

Relations for several subfamilies of Rabotnov function-based planar harmonic mappings

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Abstract. This research focuses on the theory of harmonic univalent functions, a significant branch of complex analysis. The primary objective of this paper is to introduce a novel convolution operator incorporating the Rabotnov function and to utilize it to investigate the relationships between various subfamilies of harmonic functions defined in the open unit disk. The Rabotnov fractional exponential function, which serves as a generalization of the Mittag-Leffler function, is a crucial tool in mathematical modeling for material sciences, particularly in viscoelasticity. By applying a specific normalization to the Rabotnov function to obtain $\mathbb{R}_{\kappa, \beta}(z)$, we define a new linear operator F that acts on the analytic and co-analytic parts of harmonic mappings. The study establishes precise conditions under which the operator F maps functions from the classes of harmonic starlike functions $S_{\mathcal{H}}^{*,0}$ and harmonic convex functions $K_{\mathcal{H}}^0$ into the general class $W_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$. These findings are derived through rigorous coefficient inequality analysis and the application of fundamental lemmas regarding harmonic mappings. Furthermore, the paper provides several illuminating corollaries and illustrative examples to demonstrate the geometric implications of the results.

§1 Introduction

Harmonic univalent functions, a prominent class in complex analysis, have been extensively researched due to their applications in conformal mappings and geometric function theory. These functions represent a crucial subject in mathematical analysis as they expand the standard theory of univalent functions by combining both analytic and co-analytic components. In 1984, Clunie and Sheil-Small [5] introduced these functions in their modern form. Furthermore, the significance of these functions is underscored by their utility in engineering sciences and fluid dynamics, where conformal mappings are essential [7].

Coefficient bounds and growth estimates for certain subclasses of harmonic mappings are

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among the earliest results in this field. This line of research was expanded by Sheil-Small [27], who provided a systematic study of these functions, particularly the sense-preserving ones. A deeper understanding of the distortion and covering theorems for such functions was later achieved through the work of Hengartner and Schober [15].

A crucial development in the field was the introduction of various subclasses of harmonic univalent functions, such as the harmonic starlike and convex mappings. Silverman [28] explored the specific conditions under which harmonic functions exhibit geometric properties analogous to their analytic counterparts. More recently, Dziok [8] and Jahangiri et al. [17] investigated the impact of subordination and differential operators in characterizing the structural attributes of these functions. A wide range of recent studies, including those by [3], [4], [9], [18], [24], and [34], provide comprehensive research and foundational insights into harmonic function theory. These collective advances indicate that the study of harmonic univalent functions remains a dynamic and evolving area with significant mathematical and practical implications.

Assume that \mathbb{A} represents the family of functions $u(z) = z + \sum_{j=2}^{\infty} a_j z^j$, which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $u(0) = 0$ and $u'(0) = 1$. A function $u \in \mathbb{A}$ is said to be univalent in \mathbb{D} , if it is one-to-one, meaning the same value is not taken twice. In a domain $D \subset \mathbb{C}$, a function $f(z) = \mu(z) + i\eta(z)$ that is twice continuously differentiable and satisfies $\Delta f(z) = 4f_{z\bar{z}}(z) = 0$ in D is a *complex-valued harmonic function*. When D is simply connected, the canonical representation of f is

$$f(z) = u(z) + \overline{v(z)},$$

where u and v are termed the *analytic* and *co-analytic parts* of f , respectively. f is known to be *locally univalent* in D iff the Jacobian

$$J_f(z) = |u'(z)|^2 - |v'(z)|^2 \neq 0,$$

and f is *sense-preserving* if $J_f(z) > 0$ in \mathbb{D} [19]. Consider \mathcal{H} as the family of complex-valued harmonic functions f with

$$f(0) = 0,$$

and

$$f_z(0) = 1.$$

Here, the power series representations of u and v are

$$u(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad v(z) = \sum_{j=1}^{\infty} b_j z^j, \quad z \in \mathbb{D}.$$

Thus,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j + \overline{\sum_{j=1}^{\infty} b_j z^j}, \quad z \in \mathbb{D}. \tag{1}$$

Define $S_{\mathcal{H}}$ as the family of all harmonic functions in \mathcal{H} that are univalent and sense-preserving on \mathbb{D} . A related subclass, represented by $S_{\mathcal{H}}^0$, is defined as follows:

$$S_{\mathcal{H}}^0 = \{f = u + \bar{v} \in S_{\mathcal{H}} : v'(0) = b_1 = 0\}.$$

These families are first studied in [5]. We also refer to the families $S_{\mathcal{H}}^{*,0}$ and $K_{\mathcal{H}}^0$, which indicate the subfamilies of $S_{\mathcal{H}}^0$ of harmonic starlike and convex functions in \mathbb{D} , respectively [1], [2], and [16].

