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A Cauchy integral formula and the related properties for inframonogenic functions valued in the Clifford algebra depending on parameters

DU Xiao-jing[1,2] LIANG Xiao-tong[1] XIE Yong-hong[1,*]

Abstract. Firstly, we obtain a Cauchy integral formula for inframonogenic functions which are solutions to the sandwich equation $DfD = 0$ in the framework of parameter-depending Clifford-type algebras. Secondly, the properities relating to Cauchy integral operators are discussed. Finally, the decompositions of the inframonogenic function are given.

§1 Introduction

Clifford algebra [1] was established in the 19th century by Clifford. In 1982, Brackx et al. [2] generalized some results of the complex analysis to Clifford analysis. In 2008, Clifford algebra depending on parameters and their applications to partial differential equations were introduced by Tutschke and Vanegas [3]. In 2013, Balderrama et al. [4] derived some integral representation for meta-monogenic functions in Clifford algebras depending on parameters. In 2015, Ariza et al. [5] gave fundamental solutions to second order elliptic operators in Clifford-type algebras. In 2017, García et al. [6] studied a Cauchy integral formula for inframonogenic functions in Clifford analysis. In 2018, García et al. [7] studied inframonogenic functions and their applications in 3-dimensional elasticity theory, Yang et al. [8] gave the Cauchy integral formula for k-monogenic functions with α-weight. In 2020, García et al. [9] discussed the decomposition of inframonogenic functions with applications in elasticity theory, Blaya et al. [10] proved the Cauchy integral formula for infrapolymonogenic functions. In 2021, Dinh et al. [11] gave the structure of inframonogenic functions, Cuong et al. [12] gave some new results for function theory in hypercomplex analysis with parameters. In 2022, Santiesteban et al. [13] proved

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*Corresponding author.

the integral representation formulae for (φ, ψ)-inframonogenic functions in Clifford analysis. In 2023, Dinh [14] gave new proofs of the integral representation formulae for (α, β)-monogenic functions and isotonic functions and proved the series representation of polynomial Dirac equations, and Álvarez et al. [15] derived the reduced-quaternion inframonogenic functions in the ball.

In this paper we derive a Cauchy integral formula for inframonogenic functions in the framework of parameter-depending Clifford-type algebras. Secondly, the properties relating to Cauchy integral operators are discussed. Finally, the decompositions of the inframonogenic function are given.

§2 Preliminaries

See reference [4], and let $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ be a real linear, associative and non-commutative algebra which is called a Clifford algebra depending on parameters, whose basis is e_A, where $A = \{h_1, \ldots, h_r\} \in PN, 1 \leq h_1 < \cdots < h_r \leq n$, N stands for the set $\{1, \ldots, n\}$ and PN denotes the family of all order-preserving subsets of N. We denote $e_\emptyset$ by e_0 and e_A by $e_{h_1 \cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in PN$, where the basis satisfies

$$\begin{cases} e_i^2 = -\alpha_i, & i = 1, \ldots, n, \\ e_i e_j + e_j e_i = 2\gamma_{ij}, & 1 \leq i \neq j \leq n. \end{cases} \tag{1}$$

It is obvious that $\mathcal{A}_n(2, 1, 0) = \mathcal{A}_n$, where $\mathcal{A}_n$ is the classical Clifford algebra.

The involution is defined by $\overline{e_i} = -e_i$, $i = 1, \ldots, n$. If $e_A = e_{h_1 \cdots h_r} = e_{h_1} \cdots e_{h_r}$, we define $\overline{e_A} = \overline{e_{h_r}} \cdots \overline{e_{h_1}} = (-1)^r e_{h_r} \cdots e_{h_1}$. For any $\lambda = \sum\limits_A \lambda_A e_A \in \mathcal{A}_n(2, \alpha_j, \gamma_{ij})$, we define $\overline{\lambda} = \sum\limits_A \lambda_A \overline{e_A}$.

In this paper, let Ω be a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary $\partial\Omega$. The function $f : \Omega \to \mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ is denoted by $f(x) = \sum\limits_A f_A(x) e_A$, where $f_A \in \mathbf{R}$. A function f is continuous in Ω means that each component of f is continuous in Ω.

Let $C^r(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})) = \{f | f : \Omega \to \mathcal{A}_n(2, \alpha_j, \gamma_{ij}),\ f(x) = \sum\limits_A f_A(x) e_A, f_A(x)$ is r-time continuously differentiable in Ω, $r \in \mathbf{N}^*$, $\mathbf{N}^*$ is the set of positive integers$\}$.

For $f \in C^1(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij}))$, we introduce Dirac operators as follows:

$$Df = \sum_{k=1}^n e_k \frac{\partial f}{\partial x_k},\ fD = \sum_{k=1}^n \frac{\partial f}{\partial x_k} e_k,\ \overline{D} f = \sum_{k=1}^n \overline{e_k} \frac{\partial f}{\partial x_k},\ f\overline{D} = \sum_{k=1}^n \frac{\partial f}{\partial x_k} \overline{e_k}.$$

It is easy to be proved that

$$D\overline{D} = \sum_{i=1}^n \alpha_i \partial_i^2 - 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \partial_i \partial_j.$$

We consider the corresponding quadratic form

$$\sum_{i=1}^n \alpha_i \lambda_i^2 - 2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \lambda_i \lambda_j,$$

which has the coefficient matrix

$$B=\begin{pmatrix}\alpha_1 & -\gamma_{12} & \cdots & -\gamma_{1n}\\ -\gamma_{12} & \alpha_2 & \cdots & -\gamma_{2n}\\ \cdots & \cdots & \ddots & \cdots\\ -\gamma_{1n} & -\gamma_{2n} & \cdots & \alpha_n\end{pmatrix}. \tag{2}$$

Let

$$B_1=\alpha_1, B_2=\begin{pmatrix}\alpha_1 & -\gamma_{12}\\ -\gamma_{12} & \alpha_2\end{pmatrix}, B_3=\begin{pmatrix}\alpha_1 & -\gamma_{12} & -\gamma_{13}\\ -\gamma_{12} & \alpha_2 & -\gamma_{23}\\ -\gamma_{13} & -\gamma_{23} & \alpha_3\end{pmatrix}, \cdots, B_n=B.$$

According to the Sylvesters criterion (see [17] and [18]), the quadratic form of $D\overline{D}$ is positively definite if and only if

$$det(B_j)>0,\ j=1,2,\ldots,n. \tag{3}$$

In this case $D\overline{D}$ is an elliptic operator, and $D\overline{D}$ is denoted by $\widetilde{\Delta}_n$.

