Appl. Math. J. Chinese Univ.
2026, 41(2): 255-268

A novel similarity measure between classical propositions and its applications

YU Peng

Abstract. The main purpose of this paper is to establish a kind of quantitative model by means of the cosine similarity degree for distinguishing the reliability of different propositional formulas in classical logic system L. Firstly, we define the concept of cosine similarity degree between two formulas using a propositional vector, which can be used to measure the consistency of finite theories. Then we investigate the basic properties of cosine similarity degree and prove the set of all cosine similarity degrees is dense in the interval [0,1]. Finally, we propose the concept of cosine truth degree of a formula for evaluating the reliability of different formulas.

§1 Introduction

In propositional logic system L, given that two propositional formulas $p \to p$ and $\neg(p \to p)$, then the former is always a correct proposition, while the later is always a false proposition. But as for the atomic propositional formula p, it can't be said that it is correct or false, because it is half correct and half false, which naturally raises the question of whether a propositional formula is reliable. Obviously, propositional formula $p \to p$ is reliable, whereas propositional formula $\neg(p \to p)$ is unreliable, with the reliability degree of propositional formula p reaching $\frac{1}{2}$, because only half of it is correct.

In fact, as early as 1952, Roser and Turquette have proposed the question of how to distinguish the reliability of a propositional formula in the literature [1]. Subsequently, Pavelka came up with the truth value assignment method in his series of papers On fuzzy logic: I, II, III[2]. This method is a good response to the extent to which a formula is true. At the same time, Hailperin and Nilsson respectively introduced the idea of probability into two-valued logic to reflect the truth degree of formulas, thus forming the theory of probability logic [3−7].

In recent years, Wang proposed a new quantitative method to describe the reliability of propositional formulas in his series of articles[8−11]. Specifically speaking, he introduced the

Received: 2020-06-08. Revised: 2023-11-13.
MR Subject Classification: 03B05, 03B50.
Keywords: quantitative logic, approximate reasoning, classical logic system, similarity degree.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-026-4174-z.
Supported by the National Natural Science Foundation of China (12571490).

concept of truth degree of a propositional formula using the weighted average of the assignment of a propositional formula, and realized the reliability description of the propositional formula, which led to a series of grade reasoning research[12−16], such as quantitative models in rough logic[17−19], quantitative models in multi-valued model logic[20] and probabilistically quantitative models in multi-valued logic and fuzzy logic[21−24]. In literature [25,26], the author pointed out that the distance between propositional formulas in the quantitative logic was essentially the standard hamming distance between the formula fuzzy sets (which were induced by the propositional formula itself), and that the truth degree of a propositional formula A was essentially the distance between propositional formula A and tautology T.

At the same time, the cosine similarity degree between vectors is a common method of implementing text recognition by computers. In this paper, we will first introduce the concept of description vector of a propositional formula. Then by calculating the cosine similarity between vectors, we will bring up a new method to describe the reliability of the propositional formula, which is called the cosine truth degree. Last, as a specific application of the cosine truth degree, we will adopt a method of characterizing the compatibility of finite proposition sets, and make a new attempt to study the structure of the propositional set.

§2 Cosine similarity degree between two formulas

Generally speaking, there are two kinds of main approaches in the study of propositional logic. One is the syntax method, and the other is the semantic method. The so-called syntax method means the formal deduction from set A of axioms (sometimes from an additional set Γ of assumptions) using the rules of inference. In the meantime, the semantic method of a propositional logic system provides an alternative approach to evaluating the soundness of formulas by means of valuation domain I and the concept of valuation.

In this paper we only consider the semantic method of classical propositional logic system L. In system L, its valuation domain $I = \{0, 1\}$, the operations on I are defined as $\neg 0 = 1, \neg 1 = 0, x \to y = 0$ if and only if $x = 1$ and $y = 0$, and $F(S)$ denotes the set of all formulas.

Definition 2.1 (i) A homomorphism $v : F(S) \to I$ of type $(\neg, \to)$ from $F(S)$ into I, i.e., $v(\neg A) = \neg v(A), v(A \to B) = v(A) \to v(B)$, is called a valuation of $F(S)$. The set of all valuations of $F(S)$ will be denoted by Ω;

(ii) A formula $A \in F(S)$ is called a tautology if $v(A) = 1$ for every valuation $v \in \Omega$. A is called a contradiction if $v(A) = 0$ for every valuation $v \in \Omega$;

(iii) $A, B \in F(S)$, if $v(A) = v(B)$ for every valuation $v \in \Omega$, and then A and B are considered logically equivalent.

Assume that $A = A(p_1, p_2, \cdots, p_n)$ is a formula generated by propositional variables $p_1, p_2, \cdots, p_n$ through connectives $\neg$ and $\to$. Substitute x_i for p_i in A and keep the logical connectives in A unchanged but explain them as the corresponding operators defined on I. Then we attain a function $\bar{A} : \{0, 1\}^n \to \{0, 1\}$ and call $\bar{A}(x_1, \cdots, x_n)$ the truth valued function of A. It is well known that in classical logic system L, $\bar{A}(x_1, \cdots, x_n)$ is a Boolean function.

Proposition 2.1[8] Any n-ary Boolean function can be derived from a well-formed formula ($n \in N$).

Assume that $S_m = \{p_1, p_2, \cdots, p_m\}$ is a finite atomic propositional set. $F(S_m)$ represents the formulas generated by S_m, while Ω_m represents the set of all valuations of $F(S_m)$. Let $v_i = v(p_i)(i = 1, 2, \cdots, m)$, and then we obtain a vector $\tilde{v} = (v_1, \cdots, v_m)$ in I^m. Conversely, for every $\tilde{v} = (v_1, \cdots, v_m) \in I^m$, there exists only one $v \in \Omega_m$, such that $v(p_i) = v_i$. Hence there is a one-to-one mapping between v and $\tilde{v}$, and Ω_m can be expressed as $\Omega_m = \{x_1, \cdots, x_{2^m}\}(x_i \in I^m)$.

Furthermore, let $A \in F(S_m), x_i \in \Omega_m$, so we acquire a vector $(x_1(A), x_2(A), \cdots, x_{2^m}(A))$. We call vector $(x_1(A), x_2(A), \cdots, x_{2^m}(A))$ as the description vector of formula A, denoted by x_A with x_i as the valuation vector of A. For example, taking $S_2 = \{p, q\}$, in this case, the valuation vectors of $p \vee q$ are $(0, 0), (0, 1)$, $(1, 0)$,$(1, 1)$, and the description vector $x_{p\vee q}$ is $(0, 1, 1, 1)$.

Proposition 2.2 Supposing that α is a 2^n-dimensional 0-1 vector, then α is a description vector of a certain formula.

Proposition 2.3 The logically equivalent formulas in $F(S)$ have the same description vector.

