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Well-posedness and exponential stability of a dynamic frictionless contact problem with normal compliance and infinite memory

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Abstract. A model for dynamic frictionless contact between a viscoelastic body and foundation is considered. The viscoelastic constitutive law is assumed to be nonlinear and the contact is modelled with the normal compliance condition. We obtain the well-posedness using nonlinear semigroup theory arguments. Moreover, the exponential stability result of the solution is shown by using the energy method to produce a suitable Lyapunov function.

§1 Introduction

Dynamic contact problems involving deformable bodies can be frequently found in a variety of industries and everyday life such as contact between wheels and ground in a vehicle dynamics, spheres falling into a funnel, clockworks, etc. Considerable progress has been made with modelling, mathematical analysis and numerical simulations of this kind of problems. We refer the reader to the extensive bibliography on the subject in [15, 16, 22, 28, 30].

Different contact conditions have been considered to model contact problems, such as the normal compliance condition. The term normal compliance was first used in [18, 19]. In [18] Klarbing et al. proved the existence of a weak solution of frictional contact with normal compliance. This condition was first introduced in [23] by Martins and Oden in dynamic problems. It describes a deformable foundation. Moreover, it assigns a reactive normal pressure. The latter depends on the interpenetration of the asperities on the body surface and those on the foundation.

Signorini contact condition is an idealization of normal compliance. It is employed often in mathematical publications and engineering literature, see, e.g, [13, 21].

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Contact problems with normal compliance condition have received increased attention in the literature. We refer the reader to [17] for the existence of a solution for a static contact problem with limited interpenetration with an obstacle. Contact condition is given by normal compliance. In [26], Sofonea and Pătrulescu showed the weak solvability of a quasistatic contact problem without friction. The contact is modelled with a non-standard condition where normal compliance is involved. The results obtained in [6] deal with the existence and uniqueness of a weak solution of a dynamic viscoelastic problem with normal compliance and damage. In [7], Chau et al. studied a class of dynamical problem where the contact condition is modelled by normal compliance with friction, damage and heat exchange. They established an existence and uniqueness result by using classical results on evolution variational inequalities combined with fixed-point methods and monotone operators. The method of using evolution equations arguments, monotonicity and fixed point theorem is also considered in [14] in the study of dynamic contact problem. The bodies are assumed to be elastic-viscoplastic and piezoelectric. The contact is modelled with normal compliance. In [27], Sofonea and Shillor considered a frictionless quasistatic contact problem between a viscoplastic body and foundation where the contact is modelled with normal compliance with limited penetration condition. They proved the existence and uniqueness of the problem using a theory of history-dependent variational inequalities, monotonicity and a point-fixed argument. A viscoelastic sliding contact problem is studied in [29]. The contact there is modelled with multivalued normal compliance condition combined with unilateral constraint and memory term. The proof of existence and uniqueness is based on history-dependent quasi-variational inequalities.

In this work, we consider a dynamic frictionless contact problem with normal compliance condition and infinite memory. To the best of our knowledge, the considered model is new. Furthermore, we don't use here the arguments of monotonicity and convexity combined with a fixed point result, but we use nonlinear semigroup arguments to prove that the solution exists globally in time. We apply the theory presented in [2,3,8,20,24] combined with the method followed in [11,12,25]. Moreover, we establish the exponential decay of the energy solution. This decay is first studied in this paper for this kind of problem. For this purpose, we construct a Lyapunov functional L which is equivalent to the energy of our problem and by using ideas and techniques developed in [4,10] with some necessary modifications due to the nature of the considered problem, and we derive a theorem which proves that the energy solution decays exponentially under some assumptions.

The rest of the paper is structured as follows. Section 2 introduces the notation and some preliminary material. Section 3 presents a dynamic contact model with normal compliance with assumptions on problem data. Then, section 4 states the well-posedness theorem the proof follows. It is based on nonlinear semigroup theory arguments. Section 5 presents some technical lemmas that we need for our result. Finally, we prove the exponential stability result of the energy solution by introducing a suitable Lyapunov functional.

§2 Notation and preliminaries

In this section, we introduce the basic notation and preliminary which will be used through this paper.

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain with a Lipschitz boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $\Gamma_1 > 0$.

The space of second-order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$ is denoted by $\mathbb{S}^d$. The inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$ are represented respectively by '.' and $\|.\|$. The indices i and j run from 1 to d, and summation over repeated indices is implied.

