. In many nonsmooth vector optimization problems, the data under consideration is only stable at the optimal point, but not locally Lipschitz around that solution, and the well-known results in multiobjective program cannot be employed (see Theorem 4.3 [13], for instance). Within this context, the notion of the generalized subdifferentials associated to the contingent epiderivatives was introduced by Aubin and Frankowska [2] (known as the Aubin-Frankowska's generalized subdifferentials) which allows establishing necessary conditions of Kuhn-Tucker type in scalar programs. On KKT-type necessary optimality conditions, these subdifferentials are finer than the Clarke subdifferentials, even the Clarke subdifferentials are of less restrictive application than the Aubin-Frankowska's generalized subdifferentials. They only coincide in the case of locally Lipschitz convex functions. Also, in most of research papers for multiobjective programs, primal and dual necessary as well as sufficient optimality conditions via the generalized subdifferentials for efficiency are not considered. Therefore, it is important for establishing strong Karush-Kuhn-Tucker-type necessary/and sufficient optimality conditions via these associate subdifferentials for the efficient solution types of nonsmooth vector equilibrium problem with set, inequality and equality constraints involving stable functions and its special problems. This is a motivation for our present work in the literature.

The notion of contingent epiderivatives has a rich mathematical structure, which plays a crucial role in variational analysis, convex analysis and nonlinear analysis. Moreover, such concept has been used for providing primal and dual optimality conditions for weak efficiency in nonsmooth vector equilibrium problems. It is also an extension of directional derivatives in the single-valued convex sense, which was first introduced by Aubin-Frankowska [1,2], used after by other researchers such as Jahn-Rauh [11], Jiménez-Novo [12,13], Jiménez et al. [14], Luc [27], Rodríguez-Marín-Sama [36,37], Su [38,39,40] in deriving primal and Fritz John/and Kuhn-Tucker dual efficiency conditions in any nonsmooth optimization problem. For example, in vector optimization, Jahn-Rauh [11] provided some unified sufficient and necessary conditions in terms of contingent epiderivatives; Jiménez et al. [14] established primal necessary and sufficient conditions for strict minimizers in terms of contingent epiderivatives with stable/and steady functions. More recently, Su [38] obtained Fritz John/and Kuhn-Tucker dual necessary and sufficient optimality conditions for weak efficiency by combining contingent epiderivatives and contingent derivatives with stable functions; Su [39] gave Fritz John/and Kuhn-Tucker dual optimality conditions for the weakly efficient solution of nonsmooth vector equilibrium problems with generalized convex data in terms of contingent epiderivatives. However, the Aubin-Frankowska's generalized subdifferentials involving stable functions are not used in deriving optimality conditions for nonsmooth constrained vector equilibrium problems with stable conditions.

Motivated and inspired by the results due to [2,14,36,37,38,39], our paper will continue to study and develop some KKT-type necessary/and sufficient optimality conditions in terms of the Aubin-Frankowska's generalized subdifferentials for the efficient solution types to such problems. The content of this paper is organized as follows. After some preliminaries, Section 3 presents KKT-type necessary optimality conditions for the efficient solution types of nonsmooth vector equilibrium problem with set, inequality and equality constraints (CVEP) via the Aubin-Frankowska's generalized subdifferentials with stable, steady and Hadamard differentiable functions. Section 4 is devoted to investigating KKT-type sufficient optimality for this kind of problem (CVEP) under suitable assumptions on the pseudoconvexity, quasiconvexity and quasilinearly of objective and constraint functions. Section 5, an application of these results for the vector optimization problem (CVOP) and the vector variational inequality problem with common constraints (CVVI) is presented.

§2 Preliminaries

Throughout this paper, we use the standard notation of variational analysis and convex analysis, e.g., in Aubin-Frankowska [2] and Rockafellar [35]. One writes $\mathbb{R}^n$ $(n \in \mathbb{N})$ standing for a finite real Euclide space and its topological dual, whose norm is expressed as $\|\cdot\|$, denote by $\emptyset \neq C \subset \mathbb{R}^n$ instead of a nonempty subset of $\mathbb{R}^n$. By 0 one denotes the origin of any space $\mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ the inner product in any space $\mathbb{R}^n$. The nonnegative (resp., nonpositive) orthant cone of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ (resp. $\mathbb{R}^n_-$), where $\mathbb{R}^n_- := -\mathbb{R}^n_+$ and $\mathbb{R}^n_{++} := \text{int}\mathbb{R}^n_+$ (the interior of the orthant cone $\mathbb{R}^n_+$). For $r > 0$, we use $B(\overline{x}, r)$ to denote the open ball of radius r and centered at $\overline{x}$, and $t_n \to 0^+$ to denote the sequence of positive real numbers $(t_n)_{n\geq 1}$ with the limit 0. For each $A \subset \mathbb{R}^n$, let us denote, as usual, its closure, interior, relative interior and cone hull by cl$A$, int$A$, ri$A$ and cone$A$, respectively, and by $A_x := A - x$ for all $x \in \mathbb{R}^n$. Denote by $|A|$ the cardinality of A, i.e., the number of elements of A, and by A^+ and $A^\sharp$ the positive polar cone and the strictly positive polar cone of A, respectively, where $A^+ = \{\xi \in \mathbb{R}^n \mid \langle\xi, a\rangle \geq 0,\ \forall\, a \in A\}$ and $A^\sharp = \{\xi \in \mathbb{R}^n \mid \langle\xi, a\rangle > 0,\ \forall\, a \in A \setminus \{0\}\}$. Let two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we use the following notations

$$\begin{aligned}
&x \leqq y, \text{ if } x_i \leq y_i, \quad \forall\, i,\\
&x \leq y, \text{ if } x \leqq y, \quad \text{and } x \neq y,\\
&x = y, \text{ if } x_i = y_i, \quad \forall\, i,\\
&x < y, \text{ if } x_i < y_i, \quad \forall i.
\end{aligned}$$

Let a mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ and $\overline{x}, v \in \mathbb{R}^n$. We recall that the graph, the hypograph and the epigraph of f are defined respectively by graph$f := \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s | y = f(x)\big\}$, hyp$f := \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s | y \in f(x) - \mathbb{R}^s_+\big\}$ and epi$f := \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s | y \in f(x) + \mathbb{R}^s_+\big\}$.

Definition 1 ([12]) Let $\overline{x} \in \text{cl}M$ with M be subset of $\mathbb{R}^n$.

(i) The contingent cone to the set M at the point $\overline{x}$ is defined by

$$T(M, \overline{x}) = \{v \in \mathbb{R}^n \,|\, \exists t_n \to 0^+, \exists v_n \to v \text{ such that } \overline{x} + t_n v_n \in M,\ \ \forall n \in \mathbb{N}\}.$$

(ii) The cone of attainable directions to the set M at the point $\overline{x}$ is defined by

$$A(M, \overline{x}) = \{v \in \mathbb{R}^n \,|\, \forall t_n \to 0^+, \exists v_n \to v \text{ such that } \overline{x} + t_n v_n \in M,\ \ \forall n \in \mathbb{N}\}.$$

(iii) The normal cone to the set M at the point $\overline{x}$ is defined by

$$N(M, \overline{x}) = \{\xi \in \mathbb{R}^n \mid \langle\xi, v\rangle \leq 0\ \ (\forall\, v \in T(M, \overline{x}))\}.$$

Observe that $N(M,\overline{x}) = [-T(M,\overline{x})]^+$ and $A(M,\overline{x}) \subset T(M,\overline{x})$. If M is convex, then

$$A(M,\overline{x}) = T(M,\overline{x}) = \mathrm{cl}\,\mathrm{cone}(M-\overline{x}).$$

According to Aubin and Frankowska [2], the contingent derivative of f at $\overline{x}$ in the direction v, which is denoted as $\partial f(\overline{x})v$, defined by

$$w \in \partial f(\overline{x})v \Longleftrightarrow (v,w) \in T(\mathrm{graph} f, (\overline{x}, f(\overline{x}))).$$

It is evident that $\partial f(\overline{x})v$ is a closed set and the set-valued map $v \mapsto \partial f(\overline{x})v$ is positively homogeneous. The contingent epiderivative of f at $\overline{x}$ in the direction v, which is denoted as $D_{\uparrow} f(\overline{x})v$, defined by

$$w \in D_{\uparrow} f(\overline{x})v + \mathbb{R}^s_+ \Longleftrightarrow (v,w) \in T(\mathrm{epi} f, (\overline{x}, f(\overline{x}))).$$

Also, the contingent hypoderivative of f at $\overline{x}$ in the direction v, which is denoted as $D^{\downarrow} f(\overline{x})v$, defined by

$$w \in D^{\downarrow} f(\overline{x})v - \mathbb{R}^s_+ \Longleftrightarrow (v,w) \in T(\mathrm{hyp} f, (\overline{x}, f(\overline{x}))).$$

It is not difficult to verify that for each $v \in \mathbb{R}^n$, if there exists $D_{\uparrow} f(\overline{x})v$, then, one can achieve the results that $D_{\uparrow} f(\overline{x})v \in \partial f(\overline{x})v$ and $\partial f(\overline{x})v \subset D_{\uparrow} f(\overline{x})v + \mathbb{R}^s_+$. It follows from $D^{\downarrow} f(\overline{x})v = -D_{\uparrow}(-f)(\overline{x})v$ that $D^{\downarrow} f(\overline{x})v \in \partial f(\overline{x})v$ and $\partial f(\overline{x})v \subset D^{\downarrow} f(\overline{x})v - \mathbb{R}^s_+$. When $s = 1$, by [2] we have

$$D_{\uparrow} f(\overline{x})v = \inf\{w | w \in \partial(f+\mathbb{R}_+)(\overline{x}, f(\overline{x}))(v)\} = \liminf_{t\to 0^+,\, w\to v} \frac{f(\overline{x}+tw)-f(\overline{x})}{t},$$

$$D^{\downarrow} f(\overline{x})v = \sup\{w | w \in \partial(f-\mathbb{R}_+)(\overline{x}, f(\overline{x}))(v)\} = \limsup_{t\to 0^+,\, w\to v} \frac{f(\overline{x}+tw)-f(\overline{x})}{t}.$$

Definition 2 ([12]) The Hadamard derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x} \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ is given by

$$df(\overline{x})v = \lim_{(t,u)\to(0^+,v)} \frac{f(\overline{x}+tu)-f(\overline{x})}{t}.$$

The mapping f is said to be Hadamard differentiable at $\overline{x}$ in the direction v iff, $df(\overline{x})v$ exists, and f is said to be Hadamard differentiable at $\overline{x}$ iff, $df(\overline{x})v$ exists for every $v \in \mathbb{R}^n$. If f is Fréchet differentiable at $\overline{x}$, its Fréchet derivative at $\overline{x}$ is denoted by $\nabla f(\overline{x})$.

Definition 3 ([12,20,34]) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\overline{x}, v \in \mathbb{R}^n$. Then

(i) f is said to be stable (or calm) at $\overline{x}$ iff, there exist a neighborhood U of $\overline{x}$ and a stable constant $L > 0$ such that $\|f(x) - f(\overline{x})\| \le L\|x - \overline{x}\| \quad \forall\, x \in U$.

