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Approximation properties by Bézier-Baskakov-Schurer-Szász operators based on Gould-Hopper polynomials

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Abstract. In this study, we introduce a Gould-Hopper polynomial based Bézier variation of the Baskakov-Schurer-Szàsz operators. First, we examine the uniform convergence and error bound using the Ditzian-Totik modulus of smoothness. Next, we obtain the quantitative Voronovskaya and Grüss-Voronovskaya type theorems. Furthermore, we investigate the rate of convergence by using weighted modulus of continuity and for a class of Lipschitz function.

§1 Introduction

The approximation theory, which began and gained popularity in the 19th century, is one of the important research areas of mathematical analysis that has been researched by numerous mathematicians all over the world from that century to the present. The basic goal of approximation theory is to represent any function with the aid of simpler functions. The Weierstrass approximation theorem [4] is the foundation of approximation theory, and it has been proven by several mathematicians. Bernstein proved this remarkable result in 1912 by utilising a polynomial sequence and applying ideas and concepts from probability theory [19]. Due to the simple structure and many useful approximation properties of the Bernstein operator, discovery of their various generalizations and modifications in different ways have been an intensive research area. Schurer [1] generalized the Bernstein operators in the following form

$$L_{m,p}(f;x)=\sum_{j=0}^{m+p}\binom{m+p}{j}x^j(1-x)^{m+p-j}\,f\left(\frac{j}{m}\right).$$

Szász [18] extended the Bernstein operators from finite interval to infinite interval and defined the operators as follows:

$$S_m(f;x)=e^{-mx}\sum_{j=0}^{\infty}\frac{(mx)^j}{j!}\,f\left(\frac{j}{m}\right)$$

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for $x \in [0,\infty)$. Baskakov [22] defined the following sequence of linear operators:

$$L_m(f;x) = \frac{1}{(1+x)^m}\sum_{j=0}^{\infty}\binom{m+j-1}{j}\frac{x^j}{(1+x)^j}f\left(\frac{j}{m}\right)$$

for $m \in \mathbb{N}$ and $x \in [0,\infty)$, $\mathbb{N}$ being the set of positive integers.

In 2012, Varma et al. presented a connection between positive linear operators and orthogonal polynomials. They developed Szász operators for Brenke polynomials and showed that these polynomials include Gould-Hopper polynomials in [21]. Later on, many modifications of Szász operator were discovered by utilising various orthogonal polynomials(see [2,3,12,20]).

A generating function of the Gould-Hopper polynomials is given by

$$e^{ht^{d+1}}e^{xt} = \sum_{k=0}^{\infty} g_k^{d+1}(x,h)\frac{t^k}{k!}, \tag{1}$$

where

$$g_k^{d+1}(x,h) = \sum_{s=0}^{\left[\frac{k}{d+1}\right]}\frac{k!}{s!(k-(d+1)s)!}h^s x^{k-(d+1)s}, \tag{2}$$

$h \geq 0$ and $[\cdot]$ denotes the integer part.

The main objective of this article is to study the rate of convergence by using Ditzian-Totik modulus of smoothness, weighted modulus of continuity and for class of Lipschitz function. Furthermore, Voronovskaya, Grüss-Voronovskaya type theorems and difference between the operator and their derivative by modulus of continuity is also established.

§2 Construction of operators

In paper [23], authors have defined the following form of the Baskakov-Scurer-Szász operators

$$L_{n,p}(f;x) = (n+p)\sum_{k=0}^{\infty} b_{n,p}^k(x)\int_0^{\infty} f(t)s_{n,p}^k(t)dt, \tag{3}$$

where $b_{n,p}^k(x) = \binom{n+p+k-1}{k}\frac{x^k}{(1+x)^{n+p+k}}$, $s_{n,p}^k(t) = e^{-(n+p)t}\frac{((n+p)t)^k}{k!}$ and studied the approximation properties.

It is widely known that Bèzier curves are mathematically defined curves that are subsequently utilised in curve fitting, image processing and computer-aided geometric design (CAGD). The various Bèzier variant of the operators is a key topic in approximation theory. The Bézier variation of several operators has been established by many researchers, see for instant[5,6,15,25-28].

For $n \in \mathbb{N}$, $\theta \geq 1$ and all real-valued continuous and bounded functions f, we define the Bézier variant of Baskakov-Schurer-Szàsz operators based on Gould-Hopper polynomials as follows:

$$L_{n,h,p}^{d,\theta}(f;x) = (n+p-1)\sum_{k=0}^{\infty} S_{n,h,p}^{k,d,\theta}(x)\int_0^{\infty} b_{n,p}^k(t)f(t)dt, \tag{4}$$

where

$$S_{n,h,p}^{k,d,\theta}(x) = [R_{n,h,p}^{k,d}(x)]^{\theta} - [R_{n,h,p}^{k+1,d}(x)]^{\theta},$$

$$R_{n,h,p}^{k,d}(x) = \sum_{j=k}^{\infty} K_{n,h,p}^{j,d}(x),$$

$$K^{k,d}_{n,h,p}(x) = e^{-(n+p)x-h}\frac{g^{d+1}_k((n+p)x,h)}{k!}.$$

$R^{k,d}_{n,h,p}(x)$ satisfies the following properties:

(1) $R^{k,d}_{n,h,p}(x) - R^{k+1,d}_{n,h,p}(x) = K^{k,d}_{n,h,p}(x)$,

(2) $R^{0,d}_{n,h,p}(x) > R^{1,d}_{n,h,p}(x) > \cdots > R^{n,d}_{n,h,p}(x) > \cdots$, $x \in [0,\infty)$.