Moving forward, for a function v given by the Taylor series $v(z) = z + b_2z^2 + \dots = z + \sum_{j=2}^{\infty} b_jz^j$, the convolution $u * v$ is expressed by $(u * v)(z) = z + \sum_{j=2}^{\infty} a_jb_jz^j$.

We will now refer to the families $G_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$, $N_{\mathcal{H}}(\alpha, \Phi)$ established by Porwal et al. [22]. For $0 \leq \Upsilon \leq 1$, $0 \leq \Phi < 1$, and $\alpha \in \mathbb{R}$

$$\Re \left((1 + e^{i\alpha}) \frac{zf'(z)}{(1 - \Upsilon)z + \Upsilon f(z)} - e^{i\alpha} \right) > \Phi, \tag{2}$$

where $z' = \frac{\partial}{\partial \theta}(z = re^{i\theta})$, and $f'(z) = \frac{\partial}{\partial \theta}f(re^{i\theta})$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$. The inequality (2) can be written as

$$\Re \left((1 + e^{i\alpha}) \frac{z(u(z))' - \overline{z(v(z))}'}{(1 - \Upsilon)z + \Upsilon [u(z) + \overline{v(z)}]} - e^{i\alpha} \right) > \Phi, \quad z \in \mathbb{D}. \tag{3}$$

Example 1. For $\Upsilon = 0$, we obtain $G_{\mathcal{H}}(0, \alpha, \Phi) \equiv N_{\mathcal{H}}(\alpha, \Phi)$, meeting the analytical requirement

$$\Re \left((1 + e^{i\alpha}) \frac{zf'(z)}{z'} - e^{i\alpha} \right) > \Phi.$$

Example 2 (See [25]). For $\Upsilon = 1$, we have $G_{\mathcal{H}}(1, \alpha, \Phi) \equiv R_{\mathcal{H}}(\alpha, \Phi)$, meeting the analytical requirement

$$\Re \left((1 + e^{i\alpha}) \frac{zf'(z)}{f(z)} - e^{i\alpha} \right) > \Phi.$$

On the other hand, let

- $GT_{\mathcal{H}}(\Upsilon, \alpha, \Phi) = G_{\mathcal{H}}(\Upsilon, \alpha, \Phi) \cap T_{\mathcal{H}}$,
- $NT_{\mathcal{H}}(\alpha, \Phi) = N_{\mathcal{H}}(\alpha, \Phi) \cap T_{\mathcal{H}}$,
- $RT_{\mathcal{H}}(\alpha, \Phi) = R_{\mathcal{H}}(\alpha, \Phi) \cap T_{\mathcal{H}}$,

where $T_{\mathcal{H}}$ is the subfamily of \mathcal{H} including harmonic functions of the kind

$$f(z) = z - \sum_{j=2}^{\infty} |a_j|z^j + \sum_{j=1}^{\infty} |b_j|\bar{z}^j.$$

Next, we shall utilize some lemmas to demonstrate our theorems.

Lemma 3 (See [22]). $f = u + \bar{v}$ belongs to $G_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$ if and only if

$$\sum_{j=2}^{\infty} \frac{2j - \Upsilon(1 + \Phi)}{1 - \Phi} |a_j| + \sum_{j=1}^{\infty} \frac{2j + \Upsilon(1 + \Phi)}{1 - \Phi} |b_j| \leq 1,$$

where $a_1 = 1$ and $0 \leq \Phi < 1$.