(ii) f is said to be steady at $\overline{x}$ in the direction v, abbreviated f is steady at $(\overline{x}, v)$, iff

$$\lim_{(t,u)\to(0^+,v)} \frac{f(\overline{x}+tu)-f(\overline{x}+tv)}{t} = 0.$$

It is said that f is steady at $\overline{x}$ iff, f is steady at $\overline{x}$ in all the directions $v \in \mathbb{R}^n$.

It is evident that f is steady at $(\overline{x}, 0) \Longleftrightarrow f$ is stable at $\overline{x}$. Especially, if f is stable at $\overline{x}$, then $D_{\uparrow} f(\overline{x})0 = 0$. We remark that the above notions are different and moreover the class of steady functions is wider than the class of Hadamard differentiable functions.

Given a vector function $g : \mathbb{R}^n \to \mathbb{R}^q$. We define a mapping $(f,g) : \mathbb{R}^n \to \mathbb{R}^s \times \mathbb{R}^q$ by $(f,g)(x) = (f(x), g(x))$ for every $x \in \mathbb{R}^n$, and then, a norm in $\mathbb{R}^{s+q}$ is taken as $\|(x,y)\| =$

$\|x\|+\|y\|$ for all $(x, y) \in \mathbb{R}^{s+q}$. According to Proposition 3.3 (i& ii) in [12], (f, g) is stable (resp. steady) at $\overline{x} \Longleftrightarrow f$ and g are stable (resp. steady) at that point.

We recall the following definitions, which play a crucial role for our next study.

Definition 4 ([2]) Let an extended-real-valued function f defined on $\mathbb{R}^n$ and let $\overline{x}, v \in \mathbb{R}^n$. Suppose that f is stable at $\overline{x}$. Then

(i) The generalized subdifferential of f at $\overline{x}$ in the case of Aubin and Frankowska [2] is

$$\partial_0 f(\overline{x}) := \Big\{p \in \mathbb{R}^n \,|\, \forall\, v \in \mathbb{R}^n,\ \langle p, v\rangle \le D_\uparrow f(\overline{x})(v)\Big\}.$$

For each $p \in \partial_0 f(\overline{x})$ is called the subgradient and the generalized subdifferential associating to the contingent epiderivative is called **the Aubin-Frankowska's generalized subdifferential**.

(ii) If, in addition, f is Lipschitz around a point $\overline{x} \in X$, then the generalized subgradient of f at $\overline{x}$ in the case of Clarke [6] (or the Clarke's generalized subgradient) is

$$\partial_C f(\overline{x}) := \Big\{p \in \mathbb{R}^n \,|\, \forall\, v \in \mathbb{R}^n,\ \langle p, v\rangle \le C_\uparrow f(\overline{x})(v)\Big\},$$

where

$$C_\uparrow f(\overline{x})(v) := \limsup_{t\to 0^+,\, x\to\overline{x}} \frac{f(x+tv)-f(x)}{t}.$$

Naturally, we have the following inclusion

$$\partial_0 f(\overline{x}) \subset \partial_C f(\overline{x}). \tag{1}$$

In the case f is convex, it follows from the well-known result in Ref. [2] that the contingent epiderivative and the Clarke's directional derivative coincide and are equal to

$$D_\uparrow f(\overline{x})(v) := \liminf_{u\to v} \Big(\inf_{t>0} \frac{f(\overline{x}+tu)-f(\overline{x})}{t}\Big).$$

Therefore, the subdifferential of f at $\overline{x}$ in the sense of convex analysis

$$\partial_{Co} f(\overline{x}) = \partial_0 f(\overline{x}) = \partial_C f(\overline{x}). \tag{2}$$

Let us consider the vector bifunction $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ such that $F(x, x) = 0$ for every $x \in \mathbb{R}^n$, and consider the constraint mappings $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^r$. Then, it can be written as $F = (F_1, \dots, F_p)$, $g = (g_1, \dots, g_m)$ and $h = (h_1, \dots, h_r)$, where F_i is an extended-real-valued bifunction defined on $\mathbb{R}^n \times \mathbb{R}^n$ for every $i \in I := \{1, \dots, p\}$, g_j, h_k are extended-real-valued functions defined on $\mathbb{R}^n$ for all $j \in J := \{1, \dots, m\}$ and $k \in R := \{1, \dots, r\}$. Let us call $J(\overline{x}) := \{j \in J \,|\, g_j(\overline{x}) = 0\}$ (the set of active constraints). Set $H := h^{-1}(0) = \big\{x \in \mathbb{R}^n \,|\, h(x) = 0\big\}$ and $K(H) := \big\{x \in \mathbb{R}^n \,|\, \langle \nabla h(\overline{x}),\ x\rangle = 0\big\}$. For each $\overline{x} \in \mathbb{R}^n$, one writes $F_{\overline{x}} : \mathbb{R}^n \to \mathbb{R}^p$ in which $F_{\overline{x}}(x) = F(\overline{x}, x)$ for any $x \in \mathbb{R}^n$. Then, one can reach that for all $i \in I$, $F_{i,\overline{x}} = F_i(\overline{x}, .)$ and furthermore $F_{i,\overline{x}}(\overline{x}) = 0$. For shortly, taking $F_{\overline{x}}(K) := \bigcup\{F_{\overline{x}}(x) \,|\, x \in K\}$ and $M_F := \prod_{i\in I} D_\uparrow F_{i,\overline{x}}(\overline{x})(C-\overline{x}) \times \prod_{j\in J(\overline{x})} D_\uparrow g_j(\overline{x})(C-\overline{x})$.

In the present paper, we shall be concerned with the following constrained vector equilibrium problem (in short, (CVEP)): finding $\overline{x} \in K$ such that

$$F_{\overline{x}}(x) \notin -\mathrm{int}\mathbb{R}^p_+ \quad \forall\, x \in K, \tag{3}$$

where $K := \{x \in C \,|\, g(x) \leqq 0,\ h(x) = 0\}$ denotes the feasible set of problem (CVEP). A vector $\overline{x}$ solves (3) is said to be a weakly efficient solution for the problem (CVEP). If there exists a neighborhood U of $\overline{x}$ such that (3) holds for all $x \in K \cap U$, we say that $\overline{x}$ is a local weakly efficient solution for the problem (CVEP). A vector $\overline{x} \in K$ is a local weakly efficient solution for the problem (CVEP) iff there exists a real number $\delta > 0$ such that there is no $x_0 \in K \cap B(\overline{x}; \delta)$ with $F_{\overline{x}}(x_0) < 0$. We recall (see [27]) that a cone Q has a convex base B iff, $0 \notin \text{cl}Q$ and $Q = \text{cone}(B)$. Especially, if a cone Q has a convex base B, then $Q^\sharp \neq \emptyset$. Let B is a bounded closed convex base of $Q := \mathbb{R}^p_+$, we set

$$Q^\Delta(B) := \{\xi \in Q^\sharp \,|\, \exists\, t > 0 \text{ such that } \langle \xi, b\rangle \geq t \ \ (\forall\, b \in B)\}.$$

We always have

$$Q^\Delta(B) = \text{int}\mathbb{R}^p_+ = \mathbb{R}^p_{++}.$$

Additionally, $0 \notin \text{cl}\, B$. By invoking the standard separation theorem of two disjoint convex sets $\{0\}$ and $cl\, B$ in Rockafellar [35] yields the existence of $\xi \in \mathbb{R}^p \setminus \{0\}$ such that

$$r = \inf\Big\{ \langle \xi, b\rangle \ |\, b \in B\Big\} > \langle \xi, 0\rangle = 0.$$

We define an open absolutely convex neighborhood of the origin in $\mathbb{R}^p$ by

$$V_B := \Big\{y \in \mathbb{R}^p \,|\, |\langle \xi, y\rangle| < \frac{r}{2}\Big\},$$

where the notion V_B will be used throughout this article. According to Luc [27], for any convex neighborhood U of the origin in $\mathbb{R}^p$ such that $U \subset V_B$, the cone hull cone$(U + B)$ is pointed and convex satisfying $\mathbb{R}^p_+ \setminus \{0\} \subset \text{int}\,\text{cone}(U + B)$.

Definition 5 ([17,18,26]) The vector $\overline{x} \in K$ is said to be

(i) a globally efficient solution of problem (CVEP) iff there exists a pointed and convex cone $H \subset \mathbb{R}^p$ with $\mathbb{R}^p_+ \setminus \{0\} \subset \text{int}\, H$ such that $F_{\overline{x}}(K) \cap (-H \setminus \{0\}) = \emptyset$.

(ii) a Henig efficient solution of problem (CVEP) iff there exists some absolutely convex neighborhood U of zero with $U \subset V_B$ such that cone $F_{\overline{x}}(K) \cap (-\text{int}\,\text{cone}(U + B)) = \emptyset$.

(iii) a superefficient solution of problem (CVEP) iff for each neighborhood V of zero in $\mathbb{R}^p$, there exists some neighborhood U of the origin such that cone $F_{\overline{x}}(K) \cap (U - \mathbb{R}^p_+) \subset V$.

In previous Definitions, replacing K by $K \cap B(\overline{x}, \delta)$ for some $\delta > 0$, we obtain the notations of local globally efficient, local Henig efficient and local superefficient solutions for problem (CVEP), respectively. Finally, we recall some definitions on pseudoconvexity, quasiconvexity and quasilinearly, which the reader can be found in Refs. [2,19].

Definition 6 ([2]) An extended-real-valued function f defined on $\mathbb{R}^n$ is called pseudoconvex at $\overline{x}$ if its epigraph is pseudoconvex at $(\overline{x}, f(\overline{x}))$, i.e., iff: for every $x \in \mathbb{R}^n$,

$$D_\uparrow f(\overline{x})(x - \overline{x}) \leq f(x) - f(\overline{x}).$$

Definition 7 ([19]) An extended-real-valued function f defined on a convex subset C is said to be quasiconvex at $\overline{x} \in C$ with respect to C iff: for each $x \in C$,

$$f(x) \leq f(\overline{x}) \Longrightarrow \forall\, t \in]0, 1[,\ \ f(tx + (1 - t)\overline{x}) \leq f(\overline{x}).$$

It is said that f is quasiconvex on C iff, it is quasiconvex at each $\overline{x} \in C$. It is said that f is quasilinear at $\overline{x} \in C$ with respect to C iff, $\pm f$ are quasiconvex at $\overline{x}$ with respect to C.

Finally, we make an assumption which is needful to treat KKT-type necessary optimality conditions for weak efficiency to such a problem.

Assumption 1 Let a feasible solution $\overline{x}$. For all $i \in I$ and $j \in J(\overline{x})$, the real-valued functions $F_{i,\overline{x}}$ and g_j are steady at $\overline{x}$ and at least one of the functions $F_{\overline{x}}$ and g is Hadamard differentiable at $\overline{x}$; for all $j \in J \setminus J(\overline{x})$, the real-valued function g_j is continuous at $\overline{x}$; the mapping h is Fréchet differentiable at $\overline{x}$ and continuous on a neighborhood of $\overline{x}$; the regularity condition (RC2) holds, where

$$0 \in \sum_{k \in T} \gamma_k \nabla h_k(\overline{x}) + N(C, \overline{x}) \Longrightarrow \gamma = 0. \tag{RC2}$$

Under this assumption, it can be seen that

$$K(H) \cap ri(C - \overline{x}) \neq \emptyset.$$

§3 KKT-type necessary optimality conditions for efficiency

In this section, we establish KKT-type necessary optimality conditions in terms of the generalized subdifferentials with steady functions for the local efficient solutions of problem (CVEP). To begin with, we provide some the following regularity conditions.