In case $\theta = 1$, operator (4) reduces to operator

$$L^d_{n,h,p}(f;x) = (n+p-1)\sum_{k=0}^{\infty} K^{k,d}_{n,h,p}(x)\int_0^{\infty} b^k_{n,p}(t)f(t)dt. \tag{5}$$

In the sequel we will find moments and central moments of the operators (4)

Lemma 2.1. *For $e_i(t) = t^i$, $i \in \{0,1,2,3,4\}$, then*

(1) $L^d_{n,h,p}(e_0;x) = 1$,

(2) $L^d_{n,h,p}(e_1;x) = \frac{(n+p)x+h(d+1)+1}{n+p-2}$,

(3) $L^d_{n,h,p}(e_2;x) = \frac{1}{(n+p-2)(n+p-3)}\Big((n+p)^2x^2 + (n+p)x\{2h(d+1)+4\} + h(h+1)(d+1)^2 + 3h(d+1)+2\Big)$,

(4) $L^d_{n,h,p}(e_3;x) = \frac{1}{(n+p-2)(n+p-3)(n+p-4)}\Big((n+p)^3x^3 + 3(n+p)^2x^2\{h(d+1)+3\} + 3(n+p)x\{h(h+1)(d+1)^2+5h(d+1)+6\}+h(d+1)^3(h^2+3h+1)+6h(h+1)(d+1)^2+11h(d+1)+6\Big)$,

(5) $L^d_{n,h,p}(e_4;x) = \frac{1}{(n+p-2)(n+p-3)(n+p-4)(n+p-5)}\Big((n+p)^4x^4 + 2(n+p)^3x^3\{2h(d+1)+8\} + (n+p)^2x^2\{6h^2(d+1)^2+6h(d+1)(d+3)+30h(d+1)+72\}+(n+p)x\{6h^2(d+1)^2(2d+3)+2h(d+1)(2d^2+7d+7)+4h^3(d+1)^3+30h^2(d+1)^2+30h(d+1)(d+2)+70h(d+1)+46\}+h(d+1)^4(h^3+6h^2+7h+1)+10h(d+1)^3(h^2+3h+1)+35h(h+1)(d+1)^2+50h(d+1)+24\Big)$.

Proof. Using the generating functions (1) and properties of Beta function, we get above results. □

Lemma 2.2. *Central moments for the operator $L^d_{n,h,p}$ are*

(1) $L^d_{n,h,p}((t-x);x) = \frac{2x+h(d+1)+1}{(n+p-2)}$,

(2) $L^d_{n,h,p}((t-x)^2;x) = \frac{1}{(n+p-2)(n+p-3)}\Big(x^2(n+p+6) + 2x\{(n+p)+3h(d+1)+3\} + h(h+1)(d+1)^2 + 3h(d+1)+2\Big)$,

(3) $L^d_{n,h,p}((t-x)^4;x) = \frac{1}{(n+p-2)(n+p-3)(n+p-4)(n+p-5)}\Big(x^4\{-97(n+p)^2+86(n+p)-120\} + x^3\{12(n+p)^2+(n+p)(52h(d+1)+292)+240(h(d+1)+1)\}+x^2\{12(n+p)^2+(n+p)(6h(h+1)(d+1)^2+138h(d+1)+252)+120(h(h+1)(d+1)^2+3h(d+1)+2)\}+x\{(n+p)(4h^3(d+1)^3+12h^2(d+1)^2(d+2)+2h(d+1)(2d^2+10d+38)-99)+110(h(h+1)(d+1)^2)+11h(d+1)+6\}+h(d+1)^4(h^3+6h^2+7h+1)+10h(d+1)^3(h^2+3h+1)+35h(h+1)(d+1)^2+50h(d+1)+24\Big).$

Proof. From the linearity of $L^d_{n,h,p}$ operators and using Lemma 2.1, one can find the result. □

Lemma 2.3. *For* $e_i(t) = t^i$, $i \in \{0,1,2,3,4\}$, *we have*

(1) $L^{d,\theta}_{n,h,p}(e_0;x) = 1$,

(2) $L^{d,\theta}_{n,h,p}(e_1;x) \le \theta\frac{(n+p)x+h(d+1)+1}{n+p-2}$,

(3) $L^{d,\theta}_{n,h,p}(e_2;x) \le \frac{\theta}{(n+p-2)(n+p-3)}\Big((n+p)^2x^2+(n+p)x\{2h(d+1)+4\}+h(h+1)(d+1)^2+3h(d+1)+2\Big)$,

(4) $L^{d,\theta}_{n,h,p}(e_3;x) \le \frac{\theta}{(n+p-2)(n+p-3)(n+p-4)}\Big((n+p)^3x^3+3(n+p)^2x^2\{h(d+1)+3\}+3(n+p)x\{h(h+1)(d+1)^2+5h(d+1)+6\}+h(d+1)^3(h^2+3h+1)+6h(h+1)(d+1)^2+11h(d+1)+6\Big)$,

(5) $L^{d,\theta}_{n,h,p}(e_4;x) \le \frac{\theta}{(n+p-2)(n+p-3)(n+p-4)(n+p-5)}\Big((n+p)^4x^4+2(n+p)^3x^3\{2h(d+1)+8\}+(n+p)^2x^2\{6h^2(d+1)^2+6h(d+1)(d+3)+30h(d+1)+72\}+(n+p)x\{6h^2(d+1)^2(2d+3)+2h(d+1)(2d^2+7d+7)+4h^3(d+1)^3+30h^2(d+1)^2+30h(d+1)(d+2)+70h(d+1)+46\}+h(d+1)^4(h^3+6h^2+7h+1)+10h(d+1)^3(h^2+3h+1)+35h(h+1)(d+1)^2+50h(d+1)+24\Big).$

