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A plethora of optical solitons and other exact solutions for NLSE with Kudryashov's refractive index using improved modified extended tanh function method

Islam Samir[1] Hamdy M. Ahmed[2,*] Mohammad Mirzazadeh[3]

Abstract. The nonlinear Schrödinger's equation (NLSE) with Kudryashov's refractive index which describes the wave dynamics through optical fibers is investigated. The literature review of the previous works are introduced to exhibit the motivation and the importance of this work. In addition, the integration technique of the improved modified extended tanh function scheme is applied on the investigated model to derive various and novel solutions. These solutions are bright, dark and singular solitons. In addition, Weierstrass elliptic, singular periodic, exponential and rational type solutions are derived. It needs to be noted that this governing model can be useful for description of physical processes in nonlinear optics and also the results revealed for the equation are new and originally reported in the present work for the first time. Moreover, the graphical representations of some solutions are depicted to show the powerful and the characteristics of them.

§1 Introduction

Many physical phenomena are modeled using nonlinear partial differential equations (NPDEs), such as solid physics, fluid dynamics, plasma physics, chemical dynamics, atmospheric phenomena and others [1-6]. Recently, various mathematical techniques have been developed to obtain exact solutions for NPDEs such as unified Riccati equation [7-8], Laplace–Adomian decomposition [9], improved Adomian decomposition scheme [10-11], Sine–Gordon expansion scheme [12-14], auxiliary equation method [15], $exp(-\phi(\psi))$-expansion method [16], tanh–coth approach [17] and (G'/G) expansion scheme [18].

In the past years, there has been a lot of interest in researching the optical solitons dynamics through optical fibers[19-28]. Solitons are regarded as the primary carriers of information, so investigating and comprehending the dynamics of these waves will aid in the development of the communication systems. NLSEs are one of the many mathematical models that can be used to show how optical solitons propagate through optical fibers [29-34].

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*Corresponding author.

Nonlinear refractive index has different forms including cubic , quadratic-cubic, and anti-cubic nonlinearities. NLSE with power law and dual-power law nonlinearities were investigated with the aid of Lie group analysis [35]. NLSE with Kerr law, power law, parabolic law and dual-power law was investigated in ref. [36-37]. In addition, numerical simulations were provided to show the nature of the extracted solutions for NLSE with Kerr law of nonlinearity [38]. The Kudryashov's form (KF) is also one of the recent forms of refractive index. NLSE with KF and dual form of generalized nonlinearity have recently received a lot of attention. This model can be expressed mathematically as [39]

$$\begin{aligned} &i\Psi_t + c\Psi_{xx} + \Psi(s_1 + s_2|\psi|^m + s_3|\psi|^{2m} + s_4|\psi|^{3m} + s_5|\psi|^{4m} + s_6(|\psi|^m)_{xx} + s_7(|\psi|^{2m})_{xx}) \\ &= 0, \end{aligned} \tag{1}$$

where the wave profile represented by $\Psi(x,t)$ and t and x represent the temporal and spatial coordinates respectively. The group velocity dispersion is represented by $c\psi_{xx}(x,t)$ and s_j for $j = 1, 2, 3, 4, 5$ are the coefficients of the self phase modulation effects as presented by Kudryashov [40]. $\psi(|\psi|^m)_{xx}$ and $\psi(|\psi|^{2m})_{xx}$ represent the dual form of generalized nonlinearity [41-42]. Mathematical model of optical fiber with power nonlinearities has been investigated by Kudryashov [32]. F-expansion and trial function methods have been implemented to obtain optical solitons for KF [43–44]. In literature [45], the Lie symmetry scheme has been applied to obtain singular, dark and bright solitons for KF. Highly dispersive optical soliton perturbation with Kudryashov's sextic-power law of nonlinear refractive index was investigated in ref. [46].