Definition 8 Let $\overline{x} \in K$ and an index $s \in I$. It is said that

(i) the regularity condition of the (RC1) type holds (at $\overline{x}$) iff:

$$\Big\{v \in \mathbb{R}^n : D_{\uparrow} g_j(\overline{x})v < 0 \, (\forall\, j \in J(\overline{x})), \, \langle \nabla h_k(\overline{x}, v) \rangle = 0 \, (\forall\, k \in R)\Big\} \bigcap C_{\overline{x}} \neq \emptyset. \tag{RC1}$$

(ii) the regularity condition of the (RC3-s) type at $\overline{x}$ is fulfilled iff: there exists a direction $v_0 \in C - \overline{x}$ such that

$$\begin{cases} D_{\uparrow} F_{i,\overline{x}}(\overline{x}) v_0 < 0, \quad \forall\, i \in I \setminus \{s\}, \\ D_{\uparrow} g_j(\overline{x}) v_0 < 0, \quad \forall\, j \in J(\overline{x}), \\ \langle \nabla h_k(\overline{x}), v_0 \rangle = 0, \quad \forall\, k \in R. \end{cases} \tag{RC3-s}$$

Next, we state the extension of classical Ljusternik theorem, which can be found in Ref. [13].

Proposition 1 Let $\overline{x} \in C \cap H$. Suppose that

(i) C is a convex set;

(ii) h is continuous in a neighborhood of $\overline{x}$ and Fréchet differentiable at $\overline{x}$;

(iii) the regularity condition of the (RC2) type is valid. Then

$$A(H \cap C, \overline{x}) = T(H \cap C, \overline{x}) = K(H) \cap T(C, \overline{x}) = \mathrm{cl}\Big[K(H) \cap \mathrm{cone}(C - \overline{x})\Big].$$

Theorem 1 (*Weak KKT-type necessary optimality*) Assume that all the hypotheses of Assumption 1 are fulfilled and C is convex. Suppose also that $\overline{x} \in K$ is a local weakly efficient solution for the problem (CVEP) and the regularity condition of the (RC1) type at $\overline{x}$ is fulfilled. Then, there exist $(\lambda, \mu) \in (\mathbb{R}^p_+ \setminus \{0\}) \times \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}), \tag{4}$$

$$\mu_j g_j(\overline{x}) = 0, \quad \forall\, j \in J. \tag{5}$$

Proof. By reasons of similarity, we prove only for the hypotheses that g is Hadamard differential at $\overline{x}$. For this hypotheses, it should be pointed out that if $\overline{x} \in K$ is a local weakly efficient solution of problem (CVEP), then the following system is impossible

$$D_{\uparrow} F_{i,\overline{x}}(\overline{x})v < 0, \quad \forall\, i \in I, \tag{6}$$

$$D_{\uparrow} g_j(\overline{x})v < 0, \quad \forall\, j \in J(\overline{x}), \tag{7}$$

$$\langle \nabla h_k(\overline{x}), v \rangle = 0, \quad \forall\, k \in R, \tag{8}$$

$$v \in C - \overline{x}. \tag{9}$$

Arguing by contradiction, assume that the system from (6) to (9) has at least a solution $v_0 \in \mathbb{R}^n$. Since the set C is convex, it results in $C - \overline{x} \subset T(C, \overline{x})$, which combined with (9) guarantees that $v_0 \in T(C, \overline{x})$. It follows from the previous equalities (8) that $v_0 \in K(H)$, or equivalently, $v_0 \in K(H) \cap T(C, \overline{x})$. Under the regularity condition of the (RC2) type, which together with Proposition 1, it holds that

$$v_0 \in T(C \cap H, \overline{x}) = A(C \cap H, \overline{x}). \tag{10}$$

By invoking the concept of contingent epiderivative, for any $i \in I$, the contingent epiderivative $D_{\uparrow} F_{i,\overline{x}}(\overline{x})v_0$ can be described as a limit of differential quotients (see [2,36,37], for instance), which means that

$$D_{\uparrow} F_{i,\overline{x}}(\overline{x})v_0 = \liminf_{t \to 0^+,\ u \to v_0} \frac{F_{i,\overline{x}}(\overline{x} + tu) - F_{i,\overline{x}}(\overline{x})}{t}.$$

Therefore, for every $i \in I$, there exist sequences $t_n \to 0^+$ and $v'_n \to v_0$ such that

$$\lim_{n \to +\infty} \frac{F_{i,\overline{x}}(\overline{x} + t_n v'_n) - F_{i,\overline{x}}(\overline{x})}{t_n} = D_{\uparrow} F_{i,\overline{x}}(\overline{x})v_0 < 0. \tag{11}$$

Together (11) with the steadiness of $F_{i,\overline{x}}$ at $\overline{x}$, we obtain that

$$\lim_{n \to +\infty} \frac{F_{i,\overline{x}}(\overline{x} + t_n v''_n) - F_{i,\overline{x}}(\overline{x})}{t_n} = D_{\uparrow} F_{i,\overline{x}}(\overline{x})v_0 \tag{12}$$

holds for any sequence v''_n converges to v_0. For the preceding real numbers sequence $(t_n)_{n \geq 1}$, as $v_0 \in A(C \cap H, \overline{x})$ (see (10)), thus, there exists a sequence $v_n \to v_0$ such that

$$\overline{x} + t_n v_n \in H \cap C, \quad \forall\, n \geq 1. \tag{13}$$

From (12) and the fact $D_{\uparrow} F_{i,\overline{x}}(\overline{x})v_0 < 0$, it follows that there exists $N_1 > 0$ such that

$$F_{i,\overline{x}}(\overline{x} + t_n v_n) - F_{i,\overline{x}}(\overline{x}) < 0, \quad \forall\, n \geq N_1, \ \forall\, i \in I. \tag{14}$$

In other words, since g_j is Hadamard differentiable at $\overline{x}$ for all $j \in J(\overline{x})$, similarly to the preceding argument, it holds that

$$D_{\uparrow} g_j(\overline{x})v_0 := \liminf_{t \to 0^+,\ u \to v_0} \frac{g_j(\overline{x} + tu) - g_j(\overline{x})}{t} = dg_j(\overline{x})v_0, \quad \forall\, j \in J(\overline{x}).$$

Therefore for each $j \in J(\overline{x})$, it ensures that

$$D_{\uparrow} g_j(\overline{x})v_0 = \lim_{n \to +\infty} \frac{g_j(\overline{x} + t_n v_n) - g_j(\overline{x})}{t_n} < 0,$$

which allows us to conclude that there exists a natural number $N_2 \geq N_1$ such that

$$g_j(\overline{x} + t_n v_n) < 0, \quad \forall\, n \geq N_2, \ \ \forall\, j \in J(\overline{x}).$$

Taking into account the continuity of g_j ($j \in J \setminus J(\overline{x})$), there exists a natural number $N_3 \geq N_2$ such that for every $n \geq N_3$, one can achieve the strict inequality that

$$g_j(\overline{x} + t_n v_n) < 0, \ \ \forall\, j \in J \setminus J(\overline{x}).$$

Consequently,

$$g_j(\overline{x} + t_n v_n) < 0, \ \ \forall\, n \geq N_3, \ \ \ \forall\, j \in J. \tag{15}$$

Combining (13)-(15) yields that $\overline{x} + t_n v_n \in K$ for all $n \geq N_3$. This is a contradiction with inequality (14), means that $\overline{x}$ is not a local weakly efficient solution for the problem (CVEP). So, the system from (6) to (9) has no solution in $\mathbb{R}^n$. In view of the known result in Rockafellar [35], which can be used since $K(H) \cap ri(C - \overline{x}) \neq \emptyset$, one finds $(\lambda, \mu', \eta) \in \mathbb{R}^p \times \mathbb{R}^{|J(\overline{x})|} \times \mathbb{R}^r$ with $(\lambda, \mu') \neq (0, 0)$ such that $(\lambda, \mu') \geq 0$ and moreover,

$$\sum_{i=1}^{p} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x})v + \sum_{j \in J(\overline{x})} \mu'_j D_{\uparrow} g_j(\overline{x})v + \sum_{k=1}^{r} \eta_k \langle \nabla h_k(\overline{x}), v\rangle \geq 0, \quad \forall\, v \in C - \overline{x}.$$

Under the regularity condition of the (RC1) type, $\lambda \neq 0$. Indeed, if it was not so, then $\lambda = 0$ and so $\mu' \neq 0$. Take $\hat{v} \in K(H)$ such that $\hat{v} \in C_{\overline{x}}$ and $D_{\uparrow} g_j(\overline{x})\hat{v} < 0 \ \forall\, j \in J(\overline{x})$. If some $\mu'_j > 0$, then we have that $\sum_{j=1}^{m} \mu'_j D_{\uparrow} g_j(\overline{x})\hat{v} + \sum_{k=1}^{r} \eta_k \langle \nabla h_k(\overline{x}), \hat{v}\rangle < 0$, and this is a contradiction. We set $\mu = (\mu_j)_{j \in J}$, where

$$\mu_j = \begin{cases} \mu'_j, & \text{if } j \in J(\overline{x}), \\ 0, & \text{otherwise.} \end{cases}$$

Evidently, $(\lambda, \mu) \geq 0$ with $\lambda \neq 0$ and $\eta \in \mathbb{R}^r$ satisfying (5) and the following inequality

$$\sum_{i=1}^{p} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x})v + \sum_{j=1}^{m} \mu_j D_{\uparrow} g_j(\overline{x})v + \sum_{k=1}^{r} \eta_k \langle \nabla h_k(\overline{x}), v\rangle \geq 0 \quad \forall v \in C_{\overline{x}}. \tag{16}$$

Thus, $0 \in C_{\overline{x}}$ is a minimum on the convex set $C_{\overline{x}}$ of the convex function

$$\varphi(v) = \sum_{i=1}^{p} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x})v + \sum_{j=1}^{m} \mu_j D_{\uparrow} g_j(\overline{x})v + \sum_{k=1}^{r} \eta_k \langle \nabla h_k(\overline{x}), v\rangle.$$

Consequently,

$$\begin{aligned} 0 &\in \partial_{Co}\varphi(0) + N(C, \overline{x}) \\ &= \sum_{i=1}^{p} \lambda_i \partial_{Co} D_{\uparrow} F_{i,\overline{x}}(\overline{x})(.)(0) + \sum_{j=1}^{m} \mu_j \partial_{Co} D_{\uparrow} g_j(\overline{x})(.)(0) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}), \end{aligned}$$

which is equivalent to the relation (4), as it was shown. □

Remark 1 It is worth mentioning that if $f : \mathbb{R}^n \to \mathbb{R}$ is stable at $\overline{x}$, then $\partial_{\uparrow} f(\overline{x})0 = 0$. Thus the associate subdifferential to the contingent epiderivative of f at $\overline{x}$ is given by

$$\partial_0 f(\overline{x}) = \partial_{Co} D_{\uparrow} f(\overline{x})(.)(0),$$

and it is contained, by (6.7) in Ref. [2], in the Clarke subdifferential $\partial_C f(\overline{x})$.