In this paper, the condition (3) is satisfied. Then the matrix B in (2) has the inverse A given in the following form

$$A=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \ddots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}, \tag{4}$$

where $a_{ij}=a_{ji}$.

For two points $x=(x_1,\ldots,x_n)$ and $\xi=(\xi_1,\ldots,\xi_n)$ in $\mathbf{R}^n$, we define a non-Euclidean distance ρ by

$$\rho^2:=\rho^2(x,\xi)=\sum_{i,j=1}^{n}a_{ij}(x_i-\xi_i)(x_j-\xi_j). \tag{5}$$

The usual Euclidean distance is denoted by $r:=|x-\xi|$ for $x\neq\xi$. We can write $x-\xi=rx^*$ for some $x^*\in\mathbf{R}^n$ with $|x^*|=1$. Note that the infimum of the non-Euclidean distance between x^* and $(0,0,\ldots,0)$ over all x^* with $|x^*|=1$ is positive, and there exists a positive constant $c_0>0$ such that $\rho^2(x^*,0)\geq c_0$. From this we conclude $\rho^2\geq c_0r^2$. If $x\neq\xi$, we let

$$\widetilde{K}(x,\xi)=\frac{1}{\omega_n}\begin{cases}\ln\rho, & n=2,\\ \dfrac{-1}{n-2}\dfrac{1}{\rho^{n-2}}, & n\geq 3,\end{cases} \tag{6}$$

where ρ is the distance given by (5) and ω_n is the surface measure of the unit sphere.

For $x,\xi\in\Omega$, $x\neq\xi$, we have

$$E(x,\xi)=\widetilde{K}(x,\xi)\overline{D}=\overline{D}\widetilde{K}(x,\xi)=\frac{1}{\omega_n}\frac{1}{\rho^n}\sum_{i,j=1}^{n}\overline{e_i}a_{ij}(x_j-\xi_j)=\frac{1}{\omega_n}\frac{H(x,\xi)}{\rho^n}, \tag{7}$$

where

$$H(x,\xi)=\sum_{i,j=1}^{n}\overline{e_i}a_{ij}(x_j-\xi_j). \tag{8}$$

Lemma 2.1. [3] The function $E(x,\xi)$ satisfies $E(x,\xi)D=DE(x,\xi)=0$.

Definition 2.2. [3] If $f \in C^1(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij}))$ satisfies $Df(x) = 0\,(f(x)D = 0)$ in Ω, we say that f is a left (right) monogenic function in Ω.

Definition 2.3. If $f \in C^2(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij}))$ satisfies $D\big(f(x)D\big) = \big(Df(x)\big)D = 0$ in Ω, we say that f is an inframonogenic function in Ω.

Similar to the proof in the reference [16], we have the following proposition.

Proposition 2.4. If $f,\, g \in C^1(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij}))$, then

$$D\big(f(x)g(x)\big) = \big(Df(x)\big)g(x) + \sum_{i=1}^{n} e_i f(x)\frac{\partial g(x)}{\partial x_i},$$

$$\big(f(x)g(x)\big)D = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} g(x)e_i + f(x)\big(g(x)D\big).$$

Let $d\sigma = \sum\limits_{i=1}^{n} N_i e_i d\mu$, where $d\mu$ is the measure element of $\partial\Omega$, and the outer unit normal at $x \in \partial\Omega$ is $N(x) = (N(x_1), \ldots, N(x_n))$.

Lemma 2.5. [3] If $f,\, g \in C^1\big(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big) \bigcap C\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$, then

$$\int_{\partial\Omega} f(x)d\sigma_x g(x) = \int_{\Omega} \Big[\big(f(x)D\big)g(x) + f(x)\big(Dg(x)\big)\Big]dx.$$

Lemma 2.6. [3] (Cauchy-Pompeiu integral formula) If $f \in C^1\big(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big) \bigcap C\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$, then for any interior point $\xi \in \Omega$, we have

$$f(\xi)c(\alpha_j, \gamma_{ij}) = \mathcal{C}^r_{\partial\Omega} f(\xi) + \mathcal{T}^r_{\Omega}\big[f(\xi)D\big],$$

where $c(\alpha_j, \gamma_{ij}) = -\int\limits_{|x-\xi|=\varepsilon} d\sigma_x E(x, \xi)$ is a Clifford number, and it does not depend on ε,

$$\mathcal{C}^r_{\partial\Omega} f(\xi) = \int_{\partial\Omega} f(x)d\sigma_x E(x, \xi), \quad \mathcal{T}^r_{\Omega} f(\xi) = -\int_{\Omega} f(x)E(x, \xi)dx.$$

By the reference [3] or similar to the reference [3], we have the following three corollaries.

Corollary 2.7. (Cauchy integral formula) If $f \in C^1\big(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big) \bigcap C\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$ is a right monogenic function in Ω, then for any interior point $\xi \in \Omega$, we have

$$f(\xi)c(\alpha_j, \gamma_{ij}) = \mathcal{C}^r_{\partial\Omega} f(\xi).$$

Corollary 2.8. (Cauchy-Pompeiu integral formula) If $f \in C^1\big(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big) \bigcap C\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$, then for any interior point $\xi \in \Omega$, we have

$$c_1(\alpha_j, \gamma_{ij})f(\xi) = \mathcal{C}^l_{\partial\Omega} f(x) + \mathcal{T}^l_{\Omega}\big[Df(\xi)\big],$$

where $c_1(\alpha_j, \gamma_{ij}) = -\int\limits_{|x-\xi|=\varepsilon} E(x, \xi)d\sigma_x$ is a Clifford number, and it does not depend on ε,

$$\mathcal{C}^l_{\partial\Omega} f(\xi) = \int_{\partial\Omega} E(x, \xi)d\sigma_x f(x), \quad \mathcal{T}^l_{\Omega} f(\xi) = -\int_{\Omega} E(x, \xi)f(x)dx.$$

Corollary 2.9. (Cauchy integral formula) If $f \in C^1\big(\Omega, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big) \bigcap C\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$ is a left monogenic function in Ω, then for any interior point $\xi \in \Omega$, we have

$$c_1(\alpha_j, \gamma_{ij})f(\xi) = \mathcal{C}^l_{\partial\Omega} f(\xi).$$

Remark 2.10. By Remark 2.6 in reference [4], we know that $c(\alpha_j, \gamma_{ij})$ is a Clifford number, which has a single inverse element determined by the corresponding algebra. And $c_1(\alpha_j, \gamma_{ij})$ is similar.