In this context, the improved modified extended tanh function method, briefly described in the following section, is used to investigate the proposed model to derive different and novel solutions to Eq. (1). These solutions involving bright, dark and singular solitons. In addition, singular periodic, exponential, Weierstrass elliptic types solutions are extracted. Furthermore, 3D and 2D graphical representations of some solutions are depicted to demonstrate the physical nature of them.

§2 Revisitation of the technique

This part introduces the improved modified extended tanh function scheme as below [47-48].

Consider the following nonlinear evolution equation

$$H(\Psi, \Psi_t, \Psi_x, \Psi_{xx}, \Psi_{tx}, ...) = 0, \tag{2}$$

where H is polynomial in $\Psi(x,t)$ and its partial derivatives with respect to $x,\ t$.

Step(1): The following wave transformation is employed

$$\psi(x,t) = g(\xi),\ \xi = x - v\ t, \tag{3}$$

where v is a constant which need to be evaluated later. Then, NPDE in Eq. (2) is converted to an nonlinear ODE

$$F(g, g', g'', g''', ...) = 0. \tag{4}$$

Step(2): Assume that the solution of Eq. (4) as

$$g(\xi) = \sum_{j=0}^{N} a_j Y^j(\xi) + \sum_{j=-1}^{-N} b_{-j} Y^j(\xi), \tag{5}$$

where $Y(\xi)$ holds

$$Y'(\xi)=\sqrt{d_0+d_1Y(\xi)+d_2Y^2(\xi)+d_3Y^3(\xi)+d_4Y^4(\xi)}. \tag{6}$$

Numerous types of fundamental solutions can be provided from the last equation (6). So, many exact solutions can be revealed to Eq. (2).

Step (3): By applying the balance rule on Eq. (4), the integer N can be estimated.

Step (4): Substituting Eq. (5) and Eq. (6) into (4), then, a polynomial in $g(\xi)$ is recovered. Adding all terms with the same powers together and equating them to zero, we obtain a set of nonlinear equations that can be solved by Mathematica software packages to get $a_j,\ b_j$, and v.

§3 Application to KF

Our goal is to obtain the following solution structure for Eq. (1)

$$\Psi(x,t)=g(\xi)e^{i(\theta+t\omega-\kappa x)}. \tag{7}$$

Substitute by (7) into (1). The real part provides

$$\begin{aligned}&m^2g^2\left(-c\kappa^2+s_1-\omega\right)+c\ mg''g-c(m-1)(g')^2+m^2s_5g^6+m^2s_4g^5+m^2s_3g^4+m^2s_2g^3\\&+2m^2s_7g''g^3+2m^2s_7(g')^2g^2+m^2s_6g''g^2=0,\end{aligned} \tag{8}$$

while the imaginary part yields

$$v=-2c\ \kappa. \tag{9}$$

Balancing g^6 with $g''g^3$, one can provide $N=1$. Then, the solution can be expressed as follows:

$$g(\xi)=a_0+a_1Y(\xi)+\frac{b_1}{Y(\xi)}. \tag{10}$$

Executing step (4) which introduced in the last section, one can obtain the next results

Case 1. $d_0=d_1=d_3=0$

Result (1)

$$d_4=-\frac{a_1^2s_5}{6s_7},\ s_1=\frac{-cd_2+c\kappa^2m^2+m^2\omega}{m^2},\ d_2=-\frac{s_2}{s_6},\ s_3=\frac{cms_5+cs_5-24d_2m^2s_7^2}{6m^2s_7},$$
$$s_4=\frac{s_5s_6}{3s_7},\ a_0=b_1=0.$$

Then, the resulted solutions of Eq. (1) are

$$\Psi(x,t)=\sqrt{6}\sqrt{-\frac{s_2s_7}{s_5s_6}}\text{sech}\left(\sqrt{-\frac{s_2}{s_6}}(x-tv)\right)\times e^{i(-\kappa x+\omega t+\theta)},\quad s_2s_6<0,\quad s_5s_7>0. \tag{11}$$