Theorem 2 (*Weak KKT-type necessary optimality*) Under all the hypotheses of Theorem 1, we have the following statements:

(i) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $(\lambda, \mu) \in (\mathbb{R}^p_+ \setminus \{0\}) \times \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$

such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_C F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_C g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}), \tag{17}$$

$$\mu_j g_j(\overline{x}) = 0, \quad \forall\, j \in J; \tag{18}$$

(ii) If $F_{\overline{x}}$, g are convex on C, then there exist $(\lambda, \mu) \in (\mathbb{R}^p_+ \setminus \{0\}) \times \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_{C_0} F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_{C_0} g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}), \tag{19}$$

$$\mu_j g_j(\overline{x}) = 0, \quad \forall\, j \in J. \tag{20}$$

Proof. (i) Using the Lipschitzness of $F_{\overline{x}}$, g around $\overline{x}$, which ensures the Lipschitzness of $F_{i,\overline{x}}$, g_j $(i \in I,\ j \in J)$ around $\overline{x}$. This along with the inclusion (1) and Theorem 1 guarantees that there exist $(\lambda, \mu) \in (\mathbb{R}^p_+ \setminus \{0\}) \times \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that (17) and (18) are fulfilled.

(ii) Analogously to the proof of case (i), by applying the equality (2) guarantees that there exist $(\lambda, \mu) \in (\mathbb{R}^p_+ \setminus \{0\}) \times \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that (19) and (20) are satisfied too, which completes the proof. □

The following results can be viewed as a directly consequence from Theorems 1 and 2.

Corollary 1 Suppose that $C = \mathbb{R}^n$, under the hypotheses of Theorem 1 in which the regularity condition of the (RC2) type is replaced by the linear independence of the system $\{\nabla h_1(\overline{x}), \nabla h_2(\overline{x}), \ldots, \nabla h_r(\overline{x})\}$. Suppose also that $\overline{x} \in K$ is a local weakly efficient solution for the problem (CVEP). We have the following statements:

(i) There exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ with $(\lambda, \eta) \neq (0, 0)$ satisfying (4)-(5);

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ with $(\lambda, \eta) \neq (0, 0)$ satisfying (17)-(18);

(iii) If $F_{\overline{x}}$, g are convex on C, then there exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ with $(\lambda, \eta) \neq (0, 0)$ satisfying (19)-(20).

Additionally, $\lambda \neq 0$ under the regularity condition of the (RC1) type.

Proof. First, we need to prove that the regularity condition of the (RC2) type at $\overline{x}$ is valid. Indeed, if $0 \in \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x})$, then one can achieve from the initial hypotheses $C = \mathbb{R}^n$ the right equality $T(C, \overline{x}) = \operatorname{cl}\operatorname{cone}(C - \overline{x}) = \operatorname{cl}\operatorname{cone}(C - \overline{x}) = C$, which leads to $N(C, \overline{x}) = \{0\}$. Consequently, $\sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) = 0$. Second, from the linear independence of the system $\{\nabla h_1(\overline{x}), \ldots, \nabla h_r(\overline{x})\}$, it yields that $\eta_1 = \ldots = \eta_r = 0$, which means that the regularity condition (RC2) at $\overline{x}$ is satisfied. Finally, by taking into account Theorem 1 and Theorem 2, we get the desired conclusion. □

Corollary 2 Assume that all the hypotheses of Assumption 1 are fulfilled and the set C is convex. Suppose also that $\overline{x} \in K$ is a local weakly efficient solution of problem (CVEP). Then, we have the following assertions:

(i) There exist $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+$, not all zero, satisfying (5) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_0 g_j(\overline{x}) + N(C \cap H, \overline{x}); \tag{21}$$

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+$, not all zero, satisfying (5) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_C F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_C g_j(\overline{x}) + N(C \cap H, \overline{x}); \tag{22}$$

(iii) If $F_{\overline{x}}$, g are convex on C, then there exist $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+$, not all zero, satisfying (5) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_{C_0} F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_{C_0} g_j(\overline{x}) + N(C \cap H, \overline{x}). \tag{23}$$

Moreover, $\lambda \neq 0$ under the regularity condition of the (RC1) type.

Proof. Arguing similarity as for proving Theorem 1, we assert that there would exists a pair $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+$, not all zero, satisfying (5) and the following inequality

$$\sum_{i=1}^{p} \lambda_i D_\uparrow F_{i,\overline{x}}(\overline{x})v + \sum_{j=1}^{m} \mu_j D_\uparrow g_j(\overline{x})v \geq 0 \ \ \forall v \in K(H) \cap C_{\overline{x}}.$$

Since the single-valued mapping $L : \mathbb{R}^n \rightrightarrows \mathbb{R}$ given by

$$v \mapsto D_\uparrow F_{i,\overline{x}}(\overline{x})v$$

is positively homogeneous ($i \in I$), which guarantees that

$$\sum_{i=1}^{p} \lambda_i D_\uparrow F_{i,\overline{x}}(\overline{x})v + \sum_{j=1}^{m} \mu_j D_\uparrow g_j(\overline{x})v \geq 0 \ \ \forall v \in K(H) \cap T(C, \overline{x}). \tag{24}$$

Thanks to Proposition 1, it follows from (24) that

$$\sum_{i=1}^{p} \lambda_i D_\uparrow F_{i,\overline{x}}(\overline{x})v + \sum_{j=1}^{m} \mu_j D_\uparrow g_j(\overline{x})v \geq 0 \ \ \forall v \in T(C \cap H, \overline{x}). \tag{25}$$

By hypotheses, the cone $T(C \cap H, \overline{x})$ is convex, then, in a similar idea to the proof of Theorem 1, one can achieve from the inequality (25) the result (21). We observe that $\lambda \neq 0$ is due to the regularity condition of the (RC1) type at $\overline{x}$ is satisfied. For the remain cases, by directly applying the results obtained in Theorem 2, which combined with the previous similar arguments, we get the desired conclusion. □

Remark 2 Evidently, if $\overline{x} \in K$ is a weakly efficient solution of (CVEP), then it is also a local weakly efficient solution of that problem. Thus all the preceding obtained results are still true in the sense of a local weakly efficient solution is replaced by a weakly efficient solution.

Example 1 Let us consider the problem (CVEP) in which $n = 3$, $p = 3$, $m = 2$, $r = 2$, $C = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-1, 1]$ and $\overline{x} = (0, 0, 0)$. For the illustration, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let us may take the mapping $F_{\overline{x}}(.) := (F_{1,\overline{x}}(.), F_{2,\overline{x}}(.), F_{3,\overline{x}}(.)) : \mathbb{R}^3 \to \mathbb{R}^3$, be given by

$$F_{1,\overline{x}}(x) = dist(x_1, A) + dist(x_2, B) + x_3,$$
$$F_{2,\overline{x}}(x) = x_1 + x_2^2 + x_3^3, \quad F_{3,\overline{x}}(x) = x_1^2 + x_2 - x_3^2,$$

where $A := \Big\{0, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\Big\}$ and $B := \Big\{0\Big\} \bigcup \Big[\frac{7}{16}, \frac{5}{8}\Big] \bigcup \Big[\frac{7}{32}, \frac{5}{16}\Big] \bigcup \dots$. Consider the mapping $g := (g_1, g_2) : \mathbb{R}^3 \to \mathbb{R}^2$ is given for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$g_1(x) = x_1^2 + x_2^2 + x_3^2 - x_3; \quad g_2(x) = x_1 + x_2 + x_3 - 0.5,$$

and the mapping $h := (h_1, h_2) : \mathbb{R}^3 \to \mathbb{R}^2$ given for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$h_1(x) = x_1 - 2x_2 - x_3, \quad h_2(x) = 2x_1 - 4x_2 + x_3.$$

It can be easily seen that $F_{1,\overline{x}}, F_{2,\overline{x}}$ and $F_{3,\overline{x}}$ are steady at $\overline{x}$, g_1 is Hadamard differentiable at $\overline{x}$, g_2 is continuous at $\overline{x}$, h_1, h_2 are Fréchet differentiable at $\overline{x}$ with $\nabla h_1(\overline{x}) = (1, -2, -1)$ and $\nabla h_2(\overline{x}) = (1, -4, 1)$, and further the set M_F is convex. By directly calculating, one can achieve the result that $I = \{1, 2, 3\}$, $J = \{1, 2\}$, $R = \{1, 2\}$, $J(\overline{x}) = \{1\}$, $J \setminus J(\overline{x}) = \{2\}$, $K(H) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 \,|\, u_1 = 3u_2,\ u_2 = u_3\}$ and $T(C, \overline{x}) = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ which guarantees that $N(C, \overline{x}) = \mathbb{R}_- \times \mathbb{R}_- \times \{0\}$. Thus, the regularity condition (RC2) is valid and the set C is convex, which proves all the hypotheses of Assumption 1. We easy to verify that the feasible set of (CVEP) is of the form $K = \big\{x = (x_1, x_2, x_3) \in C \,|\, x_1^2 + x_2^2 + x_3^2 \le x_3,\ \ x_1 + x_2 + x_3 \le \frac{1}{2}\big\}$. Thus, $\overline{x} = (0, 0, 0)$ is a local weakly efficient solution of problem (CVEP). Furthermore, for all $u = (u_1, u_2, u_3) \in C_{\overline{x}}$ it results $D_\uparrow F_{1,\overline{x}}(\overline{x})u = u_3$, $D_\uparrow F_{2,\overline{x}}(\overline{x})u = u_1$, $D_\uparrow F_{3,\overline{x}}(\overline{x})u = u_2$, $D_\uparrow g_1(\overline{x})u = -u_3$, $D_\uparrow g_2(\overline{x})u = u_1 + u_2 + u_3$. Therefore, $\partial_0 F_{1,\overline{x}}(\overline{x}) = \{(0, 0, 1)\}$, $\partial_0 F_{2,\overline{x}}(\overline{x}) = \{(1, 0, 0)\}$, $\partial_0 F_{3,\overline{x}}(\overline{x}) = \{(0, 1, 0)\}$, $\partial_0 g_1(\overline{x}) = \{(0, 0, -1)\}$ and moreover $\partial_0 g_2(\overline{x}) = \{(1, 1, 1)\}$. We verify that the regularity condition of the (RC1) type at $\overline{x}$ holds. Then a weak KKT-type necessary optimality condition is described by (4)-(5) holds with $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 7$, $\mu_1 = 1$, $\mu_2 = 0$, $\eta_1 = 1$ and $\eta_2 = 1$. In fact, in this setting, one can reach that $(\lambda, \mu) \ge 0$, $\lambda \ne 0$, $\mu_1 g_1(\overline{x}) = 0$, $\mu_2 g_2(\overline{x}) = 0$ and moreover

$$0 \in \sum_{i=1}^{3} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{2} \mu_j \partial_0 g_j(\overline{x}) v + \sum_{k=1}^{2} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}) = (3, 1, 0) + \mathbb{R}_-^2 \times \{0\}.$$

It should be to mention that $dist(x, M) := \inf_{y \in M} \|x - y\|$, which proves that $F_{1,\overline{x}}(x) \ge x_3$ for all $x = (x_1, x_2, x_3) \in C$, which terminates the check.