Definition 2.2 Letting $A(p_1, \cdots, p_n), B(p_1, \cdots, p_n) \in F(S) - \{\bar{0}\}$, define

$$\xi_{Cos}(A, B) = \frac{x_A \cdot x_B}{||x_A|| \cdot ||x_B||},$$

where $x_A \cdot x_B$ denotes the vector inner-product between x_A and x_B, $||x_A||$ is the norm of x_A. Then $\xi_{Cos}(A, B)$ is called the cosine propositional similarity degree between propositional formulas A and B, and is referred to as the cosine similarity degree.

Remark 2.1 (i) In the above definition, if formulas A and B are either a contradiction or a non-tautological satisfiable formula, then the cosine similarity degree between A and B is defined as $\xi_{Cos}(A, B) = \xi_{Cos}(\neg A, \neg B)$. If formulas A and B are either a contradiction or a tautology, then the cosine similarity degree between A and B is defined as $\xi_{Cos}(A, B) = 0$.

(ii) The reason why $\xi_{Cos}(A, B)$ is called as the cosine similarity degree is that $\xi_{Cos}(A, B)$ is essentially the cosine of the angle between vectors described by formulas A and B. Using the cosine of the angle between vectors to represent the similarity between vectors is a common method of defining similarity. Introducing it into the study of mathematical logic to reflect the differences between different formulas undoubtedly injects new vitality into the study of quantitative logic. Therefore, this is research of great significance.

(iii) The definition of the cosine similarity degree $\xi_{Cos}(A, B)$ between formulas A and B is different from that of the similarity degree in Ref.[8]. In Ref.[8], the similarity degree is the number of truth-assignments satisfying $v(A) = v(B)$, divided by 2^n. In comparison, the cosine similarity degree between formulas A and B in this paper is just the number of truth-assignments satisfying $A \wedge B$, divided by the product of the square roots of the truth-assignments satisfying A and B respectively.

(iv) The cosine similarity defined in this paper originates from the structure of the two formulas themselves, rather than the specific meaning represented by them. Therefore, the similarity between the formulas defined in this paper differs from the Non-metric Propositional Similarity proposed in Ref. [27], which focuses on a notion of similarity among propositions based on similarity neither of linguistic expression nor of subject matter but of truth-conditions, avoiding any metric assumptions.

Example 2.1 (i) Calculate the cosine similarity degree between formulas p and q;

(ii) Calculate the cosine similarity degree between formulas $p \vee q \vee r$ and $(p \vee \neg q) \wedge r$.

Solution (i) To compute the cosine similarity degree between p and q, the formulas p and q must be converted into logically equivalent formulas. Since $p \approx p \wedge (q \vee \neg q), q \approx q \wedge (p \vee \neg p)$, then $\xi_{Cos}(p, q) = \xi_{Cos}(p \wedge (q \vee \neg q), q \wedge (p \vee \neg p)) = \frac{1}{2}$.

(ii) The valuation vectors of formulas $p \vee q \vee r$ and $(p \vee \neg q) \wedge r$ are $(0, 0, 0)$, $(0, 0, 1), (0, 1, 0)$, $(0, 1, 1)$,$(1, 0, 0)$, $(1, 0, 1), (1, 1, 0)$, and $(1, 1, 1)$, and its corresponding description vectors are $(0, 1, 1, 1,\ 1, 1, 1, 1)$, and $(0, 1, 0, 0, 0, 1, 0, 1)$. Hence, $\xi_{Cos}(p \vee q \vee r, (p \vee \neg q) \wedge r) = \frac{3}{\sqrt{21}}$.

As below, $|A^{-1}(1)|$ and $|A^{-1}(0)|$ represent the number of components 1 and 0 in the description vector x_A, respectively.

Proposition 2.4 Suppose that $A, B, C \in F(S)$, and then

(i) $0 \leq \xi_{Cos}(A, B) \leq 1$;

(ii)$\xi_{Cos}(A, B) = \xi_{Cos}(B, A)$;

(iii)$\xi_{Cos}(A, B) = 1$ if and only if $A \approx B$;

(iv)If $A \approx \neg B$, then $\xi_{Cos}(A, B) = 0$, but not vice versa;

(v)If $A \to B$ and $B \to C$ are tautologies, then $\xi_{Cos}(A, C) \leq min\{\xi_{Cos}(A, B), \xi_{Cos}(B, C)\}$;

(vi)If $A \to C$ and $B \to C$ are tautologies, then $\xi_{Cos}(A \wedge B, C) \leq min\{\xi_{Cos}(A,\ A \wedge B), \xi_{Cos}(A, C), \xi_{Cos}(B, A \wedge B), \xi_{Cos}(B, C)\}$.

Proof (i)(ii) It immediately follows from Definition 2.4.

(iii) On the one hand, if $A \approx B$, then $x_A = x_B$. We have $x_{A \wedge B} = x_A = x_B$, and $\xi_{Cos}(A, B) = \frac{|(A \wedge B)^{-1}(1)|}{\sqrt{|A^{-1}(1)|}\sqrt{|B^{-1}(1)|}} = \frac{|A^{-1}(1)|}{\sqrt{|A^{-1}(1)|}\sqrt{|A^{-1}(1)|}} = 1$.

On the other hand, suppose that $\xi_{Cos}(A, B) = 1$, but $A \approx B$ does not holds. Then at least one of $|(A \wedge B)^{-1}(1)| < |A^{-1}(1)|$ and $|(A \wedge B)^{-1}(1)| < |B^{-1}(1)|$ holds. Without loss of generality, let $|(A \wedge B)^{-1}(1)| < |A^{-1}(1)|$, and then $(|(A \wedge B)^{-1}(1)|)^2 < |A^{-1}(1)| \cdot |B^{-1}(1)|$. We obtain $\xi_{Cos}(A, B) < 1$, contradiction!

(iv)If $A \approx \neg B$, then $x_A = x_{\neg B}$. In this case, A and B have the opposite description vectors, that is, the component with value 1 of vector x_A is 0 relative to vector x_B, so $\xi_{Cos}(A, B) = x_A \cdot x_B = 0$, and the opposite direction does not holds. Taking q and $p \wedge \neg q$ for example, although $\xi_{Cos}(q, p \wedge \neg q) = 0$, but they are not logically equivalent.

(v)Since $A \to B$ and $B \to C$ are tautologies, we have $A \wedge C \approx A$,$|A^{-1}(1)| \leq |B^{-1}(1)|$ and $|B^{-1}(1)| \leq |C^{-1}(1)|$.

$$\xi_{Cos}(A, C) \quad = \quad \xi_{Cos}(A, B),$$

Or, $\xi_{Cos}(A, C) \leq \sqrt{\frac{|B^{-1}(1)|}{|C^{-1}(1)|}} = \xi_{Cos}(B,\ C)$.