Eq. (11) is a bright soliton solution.

$$\Psi(x,t)=\sqrt{6}\sqrt{-\frac{s_2s_7}{s_5s_6}}\sec\left(\sqrt{\frac{s_2}{s_6}}(x-tv)\right)\times e^{i(-\kappa x+\omega t+\theta)},\quad s_2s_6>0,\quad s_5s_7<0. \tag{12}$$

Eq. (12) gives a singular periodic solution

Result (2)

$$a_0 = b_1 = s_2 = 0,\ d_4 = -\frac{a_1^2 s_4}{2s_6},\ s_7 = \frac{s_5 s_6}{3s_4},\ c = -\frac{a_1^2 m^2 s_3}{d_4(m+1)},$$
$$\omega = \frac{a_1^2 \kappa^2 m^2 s_3 + d_4 m s_1 + d_4 s_1}{d_4(m+1)}, d_2 = 0.$$

Then, the resulted solution of Eq. (1) is

$$\Psi(x,t) = -\frac{\sqrt{-2s_6}}{\sqrt{s_4}(x-tv)} \times e^{i(-\kappa x+\omega t+\theta)},\quad s_4 s_6 < 0. \tag{13}$$

Eq. (13) is a rational type solution.

Case 2. $d_1 = d_3 = 0, d_0 = \frac{d_2^2}{4d_4}$

Result(1)

$$d_2 = \frac{a_0^2 s_5}{3s_7},\ d_4 = -\frac{a_1^2 s_5}{6s_7},\ s_1 = \frac{2a_0^2 c s_5 + 3c\kappa^2 m^2 s_7 + 3m^2 s_7 \omega}{3m^2 s_7},$$
$$s_2 = \frac{a_0 s_5 \left(2a_0 m^2 s_6 - cm - 2c\right)}{3m^2 s_7}, s_3 = \frac{s_5\left(-6a_0 m^2 s_6 + 16a_0^2 m^2 s_7 + cm + c\right)}{6m^2 s_7},$$
$$a_0 = \frac{s_5 s_6 - 3s_4 s_7}{10 s_5 s_7},\ b_1 = 0.$$

Then, the resulted solution for (1) is

$$\Psi(x,t) = \frac{(s_5 s_6 - 3s_4 s_7)\left(\tanh\left(\frac{(s_5 s_6 - 3s_4 s_7)(x-tv)}{10\sqrt{6}\sqrt{-s_5 s_7^3}}\right)+1\right)}{10 s_5 s_7} \times e^{i(-\kappa x+\omega t+\theta)},\quad s_5 s_7 < 0. \tag{14}$$

Eq. (14) is a dark soliton solution.

Result(2)

$$d_2 = \frac{a_0^2 s_5}{3s_7},\ d_4 = -\frac{a_0^4 s_5}{6b_1^2 s_7},\ s_1 = \frac{2a_0^2 c s_5 + 3c\kappa^2 m^2 s_7 + 3m^2 s_7 \omega}{3m^2 s_7},$$
$$s_2 = \frac{a_0 s_5 \left(2a_0 m^2 s_6 - cm - 2c\right)}{3m^2 s_7}, s_3 = \frac{s_5\left(-6a_0 m^2 s_6 + 16a_0^2 m^2 s_7 + cm + c\right)}{6m^2 s_7},$$
$$a_0 = \frac{s_5 s_6 - 3s_4 s_7}{10 s_5 s_7},\ a_1 = 0.$$

Then, the resulted solution for (1) is

$$\Psi(x,t) = \frac{(s_5 s_6 - 3s_4 s_7)\left(\coth\left(\frac{(s_5 s_6 - 3s_4 s_7)(x-tv)}{10\sqrt{6}\sqrt{-s_5 s_7^3}}\right)+1\right)}{10 s_5 s_7} \times e^{i(-\kappa x+\omega t+\theta)},\quad s_5 s_7 < 0. \tag{15}$$

Eq. (15) provides a singular soliton solution.