Theorem 3 (*KKT-type necessary optimality*) Assume that all the hypotheses of Assumption 1 are fulfilled, C is convex and the regularity condition of the (RC3-s) type at $\overline{x}$ holds (for some $s \in I$). Suppose also that $\overline{x}$ is a local weakly efficient solution for the problem (CVEP). Then

(i) There exist $(\lambda, \mu) \in \mathbb{R}_+^p \times \mathbb{R}_+^m$ with $\lambda_s > 0$ and $\eta \in \mathbb{R}^r$ such that the conditions (4)-(5) are valid;

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $(\lambda, \mu) \in \mathbb{R}_+^p \times \mathbb{R}_+^m$ with $\lambda_s > 0$ and $\eta \in \mathbb{R}^r$ such that the conditions (17)-(18) are satisfied;

(iii) If $F_{\overline{x}}$, g are convex on the convex set C, then there exist $(\lambda, \mu) \in \mathbb{R}_+^p \times \mathbb{R}_+^m$ with $\lambda_s > 0$ and $\eta \in \mathbb{R}^r$ such that the conditions (19)-(20) are fulfilled.

Proof. (i) By invoking the proof of Theorem 1, there would exist $(\lambda, \mu) \in \mathbb{R}_+^p \times \mathbb{R}_+^m \setminus \{(0, 0)\}$ and $\eta \in \mathbb{R}^r$ such that conditions (5) and (16) are satisfied. Under the regularity condition (RC3-s), where the index $s \in I$, one can find the direction $v_0 \in C - \overline{x}$ such that

$$D_{\uparrow} F_{i,\overline{x}}(\overline{x}) v_0 < 0, \quad \forall\, i \in I \setminus \{s\},$$
$$D_{\uparrow} g_j(\overline{x}) v_0 < 0, \quad \forall\, j \in J(\overline{x}),$$
$$\langle \nabla h_k(\overline{x}), v_0 \rangle = 0, \quad \forall\, k \in R.$$

Let us see that $\lambda_s > 0$. In fact, if it was not so, then $\lambda_s = 0$. It is evident that

$$\sum_{i \in I \setminus \{s\}} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x}) v_0 + \sum_{j \in J(\overline{x})} \mu_j D_{\uparrow} g_j(\overline{x}) v_0 + \sum_{k \in R} \eta_k \langle \nabla h_k(\overline{x}), v_0 \rangle < 0. \tag{26}$$

Using the inequality (16) for the case $v = v_0 \in C - \overline{x}$, it yields

$$\sum_{i \in I} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x}) v_0 + \sum_{j \in J(\overline{x})} \mu_j D_{\uparrow} g_j(\overline{x}) v_0 + \sum_{k \in R} \eta_k \langle \nabla h_k(\overline{x}), v_0 \rangle \geq 0,$$

or equivalently,

$$\sum_{i \in I \setminus \{s\}} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x}) v_0 + \sum_{j \in J(\overline{x})} \mu_j D_{\uparrow} g_j(\overline{x}) v_0 + \sum_{k \in R} \eta_k \langle \nabla h_k(\overline{x}), v_0 \rangle \geq 0,$$

which conflicts with inequality (26). Hence, $\lambda_s > 0$, and then, by directly applying the achieved result in Theorem 1, it proves the conclusion of case (i).

(ii) & (iii): Analogously to the proof of case (i) with observing Theorem 2, we also get the desired conclusion. □

In order to derive the strong KKT-type necessary optimality conditions for weak efficiency of problem (CVEP) in which Lagrange multipliers are positive with respect to all the components of the objective, observe that $\mathbb{R}^p_{++} := \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p | \lambda_i > 0, \ \forall\, i \in I\}$.

Theorem 4 (*Strong KKT-type necessary optimality*) Assume that all the hypotheses of Assumption 1 are fulfilled, C is convex and the regularity condition of the (RC3-s) type at $\overline{x}$ holds for all $s \in I$. Suppose also that $\overline{x}$ is a local weakly efficient solution for the problem (CVEP). Then

(i) There exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_{++} \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that the conditions (4)-(5) are valid.

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_{++} \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that the conditions (17)-(18) are satisfied.

(iii) If $F_{\overline{x}}$, g are convex on the convex set C, then there exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_{++} \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that the conditions (19)-(20) are fulfilled.

Proof. (i) By an argument analogous to that used for the proof of Theorem 3, we deduce that for each $s \in I$, there exist a triple $(\lambda^{(s)}, \mu^{(s)}, \eta^{(s)}) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^r$ with $\lambda^{(s)}_s > 0$ such that conditions (27)-(28) hold, where

$$\sum_{i=1}^{p} \lambda_i^{(s)} D_{\uparrow} F_{i,\overline{x}}(\overline{x}) v + \sum_{j=1}^{m} \mu_j^{(s)} D_{\uparrow} g_j(\overline{x}) v + \sum_{k=1}^{r} \eta_k^{(s)} \nabla h_k(\overline{x}) v \geq 0, \ \ \forall v \in C_{\overline{x}}, \tag{27}$$

$$\mu_j^{(s)} g_j(\overline{x}) = 0, \quad \forall\, j \in J, \tag{28}$$

in which $\lambda^{(s)} := \big(\lambda_1^{(s)}, \ldots, \lambda_p^{(s)}\big)$, $\mu^{(s)} := \big(\mu_1^{(s)}, \ldots, \mu_m^{(s)}\big)$ and $\eta^{(s)} := \big(\eta_1^{(s)}, \ldots, \eta_r^{(s)}\big)$. By making use of the preceding achieved results, let us may pick $\lambda_i = \sum_{s=1}^{p} \lambda_i^{(s)} > 0$ $(\forall\, i \in I)$, $\mu_j =$

$\sum_{s=1}^{p} \mu_j^{(s)} \geq 0$ $(\forall j \in J(\overline{x}))$ and $\eta_k = \sum_{s=1}^{p} \eta_k^{(s)} \in \mathbb{R}$ $(\forall k \in R)$ and then take $s = 1, 2, \ldots, p$ in (27)-(28) and adding up both sides of the obtained inclusions, it follows that there exist $(\lambda, \mu, \eta) \in \mathbb{R}^p_{++} \times \mathbb{R}^m_+ \times \mathbb{R}^r$ such that conditions (5) and (16) are valid. Then, in a similar way to the proof of Theorem 1 once more, we have the desired conclusion.

(ii) & (iii): Repeating the proof of case (i), we also arrive at the desired conclusion. □

Theorem 5 (*Strong KKT-type necessary optimality for the rest solutions*) Assume that all the hypotheses of Assumption 1 are fulfilled, C is convex and the regularity condition of the (RC1) type at $\overline{x}$ holds. Suppose also that the convex set B is a closed and bounded base of $\mathbb{R}^p_+$ and $\overline{x} \in K$ is a local Henig efficient (resp. globally efficient, superefficient) solution for the problem (CVEP). Then, we have the following conclusions

(i) There exists $\lambda \in Q^{\Delta}(B)$ (resp. $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ satisfying

$$\mu_j g_j(\overline{x}) = 0 \ \ \forall j \in J, \tag{29}$$

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{30}$$

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exists $\lambda \in Q^{\Delta}(B)$ (resp. $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ satisfying (29) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_C F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_C g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{31}$$

(iii) If $F_{\overline{x}}$, g are convex on C, then there exists $\lambda \in Q^{\Delta}(B)$ (resp. $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ satisfying (29) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_{C_0} F_{i,\overline{x}}(\overline{x}) + \sum_{j=1}^{m} \mu_j \partial_{C_0} g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{32}$$

Proof. We note that if (30) holds, then it guarantees that (31) and (32) hold under suitable assumptions. Let us now assume that $\overline{x}$ be a local Henig efficient solution of problem (CVEP). Then there exist some absolutely open convex neighborhood U of the origin with $U \subset V_B$ and a real number $\delta > 0$ such that

$$\operatorname{cone}\Big(F_{\overline{x}}(K \cap B(\overline{x}, \delta))\Big) \cap \Big(- \operatorname{int} \operatorname{cone}(U + B)\Big) = \emptyset,$$

which is equivalent to

$$F_{\overline{x}}(K \cap B(\overline{x}, \delta)) \cap \Big(- \operatorname{int} \operatorname{cone}(U + B)\Big) = \emptyset.$$

We confirm that $\operatorname{int} \operatorname{cone}(U + B) + \operatorname{cone}(U + B) = \operatorname{int} \operatorname{cone}(U + B)$, and further,

$$\mathbb{R}^p_+ \setminus \{0\} \subset \operatorname{int} \operatorname{cone}(U + B).$$

Arguing similarly as for proving Theorem 1 with applying the cone hull $\operatorname{cone}(U + B)$ instead of the nonnegative orthant cone $\mathbb{R}^p_+$, there exist $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ satisfying (29) and (30). It follows from the reached result in Ref. [26] that $\lambda \in Q^{\Delta}(B)$.

Next, we assume that $\overline{x} \in K$ is a local globally efficient solution of problem (CVEP). Then there exist a pointed convex cone $H \subset \mathbb{R}^p$ and a real number $\delta > 0$ such that

$$F_{\overline{x}}(K \cap B(\overline{x}, \delta)) \cap (-H \setminus \{0\}) = \emptyset,$$

which ensures that $F_{\overline{x}}(K \cap B(\overline{x}, \delta)) \cap (-\operatorname{int} H) = \emptyset$, where

$$\mathbb{R}^p_+ \setminus \{0\} \subset \operatorname{int} H, \tag{33}$$

Analogously to the proof of Theorem 1 with applying a cone H stands for a cone $\mathbb{R}^p_+$, one can find a triple $(\lambda, \mu, \eta) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ with $\lambda \in H^+ \setminus \{0\}$ and $\mu \in \mathbb{R}^m_+$ satisfying the conditions (29) and (30). It must be proven that $\lambda \in [\mathbb{R}^p_+]^\sharp$, i.e., for every $x \in \mathbb{R}^p_+ \setminus \{0\}$, it implies that $\langle \lambda, x \rangle > 0$. Under (33) and $\lambda \neq 0$, an obviousness.

Finally, we consider $\overline{x}$ is a local superefficient solution of problem (CVEP). Since $\mathbb{R}^p_+$ has a bounded closed base B, taking into account Gong's result [17,18], it follows that $Q^\Delta(B) = \operatorname{int} \mathbb{R}^p_+$. Besides, $\overline{x}$ is a local superefficient solution of problem (CVEP) if and only if $\overline{x}$ is a local Henig efficient solution of that problem, and the claim follows. □

Adapting the obtained result in Theorem 5, the following statement of corollaries is inspired by Corollary 1 and Corollary 2, respectively.

Corollary 3 Assume that $C = \mathbb{R}^n$, B is a bounded and closed convex base of $\mathbb{R}^p_+$ and all the hypotheses of Theorem 5 are satisfied in which the regularity condition of the (RC2) type is replaced by the linear independence of the system $\{\nabla h_1(\overline{x}), \ldots, \nabla h_r(\overline{x})\}$. Suppose also that $\overline{x} \in K$ is a local Henig efficient (resp. globally efficient, superefficient) solution for the problem (CVEP). We have the following assertions.

(i) There exist $\lambda \in Q^\Delta(B)$ (resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that the conditions (29) and (30) are fulfilled.

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $\lambda \in Q^\Delta(B)$ (resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that the conditions (29) and (31) are valid.

(iii) If $F_{\overline{x}}$, g are convex on C, then there exist $\lambda \in Q^\Delta(B)$ (resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that the conditions (29) and (32) are satisfied.