(vi)Since $A \wedge B \to A$ and $A \to C$ are tautologies, we have $\xi_{Cos}(A \wedge B, C) \leq min\{\xi_{Cos}(A, A \wedge B), \xi_{Cos}(A, C)\}$. Similarly, $\xi_{Cos}(A \wedge B, C) \leq min\{\xi_{Cos}(B, A \wedge B), \xi_{Cos}(B, C)\}$. Hence we have $\xi_{Cos}(A \wedge B, C) \leq min\{\xi_{Cos}(A, A \wedge B),\ \xi_{Cos}(A, C),\ \xi_{Cos}(B, A \wedge B), \xi_{Cos}(B, C)\}$. □

Remark 2.2 Judging from Proposition 2.4, there is a significant difference in the similarity degree between the cosine similarity degree defined in this paper and the similarity degree based on formula truth in quantitative logic. In quantitative logic $A \approx \neg B$ if and only if $\xi_{Cos}(A,B) = 0$. However, the above conclusion is no longer valid in this paper. In addition, neither is $\xi_{Cos}(A,B) + \xi_{Cos}(B,C) \leq 1 + \xi_{Cos}(A,B)$ in quantitative logic valid in this paper. For example, taking the formulas $A = p, B = p \vee q$ and $C = q$, then $\xi_{Cos}(A,B) = \dfrac{2}{\sqrt{6}}$, $\xi_{Cos}(B,C) = \dfrac{2}{\sqrt{6}}$, $\xi_{Cos}(A,C) = \dfrac{1}{2}$, $\xi_{Cos}(A,B) + \xi_{Cos}(B,C) = \dfrac{4}{\sqrt{6}} \geq 1 + \xi_{Cos}(A,C) = \dfrac{3}{2}$, making the above inequality invalid. As a result, the distance of the formula set cannot be constructed by means of $1 - \xi_{Cos}(A,B)$.

Theorem 2.1 The set $\Theta = \{\xi_{Cos}(A,B) | A, B \in F(S)\}$ is dense in interval [0,1].

Proof It is only necessary to prove that for any $\frac{l}{\sqrt{k}\sqrt{t}}(l \leq min\{k,t\}, k, t \leq 2^n, n = 1, 2, \cdots)$, $\frac{l}{\sqrt{k}\sqrt{t}} \in \Theta$ holds.

Suppose that A and B are formulas containing n atomic formulas.

When $n = 1$, $\frac{l}{\sqrt{k}\sqrt{t}}$ has the following values $0, \frac{1}{\sqrt{1}\sqrt{1}}, \frac{1}{\sqrt{2}\sqrt{1}}, \frac{1}{\sqrt{1}\sqrt{2}}$, and $\frac{2}{\sqrt{2}\sqrt{2}}$. Since $\xi_{Cos}(A,B) = \xi_{Cos}(B,A)$, hence $\frac{1}{\sqrt{2}\sqrt{1}}$ and $\frac{1}{\sqrt{1}\sqrt{2}}$ will be treated equally. We construct formulas $p \wedge \neg p, p$ and $p \vee \neg p$, then $\xi_{Cos}(p \wedge \neg p, p) = 0, \xi_{Cos}(p,p) = 1, \xi_{Cos}(p, p \vee \neg p) = \frac{1}{\sqrt{1}\sqrt{2}}$, and $\xi_{Cos}(p \vee \neg p, p \vee \neg p) = \frac{2}{\sqrt{2}\sqrt{2}}$. We have $0, \frac{1}{\sqrt{1}\sqrt{1}}, \frac{1}{\sqrt{2}\sqrt{1}}, \frac{2}{\sqrt{2}\sqrt{2}} \in \Theta$.

Assuming that $n = m$, then for any $\frac{l}{\sqrt{k}\sqrt{t}}, \frac{l}{\sqrt{k}\sqrt{t}} \in \Theta$, i.e., for each $\frac{l}{\sqrt{k}\sqrt{t}}$, there exist formulas $A, B \in F(S)$, such that $\xi_{Cos}(A,B) = \frac{l}{\sqrt{k}\sqrt{t}}$.

When $n = m+1$, we select the 2^{m+1}-dimensional vectors

$$x_A = (\underbrace{1, 1, \cdots, 1}_{k}, 0, \cdots, 0)$$

and

$$x_B = (\underbrace{0, \cdots, 0}_{k-l}, \underbrace{1, 1, \cdots, 1}_{t}, 0, \cdots, 0).$$

It is apparent to see that $x_A \wedge x_B = (\underbrace{0, \cdots, 0}_{k-l}, \underbrace{1, 1, \cdots, 1}_{l}, 0, \cdots, 0)$. If we view x_A, x_B and $x_A \wedge x_B$ as the description vectors of formulas A, B and $A \wedge B$, respectively, then $\xi_{Cos}(A,B) = \frac{|(A \wedge B)^{-1}(1)|}{\sqrt{|A^{-1}(1)|}\sqrt{|B^{-1}(1)|}} = \frac{l}{\sqrt{k}\sqrt{t}}$.

In a word, $\Theta = \{\xi_{Cos}(A,B) | A, B \in F(S)\}$ is dense in [0, 1]. □

Theorem 2.2 Supposing that $\Theta = \{\xi_{Cos}(A,B) | A, B \in F(S)\}$, then

$$\Theta = \{\frac{l}{\sqrt{k}\sqrt{t}} | l \leq min\{k,t\}, k, t \leq 2^n, n = 1, 2, \cdots\}.$$

Proof A consequence of Theorem 2.1. □

Theorem 2.3 Supposing that $A, B \in F(S), \xi_{Cos}(A,B) = a$, then there is a formula sequence $\{A_i\}$, such that $\xi_{Cos}(A_i, B) = a_i$, where a_i satisfies $a \geq a_1 \geq a_2 \geq \cdots \geq a_n = 0$. Furthermore, $A_i \to A_{i-1}$ is a tautology.

Proof Firstly, we prove that for any non-negative integers a, b, and c, if $1 \leq a \leq min\{b, c\}$, then

$$\frac{a}{\sqrt{b}\sqrt{c}} \geq \frac{a-1}{\sqrt{b}\sqrt{c-1}} \geq \frac{a-2}{\sqrt{b}\sqrt{c-2}} \geq \cdots \geq \frac{a-i}{\sqrt{b}\sqrt{c-i}} (0 \leq i \leq a).$$

In fact, when $c \geq a \geq 1$,

$$\begin{aligned} & a^2(c-1)-(a-1)^2c \\ = & -a^2+2ac-c \\ \geq & -a^2+a^2+ac-c \\ = & (a-1)c \\ \geq & 0. \end{aligned}$$

We have $\frac{a}{\sqrt{c}} \geq \frac{a-1}{\sqrt{c-1}}$. Furthermore, $\frac{a}{\sqrt{b}\sqrt{c}} \geq \frac{a-1}{\sqrt{b}\sqrt{c-1}}$. Hence, the above inequality sequence holds.