Result(3)

$$d_2 = \frac{a_0^2 s_5}{12s_7},\ d_4 = -\frac{a_0^4 s_5}{96b_1^2 s_7},\ s_1 = \frac{2a_0^2 c s_5 + 3c\kappa^2 m^2 s_7 + 3m^2 s_7 \omega}{3m^2 s_7},$$
$$s_2 = \frac{a_0 s_5 \left(2a_0 m^2 s_6 - cm - 2c\right)}{3m^2 s_7}, s_3 = \frac{s_5\left(-6a_0 m^2 s_6 + 16a_0^2 m^2 s_7 + cm + c\right)}{6m^2 s_7},$$

$$a_0 = \frac{s_5 s_6 - 3s_4 s_7}{10 s_5 s_7},\ a_1 = \frac{a_0^2}{4b_1}.$$

Then, the resulted solution for (1) is

$$\Psi(x,t) = \frac{(s_5s_6-3s_4s_7)\tanh\left(\frac{(s_5s_6-3s_4s_7)(x-tv)}{20\sqrt{6}\sqrt{-s_5s_7^3}}\right)\left(\coth\left(\frac{(s_5s_6-3s_4s_7)(x-tv)}{20\sqrt{6}\sqrt{-s_5s_7^3}}\right)+1\right)^2}{20s_5s_7} \times e^{i(-\kappa x+\omega t+\theta)}, \quad s_5 s_7 < 0.$$

Eq. (16) provides a singular soliton solution.

Result (4)

$$d_2 = -\frac{a_1 b_1 s_5}{3s_7},\ s_1 = \frac{-2\sqrt{6}b_1 c\sqrt{d_4}\sqrt{-s_5} + 3c\kappa^2 m^2\sqrt{s_7} + 3m^2\sqrt{s_7}\omega}{3m^2\sqrt{s_7}},\ b_1 = -\frac{3s_2 s_7}{2a_1 s_5 s_6},$$

$$s_3 = \frac{-16\sqrt{6}b_1\sqrt{d_4}m^2\sqrt{-s_5}s_7^{3/2} + cms_5 + cs_5}{6m^2 s_7},\ s_4 = \frac{s_5 s_6}{3s_7},\ d_4 = -\frac{a_1^2 s_5}{6s_7},\ a_0 = 0.$$

Then, the resulted solutions for (1) are

$$\Psi(x,t) = -\sqrt{6}\sqrt{\frac{s_2 s_7}{s_5 s_6}}\operatorname{csch}\left(\sqrt{-\frac{s_2}{s_6}}(x-tv)\right) \times e^{i(-\kappa x+\omega t+\theta)}, \quad s_2 s_6 < 0, \quad s_5 s_7 < 0. \tag{16}$$

Eq. (16) provides a singular soliton solution.

$$\Psi(x,t) = -\sqrt{\frac{3}{2}}\sqrt{-\frac{s_2 s_7}{s_5 s_6}}\tan\left(\frac{1}{2}\sqrt{\frac{s_2}{s_6}}(x-tv)\right)\left(\cot^2\left(\frac{1}{2}\sqrt{\frac{s_2}{s_6}}(x-tv)\right) - 1\right) \times e^{i(-\kappa x+\omega t+\theta)}, \quad s_2 s_6 < 0, \quad s_5 s_7 < 0. \tag{17}$$

Eq. (17) provides a singular periodic solution.