Proof. In a same way to the proof of Corollary 1 with applying the known result of Theorem 5 stands for Theorem 1, we arrive at the desired conclusion. □

Corollary 4 Suppose that all the hypotheses of Assumption 1 are fulfilled, C is a convex set, B is a bounded and closed convex base of $\mathbb{R}^p_+$ and the regularity condition of the (RC1) type holds. Suppose also that $\overline{x} \in K$ is a local Henig efficient (resp. globally efficient, superefficient) solution for the problem (CVEP). We have the following assertions.

(i) There exist $\lambda \in Q^\Delta(B)$ (resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ such that the conditions (5) and (21) are fulfilled.

(ii) If $F_{\overline{x}}$, g are Lipschitz around $\overline{x}$, then there exist $\lambda \in Q^\Delta(B)$ (resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ such that the conditions (5) and (22) are valid.

(iii) If $F_{\overline{x}}$, g are convex on C, then there exist $\lambda \in Q^\Delta(B)$ (resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int} \mathbb{R}^p_+$), $\mu \in \mathbb{R}^m_+$ such that the conditions (5) and (23) are satisfied.

Proof. It is straightforward from Theorem 5, and the claim follows. □

Remark 3 We observe that the previous obtained results are still true for the case of the weakly efficient, Henig efficient, globally efficient and superefficient solutions of problem (CVEP) (see Remark 2 for instance). Many authors have established weak/strong KKT-type necessary conditions for locally Lipschitz vector optimization problems without equality constraints, e.g., in [13,15,31,32]. In such papers, the extended-real-valued functions $f_i, i = 1, 2, \ldots, p$ are locally Lipschitz while these function in our paper (say $F_{i,\overline{x}}$, $i = 1, 2, \ldots, p$) may not be locally Lipschitz, so their results are different from our results in this section. Furthermore, our results are more general than those available recently.

§4 KKT-type sufficient optimality conditions for efficiency

In this section, we goal to provide Karush-Kuhn-Tucker type sufficient optimality conditions for the weakly efficient solution types of problem (CVEP) in terms of the generalized subdifferentials with stable functions at the point under consideration.

Theorem 6 (*KKT-type sufficient optimality*) Given a feasible solution $\overline{x}$ and assumming, in addition, that C is convex, the scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are stable at $\overline{x}$, and the mapping h is Fréchet differentiable at $\overline{x}$. Suppose also that

(i) There exist $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $\mu \in \mathbb{R}^{|J(\overline{x})|}_+$ and $\eta \in \mathbb{R}^r$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j \in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}); \tag{34}$$

(ii) The scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are pseudoconvex at $\overline{x}$;

(iii) The scalar functions $\pm h_k$ $(k \in R)$ are quasiconvex at $\overline{x}$ with respect to C.

Then, $\overline{x}$ is a weakly efficient solution of problem (CVEP).

Proof. Assume that all the assumptions of Theorem 6 are fulfilled. It follows from (34) that there exist $\xi_i \in \partial_0 F_{i,\overline{x}}(\overline{x}),\ i = 1, 2, \ldots, p$ and $\zeta_j \in \partial_0 g_j(\overline{x}),\ j \in J(\overline{x})$ such that

$$\sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j \in J(\overline{x})} \mu_j \zeta_j + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) \in -N(C, \overline{x}) = [T(C, \overline{x})]^+.$$

It follows the convexity of the set C that $C - \overline{x} \subset T(C, \overline{x})$. Thus, for any $x \in K$, one gets

$$x - \overline{x} \in T(C, \overline{x}),$$

which guarantees that the following inequality is true

$$\left\langle \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j \in J(\overline{x})} \mu_j \zeta_j + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}),\ x - \overline{x} \right\rangle \geq 0. \tag{35}$$

From the stability at $\overline{x}$ of $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$, it follows that the contingent epiderivative of such functions at $\overline{x}$ in the direction $x - \overline{x}$ $(\forall\, x \in K)$ always exists. Under the hypotheses (ii), they are pseudoconvex at $\overline{x}$. Thus, one can obtain for all $x \in K$ that

$$F_{i,\overline{x}}(x) - F_{i,\overline{x}}(\overline{x}) \geq D_{\uparrow} F_{i,\overline{x}}(\overline{x})(x - \overline{x}) \geq \langle \xi_i, x - \overline{x} \rangle \ \ \forall\, i \in I,$$

$$g_j(x) - g_j(\overline{x}) \geq D_{\uparrow} g_j(\overline{x})(\overline{x})(x - \overline{x}) \geq \langle \zeta_j, x - \overline{x} \rangle \ \ \forall\, j \in J(\overline{x}).$$

Under the hypotheses (iii), for any $k \in R$, it follows from the quasiconvexity of $\pm h_k$ at $\overline{x}$ with respect to C that h_k is quasilinear at $\overline{x}$ with respect to C. So, for every $x \in K$, it results that

$$\langle \nabla h_k(\overline{x}), x - \overline{x} \rangle = 0.$$

Since $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $\mu \in \mathbb{R}^{|J(\overline{x})|}_+$ and $\eta \in \mathbb{R}^r$, one can achieve for every $x \in K$ that

$$\begin{aligned}
\sum_{i=1}^{p} \lambda_i F_{i,\overline{x}}(x) &\geq \sum_{i=1}^{p} \lambda_i \Big(F_{i,\overline{x}}(x) - F_{i,\overline{x}}(\overline{x}) \Big) + \sum_{j \in J(\overline{x})} \mu_j \Big(g_j(x) - g_j(\overline{x}) \Big) + \sum_{k=1}^{r} \eta_k \Big(h_k(x) - h_k(\overline{x}) \Big) \\
&\geq \sum_{i=1}^{p} \lambda_i D_{\uparrow} F_{i,\overline{x}}(\overline{x})(x - \overline{x}) + \sum_{j \in J(\overline{x})} \mu_j D_{\uparrow} g_j(\overline{x})(x - \overline{x}) + \sum_{k=1}^{r} \eta_k \langle \nabla h_k(\overline{x}), x - \overline{x} \rangle \\
&\geq \left\langle \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j \in J(\overline{x})} \mu_j \zeta_j + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}),\ x - \overline{x} \right\rangle,
\end{aligned}$$

which combined with (35) yields that

$$\sum_{i=1}^{p} \lambda_i F_{i,\overline{x}}(x) \geq 0.$$

Since $\lambda \neq 0$, there exists $\hat{i} \in I$ such that $F_{\hat{i},\overline{x}}(x) \geq 0$, and so, $\overline{x}$ is a weakly efficient solution of (CVEP), which completes the proof. □

Corollary 5 Given a feasible solution $\overline{x} \in K$ and assumming, in addition, that the set C is convex, the scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are steady (or Hadamard differentiable) at $\overline{x}$, and the mapping h is Fréchet differentiable at $\overline{x}$. Suppose also that

(i) There exist $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$, $\mu \in \mathbb{R}^{|J(\overline{x})|}_+$ and $\eta \in \mathbb{R}^r$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j \in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x});$$

(ii) The scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are pseudoconvex at $\overline{x}$;

(iii) The scalar functions $\pm h_k$ $(k \in R)$ are quasiconvex at $\overline{x}$ with respect to C.

Then, $\overline{x}$ is a weakly efficient solution of problem (CVEP).

Proof. Since the class of stable functions is wider than the class of steady/or Hadamard differentiable functions, in view of Theorem 6, we get the desired conclusion. □

Example 2 Let $n = 1$, $p = 2$, $m = 1$, $r = 2$, $C = [\frac{1}{2}, \frac{3}{2}]$ and $\overline{x} = \frac{1}{2}$. Define the mapping $F_{\overline{x}} := \big(F_{1,\overline{x}},\ F_{2,\overline{x}}\big) : \mathbb{R} \to \mathbb{R}^2$ by $F_{1,\overline{x}}(x) = -|x| + x^2 + \frac{\overline{x}}{2}$ and $F_{2,\overline{x}}(x) = 2(x - |x|)$ for all $x \in \mathbb{R}$. For the illustration let us consider $g = g_1 : \mathbb{R} \to \mathbb{R}$ be defined by $g_1(x) = -\frac{3}{4}x + x^2 + x^3$, $\forall\, x \in \mathbb{R}$, and consider $h := (h_1, h_2) : \mathbb{R} \to \mathbb{R}^2$ be defined by $h_1(x) = \frac{1}{2} - x$ and $h_2(x) = -1 + 2x$ for all $x \in \mathbb{R}$. It is evident that the scalar functions $F_{1,\overline{x}}, F_{2,\overline{x}}$ and g_1 are stable even steady and Hadamard differentiable at $\overline{x}$, the scalar functions h_1, h_2 are quasilinear at $\overline{x}$ with respect to a convex set C, and moreover, these functions are Fréchet differentiable at $\overline{x}$ with its derivatives $\nabla h_1(\overline{x}) = -1$ and $\nabla h_2(\overline{x}) = 2$. Observe that $F_{\overline{x}}(\overline{x}) = (0, 0)$, $g(\overline{x}) = 0$, $I = \{1, 2\}$, $J = J(\overline{x}) = \{1\}$ and $R = \{1, 2\}$. Let $v \in \mathbb{R}$ be arbitrary, which proves that the relationship in (34) holds with $\lambda_1 = 1, \lambda_2 = 2$, $\mu = 3$, $\eta_1 = 3$ and $\eta_2 = 1$. In fact, by directly calculating, one can obtain the results as $T(C,\overline{x}) = \mathbb{R}_+$, $N(C,\overline{x}) = \mathbb{R}_-$, $D_\uparrow F_{1,\overline{x}}(\overline{x})v = 0$, $D_\uparrow F_{2,\overline{x}}(\overline{x})v = 0$, $D_\uparrow g_1(\overline{x})u = v$, $\langle \nabla h_1(\overline{x}), v\rangle = -v$ and $\langle \nabla h_1(\overline{x}), v\rangle = 2v$, which ensures that the generalized subdifferentials $\partial_0 F_{1,\overline{x}}(\overline{x}) = \partial_0 F_{1,\overline{x}}(\overline{x}) = \{0\}$ and $\partial_0 g_1(\overline{x}) = \{1\}$. Therefore, $0 \in \sum_{i=1}^2 \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^r \eta_k \nabla h_k(\overline{x}) + N(C,\overline{x}) =]-\infty, 2]$. With the help of Theorem 6 (or even Corollary 5), we conclude that $\overline{x}$ is a weakly efficient solution for the problem (CVEP), as it was checked.

Theorem 7 (*KKT-type sufficient optimality for the rest solutions*) Given a feasible solution $\overline{x} \in K$, B a convex base of Q and assumming, in addition, that the set C is convex, the scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are stable at $\overline{x}$, and the mapping h is Fréchet differentiable at $\overline{x}$. Suppose also that

(i) There exist $\lambda \in Q^\Delta(B)$ $\Big($resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+$ if, in addition, B is closed and bounded$\Big)$, $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that

$$0 \in \sum_{i=1}^p \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^r \eta_k \nabla h_k(\overline{x}) + N(C,\overline{x}); \tag{36}$$

(ii) The scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are pseudoconvex at $\overline{x}$;

(iii) The scalar functions $\pm h_k$ $(k \in R)$ are quasiconvex at $\overline{x}$ with respect to C.