§3 Main results

Proposition 3.1. Let $H(x) = \sum\limits_{i,j=1}^{n} \overline{e_i} a_{ij} x_j$, then the following formulae hold.

(*i*) $H(x)e_k = -e_k H(x) + 2I_k(x)$, where $I_k(x)$ is a real number, $k = 1, 2, \cdots, n$.

(*ii*) $\sum\limits_{k=1}^{n} x_k \sum\limits_{j=1}^{n} a_{jk} e_j = -H(x)$.

(*iii*)

$$D\Big[\big(f(x)D\big)\big(\sum_{j,k=1}^{n} a_{jk}x_k e_j\big)\Big] = \big[D(f(x)D)\big]\Big[\sum_{j,k=1}^{n} a_{jk}x_k e_j\Big] + \sum_{m=1}^{n} e_m\big[f(x)D\big]\Big[\sum_{j=1}^{n} a_{jm}e_j\Big].$$

Proof.

(*i*) As $\overline{e_k}e_k + e_k\overline{e_k} = 2\alpha_k \in \mathbf{R}$ and $\overline{e_i}e_k + e_k\overline{e_i} = -2\gamma_{ki} = -2\gamma_{ik} \in \mathbf{R}$,

$$\begin{aligned} & H(x)e_k + e_k H(x) \\ =& \Big[\sum_{i,j=1}^{n} \overline{e_i} a_{ij} x_j\Big] e_k + e_k\Big[\sum_{i,j=1}^{n} \overline{e_i} a_{ij} x_j\Big] = \sum_{i,j=1}^{n} a_{ij}x_j(\overline{e_i}e_k + e_k\overline{e_i}) \\ =& \sum_{j=1}^{n} a_{kj}x_j(\overline{e_k}e_k + e_k\overline{e_k}) + \sum_{i,j=1, i\neq k}^{n} a_{ij}x_j(\overline{e_i}e_k + e_k\overline{e_i}) \\ =& 2\sum_{j=1}^{n} a_{kj}x_j\alpha_k - 2\sum_{i,j=1,i\neq k}^{n} a_{ij}x_j\gamma_{ki} = 2\Big(\sum_{j=1}^{n} a_{kj}\alpha_k - \sum_{i,j=1,i\neq k}^{n} a_{ij}\gamma_{ki}\Big)x_j. \end{aligned}$$

As

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & \cdots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 & -\gamma_{12} & \cdots & -\gamma_{1k} & \cdots & -\gamma_{1n} \\ -\gamma_{12} & \alpha_2 & \cdots & -\gamma_{2k} & \cdots & -\gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\gamma_{1k} & -\gamma_{2k} & \cdots & \alpha_k & \cdots & -\gamma_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\gamma_{1n} & -\gamma_{2n} & \cdots & -\gamma_{kn} & \cdots & \alpha_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\alpha_1 - \sum\limits_{j=2}^{n} a_{1j}\gamma_{1j} & \cdots & a_{1k}\alpha_k - \sum\limits_{j=1,j\neq k}^{n} a_{1j}\gamma_{kj} & \cdots & a_{1n}\alpha_n - \sum\limits_{j=1,j\neq n}^{n} a_{1j}\gamma_{nj} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{k1}\alpha_1 - \sum\limits_{j=2}^{n} a_{kj}\gamma_{1j} & \cdots & a_{kk}\alpha_k - \sum\limits_{j=1,j\neq k}^{n} a_{kj}\gamma_{kj} & \cdots & a_{kn}\alpha_n - \sum\limits_{j=1,j\neq n}^{n} a_{kj}\gamma_{nj} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{n1}\alpha_n - \sum\limits_{j=2}^{n} a_{nj}\gamma_{1j} & \cdots & a_{nk}\alpha_k - \sum\limits_{j=1,j\neq k}^{n} a_{nj}\gamma_{kj} & \cdots & a_{nn}\alpha_n - \sum\limits_{j=1,j\neq n}^{n} a_{nj}\gamma_{nj} \end{pmatrix}$$

$$=E,$$

$$H(x)e_k + e_k H(x) = 2\Big(\sum_{j=1}^{n} a_{kj}\alpha_k - \sum_{i,j=1,i\neq k}^{n} a_{ij}\gamma_{ki}\Big)x_j$$

$$=2\Big(a_{k1}\alpha_k - \sum_{i=1,i\neq k}^{n} a_{i1}\gamma_{ki}\Big)x_1 + \cdots + 2\Big(a_{kk}\alpha_k - \sum_{i=1,i\neq k}^{n} a_{ik}\gamma_{ki}\Big)x_k$$

$$+ \cdots + 2\Big(a_{kn}\alpha_k - \sum_{i=1,i\neq k}^{n} a_{in}\gamma_{ki}\Big)x_n = 2x_k.$$

(ii) After direct calculation, we have

$$\sum_{k=1}^{n} x_k\Big(\sum_{j=1}^{n} a_{jk}e_j\Big) = \sum_{j,k=1}^{n} e_j a_{jk}x_k = -\sum_{j,k=1}^{n} \overline{e_j}a_{jk}x_k = -H(x).$$

(iii) By Lemma 2.4, we have

$$D\Big[\big(f(x)D\big)\Big(\sum_{j,k=1}^{n} a_{jk}x_k e_j\Big)\Big]$$

$$=[D(f(x)D)]\Big[\sum_{j,k=1}^{n} a_{jk}x_k e_j\Big] + \sum_{m=1}^{n} e_m\big(f(x)D\big)\frac{\partial\Big(\sum_{j,k=1}^{n} a_{jk}x_k e_j\Big)}{\partial x_m}$$

$$=[D(f(x)D)]\Big[\sum_{j,k=1}^{n} a_{jk}x_k e_j\Big] + \sum_{m=1}^{n} e_m\big(f(x)D\big)\Big[\sum_{j=1}^{n} a_{jm}e_j\Big].$$

Corollary 3.2. If $x \neq \xi$, then the following formulae hold.