Proof. Let us start from

$$\begin{aligned}
L^{d,\theta}_{n,h,p}(e_0;x) &= (n+p-1)\sum_{k=0}^{\infty} S^{k,d,\theta}_{n,h,p}(x)\int_0^{\infty} b^k_{n,p}(t)dt \\
&= \sum_{k=0}^{\infty} S^{k,d,\theta}_{n,h,p}(x) \\
&= \sum_{k=0}^{\infty}\Big\{(R^{k,d}_{n,h,p}(x))^{\theta} - (R^{k,d}_{n,h,p}(x))^{\theta}\Big\} \\
&= \Big(\sum_{j=0}^{\infty} K^{j,d}_{n,h,p}(x)\Big)^{\theta} \\
&= 1.
\end{aligned}$$

Since

$$L^{d,\theta}_{n,h,p}(f;x) = (n+p-1)\sum_{k=0}^{\infty} S^{k,d,\theta}_{n,h,p}(x)\int_0^{\infty} b^k_{n,p}(t)f(t)dt,$$

using the inequality $|a^\beta - b^\beta| \leq \beta|a-b|$ with $0 \leq a, b \leq 1,\ \ \beta \geq 1$ and property (1) of $R^{k,d}_{n,h,p}(x)$, we have

$$\begin{aligned} L^{d,\theta}_{n,h,p}(f;x) &\leq \theta\Big\{(n+p-1)\sum_{k=0}^{\infty} K^{k,d}_{n,h,p}(x)\int_0^\infty b^k_{n,p}(t)f(t)dt\Big\} \\ &\leq \theta L^d_{n,h,p}(f;x). \end{aligned}$$

□

Lemma 2.4. *Let $L^{d,\theta}_{n,h,p}$ be the operator defined by 4, then*

(1) $L^{d,\theta}_{n,h,p}((t-x);x) \leq \theta\frac{2x+h(d+1)+1}{(n+p-2)}$,

(2) $L^{d,\theta}_{n,h,p}((t-x)^2;x) \leq \frac{\theta}{(n+p-2)(n+p-3)}\Big(x^2(n+p+6) + 2x\{(n+p)+3h(d+1)+3\} + h(h+1)(d+1)^2 + 3h(d+1) + 2\Big)$,

(3) $L^{d,\theta}_{n,h,p}((t-x)^4;x) \leq \frac{\theta}{(n+p-2)(n+p-3)(n+p-4)(n+p-5)}\Big(x^4\{-97(n+p)^2+86(n+p)-120\} + x^3\{12(n+p)^2+(n+p)(52h(d+1)+292)+240(h(d+1)+1)\}+x^2\{12(n+p)^2+(n+p)(6h(h+1)(d+1)^2+138h(d+1)+252)+120(h(h+1)(d+1)^2+3h(d+1)+2)\}+x\{(n+p)(4h^3(d+1)^3+12h^2(d+1)^2(d+2)+2h(d+1)(2d^2+10d+38)-99)+110(h(h+1)(d+1)^2)+11h(d+1)+6\}+h(d+1)^4(h^3+6h^2+7h+1)+10h(d+1)^3(h^2+3h+1)+35h(h+1)(d+1)^2+50h(d+1)+24\Big)$.

Proof. Using Lemma 2.3, the proof is established. □

Let us denote by $C_B[0,\infty)$ the space of all bounded continuous functions in $[0,\infty)$ with norm $||f|| = \sup_{x\in[0,\infty)}|f(x)|$.

Proposition 2.1. *Let $f \in C_B(0,\infty)$, then $||L^{d,\theta}_{n,h,p}(f)|| \leq ||f||$.*

Proof. From definition of the operators (4) and Lemma 2.3 we get

$$\begin{aligned} &\left|L^{d,\theta}_{n,h,p}(f;x)\right| \\ =&\left|(n+p-1)\sum_{k=0}^{\infty} S^{k,d,\theta}_{n,h,p}(x)\int_0^\infty b^k_{n,p}(t)f(t)dt\right| \\ \leq& ||f|| \cdot \left|(n+p-1)\sum_{k=0}^{\infty} S^{k,d,\theta}_{n,h,p}(x)\int_0^\infty b^k_{n,p}(t)dt\right| \leq ||f||\left|L^{d,\theta}_{n,h,p}(1;x)\right| = ||f||. \end{aligned}$$

□

Lemma 2.5. *For fixed p, we have*

(1) $\lim_{n\to\infty}(n+p)L^{d,\theta}_{n,h,p}((t-x);x) \leq \theta(2x+h(d+1)+1)$,

(2) $\lim_{n\to\infty}(n+p)L^{d,\theta}_{n,h,p}((t-x)^2;x) \leq \theta\cdot(x^2+2x)$,

(3) $\lim_{n\to\infty}(n+p)^2L^{d,\theta}_{n,h,p}((t-x)^4;x) \leq \theta(-97x^4+12x^3+12x^2)$.

Proof. Proof follows immediately from Lemma 2.4. $\square$

The Korovkin type theorem instead of Weierstrass theorem gives very concrete result in which we can prove where on polynomial can be approximated by function or not. This is achieved by proving approximation of the given function by test functions. This problem was treated by many authors in many papers (see for example [7-11,13,14]). In what follows we will prove Korovkin type theorem for operators given by relation (4).

Theorem 2.6. *For any* $f \in C_B[0,\infty)$, *we have*

$$\lim_{n+p\to\infty} ||L_{n,h,p}^{d,\theta} f - f|| = 0,$$

uniformly in every compact subset $[a,b]$ *of* $[0,\infty)$.