Now, we prove Theorem 2.3. Assume that

$$x_A = (\underbrace{0,\cdots,0}_{k-l},\underbrace{1,1,\cdots,1}_{t},0,\cdots,0),$$
$$x_B = (\underbrace{1,1,\cdots,1}_{k},0,\cdots,0).$$

Then $\xi_{Cos}(A,B) = a = \frac{l}{\sqrt{k}\sqrt{t}}$. We select 2^n-dimensional vectors $c_i = (c_1,\cdots,c_{2^n})(i=1,\cdots,l)$ as follows

$$c_i(j) = \begin{cases} 0, & j=1,\cdots,k-l+i,k+t-l+1,\cdots,2^n, \\ 1, & j=k-l+i+1,\cdots,k+t-l. \end{cases}$$

Then the vcetor c_i satisfies $||c_i|| = t-i$, and $c_i \cdot x_B = l-i$. In this case, each component $c_i(j)$ of c_i satisfies $c_i(j) \leq c_{i-1}(j)$. If c_i is the description vector of A_i, then $\xi_{Cos}(A_i,B) = \frac{c_i \cdot x_B}{||c_i||\cdot||x_B||} = \frac{l-i}{\sqrt{k}\sqrt{t-i}}$ and $A_i \to A_{i-1}$ is a tautology. Let $\frac{l-i}{\sqrt{k}\sqrt{t-i}} = a_i$, and then $a \geq a_1 \geq a_2 \geq \cdots \geq a_n = 0(n=l)$. □

Theorem 2.4 Supposing that $A,B \in F(S), \xi_{Cos}(A,B) = a$, then there is a formulas sequence $\{A_i\}$ such that $\xi_{Cos}(A_i,B) = a_i$, where a_i satisfies the conditions $a \geq a_1 \geq a_2 \geq \cdots \geq a_n$, and $\xi_{Cos}(A_1,T) = \xi_{Cos}(A_2,T) = \cdots = \xi_{Cos}(A_n,T)$.

Proof The proof is similar to Theorem 2.3. □

Theorem 2.4 shows that for a finite formula set Λ, we can use the cosine similarity degree to give a division of Λ from tautology to contradiction, thus naturally forming a kind of division of the formula set Λ.

§3 Application of cosine similarity degree in describing the consistency of finite theories

In this section, we will use the cosine similarity degree to measure the consistency of finite theories.

Definition 3.1 Suppose that Γ is a theory of $F(S)$. If contradiction is a conclusion of Γ, we call Γ as an inconsistent theory otherwise Γ is a consistent theory.

Definition 3.2 Supposing that Γ is a theory of $F(S)$, and $D(\Gamma)$ denotes the set of all conclusions of Γ, then

$$dim(\Gamma) = 1 - inf\{\xi_{Cos}(A,B)|A,B \in D(\Gamma)\},$$

is called the divergence degree of Γ, and Γ is said to be fully divergent if $dim(\Gamma) = 1$.

Remark 3.1 (i) In above definition, since the tautology T belongs to each $D(\Gamma)$, we have the cosine similarity degree between tautology and contradiction is 0. Therefore, if contradiction is the conclusion of $D(\Gamma)$, then $dim(\Gamma) = 1$.

(ii) When Γ contains only one formula A, $D(\Gamma)$ is abbreviated as $D(A)$.

Theorem 3.1 Supposing that $\Gamma = \{A_1, \cdots, A_n\}$, and $A_1, \cdots, A_n$ contains n different atomic formulas, then

$$dim(\Gamma) = 1 - \frac{2|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)|}{2^n + |(A_1 \wedge \cdots \wedge A_n)^{-1}(1)|}.$$

In order to prove this theorem, we need to prove Lemma 3.1 and Lemma 3.2 separately.

Lemma 3.1 Suppose that $\Gamma = \{A_1, \cdots, A_n\} \subseteq F(S)$. $D(\Gamma)|_{p_1,\cdots,p_n}$ denotes the limit of $D(\Gamma)$ on atomic formulas set $\{p_1, \cdots, p_n\}$. Taking A from $D(\Gamma)|_{p_1,\cdots,p_n}$, then $\forall B \in D(\Gamma)|_{p_1,\cdots,p_n}$, $\xi_{Cos}(\neg A \vee (A_1 \wedge \cdots \wedge A_n), A) \leq \xi_{Cos}(A, B)$.

Proof Suppose that $|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)| = a$, $A, B \in D(\Gamma)|_{p_1,\cdots,p_n}$. We have

$$|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)| \leq |A^{-1}(1)|, |(A_1 \wedge \cdots \wedge A_n)^{-1}(1)| \leq |B^{-1}(1)|.$$

Furthermore, let $|A^{-1}(1)| = a + x (1 \leq x \leq 2^n - a)$, $|B^{-1}(1)| = a + y (1 \leq y \leq 2^n - a)$.

In this case,

$\xi_{Cos}(\neg A \vee (A_1 \wedge \cdots \wedge A_n), A) = \frac{|(A \wedge (\neg A \vee (A_1 \wedge \cdots \wedge A_n))^{-1}(1)|}{\sqrt{|A^{-1}(1)|}\sqrt{|(\neg A \vee (A_1 \wedge \cdots \wedge A_n))^{-1}(1)|}} = \frac{a}{\sqrt{a+x}\sqrt{2^n - x}}$.

$\xi_{Cos}(A, B) = \frac{a+z}{\sqrt{a+x}\sqrt{a+y}} (0 \leq z \leq min\{x, y\})$, where z represents the number of $|(A \wedge B)^{-1}|$ minus the number of $|(A_1 \wedge \cdots \wedge A_n)^{-1}|$, i.e, $z = |(A \wedge B)^{-1}| - |(A_1 \wedge \cdots \wedge A_n)^{-1}|$.

We need to consider the following two cases:

(i) $z = 0$. In this case, there must be $y \leq 2^n - a - x$, otherwise $z > 1$.

$$\begin{aligned} & \xi_{Cos}(A, B) \\ = & \frac{a}{\sqrt{a+x}\sqrt{a+y}} \\ \geq & \frac{a}{\sqrt{a+x}\sqrt{a+2^n-a-x}} \\ = & \frac{a}{\sqrt{a+x}\sqrt{2^n-x}} \\ = & \xi_{Cos}(\neg A \vee (A_1 \wedge \cdots \wedge A_n), A). \end{aligned}$$

(ii)$z \geq 1$. If $B \to A$ is a tautology, then $z = y, (A \wedge B)^{-1} = a + z$, $\xi_{Cos}(A, B) = \frac{a+z}{\sqrt{a+x}\sqrt{a+z}}$. Since $2^n - x \geq a$, we have $(a + z)\sqrt{2^n - x} - a\sqrt{a + z} \geq 0$.

Furthermore,

$$\begin{aligned} & \xi_{Cos}(A, B) - \xi_{Cos}(\neg A \vee (A_1 \wedge \cdots \wedge A_n), A) \\ = & \frac{a+z}{\sqrt{a+x}\sqrt{a+z}} - \frac{a}{\sqrt{a+x}\sqrt{2^n-x}} \\ = & \frac{(a+z)\sqrt{2^n-x} - a\sqrt{a+z}}{\sqrt{a+x}\sqrt{a+z}\sqrt{2^n-x}} \\ \geq & 0. \end{aligned}$$

We have $\xi_{Cos}(\neg A \vee (A_1 \wedge \cdots \wedge A_n), A) \leq \xi_{Cos}(A, B)$.