(i) $E(x,\xi)e_k = -e_k E(x,\xi) + 2J_k(x,\xi)$, where $J_k(x,\xi)$ is a real number.

(ii) $\displaystyle\sum_{k=1}^{n} J_k(x,\xi)\sum_{j=1}^{n} a_{jk}e_j = \sum_{k=1}^{n}\frac{1}{\omega_n}\frac{x_k-\xi_k}{\rho^n}\sum_{j=1}^{n} a_{jk}e_j = -E(x,\xi).$

Proof. (i) By Proposition 3.1, we have

$$E(x,\xi)e_k + e_k E(x,\xi) = \frac{1}{\omega_n}\frac{H(x,\xi)e_k + e_k H(x,\xi)}{\rho^n} = \frac{1}{\omega_n}\frac{x_k-\xi_k}{\rho^n} = 2J_k(x,\xi).$$

(ii) By Proposition 3.1, we obtain

$$\sum_{k=1}^{n} J_k(x,\xi)\sum_{j=1}^{n} a_{jk}e_j = \frac{1}{\omega_n}\frac{1}{\rho^n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_j a_{jk}\big(x_k-\xi_k\big) = \frac{1}{\omega_n}\frac{-H(x,\xi)}{\rho^n} = -E(x,\xi).$$

In this paper, we let

$$\mathcal{C}^0_{\partial\Omega}f(\xi) = \int_{\partial\Omega} E(x,\xi)d\sigma_x f(x)\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big],$$

$$\mathcal{C}^1_{\partial\Omega}f(\xi) = -\sum_{m=1}^{n} e_m\int_{\partial\Omega}\widetilde{K}(x,\xi)d\sigma_x f(x)\Big[\sum_{j=1}^{n} a_{jm}e_j\Big],$$

$$\mathcal{T}^0_{\Omega}f(\xi) = -\int_{\Omega} E(x,\xi)f(x)\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big]dx,$$

$$\mathcal{T}_{\Omega}^{1} f(\xi)=\sum_{m=1}^{n} e_m \int_{\Omega} \widetilde{K}(x,\xi) f(x) dx \Big[\sum_{j=1}^{n} a_{jm} e_j\Big],$$

$$\mathcal{C}_{\partial\Omega}^{infra} f(\xi)=\frac{1}{2}\big[\mathcal{C}_{\partial\Omega}^{0} f(\xi)+\mathcal{C}_{\partial\Omega}^{1} f(\xi)\big], \quad \mathcal{T}_{\Omega}^{infra} f(\xi)=\frac{1}{2}\big[\mathcal{T}_{\Omega}^{0} f(\xi)+\mathcal{T}_{\Omega}^{1} f(\xi)\big].$$

Theorem 3.3. (Cauchy-Pompeiu integral formula) If $f \in C^2\big(\overline{\Omega}, \mathcal{A}_n(2,\alpha_j,\gamma_{ij})\big)$, then for any $\xi \in \Omega$, we have

$$\begin{aligned} f(\xi) c(\alpha_j,\gamma_{ij}) =& \mathcal{C}_{\partial\Omega}^{r} f(\xi)+\mathcal{C}_{\partial\Omega}^{infra}\big[D f(\xi)\big]+\mathcal{T}_{\Omega}^{infra}\big[D\big(f(\xi) D\big)\big] \\ =& \frac{1}{2}\int_{\partial\Omega} E(x,\xi) d\sigma_x \big[f(x) D\big]\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big] \\ & -\frac{1}{2}\sum_{m=1}^{n} e_m \int_{\partial\Omega} \widetilde{K}(x,\xi) d\sigma_x \big[f(x) D\big]\Big[\sum_{j=1}^{n} a_{jm} e_j\Big] \\ & +\int_{\partial\Omega} f(x) d\sigma_x E(x,\xi)+\frac{1}{2}\sum_{m=1}^{n} e_m \int_{\Omega} \widetilde{K}(x,\xi)\big[D\big(f(x) D\big)\big] dx \Big[\sum_{j=1}^{n} a_{jm} e_j\Big] \\ & -\frac{1}{2}\int_{\Omega} E(x,\xi)\big[D\big(f(x) D\big)\big]\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big] dx. \end{aligned} \tag{9}$$

Proof. For any $\xi \in \Omega$, we denote $B(\xi,\varepsilon):=\{x \in \mathbf{R}^n : |x-\xi|<\varepsilon\}$, $\Omega_{\varepsilon}=\Omega \backslash \overline{B(\xi,\varepsilon)}$. Suppose

$$I^{\varepsilon}=\int_{\Omega_{\varepsilon}} E(x,\xi)\Big\{D\Big[\big(f(x) D\big)\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big)\Big]\Big\} dx,$$

and

$$J^{\varepsilon}=\int_{\Omega_{\varepsilon}} \sum_{m=1}^{n} E(x,\xi) e_m\big[f(x) D\big]\Big(\sum_{j=1}^{n} a_{jm} e_j\Big) dx.$$

By Propsition 3.1, we have

$$I^{\varepsilon}-J^{\varepsilon}=\int_{\Omega_{\varepsilon}} E(x,\xi)\big[D\big(f(x) D\big)\big]\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big] dx. \tag{10}$$

By Lemma 2.5, as $E(x,\xi)$ satisfies $E(x,\xi) D=0$, we have

$$\begin{aligned} I^{\varepsilon} &=\int_{\Omega_{\varepsilon}} E(x,\xi)\Big\{D\Big[\big(f(x) D\big)\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big)\Big]\Big\} dx \\ &=\int_{\partial\Omega_{\varepsilon}} E(x,\xi) d\sigma_x\Big[\big(f(x) D\big)\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big)\Big] \\ &\quad -\int_{\Omega_{\varepsilon}}\big(E(x,\xi) D\big)\Big[\big(f(x) D\big)\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big)\Big] dx \\ &=\int_{\partial\Omega_{\varepsilon}} E(x,\xi) d\sigma_x\Big[\big(f(x) D\big)\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k) e_j\Big)\Big]. \end{aligned} \tag{11}$$