Proof. From basic Korovkin theorem, it is enough to prove that

$$\lim_{n+p\to\infty} ||L_{n,h,p}^{d,\theta} f - f|| = 0,$$

for $f(t) = t^i$ with $i \in \{0,1,2\}$. Using into consideration Lemma 2.3, we obtain

$$\lim_{n+p\to\infty} ||L_{n,h,p}^{d,\theta} e_j - e_j|| = 0,$$

for $j = 0,1,2$, which completes the proof. $\square$

§3 Some direct results

Let $\xi(y) = \sqrt{y(1+y)}$ and $r \in C[0,1]$. For $0 \le \tau \le 1$, define

$$\omega_{\xi^\tau}(r,y) = \sup_{0\le v\le y}\ \sup_{y\pm\frac{v\xi^\tau(y)}{2}\in[0,1]} \left| r\left(y+\frac{v\xi^\tau(y)}{2}\right) - r\left(y-\frac{v\xi^\tau(y)}{2}\right)\right| \tag{6}$$

and the K-functional (see [17])

$$K_{\xi^\tau}(r,j) = \inf_{i\in W_\tau}\left\{||r-i|| + j||\xi^\tau i^{'}||\right\},$$

where

$$W_\tau = \left\{i : i \in AC_{loc}; ||\xi^\tau i^{'}|| < \infty\right\}.$$

Theorem 3.1. *Let* $r \in C_B[0,\infty)$. *Then*

$$|L_{n,h,p}^{d,\theta}(r;y) - r(y)| \le C_3\omega_{\xi^\tau}\left(r, \frac{\xi^{1-\gamma\tau}(y)}{n+p-2}\right),$$

where C_3 *is a positive constant and* $\gamma, \tau \in (0,1)$.

Proof. From Proposition 2.1, we have

$$\begin{aligned}|L_{n,h,p}^{d,\theta}(r;y) - r(y)| &\le |L_{n,h,p}^{d,\theta}(r-g;y)| + \|r(y)-g(y)\| + |L_{n,h,p}^{d,\theta}(g;y) - g(y)|\\ &\le 2||r-g|| + |L_{n,h,p}^{d,\theta}(g;y) - g(y)|.\end{aligned}$$

For $i \in W_\tau$ and using Hölder inequality, we obtain

$$\left|\int_y^t i^{'}(v)dv\right| \le 2^\tau C(y(y+1))^{-\gamma\tau}|t-y| \cdot ||\xi^\tau i^{'}||.$$

From last relation, Proposition 2.1 and Lemma 2.2, we have

$$\begin{aligned}|L_{n,h,p}^{d,\theta}(r;y) - r(y)| &\le (1+\theta)||r-g|| + |L_{n,h,p}^{d,\theta}(g;y) - g(y)|\\ &\le (1+\theta)||r-g|| + 2^\tau C(y(y+1))^{-\gamma\tau}||\xi^\tau i^{'}|| \cdot L_{n,h,p}^{d,\theta}(|t-y|;y).\end{aligned} \tag{7}$$

Applying Cauchy-Schwartz inequality and Lemma 2.4, it follows

$$\begin{aligned}L^{d,\theta}_{n,h,p}(|t-y|;y) \leq & \sqrt{L^{d,\theta}_{n,h,p}(1;y)}\cdot\sqrt{L^{d,\theta}_{n,h,p}(|t-y|^2;y)}\\ \leq & \bigg(\frac{\theta}{(n+p-2)(n+p-3)}\bigg(y^2(n+p+6)+2y\{(n+p)+3h(d+1)+3\}\\ & +h(h+1)(d+1)^2+3h(d+1)+2\bigg)\bigg)^{\frac{1}{2}}.\end{aligned}$$

From last relation it yields

$$\begin{aligned}L^{d,\theta}_{n,h,p}(|t-y|;y) \leq & \bigg(\frac{\theta}{(n+p-2)(n+p-3)}\bigg(y^2(n+p+6)+2y\{(n+p)+3h(d+1)+3\}\\ & +h(h+1)(d+1)^2+3h(d+1)+2\bigg)\bigg)^{\frac{1}{2}}\\ \leq & \bigg(\frac{\theta}{(n+p-2)(n+p-3)}\bigg(2y^2(n+p+6)+2y\{(n+p)+3h(d+1)+3\}\\ & +2y\{h(h+1)(d+1)^2+3h(d+1)+2\}\bigg)\bigg)^{\frac{1}{2}},\end{aligned}$$

and for $n+p$ large we get

$$L^{d,\theta}_{n,h,p}(|t-y|;y) \leq 2C_1\theta\frac{(y+y^2)}{n+p-2},$$

for some $C_1>0$.

Then

$$\begin{aligned}|L^{d,\theta}_{n,h,p}(r;y)-r(y)| &\leq 2||r-g||+2^{\tau}C(y(y+1))^{-\gamma\tau}||\xi^{\tau}i'||\cdot L^{d,\theta}_{n,h,p}(|t-y|;y)\\ &\leq 2||r-g||+2^{\tau}C(y(y+1))^{-\gamma\tau}||\xi^{\tau}i'||\cdot\left|2C_1\theta\frac{(y+y^2)}{n+p-2}\right|\\ &\leq 2||r-g||+C_2\frac{(y(y+1))^{1-\gamma\tau}}{n+p-2}||\xi^{\tau}i'||.\end{aligned}$$

For every $i\in W_{\tau}$, it follows

$$|L^{d,\theta}_{n,h,p}(r;y)-r(y)| \leq C_3K_{\xi^{\tau}}\left(r,\frac{\xi^{1-\gamma\tau}(y)}{n+p-2}\right).$$

Thus, the rest of the proof follows immediately from Theorem 2.1 in [24]. □

Remark 3.2. *In case where* $\tau=0$, *we obtain error estimation in terms of the modulus of continuity as follows:*

$$|L^{d,\theta}_{n,h,p}(r;y)-r(y)| \leq C_3\omega\left(r,\frac{\xi(y)}{n+p-2}\right).$$

§4 Voronovskaya type theorem

One of the problem in approximation theory is approximation of the function with certain type of polynomials with their derivatives. This results is known as Voronovskaya type theorem. Such theorems have been extensively studied in the literature in last decades (see [29-33]). In what follows we will give it for polynomials operators defined by (4).