Then, $\overline{x}$ is a Henig efficient (resp. globally efficient, superefficient) solution of problem (CVEP).

Proof. By an argument analogous to that used for the proof of Theorem 6 with observing that $\lambda \in Q^\Delta(B) \subset \mathbb{R}^p_+ \setminus \{0\}$, $\mu \in \mathbb{R}^{|J(\overline{x})|}_+$ and $\eta \in \mathbb{R}^r$, one can reach the following inequality

$$\langle \lambda, F_{\overline{x}}(x)\rangle := \sum_{i=1}^p \lambda_i F_{i,\overline{x}}(x) \geq 0, \ \ \forall\, x \in K. \tag{37}$$

Combining Gong's result [17] with $\lambda \in Q^\Delta(B)$ yields there exists some open absolutely convex neighborhood U of zero in $\mathbb{R}^p$ with $U \subset V_B$ such that $\lambda \in [\operatorname{cone}(U+B)]^+ \setminus \{0\}$. Thus, in view of the inequality (37) it guarantees that $F_{\overline{x}}(K) \cap \Big(-\operatorname{int}\operatorname{cone}(U+B)\Big) = \emptyset$, which is equivalent to $\operatorname{cone} F_{\overline{x}}(K) \cap \Big(-\operatorname{int}\operatorname{cone}(U+B)\Big) = \emptyset$. So, $\overline{x}$ is a Henig efficient solution of problem (CVEP). For the case $\lambda \in Q^\sharp$, $\mu \in \mathbb{R}^{|J(\overline{x})|}_+$ and $\eta \in \mathbb{R}^r$ satisfying (37). Taking a convex pointed cone $H \subset \mathbb{R}^p$ with $H := \{y \in \mathbb{R}^p : \langle \lambda, y\rangle > 0\} \cup \{0\}$ such that $\mathbb{R}^p_+ \setminus \{0\} \subset \operatorname{int} H$ holds true. Under inequality (37), one has $F_{\overline{x}}(K) \cap \Big(-H \setminus \{0\}\Big) = \emptyset$, which means that $\overline{x}$ is a globally efficient solution for the problem (CVEP).

Finally, let us assume that $\lambda \in \operatorname{int}\mathbb{R}^p_+$, $\mu \in \mathbb{R}^{|J(\overline{x})|}_+$ and $\eta \in \mathbb{R}^r$ satisfying (37). Because B is a bounded closed convex base of Q, taking into acount the result of Gong [18], $\operatorname{int}(\mathbb{R}^p_+) = Q^\Delta(B)$, and moreover, $\overline{x}$ is a superefficient solution of problem (CVEP) if and only if $\overline{x}$ is also a Henig efficient solution of that problem, which completes the proof. □

A direct consequence of Theorem 7, its easy proof can be omitted.

Corollary 6 Given a feasible solution $\overline{x} \in K$, B is a convex base of Q and assumming, in addition, that the set C is convex, the scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are steady (or Hadamard differentiable) at $\overline{x}$, the mapping h is Fréchet differentiable at $\overline{x}$. Suppose also that

(i) There exist $\lambda \in Q^{\Delta}(B)$ $\Big($resp. $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+$ if, in addition, B is closed and bounded$\Big)$, $\mu \in \mathbb{R}^m_+$ and $\eta \in \mathbb{R}^r$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 F_{i,\overline{x}}(\overline{x}) + \sum_{j \in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}); \tag{38}$$

(ii) The scalar functions $F_{i,\overline{x}}$ $(i \in I)$ and g_j $(j \in J(\overline{x}))$ are pseudoconvex at $\overline{x}$;

(iii) The scalar functions $\pm h_k$ $(k \in R)$ are quasiconvex at $\overline{x}$ with respect to C.

Then, $\overline{x}$ is a Henig efficient (resp. globally efficient, superefficient) solution of problem (CVEP).

We close this section by making some comparisons between the results obtained in the paper and the existing one in the literature.

Remark 4 As far as we know, there have not been results on KKT-type necessary/sufficient optimality conditions via the Aubin-Frankowska's generalized subdifferentials for the efficient solution types of a nonsmooth vector equilibrium problem with set, equality and inequality constraints (CVEP). In fact, in our paper, some weak/strong KKT-type necessary/and sufficient optimality conditions for these solutions to such problem in terms of the Aubin-Frankowska's generalized subdifferentials with the class of stable even steady and Hadamard differentiable functions are established in finite real spaces, while Rodríguez-Marín and Sama [36,37] studied the results on existence, uniqueness and some basic characterizations for the contingent epiderivatives with stable functions. The authors [36,37] obtained some applications of the results for mth-order local strict minimizers in general vector optimization problems; Jiménez-Novo-Sama [14] derived dual and primal necessary and sufficient optimality conditions for the local strict minimizers of vector optimization problem with set constraint using the different notions of graphical/ or epigraphical derivatives; Jiménez - Novo [13] gave KKT- type necessary optimality conditions for weak efficiency in nonsmooth vector optimization problem with constraints in terms of the Clarke subdifferentials with locally Lipschitz functions in finite real spaces; Constantin [10] presented first-order necessary conditions for local efficiency in terms of Clarke subdifferetials for multiobjective optimization problems with inequality, equality and an arbitrary set constraint; Luu [30-32] established KKT-type necessary conditions for vector equilibrium problems in terms of convexificators and Michel-Penot's subdifferentials; Jourani [15,16] provided Kuhn-Tucker-type optimality conditions for non-differentiable programming problems through the Lagrange multipliers.

§5 Applications to vector variational inequality and vector optimization problem with constraints

In this section, we derive some weak/strong KKT-type necessary optimality conditions and weak/strong KKT-type sufficient optimality conditions for efficiency of problems (CVOP) and (CVVI) via the generalized subdifferentials with stable functions. We always assume that B is a bounded closed convex base of $Q := \mathbb{R}^p_+$.

Consider the vector-valued mapping $T : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^p)$, is defined by

$$Tx = ((Tx)_1, \ldots, (Tx)_p) \quad \forall\, x \in \mathbb{R}^n,$$

where $(Tx)_1, \ldots, (Tx)_p$ are real-valued bounded linear mappings defined on $\mathbb{R}^n$, and $L(\mathbb{R}^n, \mathbb{R}^p)$ is the space of all bounded linear mappings from $\mathbb{R}^n$ into $\mathbb{R}^p$.

A special case for the problem (CVEP) is called the constrained vector variational inequality problem (in short, (CVVI)) concerning

$$F_{i,x}(y) = \langle (Tx)_i, y - x\rangle \quad \forall\, x, y \in \mathbb{R}^n, \ \ \forall\, i \in I.$$

For this sense, if $\overline{x} \in K$ is a (local) weakly (resp., Henig efficient, globally efficient, superefficient) solution of problem (CVEP) then $\overline{x} \in K$ is said to be a (local) weakly (resp., Henig efficient, globally efficient, superefficient) solution of problem (CVVI).

In a similar way, for $f = (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ is a vector-valued mapping in which the scalar functions $f_1, \ldots, f_p : \mathbb{R}^n \to \mathbb{R}$ and $f(x) = \big(f_1(x), \ldots, f_p(x)\big)$ for all $x \in \mathbb{R}^n$.

An other special case for the problem (CVEP) is called the constrained vector optimization problem (in short, (CVOP)) that involving

$$F_{i,x}(y) = f_i(y) - f_i(x), \quad \forall\, x, y \in \mathbb{R}^n, \ \ \forall\, i \in I.$$

For this sense, if $\overline{x} \in K$ is a (local) weakly (resp., Henig efficient, globally efficient, superefficient) solution of problem (CVEP) then $\overline{x} \in K$ is called a (local) weakly (resp., Henig efficient, globally efficient, superefficient) solution of problem (CVOP).

Theorem 8 (*KKT-type necessary optimality)* Assume that all the hypotheses of Assumption 1 are fulfilled and the set C is convex in which the functions $F_{i,\overline{x}}$ $(\forall\, i \in I)$ and $F_{\overline{x}}$ are replaced by f_i $(i \in I)$ and f, respectively. Suppose also that the vector $\overline{x}$ is a local weakly (resp. Henig, globally and super-) efficient solution for the problem (CVOP) with $F_{\overline{x}}(x) = f(x) - f(\overline{x})$ $(\forall\, x \in \mathbb{R}^n)$ and the regularity condition of the (RC1) type at $\overline{x}$ holds. Then

(i) There exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}), \tag{39}$$

$$\mu_j g_j(\overline{x}) = 0, \quad \forall\, j \in J. \tag{40}$$

(ii) If f and g are Lipschitz around $\overline{x}$, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_C f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_C g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{41}$$

(iii) If f and g are convex on C, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_{C_0} f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_{C_0} g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{42}$$

Proof. Let us consider $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ be defined by $F(x, y) = f(y) - f(x)$ for all $x, y \in \mathbb{R}^n$. It is evident that $F(\overline{x}, \overline{x}) = 0$. For any $i \in I$, $v \in C_{\overline{x}} := C - \overline{x}$, one gets $D_{\uparrow} F_{i,\overline{x}}(\overline{x})v = D_{\uparrow} f_i(\overline{x})v$. Consequently, $\partial_0 F_{i,\overline{x}}(\overline{x}) = \partial_0 f_i(\overline{x})$ for every $i \in I$. Notice that the stability of f_i $(\forall\, i \in I)$ at $\overline{x}$ yields the stability of $F_{i,\overline{x}}$ $(\forall\, i \in I)$ at $\overline{x}$, and moreover, the steadiness and Hadamard differentiability of f at $\overline{x}$ entails the steadiness and Hadamard differentiability of $F_{\overline{x}}$ at $\overline{x}$, respectively. Therefore, all the hypotheses of Theorem 1 and Theorem 5 are fulfilled to the problem (CVOP) with $F(x, y) = f(y) - f(x)$. By directly applying both these theorems to the problem (CVOP), we get the desired conclusion. □

Example 3 Let us consider the problem (CVOP) in which $n = 3$, $p = 3$, $m = 2$, $r = 2$, $C = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-1, 1]$ and $\overline{x} = (0, 0, 0)$. For the illustration let us may take A, B, g, h being given as in Example 1 and the mapping $f := (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be defined for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$f_1(x) = x_3 + dist(x_2, B) + dist(x_1, A),$$
$$f_2(x) = x_1, \quad f_3(x) = x_2.$$