Lemma 4 (See [7]). If $f = u + \bar{v} \in K_{\mathcal{H}}^0$ with $b_1 = 0$, then

$$|a_j| \leq \frac{j+1}{2}, \quad |b_j| \leq \frac{j-1}{2}, \quad j \geq 2.$$

Lemma 5 (See [7]). *If $f = u + \bar{v} \in S_{\mathcal{H}}^{*,0}$ with $b_1 = 0$, then*

$$|a_j| \leq \frac{(2j+1)(j+1)}{6}, \quad |b_j| \leq \frac{(2j-1)(j-1)}{6}, \quad j \geq 2.$$

We also remember some remarks.

Remark 6 (See [22]). *$f \in WT_{\mathcal{H}}(\mathcal{T}, \alpha, \Phi)$ iff*

$$|a_j| \leq \frac{1-\Phi}{2j-\Upsilon(1+\Phi)}, \quad j \geq 2,$$

$$|b_j| \leq \frac{1-\Phi}{2j+\Upsilon(1+\Phi)}, \quad j \geq 1.$$

Remark 7. *Assume $0 \leq \Phi < 1$ and $\alpha \in \mathbb{R}$. Then, the function $f \in NT_{\mathcal{H}}(\alpha, \Phi)$ if and only if the following criteria are satisfied*

$$|a_j| \leq \frac{1-\Phi}{2j} \quad \text{and} \quad |b_j| \leq \frac{1-\Phi}{2j}, \quad j \geq 2.$$

Remark 8 (See [25]). *Let $0 \leq \Phi < 1$ and $\alpha \in \mathbb{R}$. Then $f \in RT_{\mathcal{H}}(\alpha, \Phi)$ if and only if*

$$|a_j| \leq \frac{1-\Phi}{2j-1-\Phi} \quad \text{and} \quad |b_j| \leq \frac{1-\Phi}{2j+1+\Phi}, \quad j \geq 2.$$

§2 Preliminary results

The Mittag-Leffler function, denoted as $E_K(z)$, is a widely used special function in fractional calculus, defined as $E_K(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(Kj+1)}$, ($K, z \in \mathbb{C}$, $\Re(K) > 0$) [20]. This function is further generalized to include two parameters given by [33]

$$E_{K,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(Kj+\beta)}, \quad (K, \beta, z \in \mathbb{C}, \Re(K) > 0, \Re(\beta) > 0).$$

Here, Γ represents the Gamma function. It has since been generalized in various ways over time [6], [10], [14], and [29].

Rabotnov, a prominent researcher in solid mechanics, introduced a remarkable function applied in viscoelasticity [26], contributing significantly to the mathematical modeling of elastic and plastic behavior of materials. This function, now widely recognized as the Rabotnov fractional exponential function (or simply the Rabotnov function), is expressed by

$$R_{K,\beta}(z) = z^K \sum_{j=0}^{\infty} \frac{\beta^j}{\Gamma((j+1)(1+K))} z^{j(1+K)}, \quad (K, \beta, z \in \mathbb{C}).$$

This series converges for all values of z . Notably, when $K = 0$, the Rabotnov function reduces to the standard exponential function $\exp(\beta z)$. Moreover, the Rabotnov function can be expressed as a specific case of the Mittag-Leffler function, a cornerstone in fractional calculus, through the relation

$$R_{K,\beta}(z) = z^K E_{1+K,1+K}(\beta z^{1+K}).$$

The Rabotnov function $R_{K,\beta}(z)$ clearly does not belong to the class \mathbb{A} . Consequently, it

makes sense to take into account the subsequent normalization [10]

$$\mathbb{R}_{K,\beta}(z) = z^{\frac{1}{1+K}} \Gamma(1+K) R_{K,\beta} \left(z^{\frac{1}{1+K}} \right) = z + \sum_{j=2}^{\infty} \frac{\beta^{j-1} \Gamma(1+K)}{\Gamma((1+K)j)} z^j. \tag{4}$$

Utilizing the concept of convolution, the family of analytic functions is expressed as

$$\mathcal{R} = \mathbb{A} * \mathbb{R}_{K,\beta} = \left\{ \mathcal{RF} : \mathcal{RF}(z) = (f * \mathbb{R}_{K,\beta})(z) = z + \sum_{j=2}^{\infty} \frac{\beta^{j-1} \Gamma(1+K)}{\Gamma((1+K)j)} a_j z^j, f \in \mathbb{A} \right\}.$$

While formula (4) is valid for complex-valued K, β , and $z \in \mathbb{C}$, in this paper, we focus exclusively on the case where $K \geq 0, \beta > 0$, and $z \in \mathbb{D}$.