Theorem 4.1. *For $g \in C_B[0,\infty)$ and fixed p, relation*

$$\left[g'(v)(2v+h(d+1)+1)+\frac{g''(v)}{2}(v^2+2v)\right] \leq \lim_{(n+p)\to\infty}(n+p)\big[L_{n,h,p}^{d,\theta}\big(g,v\big)-g(v)\big]$$
$$\leq \theta\left[g'(v)(2v+h(d+1)+1)+\frac{g''(v)}{2}(v^2+2v)\right],$$

holds true for every $v \in [0,M]$, where M is a finite number.

Proof. By Taylor's theorem, for any $g \in C_B[0,\infty)$ we have

$$g(v)=g(u)+(v-u)g'(u)+\frac{1}{2}(v-u)^2g''(u)+(v-u)^2\phi_g(v),$$

where

$$\phi_g(v)=\begin{cases}\dfrac{g(v)-g(u)-(v-u)g'(u)-\frac{1}{2}(v-u)^2g''(u)}{(v-u)^2}, & (v\neq u),\\ 0, & (v=u),\end{cases}$$

$\phi_g(\cdot)\in C_B[0,\infty)$ and $\phi_g(y)\to 0$ as $y\to u$. Then it follows

$$(n+p)\left[L_{n,h,p}^{d,\theta}\big(g;v\big)-g(v)\right]=g'(u)(n+p)L_{n,h,p}^{d,\theta}\big((v-u);v\big)+\frac{g''(u)}{2}(n+p)L_{n,h,p}^{d,\theta}\big((v-u)^2;v\big)$$
$$+(n+p)L_{n,h,p}^{d,\theta}\big((v-u)^2\phi_g(v);v\big).$$

Then

$$\begin{aligned}&\lim_{(n+p)\to\infty}(n+p)\big[L_{n,h,p}^{d,\theta}\big(g,v\big)-g(v)\big]\\ &\leq \lim_{(n+p)\to\infty}\theta g'(u)(n+p)\left(\frac{2v+h(d+1)+1}{(n+p-2)}\right)\\ &+\lim_{(n+p)\to\infty}\theta\frac{g''(u)}{2}\cdot\frac{n+p}{(n+p-2)(n+p-3)}\bigg(v^2(n+p+6)+2v\{(n+p)\\ &+3h(d+1)+3\}+h(h+1)(d+1)^2+3h(d+1)+2\bigg)\\ &+\lim_{(n+p)\to\infty}(n+p)L_{n,h,p}^{d,\theta}\big((v-u)^2\phi_g(v),v\big).\end{aligned}$$

Applying Cauchy-Schwarz inequality in the following relation

$$(n+p)L_{n,h,p}^{d,\theta}\big((v-u)^2\phi_g(v);v\big)\leq\big\{(n+p)^2L_{n,h,p}^{d,\theta}\big((v-u)^4;v\big)\big\}^{\frac{1}{2}}\big\{L_{n,h,p}^{d,\theta}\big(\phi_g^2(v);v\big)\big\}^{\frac{1}{2}}.$$

By Theorem 2.6 and Lemma 2.5, it yields

$$\big\{L_{n,h,p}^{d,\theta}\big(\phi_g^2(v);v\big)\big\}^{\frac{1}{2}}\to 0$$

as $(n+p)\to\infty$. Then, by using the last relations and Lemma 2.5 it follows

$$\begin{aligned}&\left[g'(v)(2v+h(d+1)+1)+\frac{g''(v)}{2}(v^2+2v)\right]\\ &\leq\lim_{(n+p)\to\infty}(n+p)\big[L_{n,h,p}^{d,\theta}\big(g,v\big)-g(v)\big]\\ &\leq\theta\left[g'(v)(2v+h(d+1)+1)+\frac{g''(v)}{2}(v^2+2v)\right].\end{aligned}$$

□

The Grüss-Voronovskaya type theorem (see [16]) gives behavior of the product of two functions with their derivatives and we prove it for the operators (4).

Theorem 4.2. *Let* $g', g'', k', k'' \in C_B[0,\infty)$ *and* p *be a fixed value. Then*

$$\begin{aligned}&(2v + h(d+1)+1)g'(v)k'(v)\\ \leq &\lim_{n+p\to\infty}(n+p)\left|L^{d,\theta}_{n,h,p}(gk;v) - L^{d,\theta}_{n,h,p}(g,v)L^{d,\theta}_{n,h,p}(k;v)\right|\\ \leq &\theta(2v + h(d+1)+1)g'(v)k'(v),\end{aligned}$$

for each $v \in [0,M]$, *for finite* M.

Proof. The following relation

$$\begin{aligned}&(n+p)\left(L^{d,\theta}_{n,h,p}(gk;v) - L^{d,\theta}_{n,h,p}(g;v)L^{d,\theta}_{n,h,p}(k;v)\right)\\ =&\left[(n+p)\left(L^{d,\theta}_{n,h,p}(gk;v) - gk\right) - \theta(2v+h(d+1)+1)(gk)'(v) - \theta(2v+v^2)\frac{(gk)''(v)}{2}\right]\\ &-k(v)\left[(n+p)\left(L^{d,\theta}_{n,h,p}(g;v) - g(v)\right) - \theta(2v+h(d+1)+1)g'(v) - \theta(2v+v^2)\frac{g''(v)}{2}\right]\\ &-L^{d,\theta}_{n,h,p}(g;v)\left[(n+p)\left(L^{d,\theta}_{n,h,p}(k;v) - k(v)\right) - \theta(2v+h(d+1)+1)k'(v) - \theta(2v+v^2)\frac{k''(v)}{2}\right]\\ &+\theta(2v+h(d+1)+1)g'(v)k'(v) + \theta(2v+v^2)\frac{k''(v)}{2}[g(v) - L^{d,\theta}_{n,h,p}(g;v)]\\ &+\theta(2v+h(d+1)+1)k'(v)[g(v) - L^{d,\theta}_{n,h,p}(g;v)],\end{aligned}$$

is true.