Then f_1, f_2 and f_3 are steady at $\overline{x}$, g_1 is Hadamard differentiable at $\overline{x}$, g_2 is continuous at $\overline{x}$, h_1, h_2 are Fréchet differentiable at $\overline{x}$ with $\nabla h_1(\overline{x}) = (1, -2, -1)$ and $\nabla h_2(\overline{x}) = (1, -4, 1)$, M_F and C are convex, $I = \{1, 2, 3\}$, $J = \{1, 2\}$, $R = \{1, 2\}$, $J(\overline{x}) = \{1\}$, $J \setminus J(\overline{x}) = \{2\}$, $K(H) = \{(u_1, u_2, u_3) \in \mathbb{R}^3 \,|\, u_1 = 3u_2,\ u_2 = u_3\}$, $T(C, \overline{x}) = \mathbb{R}^2_+ \times \mathbb{R}$ and $N(C, \overline{x}) = \mathbb{R}^2_- \times \{0\}$, which guarantees that the regularity condition (RC2) is fulfilled. Hence, all the hypotheses of Assumption 1 are satisfied. It is known from Example 1 that the feasible set of problem (CVOP) is of the form $K = \{x := (x_1, x_2, x_3) \in C \,|\, x_1^2 + x_2^2 + x_3^2 \le x_3,\ x_1 + x_2 + x_3 \le \frac{1}{2}\}$. We observe that the closed pointed convex cone $Q := \mathbb{R}^3_+$ has the bounded closed convex base $B = \{(b_1, b_2, b_3) \in \mathbb{R}^3_+ | b_1 + b_2 + b_3 = 1\}$. Take $U = \{(u_1, u_2, u_3) \in \mathbb{R}^3 | u_1^2 + u_2^2 + u_3^2 < \frac{1}{16}\}$. Then U is an absolutely convex neighborhood of the origin in $\mathbb{R}^3$ satisfying

$$\mathrm{cone} f(K) \cap (U - B) = \emptyset.$$

According to Ref. [18], $\overline{x} = (0, 0, 0)$ is a locally Henig efficient solution of problem (CVOP). Additionally, for every $u = (u_1, u_2, u_3) \in C_{\overline{x}}$ it results $D_{\uparrow} f_1(\overline{x})u = u_3$, $D_{\uparrow} f_2(\overline{x})u = u_1$, $D_{\uparrow} f_3(\overline{x})u = u_2$, $D_{\uparrow} g_1(\overline{x})u = -u_3$, $D_{\uparrow} g_2(\overline{x})u = u_1 + u_2 + u_3$. Thus, $\partial_0 f_1(\overline{x}) = \{(0, 0, 1)\}$, $\partial_0 f_2(\overline{x}) = \{(1, 0, 0)\}$, $\partial_0 f_3(\overline{x}) = \{(0, 1, 0)\}$, $\partial_0 g_1(\overline{x}) = \{(0, 0, -1)\}$ and further $\partial_0 g_2(\overline{x}) = \{(1, 1, 1)\}$. We verify that the regularity condition of the (RC1) type at $\overline{x}$ holds. Then a weak KKT-type necessary optimality condition described by (39)-(40) holds with $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 7$, $\mu_1 = 1$, $\mu_2 = 0$, $\eta_1 = 1$ and $\eta_2 = 1$. In fact, in this setting, one can achieve that

$$Q^{\triangle}(B) = [\mathbb{R}^3_+]^{\sharp} = \mathrm{int}\, \mathbb{R}^3_+ = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 | \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0\}$$

and thus $\lambda \in Q^{\triangle}(B)$, $\mu \in \mathbb{R}^2_+$, $\mu_1 g_1(\overline{x}) = 0$, $\mu_2 g_2(\overline{x}) = 0$ satisfy

$$0 \in \sum_{i=1}^{3} \lambda_i \partial_0 f_i(\overline{x}) + \sum_{j=1}^{2} \mu_j \partial_0 g_j(\overline{x})v + \sum_{k=1}^{2} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}) = (3, 1, 0) + \mathbb{R}^2_- \times \{0\},$$

which terminates the check.

Corollary 7 Assume that $C = \mathbb{R}^n$, all the hypotheses of Theorem 8 are fulfilled without (RC2). Suppose also that the linear independence of the system $\{\nabla h_1(\overline{x}), \ldots, \nabla h_r(\overline{x})\}$ is fulfilled and $\overline{x} \in K$ is a local weakly (resp. Henig, globally and super-) efficient solution for the problem

(CVOP) with $F_{\overline{x}}(x) = f(x) - f(\overline{x})$ $(\forall\, x \in \mathbb{R}^n)$. Then

(i) There exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ such that (39), (40) are satisfied.

(ii) If f and g are Lipschitz around $\overline{x}$, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (39), (41).

(iii) If f and g are convex on C, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (39), (42).

Proof. By the initial hypotheses, $C = \mathbb{R}^n$, which together with the linear independence of the system $\{\nabla h_1(\overline{x}), \ldots, \nabla h_r(\overline{x})\}$, we confirm that the regularity condition of the (RC2) type holds true too. Also, all the hypotheses of Theorem 8 are valid to the problem (CVOP) with $F(x, y) = f(y) - f(x)$. By directly applying this theorem to the problem (CVOP), we arrive at the desired conclusion. □

Theorem 9 (*KKT-type sufficient optimality)* Given a feasible solution $\overline{x} \in K$ with $F_{\overline{x}}(x) = f(x) - f(\overline{x})$ $(\forall\, x \in \mathbb{R}^n)$, and assumming, in addition, that C is convex, the scalar functions f_i $i \in I)$ and g_j $(j \in J)$ are stable (or steady, Hadamard differentiable) at $\overline{x}$, the mapping h is Fréchet differentiable at $\overline{x}$. Suppose also that

(i) There exist $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu \in \mathbb{R}^m_+$, $\eta \in \mathbb{R}^r$ satisfying

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}).$$

(ii) The scalar functions f_i $(i \in I)$ and g_j $(j \in J)$ are pseudoconvex at $\overline{x}$.

(iii) The scalar functions $\pm h_k$ $(k \in R)$ are quasiconvex at $\overline{x}$ with respect to C.

Then, $\overline{x}$ is a weakly efficient (resp. Henig efficient, globally efficient, superefficient) solution of problem (CVOP).

Proof. Arguing similarly as for proving Theorem 8 with observing Theorems 6 &7 and Corollary 6 in Section 4, we arrive at the desired conclusion. □

Theorem 10 (*KKT-type necessary optimality)* Let $\overline{x} \in K$ be a local weakly efficient (resp. Henig efficient, globally efficient and superefficient) solution for the problem (CVVI) with $F_{\overline{x}}(x) = \langle T\overline{x}, x - \overline{x}\rangle$ $(\forall\, x \in \mathbb{R}^n)$. Suppose that g_j $(\forall\, j \in J(\overline{x}))$ are steady at $\overline{x}$, g_j $(\forall\, j \notin J(\overline{x}))$ are continuous at $\overline{x}$, h is Fréchet differentiable at $\overline{x}$ and continuous on a neighborhood of that point and the regularity conditions (RC1) and (RC2) at $\overline{x}$ hold. Then

(i) There exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 (T\overline{x})_i(\cdot - \overline{x})(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{43}$$

(ii) If g is Lipschitz around $\overline{x}$, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_C (T\overline{x})_i(\cdot - \overline{x})(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_C g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{44}$$

(iii) If g is convex on C, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) and

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_{C_0} (T\overline{x})_i(\cdot - \overline{x})(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \partial_{C_0} g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x}). \tag{45}$$

Proof. Consider $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is formulated by $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in \mathbb{R}^n$. It is easy to verify that $F(\overline{x}, \overline{x}) = 0$, $D_{\uparrow}F_{i,\overline{x}}(\overline{x})v = (T(\overline{x}))_i v$ for all $i \in I$ and $v \in C_{\overline{x}}$. In consequence, $\partial_0 F_{i,\overline{x}}(\overline{x}) = \partial_0 (T\overline{x})_i(\cdot - \overline{x})(\overline{x})$ for all $i \in I$. Evidently, the stability (or steadiness, Hadamard differentiability) of $(T(\overline{x}))_i(\cdot - \overline{x})$ ($\forall\, i \in I$) at $\overline{x}$ is satisfied automatically, which yields that the stability (or steadiness, Hadamard differentiability) of $F_{i,\overline{x}}$ ($\forall\, i \in I$) at $\overline{x}$ is satisfied too. Also, all the assumptions of Theorems 1 and 5 are fulfilled to the problem (CVVI) with $F(x, y) = \langle Tx, y - x\rangle$. Directly applying such theorems to the problem (CVVI) again proves the claim. □

Corollary 8 Assume that $C = \mathbb{R}^n$, all the hypotheses of Theorem 10 are fulfilled without (RC2). Suppose also that the linear independence of the system $\{\nabla h_1(\overline{x}), \ldots, \nabla h_r(\overline{x})\}$ is fulfilled and $\overline{x} \in K$ is a local weakly efficient (resp. Henig efficient, globally efficient and superefficient) solution for the problem (CVVI) with $F_{\overline{x}}(x) = \langle T\overline{x}, x - \overline{x}\rangle$ ($\forall\, x \in \mathbb{R}^n$). Then

(i) There exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) & (43).

(ii) If g is Lipschitz around $\overline{x}$, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) & (44).

(iii) If g is convex on C, then there exist $\lambda = (\lambda_i)_{i\in I} \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^\sharp$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu = (\mu_j)_{j\in J} \in \mathbb{R}^m_+$ and $\eta = (\eta_k)_{k\in R} \in \mathbb{R}^r$ satisfying (40) & (45).

Proof. By an argument analogous to that used for the proof of Corollary 7 with noting that

$$Q^{\Delta}(B) = \operatorname{int}\mathbb{R}^p_+ = \operatorname{int}\mathbb{R}^p_{++},$$

the claim follows. □

Theorem 11 (*KKT-type sufficient optimality)* Given a feasible solution $\overline{x} \in K$ with $F_{\overline{x}}(x) = \langle T\overline{x}, x - \overline{x}\rangle$ ($\forall\, x \in \mathbb{R}^n$), and assumming, in addition, that C is convex, g_j ($j \in J(\overline{x})$) are stable (or steady, Hadamard differentiable) at $\overline{x}$ and h is Fréchet differentiable at $\overline{x}$. Suppose also that

(i) There exist $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$ $\Big($resp. $Q^{\Delta}(B)$, $[\mathbb{R}^p_+]^{\sharp}$, $\operatorname{int}\mathbb{R}^p_+\Big)$, $\mu \in \mathbb{R}^m_+$, $\eta \in \mathbb{R}^r$ satisfying

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_0 (T\overline{x})_i(\cdot - \overline{x})(\overline{x}) + \sum_{j \in J(\overline{x})} \mu_j \partial_0 g_j(\overline{x}) + \sum_{k=1}^{r} \eta_k \nabla h_k(\overline{x}) + N(C, \overline{x});$$

(ii) The scalar function g_j $(j \in J(\overline{x}))$ is pseudoconvex at $\overline{x}$;

(iii) The scalar functions $\pm h_k$ $(k \in R)$ are quasiconvex at $\overline{x}$ with respect to C;

Then $\overline{x}$ is a weakly efficient (resp. Henig efficient, globally efficient, superefficient) solution of problem (CVVI).

Proof. Consider the bifunction $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ given by $F(x, y) = \langle Tx, y - x\rangle$ for every $x, y \in \mathbb{R}^n$. Obviously, $F_{i,\overline{x}}$ $(i \in I)$ is stable even steady and Hadamard differentiable at $\overline{x}$ and such scalar function is pseudoconvex at $\overline{x}$, too. Arguing similarly as for proving Theorem 10 with applying Theorems 6 and 7 instead of Theorems 1 and 5, respectively, we get the desired conclusion. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]Faculty of Mathematics and Information Technology, The University of Danang - University of Science and Education, 459 Ton Duc Thang, Danang, 550000, Vietnam.

Email: tvsu@ued.udn.vn

[2]Faculty of Natural Sciences, Electric Power University, 235 Hoang Quoc Viet, Hanoi, Vietnam.

Email: hangdd@epu.edu.vn