Theorem 9. (See [10]) Let $K \geq 0$ and $\beta > 0$. If $K \geq 4\beta - 1$, then the normalized Rabotnov function $\mathbb{R}_{K,\beta}(z)$ is starlike in \mathbb{D} .

Example 10. Let $K = 0$ and $\beta = 1/4$. Next, let us consider the function

$$f_1(z) = ze^{\frac{1}{4}z}, \quad z \in \mathbb{D}.$$

The figure below illustrates the image of \mathbb{D} under $f_1(z)$.

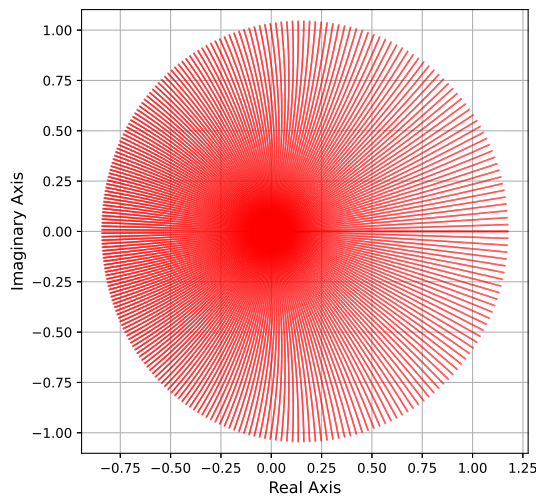


Figure 1. The transformation of \mathbb{D} under $f_1(z)$.

Theorem 11. (See [10]) Let $K \geq 0$ and $\beta > 0$. If $\frac{\beta}{1+K} < 0.199496$, then the normalized Rabotnov function $\mathbb{R}_{K,\beta}(z)$ is convex in \mathbb{D} .

Example 12. Let $K = 1$ and $\beta = 1/4$. Next, let us consider the function

$$f_2(z) = 2\sqrt{z} \sinh \frac{\sqrt{z}}{2}, \quad z \in \mathbb{D}.$$

The figure below illustrates the image of \mathbb{D} under $f_2(z)$.

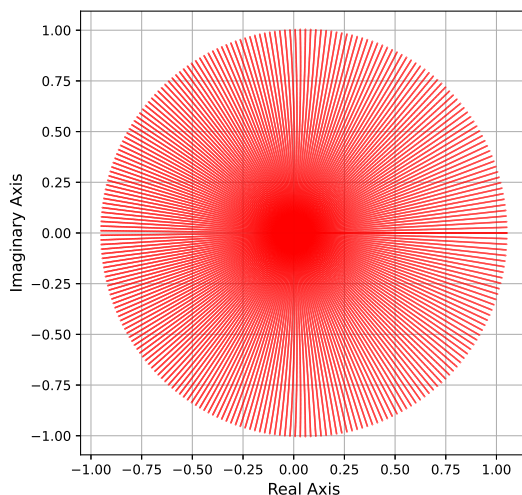


Figure 2. The transformation of \mathbb{D} under $f_2(z)$.

Convolution serves as the foundation for a novel linear operator that is defined as

$$\begin{aligned}
 F(z) = F(f)(z) &= u(z) * \mathbb{R}_{K,\beta}(z) + \overline{v(z) * \mathbb{R}_{M,\delta}(z)} \\
 &= z + \sum_{j=2}^{\infty} \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} a_j z^j + z + \overline{\sum_{j=2}^{\infty} \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)} b_j z^j}.
 \end{aligned}
 \tag{5}$$

Regarding the harmonic function families, including different linear and nonlinear operators, a number of authors have recently investigated mapping properties and inclusion results [11], [12], [13], [21], [23], [30], [31], [32], [35], and [36]. Inspired by the previously published works, we present a new convolution operator F associated with the Rabotnov function in our present paper to determine some relations between the families $S_{\mathcal{H}}^{*,0}$, $K_{\mathcal{H}}^0$, and $W_{\mathcal{H}}(\gamma, \alpha, \Phi)$.