Now our result follows from Theorems 2.6 and 4.1. □

The difference between the operators defined by (4) and their derivatives, measured by modulus of continuity are given by the following result.

Theorem 4.3. *Let* $t, t', t'' \in C_B[0,\infty)$ *and* p *fixed value. Then*

$$\begin{aligned}&\left|(n+p)L^{d,\theta}_{n,h,p}(t;y) - t(y)] - t'(y)L^{d,\theta}_{n,h,p}(v-y;y) - \frac{t''(y)}{2}L^{d,\theta}_{n,h,p}((v-y)^2;y)\right|\\ \leq &O(1)\omega_{\xi^\tau}\left(t''; \frac{1}{\sqrt{n+p}}\right),\ (n+p\to\infty)\end{aligned}$$

$\forall y \in [0,M]$, *where* M *is finite value.*

Proof. From the Taylor's formula

$$t(v) = t(y) + t'(y)(v-y) + \frac{t''(y)}{2}(v-y)^2 + R(v,y),$$

where $R(v,y) = \frac{t''(\tau)-t''(y)}{2}(v-y)^2$, for $\tau \in (v,y)$, Thus

$$\begin{aligned}&\left|L^{d,\theta}_{n,h,p}(t;y) - t(y) - t'(y)L^{d,\theta}_{n,h,p}(v-y;y) - \frac{t''(y)}{2}L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big)\right|\\ \leq &L^{d,\theta}_{n,h,p}(|R(v,y)|;y),\end{aligned}$$

from which it follows that

$$\begin{aligned}&\left|(n+p)\big[L^{d,\theta}_{n,h,p}(t;z) - t(y)\big] - t'(y)L^{d,\theta}_{n,h,p}(v-y;y) - \frac{\mathfrak{t}''(z)}{2}L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big)\right|\\ \leq &(n+p)[L^{d,\theta}_{n,h,p}\big(|R(v,y)|;y\big).\end{aligned}$$

Based on properties of the modulus of continuity, we have

$$\left|\frac{t''(v)-t''(y)}{2!}\right| \leq \frac{1}{2!}\left(1+\frac{|v-y|}{\sigma}\right)\omega_{\xi^\tau}(t'';\sigma),$$

and

$$\left|\frac{t''(v)-t''(y)}{2!}\right| \leq \begin{cases} \omega_{\xi^\tau}(t'';\sigma), & (|v-y|\leq\sigma), \\ \frac{(v-y)^2}{\sigma^2}\omega_{\xi^\tau}(t'';\sigma), & (|v-y|\geq\sigma). \end{cases}$$

For $0<\sigma<1$, it follows

$$\left|\frac{t''(v)-t''(y)}{2!}\right| \leq \omega_{\xi^\tau}(t'';\sigma)\left(1+\frac{(v-y)^2}{\sigma^2}\right),$$

which yields

$$|R(v,y)| \leq \omega_{\xi^\tau}(t'';\sigma)\left(1+\frac{(v-y)^2}{\sigma^2}\right)(v-y)^2 = \omega_{\xi^\tau}(t'';\sigma)\left((v-y)^2+\frac{(v-y)^4}{\sigma^2}\right).$$

By the linearity of $L^{d,\theta}_{n,h,p}$, we obtain

$$L^{d,\theta}_{n,h,p}(|R(v,y)|;y) \leq \omega_{\xi^\tau}(t'';\sigma)\left(L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big)+\frac{1}{\sigma^2}L^{d,\theta}_{n,h,p}\big((v-y)^4;y\big)\right).$$

Now, using into consideration Lemma 2.5, $z\in[0,M]$, we have

$$\begin{aligned} L^{d,\theta}_{n,h,p}(|R(v,y)|;y) &\leq \omega_{\xi^\tau}(t'';\sigma)\left(O\left(\frac{1}{n+p}\right)+\frac{1}{\sigma^2}\,O\left(\frac{1}{(n+p)^2}\right)\right) \\ &= O\left(\frac{1}{n+p}\right)\omega_{\xi^\tau}(t'';\sigma). \end{aligned}$$

For $\sigma=\frac{1}{\sqrt{n+p}}$, we obtain our result. ☐

In what follows we will deal with the following modulus of continuity and smoothness (see [24])

$$\omega_\eta(r;\beta) = \sup_{0<|k|\leqq\beta}\ \sup_{y,y+k\eta(y)\in[0,\infty)}\left\{\left|r\big(y+k\eta(y)\big)-r(y)\right|\right\},$$

and

$$\omega_2^\eta(r;\beta) = \sup_{0<|k|\leqq\beta}\ \sup_{y,y\pm k\eta(y)\in[0,\infty)}\left\{\left|r\big(y+k\eta(y)\big)-2r(y)+r\big(y-k\eta(y)\big)\right|\right\},$$

respectively, where $\eta(y)=\sqrt{y(1+y)}$ is weight function.

K-functional is defined as

$$K_{2,\eta(y)}(r,\beta) = \inf_{h\in W^2(\eta)}\left\{\|r-h\|_{C_B[0,\infty)}+\beta\|\eta^2 r''\|_{C_B[0,\infty)}\right\},$$

where $\beta>0$ and

$$W^2(\eta)=\{h\in C_B[0,\infty): h'\in AC_B[0,\infty),\ \eta^2h''\in C_B[0,\infty)\} \text{ and } h'\in AC_B[0,\infty)$$

means that h' is absolutely continuous on $[0,\infty)$.