By Corollary 3.2, we obtain

$$
\begin{aligned}
J^{\varepsilon} &= \int_{\Omega_\varepsilon} \sum_{m=1}^{n} E(x,\xi)e_m\big[f(x)D\big]\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)dx \\
&= \int_{\Omega_\varepsilon} \sum_{m=1}^{n} \Big(-e_mE(x,\xi)+2J_m(x,\xi)\Big)\big[f(x)D\big]\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)dx \\
&= \sum_{m=1}^{n} -e_m \int_{\Omega_\varepsilon} E(x,\xi)\big[f(x)D\big]dx\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) + \int_{\Omega_\varepsilon}\big[f(x)D\big]\Big[\sum_{m=1}^{n} 2J_m(x,\xi)\sum_{j=1}^{n} a_{jm}e_j\Big]dx \\
&= \sum_{m=1}^{n} -e_m \int_{\Omega_\varepsilon} E(x,\xi)\big[f(x)D\big]dx\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) - 2\int_{\Omega_\varepsilon}\big[f(x)D\big]E(x,\xi)dx.
\end{aligned}
$$

By Lemma 2.5, we get

$$
\begin{aligned}
\int_{\Omega_\varepsilon} E(x,\xi)\big[f(x)D\big]dx &= \int_{\Omega_\varepsilon}\big[\widetilde{K}(x,\xi)\overline{D}\big]\big[f(x)D\big]dx = -\int_{\Omega_\varepsilon}\big[\widetilde{K}(x,\xi)D\big]\big[f(x)D\big]dx \\
&= -\Big(\int_{\partial\Omega_\varepsilon}\widetilde{K}(x,\xi)d\sigma_x\big[f(x)D\big] - \int_{\Omega_\varepsilon}\widetilde{K}(x,\xi)\big[D\big(f(x)D\big)\big]dx\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
J^{\varepsilon} = &\sum_{m=1}^{n} e_m \int_{\partial\Omega_\varepsilon}\widetilde{K}(x,\xi)d\sigma_x\big[f(x)D\big]\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) \\
&- \sum_{m=1}^{n} e_m \int_{\Omega_\varepsilon}\widetilde{K}(x,\xi)\big[D\big(f(x)D\big)\big]dx\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) - 2\int_{\Omega_\varepsilon}\big[f(x)D\big]E(x,\xi)dx.
\end{aligned} \tag{12}
$$

By Equalities (10), (11) and (12), we have

$$
\begin{aligned}
I^{\varepsilon} - J^{\varepsilon} &= \int_{\Omega_\varepsilon} E(x,\xi)[D(f(x)D)]\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big]dx \\
&= \int_{\partial\Omega_\varepsilon} E(x,\xi)d\sigma_x\Big[\big(f(x)D\big)\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big)\Big] \\
&\quad - \sum_{m=1}^{n} e_m \int_{\partial\Omega_\varepsilon}\widetilde{K}(x,\xi)d\sigma_x\big[f(x)D\big]\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) \\
&\quad + \sum_{m=1}^{n} e_m \int_{\Omega_\varepsilon}\widetilde{K}(x,\xi)\big[D\big(f(x)D\big)\big]dx\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) + 2\int_{\Omega_\varepsilon}\big[f(x)D\big]E(x,\xi)dx.
\end{aligned} \tag{13}
$$

As $|x_i-\xi_i| \le |x-\xi| = \varepsilon$ and $\rho \ge c_0|x-\xi|$,

$$
|E(x,\xi)| = \frac{1}{\omega_n}\frac{|H(x,\xi)|}{\rho^n} \le \frac{c_1}{\omega_n}\frac{|x-\xi|}{\varepsilon^n} = \frac{c_1}{\omega_n}\frac{1}{\varepsilon^{n-1}},
$$

$$
|\widetilde{K}(x,\xi)| = \frac{|-1|}{(n-2)\omega_n}\frac{1}{\rho^{n-2}} \le \frac{c_2}{\varepsilon^{n-2}},
$$

where c_1 and c_2 are positive constants.

As $f \in C^2\big(\overline{\Omega}, \mathcal{A}_n(2,\alpha_j,\gamma_{ij})\big)$,

$$
\Big|\int_{\partial B(\xi,\varepsilon)} E(x,\xi)d\sigma_x\big[f(x)D\big]\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big]\Big|
$$

$$\leq \int_{\partial B(\xi,\varepsilon)} \left| \frac{c_1}{\omega_n} \frac{1}{\varepsilon^{n-1}} \right| \left| c_3 \varepsilon^{n-1} d\mu_1 \right| \left| c_4 \right| \left| c_5 \varepsilon \right| \leq c_6 \varepsilon,$$

where c_j are positive constants, $j = 3, 4, 5, 6$, so

$$\lim_{\varepsilon \to 0} \int_{\partial B(\xi,\varepsilon)} E(x,\xi) d\sigma_x \left[f(x) D \right] \Big[\sum_{j,k=1}^{n} a_{jk} (x_k - \xi_k) e_j \Big] = 0.$$

Hence

$$\lim_{\varepsilon \to 0} \int_{\partial \Omega_\varepsilon} E(x,\xi) d\sigma_x \left[f(x) D \right] \Big[\sum_{j,k=1}^{n} a_{jk} (x_k - \xi_k) e_j \Big]$$
$$= \int_{\partial \Omega} E(x,\xi) d\sigma_x \left[f(x) D \right] \Big[\sum_{j,k=1}^{n} a_{jk} (x_k - \xi_k) e_j \Big].$$

Similarily, we have

$$\lim_{\varepsilon \to 0} \int_{\partial B(\xi,\varepsilon)} \widetilde{K}(x,\xi) d\sigma_x \left[f(x) D \right] = 0.$$

Hence

$$\lim_{\varepsilon \to 0} \int_{\partial \Omega_\varepsilon} \widetilde{K}(x,\xi) d\sigma_x \left[f(x) D \right] = \int_{\partial \Omega} \widetilde{K}(x,\xi) d\sigma_x \left[f(x) D \right].$$

Let $\varepsilon \to 0$ in Equality (13), and by Lemma 2.6, we draw the conclusion.