If $A \to B$ is a tautology, then $x \leq y$, $\xi_{Cos}(A, B) = \frac{a+x}{\sqrt{a+x}\sqrt{a+y}} = \sqrt{\frac{a+x}{a+y}}$. In this case, $2^n - x = a + h$, so we have $h + x = 2^n - a \geq y$, $\xi_{Cos}(\neg A \vee (A_1 \wedge \cdots \wedge A_n), A) = \frac{a}{\sqrt{a+x}\sqrt{a+h}}$.

In addition,

$$\begin{aligned}&(\sqrt{\tfrac{a+x}{a+y}})^2-(\tfrac{a}{\sqrt{a+x}\sqrt{a+h}})^2\\=&\tfrac{(a+x)^2(a+h)-a^2(a+y)}{(a+x)(a+y)(a+h)}\\=&\tfrac{a^2(h+2x-y)+x(2ah+ax+xh)}{(a+x)(a+y)(a+h)}\\\geq&\ 0.\end{aligned}$$

We obtain the inequality

$$\xi_{Cos}(A,B)\geq\xi_{Cos}(\neg A\vee(A_1\wedge\cdots\wedge A_n),A).$$

When $A\to B$ and $B\to A$ are not tautologies, $\xi_{Cos}(A,B)=\frac{a+z}{\sqrt{a+x}\sqrt{a+y}}(z<min\{x,y\})$. In this case, $2^n-x=a+h$.

Moreover, $h+x=2^n-a, h+z\geq y$,

$$(\frac{a+z}{\sqrt{a+x}\sqrt{a+y}})^2-(\frac{a}{\sqrt{a+x}\sqrt{a+h}})^2=\frac{a^2(h+2z-y)+2azh+(a+h)z^2}{(a+x)(a+y)(a+h)}\geq 0,$$

we get $\xi_{Cos}(A,B)\geq\xi_{Cos}(\neg A\vee(A_1\wedge\cdots\wedge A_n),A)$.

In summary, for any $A,B\in D(\Gamma)|_{p_1,\cdots,p_n}$, we have $\xi_{Cos}(\neg A\vee(A_1\wedge\cdots\wedge A_n),A)\leq\xi_{Cos}(A,B)$. $\square$

Lemma 3.2 Suppose that $A=A(p_1,p_2,\cdots,p_n)$ is a formula containing n atomic formulas $p_1,p_2,\cdots,p_n$, $\tilde{A}=A\wedge(\bigwedge\limits_{i=1}^{m}(p_{n+i}\to p_{n+i}))(m>n)$,

(i) If $|A^{-1}(1)|$ is an even number, then $inf\{\xi_{Cos}(B,C)|B,C\in D(A)|_{p_1,\cdots,p_n}\}=inf\{\xi_{Cos}(B,C)|B,C\in D(\tilde{A})|_{p_1,\cdots,p_{n+m}}\}(m\geq 1)$,

(ii) If $|A^{-1}(1)|$ is an odd number, then $inf\{\xi_{Cos}(B,C)|B,C\in D(A)|_{p_1,\cdots,p_n}\}>inf\{\xi_{Cos}(B,$ T $C)|B,C\in D(\tilde{A})|_{p_1,\cdots,p_{n+m}}\}(m\geq 1)$.

Proof (i) By Lemma 3.1, we have

$$inf\{\xi_{Cos}(B,C)|B,C\in D(A)|_{p_1,\cdots,p_n}\}=inf\{\xi_{Cos}(B,\neg B\vee A)|B\in D(A)|_{p_1,\cdots,p_n}\}.$$

Let $|A^{-1}(1)|=a$ be an even number. Judging from $B\in D(\Gamma)_{p_1,\cdots,p_n}$, we acquire

$$|B^{-1}(1)|=a+x(1\leq x\leq 2^n-a),\xi_{Cos}(B,\neg B\vee A)=\frac{a}{\sqrt{a+x}\sqrt{2^n-x}}.$$

In order to make $\xi_{Cos}(B,\neg B\vee A)$ reach the minimum value, $(a+x)(2^n-x)$ is required to take the maximum value. According to the properties of the quadratic function of one variable, if $x=\frac{2^n-a}{2}$, then function $\frac{a}{\sqrt{a+x}\sqrt{2^n-x}}$ takes the maximum value. But x is an integer, so when $\frac{2^n-a}{2}$ is an integer, formula B satisfies $|B^{-1}(1)|=a+x=\frac{2^n+a}{2^n}$, and $\xi_{Cos}(B,\neg B\vee A)$ take minimum value. If $\frac{2^n-a}{2}$ is not an integer, the formula B either satisfies $|B^{-1}(1)|=a+x+\frac{1}{2}$, or the formula B satisfies $|B^{-1}(1)|=a+x-\frac{1}{2}$. Since a is an even number, we have $x=\frac{2^n-a}{2}$ as an integer, and $2^n-x=2^{n-1}+\frac{a}{2}$. Furthermore,

$$\xi_{Cos}(B,\neg B\vee A)=\frac{a}{\sqrt{2^{n-1}+\frac{a}{2}}\sqrt{2^{n-1}+\frac{a}{2}}}=\frac{2a}{2^n+a},$$

$$inf\{\xi_{Cos}(B,C)|B,C\in D(A)|_{p_1,\cdots,p_n}\}=\frac{2a}{2^n+a}.$$

Let $\tilde{A}\approx A$ and $|(\tilde{A})^{-1}(1)|=2^m a$. By $B\in D(\Gamma)_{p_1,\cdots,p_{n+m}}$, we have

$$|B^{-1}(1)|=2^m a+x(1\leq x\leq 2^{n+m}-2^m a).$$

Based on the above facts, we can see that if $x = \frac{2^{n+m}-2^m a}{2}$, $\xi_{Cos}(B, \neg B \vee A)$ takes the minimum value. Therefore,

$$\begin{aligned} & inf\{\xi_{Cos}(B,C)|B,C \in D(\tilde{A})|_{p_1,\cdots,p_{n+m}} \\ = & \frac{2^m a}{\sqrt{2^m a + \frac{2^{n+m}-2^m a}{2}}\sqrt{2^{n+m}-(\frac{2^{n+m}-2^m a}{2})}} \\ = & \frac{2a}{2^n+a}. \end{aligned}$$

When $m \geq 1$, we have

$$inf\{\xi_{Cos}(B,C)|B,C \in D(A)|_{p_1,\cdots,p_n}\} = inf\{\xi_{Cos}(B,C)|B,C \in D(\tilde{A})|_{p_1,\cdots,p_{n+m}}\}.$$