Theorem 4.4. *Let $r\in C_B[0,\infty)$ and $y\in(0,\infty)$, then*

$$\left|L^{d,\theta}_{n,h,p}(r;y)-r(y)\right| \leq K_{2,\eta(y)}\left(r,\frac{M^{B,\alpha,\beta}_{s+p,\theta}\big((v-y)^2;y\big)+\left(\frac{y}{n+p-2}\right)^2}{4\rho^2(y)}\right)+\omega_\eta\left(r;\frac{\frac{y}{n+p-2}}{\rho(y)}\right). \quad (8)$$

Proof. We denote by $L^{d,\theta,*}_{n,h,p}(r,y)=L^{d,\theta}_{n,h,p}(r;y)+r(y)-r\big(y+\delta(n,p,\theta,y)\big)$, where

$$\delta(n,p,\theta,y)=\theta\frac{2y+h(d+1)+1}{n+p-2}.$$

Then it follows that

$$L^{d,\theta,*}_{n,h,p}(1;y)=1$$

and

$$L^{d,\theta,*}_{n,h,p}\big((v-y);y\big)\le\theta\frac{2y+h(d+1)+1}{n+p-2}-\delta(n,p,\theta,y)=0.$$

From linearity and positivity of the operators (4) and above relation we get

$$L^{d,\theta,*}_{n,h,p}\big((v-y);y\big)=0.$$

For $k\in W^2(\eta)$ and Taylor's formula, we have

$$k(v)=k(y)+k'(y)(v-y)+\int_y^v(v-w)k''(w)\,\mathrm{d}w,\qquad \big(v\in[0,\infty)\big),$$

which implies

$$\begin{aligned}L^{d,\theta,*}_{n,h,p}(k;y)-k(y)=&L^{d,\theta}_{n,h,p}\left(\int_y^v(v-w)k''(w)\,\mathrm{d}w;y\right)\\&-\int_y^{y+\delta(n,p,\theta,y)}[y+\delta(n,p,\theta,y)-w]k''(w)\,\mathrm{d}w.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\big|L^{d,\theta,*}_{n,h,p}(k;y)-k(y)\big|\\\le&L^{d,\theta}_{n,h,p}\left(\left|\int_y^v(v-w)k''(w)\,\mathrm{d}w\right|;y\right)\\&+\int_y^{y+\delta(n,p,\theta,y)}|y+\delta(n,p,\theta,y)-w|\cdot|k''(w)|\,\mathrm{d}w\\\le&\big\|\eta^2k''\big\|\ L^{d,\theta}_{n,h,p}\left(\left|\int_y^v\frac{|v-w|}{\eta^2(w)}\,\mathrm{d}w\right|;y\right)\\&+\big\|\eta^2k''\big\|\cdot\left|\int_y^{y+\delta(n,p,\theta,y)}\frac{|y+\delta(n,p,\theta,y)-w|}{\eta^2(w)}\,\mathrm{d}w\right|.\end{aligned}$$

In $[0,\infty)$ function η^2 is increasing and from $0<y<w<v$, we have $\frac{|v-w|}{\eta^2(w)}\le\frac{|y-v|}{\eta^2(y)}$ and

$$\begin{aligned}\big\|L^{d,\theta,*}_{n,h,p}k-k\big\|\le&\frac{\|\eta^2k''\|}{\eta^2(y)}\ L^{d,\theta}_{n,h,p}\left(\left|\int_y^v|y-v|\,\mathrm{d}w\right|;y\right)\\&+\frac{\|\eta^2k''\|}{\eta^2(y)}\cdot\left|\int_y^{y+\delta(n,p,\theta,y)}|y+\delta(n,p,\theta,y)-w|\,\mathrm{d}w\right|\\\le&\frac{\|\eta^2k''\|}{\eta^2(y)}\left(L^{d,\theta}_{n,h,p}((v-y)^2;y)+\delta^2(n,p,\theta,y)\right).\end{aligned}$$

Then

$$\begin{aligned}&|L^{d,\theta,*}_{n,h,p}(r;y)-r(y)|\\\le&\big\|L^{d,\theta,*}_{n,h,p}(r-k)\big\|+\big\|L^{d,\theta,*}_{n,h,p}(k)-k\big\|+\|r-k\|+\big|r\big(y+\delta(n,p,\theta,y)\big)-r(y)\big|\end{aligned}$$

$$\leq 4\|r-k\| + \frac{\|\eta^2 k''\|}{\eta^2(y)}\left[L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big) + \delta^2(n,p,\theta,y)\right] + \big|r(y+\delta(n,p,\theta,y)) - r(y)\big|.$$

From relation

$$\begin{aligned}|r(y+\delta(n,p,\theta,y)) - r(y)| &\leq \left|r\left(y+\rho(y)\frac{L^{d,\theta}_{n,h,p}\big((v-y);y\big)}{\rho(y)}\right) - r(y)\right| \\ &\leq \omega_\eta\left(r;\frac{\delta(n,p,\theta,y)}{\rho(y)}\right),\end{aligned}$$

we obtain

$$\begin{aligned}\big|L^{d,\theta}_{n,h,p}(r,y) - r(y)\big| \leq & 4K_{2,\eta(y)}\left(r,\frac{L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big) + \delta^2(n,p,\theta,y)}{4\eta^2(y)}\right) \\ &+ \omega_\eta\left(r;\frac{\delta(n,p,\theta,y)}{\rho(y)}\right). \end{aligned} \tag{9}$$

For $n,p>1$, we have

$$\delta(n,p,\theta,y) = \theta\frac{2y+h(d+1)+1}{n+p-2} \leq C_2\frac{y}{n+p-2},$$

for some constant C_2. From properties of the K-functional, it yields

(1)

$$\begin{aligned}&K_{2,\eta(y)}\left(r,\frac{L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big) + \delta^2(n,p,\theta,y)}{4\eta^2(y)}\right) \\ \leq & K_{2,\eta(y)}\left(r,\frac{L^{d,\theta}_{n,h,p}\big((v-y)^2;y\big) + \left(\frac{y}{n+p-2}\right)^2}{4\eta^2(y)}\right),\end{aligned}$$

(2)

$$\omega_\eta\left(r;\frac{\delta(n,p,\theta,y)}{\rho(y)}\right) \leq \omega_\eta\left(r;\frac{\frac{y}{n+p-2}}{\rho(y)}\right)$$

for every $y\in[0,\infty)$.