Corollary 3.4. (Cauchy integral formula) If $f \in C^2\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$ is an inframonogenic function in Ω, then for any $\xi \in \Omega$, we have

$$\begin{aligned} f(\xi) c(\alpha_j, \gamma_{ij}) =& \mathcal{C}^r_{\partial\Omega} f(\xi) + \mathcal{C}^{infra}_{\partial\Omega} \left[D f(\xi) \right] \\ =& \frac{1}{2} \int_{\partial \Omega} E(x,\xi) d\sigma_x \left[f(\xi) D \right] \Big[\sum_{j,k=1}^{n} a_{jk} (x_k - \xi_k) e_j \Big] \\ & - \frac{1}{2} \sum_{m=1}^{n} e_m \int_{\partial\Omega} \widetilde{K}(x,\xi) d\sigma_x \left[f(x) D \right] \Big(\sum_{j=1}^{n} a_{jm} e_j \Big) + \int_{\partial \Omega} f(x) d\sigma_x E(x,\xi). \end{aligned}$$

Remark 3.5. If we take $\alpha_j = 1$ and $\gamma_{ij} = 0$ for $i \neq j = 1, 2, \ldots, n$ in $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$, we get the Clifford algebra $\mathcal{A}_n$. In this case $B = A = E$, where $a_{ij} = \delta_{ij}$, $c(1,0) = 1$, Equalities (5) and (6) become

$$\widetilde{K}(x,\xi) = \frac{1}{\omega_n} \frac{-1}{n-2} \frac{1}{|x-\xi|^{n-2}}, \; E(x,\xi) = \frac{1}{\omega_n} \frac{\overline{x} - \overline{\xi}}{|x-\xi|^n}, \tag{14}$$

where $E(x,\xi)$ is a fundamental solution to the classical operator D in $\mathcal{A}_n$. And Corollary 3.4 is reduced to the following lemma.

Lemma 3.6. [6] (Cauchy integral formula) If $f \in C^2\big(\overline{\Omega}, \mathcal{A}_n(2, \alpha_j, \gamma_{ij})\big)$ is an inframonogenic function in Ω, then for any $\xi \in \Omega$, we have

$$\begin{aligned} f(\xi) =& \Big[\frac{1}{2} \int_{\partial\Omega} \frac{1}{\omega_n} \frac{\overline{x} - \overline{\xi}}{|x-\xi|^n} d\sigma_x \left[f(\xi) D \right] (x - \xi) \Big] \\ & + \Big[\frac{1}{2} \sum_{m=1}^{n} e_m \int_{\partial\Omega} \frac{1}{\omega_n} \frac{1}{n-2} \frac{1}{|x-\xi|^{n-2}} d\sigma_x \left[f(x) D \right] e_m \Big] \\ & + \Big[\int_{\partial\Omega} f(x) d\sigma_x \frac{1}{\omega_n} \frac{\overline{x} - \overline{\xi}}{|x-\xi|^n} \Big]. \end{aligned}$$

Proposition 3.7. For $x \neq \xi$, the following formulae hold.

(*i*)

$$D_\xi\Big[E(x,\xi)M\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big)\Big] = -\sum_{m=1}^{n} e_m E(x,\xi)M\sum_{j=1}^{n} a_{jm}e_j.$$

(*ii*)

$$D_\xi\Big[\sum_{m=1}^{n} e_m\widetilde{K}(x,\xi)M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)\Big] = \sum_{m=1}^{n} e_m E(x,\xi)M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) + 2ME(x,\xi),$$

where M is any Clifford number.

Proof. (*i*) By Lemma 2.4, we have

$$\begin{aligned}
&D_\xi\Big[E(x,\xi)M\Big(\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big)\Big]\\
=&\Big[D_\xi\Big(E(x,\xi)M\Big)\Big]\Big[\sum_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big] + \sum_{m=1}^{n} e_m\big[E(x,\xi)M\big]\frac{\partial\Big(\sum\limits_{j,k=1}^{n} a_{jk}(x_k-\xi_k)e_j\Big)}{\partial\xi_m}\\
=&-\sum_{m=1}^{n} e_m E(x,\xi)M\sum_{j=1}^{n} a_{jm}e_j.
\end{aligned}$$

(*ii*) By Corollary 3.2, we have

$$\begin{aligned}
&D_\xi\Big[\sum_{m=1}^{n} e_m\widetilde{K}(x,\xi)M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)\Big]\\
=&\Big[D_\xi\widetilde{K}(x,\xi)\Big]\Big[\sum_{m=1}^{n} e_j M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)\Big]\\
=&-E(x,\xi)\Big[\sum_{m=1}^{n} e_m M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)\Big]\\
=&-\sum_{m=1}^{n} E(x,\xi)e_m M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)\\
=&\sum_{m=1}^{n}\Big[e_m E(x,\xi) - 2J_m(x,\xi)\Big]M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big)\\
=&\sum_{m=1}^{n} e_m E(x,\xi)M\Big(\sum_{j=1}^{n} a_{jm}e_j\Big) + 2ME(x,\xi).
\end{aligned}$$

If $f \in C^2\big(\partial\Omega, \mathcal{A}_n(2,\alpha_j,\gamma_{ij})\big)$, we define $F(\xi) = \mathcal{C}_{\partial\Omega}^{infra} f(\xi)$, where $\xi \in \mathbf{R}^n\backslash\partial\Omega$.

Theorem 3.8. If $f \in C^2\big(\partial\Omega, \mathcal{A}_n(2,\alpha_j,\gamma_{ij})\big)$, then

$$D_\xi\big[F(\xi)\big] = -\int_{\partial\Omega} d\sigma_x f(x)E(x,\xi), \tag{15}$$

and F is an inframonogenic function in $\mathbf{R}^n\backslash\partial\Omega$.