(ii) By the proof of previous point (i), if $|A^{-1}(1)| = a$ is odd, and $B \in D(A)|_{p_1,\cdots,p_n}$, then

$$|B^{-1}(1)| = a + x(1 \leq x \leq 2^n - a).$$

When $x = \frac{2^n-a}{2} - \frac{1}{2}$, $\xi_{Cos}(B, \neg B \vee A)$ takes the minimum value. In this case, $a + x = 2^{n-1} + \frac{a}{2} - \frac{1}{2}, 2^n - x = 2^{n-1} + \frac{a}{2} + \frac{1}{2}$,

$$inf\{\xi_{Cos}(B,C)|B,C \in D(A)|_{p_1,\cdots,p_n}\} = \frac{a}{\sqrt{2^{n-1}+\frac{a}{2}-\frac{1}{2}}\sqrt{2^{n-1}+\frac{a}{2}+\frac{1}{2}}}.$$

Let $\tilde{A} \approx A$ and $|(\tilde{A})^{-1}(1)| = 2^m a$. In this case, $|B^{-1}(1)| = 2^m a + x(1 \leq x \leq 2^{n+m} - 2^m a)$ for every $B \in D(\Gamma)_{p_1,\cdots,p_{n+m}}$. By the proof of previous point (i), it can be seen tha when $x = \frac{2^{n+m}-2^m a}{2}$, $\xi_{IP}(B, \neg B \vee A)$ takes the minimum value. Therefore,

$$\begin{aligned} & inf\{\xi_{Cos}(B,C)|B,C \in D(\tilde{A})|_{p_1,\cdots,p_{n+m}}\} \\ = & \frac{2^m a}{\sqrt{2^m a + \frac{2^{n+m}-2^m a}{2}}\sqrt{2^{n+m}-(\frac{2^{n+m}-2^m a}{2})}}. \end{aligned}$$

Moreover, we have $inf\{\xi_{Cos}(B,C)|B,C \in D(\tilde{A})|_{p_1,\cdots,p_{n+m}}\} = \frac{2a}{2^n+a}$.

In addition,

$$\begin{aligned} & \frac{a}{\sqrt{2^{n-1}+\frac{a}{2}-\frac{1}{2}}\sqrt{2^{n-1}+\frac{a}{2}+\frac{1}{2}}} \\ = & \frac{a}{\sqrt{(2^{n-1}+\frac{a}{2})^2-\frac{1}{4}}} \\ > & \frac{a}{\sqrt{(2^{n-1}+\frac{a}{2})^2}} \\ = & \frac{2a}{2^n+a}. \end{aligned}$$

We have

$$inf\{\xi_{Cos}(B,C)|B,C \in D(A)|_{p_1,\cdots,p_n}\} > inf\{\xi_{Cos}(B,C)|B,C \in D(\tilde{A})|_{p_1,\cdots,p_{n+m}}\}. \quad \square$$

Proof of Theorem 3.1 Since $D(\Gamma)$ is the set of all conclusions of Γ, we have

$$D(\Gamma) = D(\Gamma)|_{p_1,\cdots,p_n} \cup D(\Gamma)|_{p_1,\cdots,p_{n+1}} \cup D(\Gamma)|_{p_1,\cdots,p_{n+2}} \cup \cdots,$$

where $p_1, \cdots, p_n$ are different atomic formulas appearing in Γ, from Lemma 3.1 and Lemma 3.2, when $|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)| = a$ is an even number,

$$inf\{\xi_{Cos}(A,B)|A,B \in D(\Gamma)\} = \frac{2a}{2^n+a} = \frac{2|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)|}{2^n + |(A_1 \wedge \cdots \wedge A_n)^{-1}(1)|}.$$

When $|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)| = a$ is odd, we have

$$\begin{aligned} & inf\{\xi_{Cos}(A,B)|A,B \in D(\Gamma)|_{p_1,\cdots,p_{n+1}}\} \\ & = inf\{\xi_{Cos}(A,B)|A,B \in D(\Gamma)|_{p_1,\cdots,p_{n+2}}\} = \cdots = \frac{2a}{2^n+a}. \end{aligned}$$

Hence, $inf\{\xi_{Cos}(A,B)|A,B \in D(\Gamma)\} = \frac{2a}{2^n+a}$.

In summary, $dim(\Gamma) = 1 - \frac{2|(A_1\wedge\cdots\wedge A_n)^{-1}(1)|}{2^n+|(A_1\wedge\cdots\wedge A_n)^{-1}(1)|}$. □

Example 3.1 (i) Letting $\Gamma = \{p\}$, then $|p^{-1}(1)| = 1$. By Theorem 3.1, $dim(\Gamma) = 1 - \frac{2|p^{-1}(1)|}{2+|p^{-1}(1)|} = \frac{1}{3}$.

(ii) Let $\Gamma = \{p, q\}$, then $|(p \wedge q)^{-1}(1)| = 1$. By Theorem 3.1,

$$dim(\Gamma) = 1 - \frac{2|(p \wedge q)^{-1}(1)|}{2^2 + |(p \wedge q)^{-1}(1)|} = \frac{3}{5}.$$

In this example, $\xi_{Cos}(p \wedge q, T) = \frac{1}{2}$ is not the minimum value of $\{\xi_{Cos}(A, B)|A, B \in D(\Gamma)\}$.

(iii) Let $\Gamma = \{p_1, \cdots, p_n\}$, then $|(p_1 \wedge p_2 \wedge \cdots \wedge p_n)^{-1}(1)| = 1$. By Theorem 3.1, $dim(\Gamma) = 1 - \frac{2}{2^n+1}$.

Remark 3.2 From Example 3.1 (iii) and Definition 3.2, it can be inferred if $\Gamma = S$, then $dim(S) = \lim_{n\to\infty}(1 - \dfrac{2}{2^n + 1}) = 1$. Although the divergence degree of atomic formula set S is equal to 1, but S is a consistent theory. Therefore, we can not get the conclusion that Γ is an inconsistent theory from the fact that Γ is a full divergence.

Theorem 3.2 Supposing that $A \in F(S)$, $\Delta = \{B|\xi_{Cos}(A, B) = \alpha, a < 1, B \in F(S)\}$, then the formula set Δ is an inconsistent formula set.

Proof Without loss of generality, it is assumed $a = \frac{l}{\sqrt{k}\sqrt{t}}(l \le k, t \le 2^n)$, $x_A = (a_1, a_2, \cdots, a_{2^n})$ is the description vector of A, and

$$a_i = \begin{cases} 1, & i = 1, \cdots, k, \\ 0, & i = k + 1, \cdots, 2^n. \end{cases}$$

On the one hand, we select the vector sequence $x_{B_i} = (b^i_1, b^i_2, \cdots, b^i_{2^n})(i = 1, \cdots, l)$ as follows:

$$x_{B_i}(j) = b^i_j = \begin{cases} 1, & j = 1, \cdots, i-1, i+1, \cdots l+1, \\ 0, & j = i, \\ 1, & j = k+1, \cdots, k+t-l, \\ 0, & \text{otherwise.} \end{cases}$$

Then, every vector x_{B_i} satisfies $||x_{B_i}|| = t$, and $x_A \cdot x_{B_i} = 1$. Hence, the formula B_i obtained by vector x_{B_i} satisfies $\xi_{Cos}(A, B_i) = \alpha$.