Case (3). $d_2 = d_4 = 0$

Result (1)

$$d_3 = \frac{a_0^3 s_5}{12 b_1 s_7},\ d_1 = \frac{a_0 b_1 s_5}{4s_7},\ s_1 = \frac{a_0^2 c s_5 + 4c\kappa^2 m^2 s_7 + 4m^2 s_7\omega}{4m^2 s_7},$$

$$s_2 = \frac{a_0 s_5\left(6a_0 m^2 s_6 - 5cm - 10c\right)}{24m^2 s_7},\ s_3 = \frac{s_5\left(-15a_0 m^2 s_6 + 24a_0^2 m^2 s_7 + 4cm + 4c\right)}{24m^2 s_7},$$

$$a_0 = \frac{4\left(s_5 s_6 - 3s_4 s_7\right)}{25 s_5 s_7},\ d_0 = -\frac{b_1^2 s_5}{6s_7},\ a_1 = 0.$$

Then, a Weierstrass elliptic solution is raised for (1)

$$\Psi(x,t) = \left\{\frac{b_1}{\wp\left(\frac{2(x-tv)\sqrt{\frac{(s_5s_6-3s_4s_7)^3}{b_1}}}{125\sqrt{3}s_5s_7^2}; H_2, H_3\right)} + \frac{4\left(s_5 s_6 - 3s_4 s_7\right)}{25 s_5 s_7}\right\} \times e^{i(-\kappa x+\omega t+\theta)},$$

$$b_1\ \left(s_5 s_6 - 3s_4 s_7\right) > 0,$$

where $H_2 = -\frac{1875 b_1^2 s_5^2 s_7^2}{4(s_5s_6-3s_4s_7)^2}$ and $H_3 = \frac{15625 b_1^3 s_5^3 s_7^3}{8(s_5s_6-3s_4s_7)^3}$.

Case 4. $d_0 = d_1 = d_2 = 0$

Result (1)

$$d_4 = -\frac{a_1^2 s_5}{6s_7},\ s_1 = c\kappa^2 + \omega,\ s_2 = -\frac{cd_3(m+2)}{2a_1 m^2},\ s_3 = \frac{a_1 cms_5 + a_1 cs_5 - 9d_3 m^2 s_6 s_7}{6a_1 m^2 s_7},$$

$$d_3 = \frac{a_1 \left(s_5 s_6 - 3 s_4 s_7\right)}{15 s_7^2},\ b_1 = a_0 = 0.$$

Then, the resulted solutions of Eq. (1) are

$$\Psi(x,t) = -\frac{\left(s_5 s_6 - 3 s_4 s_7\right) \exp\left(\frac{(s_5 s_6 - 3 s_4 s_7)(x - tv)}{5\sqrt{6}\sqrt{s_5 s_7^3}}\right)}{5 s_5 s_7} \times e^{i(-\kappa x + \omega t + \theta)},\quad s_5 s_7 > 0. \tag{18}$$

Eq. (18) represents an exponential type solution.

$$\Psi(x,t) = \frac{4\left(s_5 s_6 - 3 s_4 s_7\right)}{15 s_7 \left(\frac{(s_5 s_6 - 3 s_4 s_7)^2 (x - ct)^2}{225 s_7^3} + \frac{2 s_5}{3}\right)} \times e^{i(-\kappa x + \omega t + \theta)}. \tag{19}$$

Eq. (19) is a rational type solution.

Case (5). $d_3 = d_4 = 0$

Result (1)

$$d_0 = -\frac{b_1^2 s_5}{6 s_7},\ s_1 = \frac{-c d_2 + c\kappa^2 m^2 + m^2 \omega}{m^2},\ d_2 = -\frac{s_2}{s_6},\ s_3 = \frac{c m s_5 + c s_5 - 24 d_2 m^2 s_7^2}{6 m^2 s_7},$$
$$s_4 = \frac{s_5 s_6}{3 s_7},\ a_0 = a_1 = 0,\ d_1 = 0.$$

Then, the resulted solutions for Eq. (1) are

$$\Psi(x,t) = \frac{\sqrt{6 s_2 s_7} \operatorname{csch}\left(\sqrt{\frac{s_2}{s_6}}(x - tv)\right)}{\sqrt{s_5 s_6}} \times e^{i(-\kappa x + \omega t + \theta)},\quad s_2 s_6 > 0,\quad s_5 s_7 > 0. \tag{20}$$