Proof. By Proposition 3.7, we obtain

$$
\begin{aligned}
&D_\xi\big[\mathcal{C}^0_{\partial\Omega}f(\xi)\big]\\
=&D_\xi\Big[\int_{\partial\Omega}E(x,\xi)d\sigma_x f(x)\Big(\sum_{j,k=1}^n a_{jk}x_k e_j\Big)-\int_{\partial\Omega}E(x,\xi)d\sigma_x f(x)\Big(\sum_{j,k=1}^n a_{jk}\xi_k e_j\Big)\Big]\\
=&\int_{\partial\Omega}\big[D_\xi E(x,\xi)\big]d\sigma_x f(x)\Big[\sum_{j,k=1}^n a_{jk}x_k e_j\Big]-\int_{\partial\Omega}D_\xi\Big[E(x,\xi)d\sigma_x f(x)\Big(\sum_{j,k=1}^n a_{jk}\xi_k e_j\Big)\Big]\\
=&-\int_{\partial\Omega}D_\xi\Big[E(x,\xi)d\sigma_x f(x)\Big(\sum_{j,k=1}^n a_{jk}\xi_k e_j\Big)\Big]=\int_{\partial\Omega}\sum_{m=1}^n e_m E(x,\xi)d\sigma_x f(x)\Big(\sum_{j=1}^n a_{jm}e_j\Big).
\end{aligned}
$$

By Proposition 3.7, we obtain

$$
\begin{aligned}
D_\xi\Big[\mathcal{C}^1_{\partial\Omega}f(\xi)\Big]&=-D_\xi\Big[\int_{\partial\Omega}\sum_{m=1}^n e_m\widetilde{K}(x,\xi)d\sigma_x f(x)\Big(\sum_{j=1}^n a_{jm}e_j\Big)\Big]\\
&=-\int_{\partial\Omega}D_\xi\Big[\sum_{m=1}^n e_m\widetilde{K}(x,\xi)d\sigma_x f(x)\Big(\sum_{j=1}^n a_{jm}e_j\Big)\Big]\\
&=-\int_{\partial\Omega}\sum_{m=1}^n e_m E(x,\xi)d\sigma_x f(x)\Big(\sum_{j=1}^n a_{jm}e_j\Big)-\int_{\partial\Omega}2d\sigma_x f(x)E(x,\xi).
\end{aligned}
$$

Hence

$$D_\xi\big(F(\xi)\big)=D_\xi\big[\mathcal{C}^{infra}_{\partial\Omega}f(\xi)\big]=-\int_{\partial\Omega}d\sigma_x f(x)E(x,\xi),$$

so F is an inframonogenic function in $\mathbf{R}^n\backslash\partial\Omega$.

Definition 3.9. [11] If for any $x\in\Omega_1$ and any $t\in[0,1]$, we have $tx\in\Omega_1$, then we said that the domain $\Omega_1\subseteq\mathbf{R}^n$ is a star-like domain with centre 0.

From now on, we always suppose that $\Omega_1\subseteq\mathbf{R}^n$ is a star-like domain with centre 0.

Theorem 3.10. If $f\in C^2(\Omega_1,\mathcal{A}_n(2,\alpha_j,\gamma_{ij}))$ is an inframonogenic function in Ω_1, then there exist a unique left monogenic function g and a unique right monogenic function h in Ω_1, such that

$$f(x)=h(x)-2g(x)H(x)+|x|^2_A\big(g(x)D\big),\quad\forall x\in\Omega_1,\tag{16}$$

where $|x|^2_A=\rho^2(x,0)=\sum_{i,j=1}^n a_{ij}x_ix_j$, $H(x)=\sum_{i,j=1}^n\overline{e_i}a_{ij}x_j$. Conversely, if g and h are left and right monogenic functions in Ω_1, respectively, then the function f in Equality (16) is an inframonogenic function in Ω_1.

Proof. If $f\in C^2(\Omega_1,\mathcal{A}_n(2,\alpha_j,\gamma_{ij}))$ is an inframonogenic function in Ω_1, let

$$g(x)=\frac{-1}{4}\int_0^1\big(f(tx)D\big)t^{\frac{n-2}{2}}dt,$$

we have

$$Dg(x)=\frac{-1}{4}\int_0^1 D\big[f(tx)D\big]t^{\frac{n-2}{2}}dt=\frac{-1}{4}\int_0^1\big[D(fD)\big](tx)t^{\frac{n}{2}}dt=0.$$

By Lemma 2.4 and the proof of Proposition 3.1, we have

$$
\begin{aligned}
&\Big[-2g(x)H(x)+|x|_A^2\left(g(x)D\right)\Big]D\\
=&-2g(x)\Big[H(x)D\Big]-2\sum_{k=1}^{n}\frac{\partial g(x)}{\partial x_k}H(x)e_k+\big(g(x)D\big)\Big[|x|_A^2D\Big]-|x|_A^2\big[\widetilde{\triangle}_n g(x)\big]\\
=&-2ng(x)-2\sum_{k=1}^{n}\frac{\partial g(x)}{\partial x_k}\Big[-e_kH(x)+2x_k\Big]+2\big(g(x)D\big)\Big[\sum_{j,k=1}^{n}a_{kj}x_ke_j\Big]\\
=&-2ng(x)-4\sum_{k=1}^{n}\frac{\partial g(x)}{\partial x_k}x_k\\
=&-2ng(x)+\sum_{k=1}^{n}\frac{\partial\int_0^1\big(f(tx)D\big)t^{\frac{n-2}{2}}dt}{\partial x_k}x_k\\
=&-2ng(x)+\int_0^1\sum_{k=1}^{n}x_k\frac{\partial\big(fD\big)}{\partial x_k}(tx)t^{\frac{n}{2}}dt\\
=&-2ng(x)+\int_0^1\frac{d\Big[\big(f(tx)D\big)\Big]}{dt}t^{\frac{n}{2}}dt\\
=&-2ng(x)+\Big[t^{\frac{n}{2}}\left(f(tx)D\right)\Big]\Big|_0^1-\frac{n}{2}\int_0^1\big(f(tx)D\big)t^{\frac{n-2}{2}}dt=f(x)D.
\end{aligned}
$$

Let $h(x)=f(x)-\Big[-2g(x)H(x)+|x|_A^2\left(g(x)D\right)\Big]$, that is, $f(x)=h(x)-2g(x)H(x)+|x|_A^2\left(g(x)D\right)$, then $h(x)D=0$.

Next, we will prove the uniqueness.