On the other hand, we select the vector sequence $x_{C_i} = (c^i_1, c^i_2, \cdots, c^i_{2^n})(i = 1, \cdots, t-l)$ as follows:

$$c^i_j = \begin{cases} 1, & j = 1, \cdots, l, \\ 0, & j = k+i, \\ 1, & j = k+1, \cdots, j = k+i-1, k+i+1, \cdots, k+t-l+1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the vector $x_{C_i}(i = 1, \cdots, t-l)$ satisfies $||x_{C_i}|| = t$, $\xi_{IP}(A, C_i) = \alpha$, and

$$x_{B_1} \wedge \cdots \wedge x_{B_l} \wedge x_{C_1} \wedge \cdots \wedge x_{C_{t-l}} = (0, \cdots, 0).$$

Therefore, the formula $B_1 \wedge B_2 \wedge \cdots \wedge B_n$ derived from $x_{B_1} \wedge \cdots \wedge x_{B_l} \wedge x_{C_1} \wedge \cdots \wedge x_{C_{t-l}}$ is a contradiction, and formula set Δ is an inconsistent formula set. □

Theorem 3.3 Let $\Lambda = \{A|\tau_{Cos}(A) = \alpha, A \in F(S)\}$, then Λ is an inconsistent theory.

Proof A consequence of Theorem 3.2. □

Definition 3.3 Let Γ be a theory of $F(S)$, and define

$$consit(\Gamma) = 1 - dim(\Gamma),$$

then $consit(\Gamma)$ is called the consistent degree of theory Γ.

Theorem 3.4 Let $\Gamma = \{A_1, \cdots, A_n\}$, then

$$consit(\Gamma) = \frac{2|(A_1 \wedge \cdots \wedge A_n)^{-1}(1)|}{2^n + |(A_1 \wedge \cdots \wedge A_n)^{-1}(1)|}.$$

Proof It follows from Theorem 3.1 and Definition 3.3. □

Theorem 3.5 Let $\Gamma = \{A_1, \cdots, A_n\}, \Sigma = \{B_1, \cdots, B_m\}$, and $D(\Gamma) = D(\Sigma)$ then $consit(\Gamma) = consit(\Sigma)$.

Proof Because $D(\Gamma) = D(\Sigma)$, we have $\wedge\Gamma \approx \wedge\Sigma, |(\wedge\Gamma)^{-1}(1)| = |(\wedge\Sigma)^{-1}(1)|$. By Theorem 3.4, $consit(\Gamma) = consit(\Sigma)$. □

In fact, Theorem 3.5 shows that two equivalent but distinct finite theories have the same consistency.

§4 Application of cosine similarity degree in describing the reliability of the propositional formulas

In this section, we will introduce the concept of cosine truth degree of a propositional formula via the cosine similarity degree, which can be used to distinguish the reliability of different propositional formulas.

Definition 4.1 Suppose that $A \in F(S)$ and T is a tautology, define

$$\tau_{Cos}(A) = \xi_{Cos}(A, T),$$

and then $\tau_{Cos}(A)$ is called the cosine truth degree of propositional formula A.

Remark 4.1 (i) In fact $\tau_{Cos}(A)$ is obtained by comparing the propositional formula A with the tautology T. The tautology is a well-known propositional formula in all propositional formulas. A propositional formula naturally can be expressed with its reliability degree.

(ii)Since the description vector of tautology T is $(1, \cdots, 1)$, we have $x_A \cdot x_T = |A^{-1}(1)|$, thereby $\tau_{Cos}(A) = \xi_{Cos}(A, T) = \frac{|A^{-1}(1)|}{\sqrt{|A^{-1}(1)|}\sqrt{2^n}} = \sqrt{\frac{|A^{-1}(1)|}{2^n}}$.

It is clear that $\tau_{Cos}(p \to p) = 1, \tau_{Cos}(\neg(p \to p)) = 0, \tau_{Cos}(p \vee q) = \frac{\sqrt{3}}{2}, \tau_{Cos}(p \to q) = \frac{\sqrt{3}}{2}$, and $\tau_{Cos}(\neg(p \to (q \to (r \vee s)))) = \frac{1}{4}$. The cosine truth degree τ_{Cos} can be distinguish the reliability of different formulas.

For the cosine similarity degree and cosine truth degree, the following conclusion holds.

Proposition 4.1 Suppose that $A, B \in F(S)$, then $\xi_{Cos}(A, B) = \frac{\tau^2_{Cos}(A \wedge B)}{\tau_{Cos}(A)\tau_{Cos}(B)}$.

Proof It immediately from Definition 2.4. □

In Ref.[10], Wang introduced the concept of truth degree of a propositional formula by virtue of uniform probability on $F(S)$ in classical two-valued logic system. From Remark 4.1(ii), we can see that $\tau_{Cos}(A) = \tau_{Cos}(A)^2$, with $\tau_{Cos}(A)$ defined as $\tau_{Cos}(A) = \frac{|A^{-1}(1)|}{2^n}$ in quantitative logic[10]. Hence the following facts are true.

Proposition 4.2 Suppose that $A, B, C \in F(S)$, then

$$\tau^2_{Cos}((A \to B) \wedge (B \to A)) + \tau^2_{Cos}((B \to C) \wedge (C \to B)) \leq 1 + \tau^2_{Cos}((A \to C) \wedge (C \to A)).$$

Proof By Definition 4.1 $\tau^2_{Cos}((A \to B) \wedge (B \to A)) = \frac{|(A \wedge B)^{-1}(1)| + |(A \vee B)^{-1}(0)|}{2^n}$.

And

$$\begin{aligned}(\tau^2_{Cos}((A \to B) \wedge (B \to A)) + \tau^2_{Cos}((B \to C) \wedge (C \to B)) - \tau^2_{Cos}((A \to C) \wedge (C \to A)))2^n \\ = 2^n + 2(|B^{-1}(1)| - |(A \vee B)^{-1}(1)| - |(B \vee C)^{-1}(1)| + (A \vee C)^{-1}(1)|).\end{aligned}$$

In order to prove the inequality in Proposition 4.3, we need to prove

$$|B^{-1}(1)| - |(A \vee B)^{-1}(1)| - |(B \vee C)^{-1}(1)| + |(A \vee C)^{-1}(1)| \leq 0. \qquad (*)$$

That is, to prove

$$|B^{-1}(1)| + |(A \vee C)^{-1}(1)| \leq |(A \vee B)^{-1}(1)| + |(B \vee C)^{-1}(1)|.$$

Since $|((A \vee B) \wedge (B \vee C))^{-1}(1)| \geq |B^{-1}(1)|$, we have

$$|(A \vee B)^{-1}(1)| + |(B \vee C)^{-1}(1) - (A \vee B \vee C)^{-1}| \geq |B^{-1}(1)|.$$

Furthermore, we have

$$|(A \vee B)^{-1}(1)| + |(B \vee C)^{-1}(1)| \geq |B^{-1}(1)| + |(A \vee B \vee C)^{-1}| \geq |B^{-1}(1)| + |(A \vee C)^{-1}|.$$

Therefore, inequality (*) holds and inequality

$$(\tau^2_{Cos}((A \to B) \wedge (B \to A)) + \tau^2_{Cos}((B \to C) \wedge (C \to B)) - \tau^2_{Cos}((A \to C) \wedge (C \to A)))2^n \leq 2^n$$

also holds. This completes the verification of Proposition 4.3. □

The significance of Proposition 4.2 is that we can induce a logical metric space by defining pseudo-distance $\rho(A, B) = 1 - \tau^2_{Cos}((A \to B) \wedge (B \to A))$ on $F(S)$.