The following notations

$$\mathbb{R}_{K,\beta}(1) - 1 = \sum_{j=2}^{\infty} \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)},
 \tag{6}$$

$$\mathbb{R}'_{K,\beta}(1) - 1 = \sum_{j=2}^{\infty} \frac{j\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)},
 \tag{7}$$

$$\mathbb{R}''_{K,\beta}(1) = \sum_{j=2}^{\infty} \frac{j(j-1)\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)},
 \tag{8}$$

and

$$\mathbb{R}'''_{K,\beta}(1) = \sum_{j=2}^{\infty} \frac{j(j-1)(j-2)\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} \quad (9)$$

will be used in the proofs of theorems for simplicity's sake.

§3 Main results

Theorem 13. Assume $0 \leq \Phi < 1$. If

$$\begin{aligned} & 2\mathbb{R}''_{K,\beta}(1) + (4 - \Upsilon(1 + \Phi))\mathbb{R}'_{K,\beta}(1) - \Upsilon(1 + \Phi)\mathbb{R}_{K,\beta}(1) \\ & + 2\mathbb{R}''_{M,\delta}(1) + \Upsilon(1 + \Phi)\mathbb{R}'_{M,\delta}(1) - \Upsilon(1 + \Phi)\mathbb{R}_{M,\delta}(1) \\ & \leq 2[(1 - \Upsilon)(1 + \Phi) + 2(1 - \Phi)] \end{aligned}$$

is true, then we have the inclusion $F(K_{\mathcal{H}}^0) \subset W_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$.

Proof. Assuming $b_1 = 0$, consider $f = u + \bar{v} \in K_{\mathcal{H}}^0$. As stated in (5), where $b_1 = 0$, our goal is to demonstrate that $F(z) = F(f)$ belongs to $W_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$. Therefore, Lemma 3 is sufficient to illustrate the inequality

$$\begin{aligned} f_1 &= \sum_{j=2}^{\infty} [2j - \Upsilon(1 + \Phi)] \left| \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} a_j \right| \\ &+ \sum_{j=2}^{\infty} [2j + \Upsilon(1 + \Phi)] \left| \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)} b_j \right| \leq 1 - \Phi. \end{aligned}$$

Using Lemma 4, we now establish the relationship between harmonic convex functions and $W_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$

$$\begin{aligned} f_1 &\leq \frac{1}{2} \left\{ \sum_{j=2}^{\infty} (2j - \Upsilon(1 + \Phi))(j+1) \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} \right. \\ &\quad \left. + \sum_{j=2}^{\infty} (2j + \Upsilon(1 + \Phi))(j-1) \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)} \right\}. \end{aligned}$$

Simplifying further, we have

$$\begin{aligned} f_1 &= \sum_{j=2}^{\infty} \left[j - \frac{\Upsilon}{2}(1 + \Phi) \right] (j+1) \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} \\ &+ \sum_{j=2}^{\infty} \left[j + \frac{\Upsilon}{2}(1 + \Phi) \right] (j-1) \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)}. \end{aligned}$$

After expressing $j^2 = (j-1)(j-2) + 3(j-1) + 1$ and $j = (j-1) + 1$, we arrive

$$\begin{aligned} f_1 &= \sum_{j=2}^{\infty} \left[(j-1)(j-2) + \left(4 - \frac{\Upsilon(1 + \Phi)}{2} \right) (j-1) \right] \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} \\ &+ \sum_{j=2}^{\infty} 2 \left(1 - \frac{\Upsilon(1 + \Phi)}{2} \right) \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} \end{aligned}$$

$$+ \sum_{j=2}^{\infty} \left[(j-1)(j-2) + \left(2 + \frac{\gamma(1+\Phi)}{2} \right) (j-1) \right] \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)}.$$