Suppose that $Dg(x)=0$ and $h(x)D=0$ in Ω_1 and $h(x)-2g(x)H(x)+|x|_A^2\big(2g(x)D\big)=0$, we only need to prove that $g(x)=0$ and $h(x)=0$.

$$h(x)-2g(x)H(x)+|x|_A^2\big(2g(x)D\big)=0\Rightarrow\Big[h(x)-2g(x)H(x)+|x|_A^2\big(2g(x)D\big)\Big]D=0$$

$$\Rightarrow ng(x)+2\sum_{k=1}^{n}x_k\frac{\partial g(x)}{\partial x_k}=0.$$

For a fixed element $x\in\Omega_1$, let $u(t)=g(tx)$, $t\in[0,1]$. Then $u(t)$ satisfies

$$2t\frac{du(t)}{dt}+nu(t)=0.$$

The unique solution to this equation is $u(t)=0$. So $g(x)=0$, then $h(x)=0$.

Conversely, if g is a left monogenic function and h is a right monogenic function in Ω_1, let $f(x)=h(x)-2g(x)H(x)+|x|_A^2\left(g(x)D\right)$, we have

$$
\begin{aligned}
f(x)D=&\Big[h(x)-2g(x)H(x)+|x|_A^2\left(g(x)D\right)\Big]D\\
=&\Big[-2g(x)H(x)+|x|_A^2\left(g(x)D\right)\Big]D=-2ng(x)-4\sum_{k=1}^{n}x_k\frac{\partial g(x)}{\partial x_k},
\end{aligned}
$$

so

$$D\big[f(x)D\big]=D\Big[-2ng(x)-4\sum_{k=1}^{n}x_k\frac{\partial g(x)}{\partial x_k}\Big]=-2nDg(x)-4Dg(x)-4\sum_{k=1}^{n}x_k\frac{\partial\big(Dg(x)\big)}{\partial x_k}=0.$$

Similarly, we can obtain the following theorem.

Theorem 3.11. If $f\in C^2(\Omega_1,\mathcal{A}_n(2,\alpha_j,\gamma_{ij}))$ is an inframonogenic function in Ω_1, then there exist a unique left monogenic function g_1 and a unique right monogenic function h_1 in Ω_1, such that

$$f(x)=g_1(x)-2H(x)h_1(x)+|x|_A^2\big(Dh_1(x)\big),\quad \forall x\in\Omega_1, \tag{17}$$

where $|x|_A^2=\rho^2(x,0)=\sum\limits_{i,j=1}^{n}a_{ij}x_ix_j$, $H(x)=\sum\limits_{i,j=1}^{n}\overline{e_i}a_{ij}x_j$. Conversely, if g_1 and h_1 are left and right monogenic functions in Ω_1, respectively, then the function f in Equality (17) is an inframonogenic function in Ω_1.

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] W K Clifford. *Applications of Grassman's extensive algebra*, Am J Math, 1878, 1(4): 350-358.

[2] F Brackx, R Delanghe, F Sommen. *Clifford Analysis*, Boston: Pitman Books Limits, 1982.

[3] W Tutschke, C J Vanegas. *Clifford algebras depending on parameters and their applications to partial differential equations*, In: *Some topics on value distribution and differentiability in complex and p-adic analysis*, Beijing: Science Press, 2008, 11: 430-450.

[4] C Balderrama, A Di Teodoro, A Infante. *Some integral representation for meta-monogenic function in clifford algebras depending on parameters*, Adv Appl Clifford Algebras, 2013, 23(4): 793-813.

[5] E Ariza, A Di Teodoro, A Infante, et al. *Fundamental solutions for second order elliptic operators in Clifford-type algebras*, Adv Appl Clifford Algebras, 2015, 25(3): 527-538.

[6] A M García, T M García, R A Blaya, et al. *A Cauchy integral formula for inframonogenic functions in Clifford analysis*, Adv Appl Clifford Algebras, 2017, 27(2): 1147-1159.

[7] A M García, T M García, R A Blaya, et al. *Inframonogenic functions and their applications in 3-dimensional elasticity theory*, Math Methods Appl Sci, 2018, 41(10): 3622-3631.

[8] H Yang, Y Qiao, Y Xie, et al. *Cauchy integral formula for k-monogenic function with α-weight*, Adv Appl Clifford Algebras, 2018, 28(1): 2.

[9] A M García, T M García, R A Blaya, et al. *Decomposition of inframonogenic functions with applications in elasticity theory*, Math Methods Appl Sci, 2020, 43(4): 1915-1924.

[10] R A Blaya, J B Reyes, A M García, et al. *A Cauchy integral formula for infrapolymonogenic functions in Clifford analysis*, Adv Appl Clifford Algebras, 2020, 30(2): 21.

[11] D C Dinh. *On structure of inframonogenic functions*, Adv Appl Clifford Algebras, 2021, 31(3): 50.

[12] D V Cuong, L H Son. *Some new results for function theory in hypercomplex analysis with Parameters*, Adv Appl Clifford Algebras, 2021, 31(3): 37.

[13] D A Santiesteban, R A Blaya, M P Á Alejandre. *On* (φ,ψ)*-inframonogenic functions in Clifford analysis*, Bull Braz Math Soc New Ser, 2022, 53: 605-621.

[14] D C Dinh. *Applications of endomorphisms on Clifford algebras to* (α,β)*-monogenic functions and isotonic functions*, Complex Anal Oper Theory, 2023, 17(2): 21.

[15] C Álvarez, J Morais, R M Porter. *Reduced-quaternion inframonogenic functions on the ball*, Math Methods Appl Sci, 2023, 46(18): 18935-18951.

[16] S Huang, Y Y Qiao, G C Wen. *Real and complex Clifford analysis*, New York: Springer, 2006.

[17] G T Gilbert. *Positive definite matrices and Sylvesters criterion*, Math Assoc Am, 1991, 98(1): 44-46.

[18] G Giorgi. *Various proofs of the Sylvester criterion for quadratic forms*, J Math Res, 2017, 9(6): 55-66.

[1]School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China.

[2]Department of Public Teaching, Hebei Chemical and Pharmaceutical College, Shijiazhuang 050026, China.

Email: xyh1973@126.com