Theorem 4.1 The set of cosine truth degree of all formulas of $F(S)$

$$\bar{H} = \{\sqrt{\frac{k}{2^n}} | k = 0, \cdots, 2^n; n = 1, 2, \cdots\}.$$

proof By Theorem 2.2 and Definition 4.1, $\bar{H} = \{\xi_{Cos}(A, T) | A \in F(S)\} = \{\frac{k}{\sqrt{k}\sqrt{2^n}} | k = 0, \cdots, 2^n; n = 1, 2, \cdots\} = \{\sqrt{\frac{k}{2^n}} | k = 0, \cdots, 2^n; n = 1, 2, \cdots\}$. □

In fact, Theorem 4.1 is the corollary of Theorem 2.2, and $\bar{H}$ is the true subset of Θ, i.e., $\bar{H} \subset \Theta$ holds.

Theorem 4.2 The set $\{\tau_{Cos}(A) | A \in F(S)\}$ is dense in interval [0,1].

Proof It follows from Theorem 2.1, Definition 4.1 and Theorem 4.1. □

§5 Concluding remarks

In this paper, we propose the concept of cosine similarity degree in classical two-valued logic system, and present some basic properties of it. It is shown by these properties that for a finite formula set Λ, we can use the cosine similarity degree to give a division of Λ from tautology to contradiction. As for the application of cosine similarity degree, we introduce the concept of cosine truth degree of a formula, and study the consistency issue of finite theories. The results obtained in this paper provide a possible way to further study the properties of logic systems. With the progress in research, there emerge increasing issues for further exploration.

For illustration, such questions as how to expand the approximate reasoning on $F(S)$ under the framework of this paper and how to carry on the corresponding discussion in a multi-valued logic system are the key to future research.

Acknowledgements

The author thanks the referee for his/her comments and detailed suggestions. These have significantly improved the presentation of this paper.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] J B Roser, A R Turquette. *Many-valued Logic*, Amsterdam: North-Holland Publishing Co., 1952.

[2] J Pavelka. *On fuzzy logic: I, II, III*, Zeitschrift für Mathematische Logik und Grundlagen Mathematik, 1979, 25(2): 45-52, 119-134, 447-464.

[3] T Hailperin. *Probability logic*, Notre Dame Journal of Formal Logic, 1984, 25(3): 198-212.

[4] N J Nilsson. *Probabilistic logic*, Artificial Intelligence, 1986, 28(1): 71-87.

[5] A M Frisch, P Haddawy. *Anytime deduction for probabilistic logic*, Artificial Intelligence, 1994, 69(1-2): 93-122.

[6] E W Adams. *A Primer of Probability Logic*, Stanford: CSLI Publications, 1996.

[7] D Mundici. *Averaging the truth-value in Łukasiewicz logic*, Studia Logica, 1995, 55(1): 113-127.

[8] G Wang, L Fu, J Song. *Theory of truth degrees of propositions in two-valued logic*, Science in China Series A: Mathematics, 2002, 45(9): 40-50.

[9] B Li, G Wang. *Theory of truth degrees of formulas in Łukasiewicz n-valued propositional logic and a limit theorem*, Science in China, Series F: Information Sciences, 2005, 48(6): 727-738.

[10] G Wang, H Zhou. *Quantitative logic*, Information Sciences, 2009, 179(3): 226-247.

[11] G J Wang, Y Leung. *Integrated semantics and logic metric spaces*, Fuzzy Sets and Systems, 2003, 136(1): 71-91.

[12] W Zuo. *Probability Truth Degrees of Formulas in MTL-Algebras Semantics*, Acta Electronica Sinica, 2015, 43(2): 293-298. (in Chinese)

[13] C Li, H W Liu, G J Wang. *Correction and improvement on several results in quantitative logic*, Information Sciences, 2014, 278: 555-558.

[14] X N Zhou, G J Wang. *Consistency degrees of theories in some systems of propositional fuzzy logic*, Fuzzy Sets and Systems, 2005, 152(2): 321-331.

[15] G J Wang, Y H She. *Topological description of divergency and consistency of two-valued propositional theories*, Acta Mathematica Sinica, Chinese Series, 2007, 50(4): 841-850. (in Chinese)

[16] G J Wang, Y H She. *A topological characterization of consistency of logic theories in propositional logic*, Mathematical Logic Quarterly, 2006, 52(5): 470-477.

[17] Y She, X He, G Wang. *Rough truth degrees of formulas and approximate reasoning in rough logic*, Fundamenta Informaticae, 2011, 107(1): 67-83.

[18] Y She, X He, Y Qian, et al. *A quantitative approach to reasoning about incomplete knowledge*, Information Sciences, 2018, 451-452: 100-111.

[19] Y She, X He. *Rough approximation operators on R_0-algebras (nilpotent minimum algebras) with an application in formal logic L^**, Information Science, 2014, 277: 71-89.

[20] H X Shi, G J Wang. *Quantitative Method for Multi-Value Modal Logics*, Journal of Software, 2012, 23(12): 3074-3087.

[21] H Zhou, G Wang. *Borel probabilistic and quantitative logic*, Science China Information Science, 2011, 54(9): 1843-1855.

[22] H J Zhou, S M Lan, Q Ma. *A state of art survey of probabilistically quantitative logic*, Fuzzy systems and Mathematics, 2017, 31(1): 1-17.

[23] H Zhou, B Zhao. *Stone-like representation theorems and three-valued filters in R_0-algebras(nolpotent minimum algebra)*, Fuzzy sets and systems, 2011, 162(1): 1-26.

[24] H J Zhou, Y H She. *Theory of Choquet Integral Truth Degrees of Propositions in Lukasiewicz Propositional Logic*, Acta Electronica Sinica, 2013, 41(12): 2327-2333.

[25] P Yu, B Zhao. *The Hamming Distance Representation and Decomposition Theorem of Formula's Truth Degree*, Journal of Software, 2018, 29(10): 3091-3110.(in Chinese)

[26] P Yu. *Quantitative Method Based on Cotangent Similarity Degree in Three-Valued Lukasiewicz Logic*, Chinese Journal of Electronics, 2021, 30(1): 134-144.

[27] A C Paseau. *Non-metric Propositional Similarity*, Erkenntnis, 2022, 87: 2307-2328.

School of Mathematics & Data Science, Shaanxi University of Science and Technology, Xi'an 710021, China.

Email: yupeng@sust.edu.cn