We introduce the following spaces

$$\begin{aligned} H &= \left\{ u = (u_i) \mid u_i \in L^2(\Omega) \right\}, \\ \mathcal{H} &= \left\{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \\ H_1 &= \left\{ u = (u_i) \mid u_i \in H^1(\Omega) \right\}, \end{aligned}$$

where

$$H^1(\Omega) = \left\{ u_i \in L^2(\Omega), \frac{\partial u_i}{\partial x_j} \in L^2(\Omega) \; \forall 1 \le j \le d \right\}.$$

We endow the real Hilbert spaces $H, \mathcal{H}$, and H_1 with the following canonical inner product

$$\begin{aligned} (u, v)_{L^2(\Omega)^d} &= \int_\Omega u_i v_i \, \mathrm{d}x, \\ (\sigma, \tau)_{\mathcal{H}} &= \int_\Omega \sigma_{ij} \tau_{ij} \, \mathrm{d}x, \\ (u, v)_{H_1} &= (u, v)_{L^2(\Omega)^d} + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \end{aligned}$$

where $\varepsilon : H_1 \longrightarrow \mathcal{H}$ is the deformation operator given by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}),$$

with $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Moreover, we define the function spaces

$$\begin{aligned} \mathcal{H}_1 &= \{ \sigma \in \mathcal{H} \mid \text{ Div } \sigma \in H \}, \\ V &= \{ v \in H_1 \mid v = 0 \text{ on } \Gamma_1 \}, \end{aligned}$$

where Div : $\mathcal{H}_1 \longrightarrow H$ is the divergence operator, and we recall that

$$\text{Div}\sigma = (\sigma_{ij,j}),$$

where $\sigma_{ij,j} = \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}$

The real Hilbert spaces $\mathcal{H}_1$ and V are endowed with the canonical inner products given by

$$\begin{aligned} (\sigma, \tau)_{\mathcal{H}_1} &= (\sigma, \tau)_{\mathcal{H}} + (\text{Div}\sigma, \text{ Div}\,\tau)_H, \\ (u, v)_V &= (\varepsilon(u), \varepsilon(v))_{\mathcal{H}.} \end{aligned}$$

The corresponding norms on the spaces $\mathcal{H}_1$ and V are denoted respectively by $\|\cdot\|_{\mathcal{H}_1}$ and $\|\cdot\|_V$. Since meas $\Gamma_1 > 0$, the following Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_K \|v\|_{H_1}, \quad \forall v \in V, \tag{2.1}$$

where $c_K > 0$ is a constant depending only on Ω and Γ_1.

Let ν denote the unit outer normal on Γ. For every element $v \in H_1$, we still write v to denote the trace of v on Γ. The notations v_ν and v_τ represent the normal and the tangential components of v on Γ given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

For every function $\sigma \in \mathcal{H}_1$, we denote by σ_ν and σ_τ the normal and the tangential traces of σ, and we note that when σ is a regular function, say $\sigma \in C^1\left(\overline{\Omega}; \mathbb{S}^d\right)$, then

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

And the following Green formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{Div} \sigma, v)_H = \int_\Gamma \sigma\nu \cdot v \ \mathrm{d}a, \quad \forall v \in H_1. \tag{2.2}$$

The formula (2.2) is used to obtain a variational formulation for a contact problem. We recall that there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V, \quad \forall v \in V. \tag{2.3}$$

The previous inequality follows from the Sobolev trace theorem.

The modified inner product on H used in this paper is

$$\langle u, v\rangle_H = \langle \rho u, v\rangle_{L^2(\Omega)^d}, \quad \forall u, v \in H, \tag{2.4}$$

where the mass density ρ satisfies

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^* \text{a.e } x \in \Omega. \tag{2.5}$$

Let V' denotes the dual of space V. The duality pairing V' and V will be denoted by $\langle \cdot, \cdot \rangle_{V' \times V}$. We call (V, H, V') Gelfand triple where

$$V \subset H \subset V',$$

and the embedding of V into H is continuous and dense. Then, we have

$$\|u\|_H \leq d\|u\|_V, \quad \forall u \in V, \tag{2.6}$$

where d is a strictly positive constant.