Eq. (20) is a singular soliton solution.

$$\Psi(x,t) = \frac{\sqrt{-6 s_2 s_7} \csc\left(\sqrt{\frac{s_2}{s_6}}(x - tv)\right)}{\sqrt{s_5 s_6}} \times e^{i(-\kappa x + \omega t + \theta)},\quad s_2 s_6 > 0,\quad s_5 s_7 < 0. \tag{21}$$

Eq. (21) is a singular periodic solution.

Result (2)

$$d_1 = -\frac{b_1 \left(s_5 s_6 - 3 s_4 s_7\right)}{15 s_7^2},\ s_1 = \frac{-c d_2 + c\kappa^2 m^2 + m^2 \omega}{m^2},\ a_0 = a_1 = 0,\ d_0 = \frac{d_1^2}{4 d_2},$$
$$s_2 = \frac{\left(s_5 s_6 - 3 s_4 s_7\right)\left(s_5 \left(m^2 s_6^2 - 5c(m+2) s_7\right) - 3 m^2 s_4 s_6 s_7\right)}{150 m^2 s_5 s_7^3},$$
$$s_3 = \frac{1}{150}\left(\frac{s_5 \left(25 c(m+1) s_7 + 19 m^2 s_6^2\right)}{m^2 s_7^2} + \frac{36 s_4^2}{s_5} - \frac{69 s_6 s_4}{s_7}\right),$$
$$d_2 = \frac{-s_5^2 s_6^2 + 6 s_4 s_5 s_7 s_6 - 9 s_4^2 s_7^2}{150 s_5 s_7^3}.$$

Then, a dark soliton solution is provided for Eq. (1)

$$\Psi(x,t) = \frac{b_1}{\frac{5 b_1 s_5 s_7}{3 s_4 s_7 - s_5 s_6} + \exp\left(\frac{(s_5 s_6 - 3 s_4 s_7)(x - tv)}{5\sqrt{6}\sqrt{-s_5 s_7^3}}\right)} \times e^{i(-\kappa x + \omega t + \theta)},\quad s_5 s_7 < 0. \tag{22}$$

Result (3)

$$d_2 = -\frac{a_0^2 s_5}{6 s_7},\ d_1 = -\frac{a_0 b_1 s_5}{3 s_7},\ s_1 = \frac{a_0^2 c s_5 + 6 c\kappa^2 m^2 s_7 + 6 m^2 s_7 \omega}{6 m^2 s_7},$$
$$s_2 = \frac{a_0 s_5 \left(a_0 m^2 s_6 - c m - 2c\right)}{6 m^2 s_7}, s_3 = \frac{s_5 \left(-3 a_0 m^2 s_6 + 4 a_0^2 m^2 s_7 + c m + c\right)}{6 m^2 s_7},$$

$$a_0 = \frac{s_5 s_6 - 3 s_4 s_7}{5 s_5 s_7},\ a_1 = 0,\ d_0 = \frac{d_1^2}{4 d_2}.$$

Then, a dark soliton solution is provided for Eq. (1)

$$\Psi(x,t) = \left\{ \frac{b_1}{\exp\left(\frac{(s_5 s_6 - 3 s_4 s_7)(x - tv)}{5\sqrt{6}\sqrt{-s_5 s_7^3}}\right) - \frac{5 b_1 s_5 s_7}{s_5 s_6 - 3 s_4 s_7}} - \frac{3 s_4}{5 s_5} + \frac{s_6}{5 s_7} \right\} \times e^{i(-\kappa x + \omega t + \theta)}, \quad s_5 s_7 < 0. \tag{23}$$