Now, using (6), (7) and (8), we have

$$\begin{aligned} f_1 \leq & \mathbb{R}''_{K,\beta}(1) - 2\mathbb{R}'_{K,\beta}(1) + 2\mathbb{R}_{K,\beta}(1) + \left(4 - \frac{\gamma}{2}(1+\Phi) \right) [\mathbb{R}'_{K,\beta}(1) - \mathbb{R}_{K,\beta}(1)] \\ & + 2 \left(1 - \frac{\gamma}{2}(1+\Phi) \right) [\mathbb{R}_{K,\beta}(1) - 1] + \mathbb{R}''_{M,\delta}(1) - 2\mathbb{R}'_{M,\delta}(1) + 2\mathbb{R}_{M,\delta}(1) \\ & + \left(2 + \frac{\gamma}{2}(1+\Phi) \right) [\mathbb{R}'_{M,\delta}(1) - \mathbb{R}_{M,\delta}(1)]. \end{aligned}$$

An easy computation shows that

$$\begin{aligned} f_1 = & \mathbb{R}''_{K,\beta}(1) + \left(2 - \frac{\gamma}{2}(1+\Phi) \right) \mathbb{R}'_{K,\beta}(1) - \frac{\gamma}{2}(1+\Phi)\mathbb{R}_{K,\beta}(1) \\ & + \mathbb{R}''_{M,\delta}(1) + \frac{\gamma}{2}(1+\Phi)\mathbb{R}'_{M,\delta}(1) - \frac{\gamma}{2}(1+\Phi)\mathbb{R}_{M,\delta}(1) \\ \leq & (1-\gamma)(1+\Phi) + 2(1-\Phi). \end{aligned}$$

The final expression does not exceed $1 - \Phi$ when the stated condition holds. □

Following the analytical framework of Theorem 13, we establish the characterizing conditions that govern the relationship between the class of harmonic starlike functions $S_{\mathcal{H}}^{*,0}$ and the family $W_{\mathcal{H}}(\gamma, \alpha, \Phi)$.

Theorem 14. Assume $0 \leq \Phi < 1$. If

$$\begin{aligned} & 4\mathbb{R}''_{K,\beta}(1) + 2[9 - \gamma(1+\Phi)]\mathbb{R}'_{K,\beta}(1) + [12 - 5\gamma(1+\Phi)]\mathbb{R}'_{K,\beta}(1) - \gamma(1+\Phi)\mathbb{R}_{K,\beta}(1) \\ & + 4\mathbb{R}''_{M,\delta}(1) + 2[3 + \gamma(1+\Phi)]\mathbb{R}'_{M,\delta}(1) - \gamma(1+\Phi)\mathbb{R}'_{M,\delta}(1) + \gamma(1+\Phi)\mathbb{R}_{M,\delta}(1) \\ & \leq 3[4 - (1+\gamma)(1+\Phi)] \end{aligned}$$

is fulfilled, then $F(S_{\mathcal{H}}^{*,0}) \subset W_{\mathcal{H}}(\gamma, \alpha, \Phi)$.

Proof. Assume $f = u + \bar{v} \in S_{\mathcal{H}}^{*,0}$ with $b_1 = 0$. We aim to prove that $F(z) = F(f) \in W_{\mathcal{H}}(\gamma, \alpha, \Phi)$. Using Lemma 3, it is sufficient to show that

$$\begin{aligned} f_2 = & \sum_{j=2}^{\infty} [2j - \gamma(1+\Phi)] \left| \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} a_j \right| \\ & + \sum_{j=2}^{\infty} [2j + \gamma(1+\Phi)] \left| \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)} b_j \right| \\ \leq & 1 - \Phi. \end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned} f_2 \leq & \frac{1}{3} \left\{ \sum_{j=2}^{\infty} \left(j - \frac{\gamma}{2}(1+\Phi) \right) (2j+1)(j+1) \frac{\beta^{j-1}\Gamma(1+K)}{\Gamma((1+K)j)} \right. \\ & \left. + \sum_{j=2}^{\infty} \left(j + \frac{\gamma}{2}(1+\Phi) \right) (2j-1)(j-1) \frac{\delta^{j-1}\Gamma(1+M)}{\Gamma((1+M)j)} \right\}. \end{aligned}$$