Hence

$$\big|L^{d,\theta}_{n,h,p}(r,y) - r(y)\big| \leq K_{2,\eta(y)}\left(r,\frac{M^{B,\alpha,\beta}_{s+p,\theta}\big((v-y)^2;y\big) + \left(\frac{y}{n+p-2}\right)^2}{4\rho^2(y)}\right) + \omega_\eta\left(r;\frac{\frac{y}{n+p-2}}{\rho(y)}\right), \tag{10}$$

as required. □

Behavior of the operators (4) in the Lipschitz-type space is given as follows. Let

$$\mathrm{Lip}_A(\beta) := \left\{r\in C_B[0,\infty) : |r(v)-r(y)| \leqq \mathcal{B}\frac{|v-y|^\beta}{(v+y)^{\frac{\beta}{2}}},\ y\in(0,\infty) \text{ and } v\in(0,\infty)\right\},$$

for positive $\mathcal{B}$, and $\beta\in(0,1]$.

Theorem 4.5. *Let $r\in\mathrm{Lip}_B(\beta)$. Then, for all $v,y\in(0,\infty)$ and $\beta\in(0,1]$,*

$$\big|L^{d,\theta}_{n,h,p}\big(r(v);y\big) - r(y)\big| \leq \frac{\mathcal{B}}{(v+y)^{\frac{\beta}{2}}}\left(C_1\theta(y+y^2)\right)^{\frac{\beta}{2}},$$

for positive constant C_1 and big enough $n+p$.

Proof. Let $r \in \mathrm{Lip}_B(\beta)$ and $\beta \in (0,1]$.

I. For $\beta = 1$, we have

$$\left|L_{n,h,p}^{d,\theta}\big(r(v);y\big) - r(y)\right| \leq L_{n,h,p}^{d,\theta}\big(|r(v)-r(y)|;y\big)$$
$$\leq \mathcal{B} \cdot L_{n,h,p}^{d,\theta}\left(\frac{|v-y|}{(v+y)^{\frac{1}{2}}};y\right) \leq \frac{B}{(v+y)^{\frac{1}{2}}} L_{n,h,p}^{d,\theta}\left(|v-y|;y\right).$$

From Cauchy-Schwarz inequality, we have

$$\left|L_{n,h,p}^{d,\theta}\big(r(v);y\big) - r(y)\right| \leq \frac{\mathcal{B}}{(v+y)^{\frac{1}{2}}} L_{n,h,p}^{d,\theta}\left(|v-y|;y\right) \leq \frac{\mathcal{B}}{(v+y)^{\frac{1}{2}}} \sqrt{L_{n,h,p}^{d,\theta}\big((v-y)^2;y\big)}$$
$$\leq \frac{\mathcal{B}}{(v+y)^{\frac{1}{2}}} \left(\frac{\theta}{(n+p-2)(n+p-3)}\Big(y^2(n+p+6) + 2y\{(n+p) + 3h(d+1) + 3\}\right.$$
$$\left. + h(h+1)(d+1)^2 + 3h(d+1) + 2\Big)\right)^{\frac{1}{2}} \leq \frac{\mathcal{B}}{(v+y)^{\frac{1}{2}}} \left(C_1\theta(y+y^2)\right)^{\frac{1}{2}},$$

for big enough $n+p$ and positive constant C_1.

II. For $\beta \in (0,1)$, we have

$$\left|L_{n,h,p}^{d,\theta}\big(r(v);y\big) - r(y)\right| \leq L_{n,h,p}^{d,\theta}\big(|r(v)-r(y)|;y\big)$$
$$\leq \mathcal{B} \cdot L_{n,h,p}^{d,\theta}\left(\frac{|v-y|^{\beta}}{(v+y)^{\frac{\beta}{2}}};y\right) \leq \frac{\mathcal{B}}{(v+y)^{\frac{\beta}{2}}} L_{n,h,p}^{d,\theta}\left(|v-y|^{\beta};y\right).$$

Applying Hölder inequality for $p = \frac{1}{\beta}$, $q = \frac{1}{1-\beta}$,

$$\left|L_{n,h,p}^{d,\theta}\big(r(v);y\big) - r(y)\right| \leqq \frac{\mathcal{B}}{(v+y)^{\frac{\beta}{2}}} \left[L_{n,h,p}^{d,\theta}\left(|v-y|;y\right)\right]^{\beta}$$

, and by Cauchy-Schwarz inequality, we find

$$\left|L_{n,h,p}^{d,\theta}\big(r(v);y\big) - r(y)\right| \leq \frac{\mathcal{B}}{(v+y)^{\frac{\beta}{2}}} \left[\sqrt{L_{n,h,p}^{d,\theta}\big((v-y)^2;y\big)}\right]^{\beta}$$
$$= \frac{\mathcal{B}}{(v+y)^{\frac{\beta}{2}}} \left\{C_1\theta\left(y+y^2\right)\right\}^{\frac{\beta}{2}},$$

which completes the proof. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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