Case (6). $d_0 = d_1 = 0, \quad d_3 = 2\sqrt{d_2 d_4}$

Result (1)

$$d_2 = -\frac{a_0^2 s_5}{6 s_7},\ d_3 = -\frac{a_0 a_1 s_5}{3 s_7},\ s_1 = \frac{a_0^2 c s_5 + 6 c \kappa^2 m^2 s_7 + 6 m^2 s_7 \omega}{6 m^2 s_7},$$

$$s_2 = \frac{a_0 s_5 \left(a_0 m^2 s_6 - cm - 2c\right)}{6 m^2 s_7}, s_3 = \frac{s_5 \left(-3 a_0 m^2 s_6 + 4 a_0^2 m^2 s_7 + cm + c\right)}{6 m^2 s_7},$$

$$a_0 = \frac{s_5 s_6 - 3 s_4 s_7}{5 s_5 s_7},\ b_1 = 0.$$

Then, a dark soliton solution is provided for Eq. (1)

$$\Psi(x,t) = \frac{(s_5 s_6 - 3 s_4 s_7)\left(\tanh\left(\frac{(s_5 s_6 - 3 s_4 s_7)(x - tv)}{10\sqrt{6}\sqrt{-s_5 s_7^3}}\right) + 3\right)}{10 s_5 s_7} \times e^{i(-\kappa x + \omega t + \theta)}, \quad s_5 s_7 < 0. \tag{24}$$

Result (2)

$$d_3 = \frac{a_1 (s_5 s_6 - 3 s_4 s_7)}{15 s_7^2},\ s_1 = \frac{-c d_2 + c \kappa^2 m^2 + m^2 \omega}{m^2},\ a_0 = b_1 = 0,$$

$$s_2 = \frac{(s_5 s_6 - 3 s_4 s_7)\left(s_5 \left(m^2 s_6^2 - 5c(m+2) s_7\right) - 3 m^2 s_4 s_6 s_7\right)}{150 m^2 s_5 s_7^3},$$

$$s_3 = \frac{1}{150}\left(\frac{s_5 \left(25 c (m+1) s_7 - 11 m^2 s_6^2\right)}{m^2 s_7^2} + \frac{36 s_4^2}{s_5} + \frac{21 s_6 s_4}{s_7}\right),$$

$$d_2 = \frac{-s_5^2 s_6^2 + 6 s_4 s_5 s_7 s_6 - 9 s_4^2 s_7^2}{150 s_5 s_7^3}.$$

Then, a dark soliton solution is provided for Eq. (1)

$$\Psi(x,t) = -\frac{(s_5 s_6 - 3 s_4 s_7)\left(\tanh\left(\frac{(s_5 s_6 - 3 s_4 s_7)(x - tv)}{10\sqrt{-6 s_5 s_7^3}}\right) + 1\right)}{10 s_5 s_7} \times e^{i(-\kappa x + \omega t + \theta)}, \quad s_5 s_7 < 0. \tag{25}$$

§4 Illustrations of the solutions

To highlight the nature of the obtained results, graphical simulations of some extracted results are provided. The graphical representation of a bright soliton of Eq. (11) is depicted in Fig. (1) with $s_2 = 2, \ v = s_5 = s_6 = s_7 = -2$. The graphical representation of a dark soliton of Eq. (14) is depicted in Fig. (2) with $s_4 = s_6 = -2, \ s_5 = 2, \ s_7 = -0.23, \ v = -0.075$. These solutions represent very stable solutions as these solitary waves can propagate over long distances while retaining their shape and speed due to a delicate balance which occurs between the dispersion and nonlinear effects. The graphical representation of a singular soliton of Eq. (15) is depicted in Fig. (3) with $s_4 = s_6 = s_7 = v = -2, \quad s_5 = 2$. This solution represents a rare phenomenon in nonlinear physics, characterized by a point of singularity or divergence in intensity. It captures the abrupt change at the point, offering insight into the interplay of nonlinearity and dispersion in forming exotic solitary waves. The graphical representation of a singular periodic solution of Eq. (21) is depicted in Fig. (4) with $s_2 = s_5 = s_6 = v = -2, \quad s_7 = 2$. This solution displays a periodically repeating wave with a point of singularity, showing valuable insights into the behavior of nonlinear systems with recurring singularities.