Next, expressing $j^3 = (j-1)(j-2)(j-3) + 6(j-1)(j-2) + 7(j-1) + 1$, $j^2 = (j-1)(j-2) + 3(j-1) + 1$ and $j = (j-1) + 1$, we arrive

$$\begin{aligned}
 f_2 = \frac{1}{3} & \left\{ 2 \sum_{j=2}^{\infty} (j-1)(j-2)(j-3) \frac{\beta^{j-1} \Gamma(1+K)}{\Gamma((1+K)j)} \right. \\
 & + [15 - \Upsilon(1+\Phi)] \sum_{j=2}^{\infty} (j-1)(j-2) \frac{\beta^{j-1} \Gamma(1+K)}{\Gamma((1+K)j)} \\
 & + \left[24 - \frac{9\Upsilon}{2}(1+\Phi) \right] \sum_{j=2}^{\infty} (j-1) \frac{\beta^{j-1} \Gamma(1+K)}{\Gamma((1+K)j)} \\
 & + 6 \left[1 - \frac{\Upsilon}{2}(1+\Phi) \right] \sum_{j=2}^{\infty} \frac{\beta^{j-1} \Gamma(1+K)}{\Gamma((1+K)j)} \\
 & + 2 \sum_{j=2}^{\infty} (j-1)(j-2)(j-3) \frac{\delta^{j-1} \Gamma(1+M)}{\Gamma((1+M)j)} \\
 & + [9 + \Upsilon(1+\Phi)] \sum_{j=2}^{\infty} (j-1)(j-2) \frac{\delta^{j-1} \Gamma(1+M)}{\Gamma((1+M)j)} \\
 & \left. + \left[6 + \frac{3\Upsilon}{2}(1+\Phi) \right] \sum_{j=2}^{\infty} (j-1) \frac{\delta^{j-1} \Gamma(1+M)}{\Gamma((1+M)j)} \right\}.
 \end{aligned}$$

Using (6), (7), (8) and (9), we find

$$\begin{aligned}
 f_2 \leq \frac{1}{3} & \left\{ 2 [\mathbb{R}'''_{K,\beta}(1) - 3\mathbb{R}''_{K,\beta}(1) + 6\mathbb{R}'_{K,\beta}(1) - 6\mathbb{R}_{K,\beta}(1)] \right. \\
 & + [15 - \Upsilon(1+\Phi)] [\mathbb{R}''_{K,\beta}(1) - 2\mathbb{R}'_{K,\beta}(1) + 2\mathbb{R}_{K,\beta}(1)] \\
 & + \left[24 - \frac{9\Upsilon}{2}(1+\Phi) \right] [\mathbb{R}'_{K,\beta}(1) - \mathbb{R}_{K,\beta}(1)] + 6 \left[1 - \frac{\Upsilon}{2}(1+\Phi) \right] [\mathbb{R}_{K,\beta}(1) - 1] \\
 & + 2 [\mathbb{R}'''_{M,\delta}(1) - 3\mathbb{R}''_{M,\delta}(1) + 6\mathbb{R}'_{M,\delta}(1) - 6\mathbb{R}_{M,\delta}(1)] \\
 & + [9 + \Upsilon(1+\Phi)] [\mathbb{R}''_{M,\delta}(1) - 2\mathbb{R}'_{M,\delta}(1) + 2\mathbb{R}_{M,\delta}(1)] \\
 & \left. + \left[6 + \frac{3\Upsilon}{2}(1+\Phi) \right] [\mathbb{R}'_{M,\delta}(1) - \mathbb{R}_{M,\delta}(1)] \right\},
 \end{aligned}$$

or, equivalently

$$\begin{aligned}
 f_2 = \frac{1}{3} & \left\{ 2\mathbb{R}'''_{K,\beta}(1) + [9 - \Upsilon(1+\Phi)] \mathbb{R}''_{K,\beta}(1) + \left[6 - \frac{5\Upsilon}{2}(1+\Phi) \right] \mathbb{R}'_{K,\beta}(1) \right. \\
 & - \frac{\Upsilon}{2}(1+\Phi) \mathbb{R}_{K,\beta}(1) + [-6 + 3\Upsilon(1+\Phi)] \\
 & \left. + 2\mathbb{R}'''_{M,\delta}(1) + [3 + \Upsilon(1+\Phi)] \mathbb{R}''_{M,\delta}(1) - \frac{\Upsilon}{2}(1+\Phi) \mathbb{R}'_{M,\delta}(1) + \frac{\Upsilon}{2}(1+\Phi) \mathbb{R}_{M,\delta}(1) \right\} \\
 & \leq 1 - \Phi.
 \end{aligned}$$