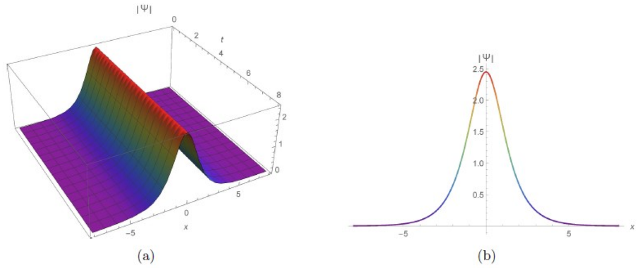

Figure 1. The graphical representation of a bright soliton solution of Eq. (11).

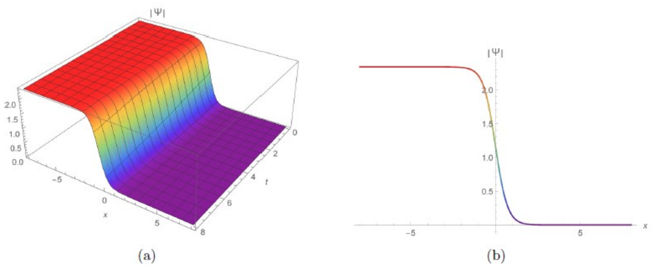

Figure 2. The graphical representation of a dark soliton solution of Eq. (14).

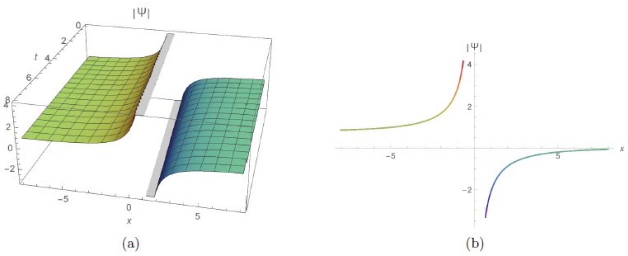

Figure 3. The graphical representation of a singular soliton solution of Eq. (15).

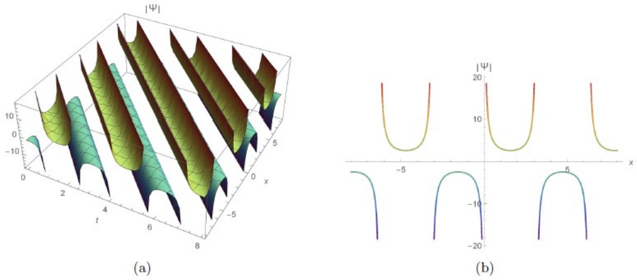

Figure 4. The graphical representation of a singular periodic solution of Eq. (21) .

§5 Conclusions

In this context, the NLSE with KF of refractive index which mimics the wave propagation through optical fiber has been studied. The study was conducted with the aid of the improved modified extended tanh function integration technique which gave various and novel types of solutions. These solutions including solitons (bright, dark and singular), singular periodic, rational, exponential and Weierstrass elliptic solutions. To show the powerful of the obtained solutions, 3D and 2D graphical representations were introduced for some of them. These solutions will be essential in the developments of telecommunication systems.

Declarations

Conflict of interest The authors declare no conflict of interest.

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[1]Department of Physics and Mathematics Engineering, Faculty of Engineering, Ain Shams University, Cairo, Egypt.

[2]Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El Shorouk Academy, Cairo, Egypt.

[3]Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, P.C. 44891-63157 Rudsar-Vajargah, Iran.

Email: hamdy_17eg@yahoo.com