This completes the proof. \square

Theorem 15. Assume $0 \leq \Phi < 1$. If

$$\mathbb{R}_{K,\beta}(1) + \mathbb{R}_{M,\delta}(1) \leq 2$$

is true, then $F(WT_{\mathcal{H}}(\Upsilon, \alpha, \Phi)) \subset W_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$.

Proof. Consider $f = u + \bar{v} \in WT_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$. Utilizing Lemma 3, it is sufficient to establish that

$$f_3 = \sum_{j=2}^{\infty} [2j - \Upsilon(1 + \Phi)] \left| \frac{\beta^{j-1}\Gamma(1 + K)}{\Gamma((1 + K)j)} a_j \right| + \sum_{j=2}^{\infty} [2j + \Upsilon(1 + \Phi)] \left| \frac{\delta^{j-1}\Gamma(1 + M)}{\Gamma((1 + M)j)} b_j \right| \leq 1 - \Phi.$$

By Remark 8, we have

$$f_3 \leq (1 - \Phi) \left\{ \sum_{j=2}^{\infty} \frac{\beta^{j-1}\Gamma(1 + K)}{\Gamma((1 + K)j)} + \sum_{j=1}^{\infty} \frac{\delta^{j-1}\Gamma(1 + M)}{\Gamma((1 + M)j)} \right\}.$$

By using equation (6), finally we get

$$f_3 \leq (1 - \Phi) [\mathbb{R}_{K,\beta}(1) - 1 + \mathbb{R}_{M,\delta}(1)],$$

which satisfies the given condition. □

Setting $\Upsilon = 0$ in Theorems 13 and 14 yields

Corollary 16. Assume $0 \leq \Phi < 1$. If

$$\mathbb{R}''_{K,\beta}(1) + 2\mathbb{R}'_{K,\beta}(1) + \mathbb{R}''_{M,\delta}(1) \leq 3 - \Phi$$

is true, then we have the inclusion $F(K_{\mathcal{H}}^0) \subset N_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$.

Corollary 17. Assume $0 \leq \Phi < 1$. If

$$\begin{aligned} &4\mathbb{R}'''_{K,\beta}(1) + 18\mathbb{R}''_{K,\beta}(1) + 12\mathbb{R}'_{K,\beta}(1) + 4\mathbb{R}'''_{M,\delta}(1) + 6\mathbb{R}''_{M,\delta}(1) \\ &\leq 3(3 - \Phi) \end{aligned}$$

is true, then $F(\mathcal{S}_{\mathcal{H}}^{*,0}) \subset N_{\mathcal{H}}(\alpha, \Phi)$.

§4 Conclusion

In this research, we have successfully established rigorous inclusion relations between the harmonic function families $W_{\mathcal{H}}(\Upsilon, \alpha, \Phi)$ and the subfamilies $\mathcal{S}_{\mathcal{H}}^{*,0}$ and $K_{\mathcal{H}}^0$. By introducing the Rabotnov-based convolution operator F , we have bridged the gap between special functions used in fractional calculus and geometric function theory.

The development of this operator framework allows for the systematic study of harmonic mappings by leveraging the properties of the normalized Rabotnov function. Specifically, Theorems 13, 14, and 15 provide the exact mathematical thresholds required for various harmonic subclasses to satisfy the inclusion criteria of the $W_{\mathcal{H}}$ family.

Our geometric insights confirm that the choice of parameters K and β in the Rabotnov function directly influences the starlikeness and convexity of the resulting harmonic transformations. These results offer a foundation for researchers to explore new relations for other harmonic families using different distribution series or derivative operators.

Research on the integration of the Rabotnov function into harmonic univalent function theory remains an active area with strong mathematical and practical implications. This is particularly relevant in engineering sciences and fluid dynamics, where conformal mappings are essential for modeling physical behaviors.

Declarations

Conflict of interest The authors declare no conflict of interest.

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