Through this paper, we put

$$\langle u, v\rangle_{V' \times V} = \langle u, v\rangle_H = \langle \rho u, v\rangle_{L^2(\Omega)^d}, \quad \forall u \in H, \ \forall v \in V. \tag{2.7}$$

We end this section by the spaces of vector-valued spaces that we shall use in the rest of the paper. For every real Banach space X, we use the classical notation for the space $L^p(\mathbb{R}_+; X)$ and $W^{k,p}(\mathbb{R}_+; X)$, where $1 \leq p \leq \infty$, $k \geq 1$ and let $C(\mathbb{R}_+; X)$ denote the space of continuous functions on $\mathbb{R}_+$ with values in X.

§3 Problem statment

In this section, we describe a model for process and list the assumptions on the problem data.

We consider a viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $\Gamma_1 > 0$. Let $[0, +\infty)$ denote the time interval of interest. In this work, we are interested in the following dynamic frictionless contact problem

$$\sigma = A\varepsilon(u_t) + G\varepsilon(u) - \int_0^{+\infty} g(s)B\varepsilon\left(u\left(t-s\right)\right)ds \quad \text{in} \quad \Omega \times (0,\infty), \tag{3.1}$$

$$\rho u_{tt} = Div\sigma + f_0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{3.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty) \tag{3.3}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2 \times (0,\infty), \tag{3.4}$$

$$-\sigma_\nu = p\left(u_\nu\right) \quad \text{on} \quad \Gamma_3 \times (0,\infty), \tag{3.5}$$

$$\sigma_\tau = 0 \quad \text{on} \quad \Gamma_3 \times (0,\infty), \tag{3.6}$$

$$u(-t) = u_0(t), u_t(0) = u_1 \quad \text{in} \quad \Omega \times [0,\infty), \tag{3.7}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$.

First, we denote by u the displacement vector, u_t the velocity vector and u_{tt} the acceleration vector, σ represents the stress field and $\varepsilon = \varepsilon(u)$ is the small strain tensor. Equation (3.1) represents the viscoelastic constitutive law with a long memory, in which A, G and B are nonlinear given functions and the positive function g is the kernel of the memory term. Equation (3.2) is the equation of motion, the process is dynamic and the body is submitted to volume forces of density f_0 which acts in Ω. Condition (3.3) is the displacement boundary condition. We assume that the body is clamped on Γ_1 and the displacement field vanishes there. The traction boundary condition is given by (3.4) where the surface traction f_2 acts on Γ_2. Next, (3.5) represents a normal compliance condition, in which σ_ν is the normal stress, u_ν is the normal displacement, p is a prescribed function which equals to zero when its argument is negative and from (3.6), we conclude that the contact is frictionless. Finally, u_0 and u_1 are given history and initial data in (3.7).

An example of the normal compliance p is

$$p(r) = r_+,$$

where $r_+ = max\left\{0, r\right\}$ is the positive part of r.

In the study of the above system, we assume that the viscosity operator A, the elasticity operator G, the operator B and the contact function p satisfy

$$\left\{\begin{array}{l} \text{(a) } A : \Omega \times \mathbb{S}^d \to \mathbb{S}^d; \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \qquad \|A\left(x,\varepsilon_1\right) - A\left(x,\varepsilon_2\right)\| \le L_A\|\varepsilon_1 - \varepsilon_2\| \quad ,\forall\varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_A > 0 \text{ such that} \\ \qquad \left(A\left(x,\varepsilon_1\right) - A\left(x,\varepsilon_2\right)\right)\cdot\left(\varepsilon_1 - \varepsilon_2\right) \ge m_A\|\varepsilon_1 - \varepsilon_2\|^2 \quad ,\forall\varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto A(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } A(x,0) = 0 \text{ a.e. } x \in \Omega. \end{array}\right. \tag{3.8}$$

$$\begin{cases} \text{(a) } G:\Omega\times\mathbb{S}^d\to\mathbb{S}^d; \\ \text{(b) There exists } L_G>0 \text{ such that} \\ \qquad \|G(x,\varepsilon_1)-G(x,\varepsilon_2)\|\le L_G\|\varepsilon_1-\varepsilon_2\| \quad ,\forall\varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega; \\ \text{(c) There exists } m_G>0 \text{ such that} \\ \qquad (G(x,\varepsilon_1)-G(x,\varepsilon_2))\cdot(\varepsilon_1-\varepsilon_2)\ge m_G\|\varepsilon_1-\varepsilon_2\|^2 \quad ,\forall\varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega; \\ \text{(d) } x\mapsto G(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } G(x,0)=0 \text{ a.e. } x\in\Omega. \end{cases} \tag{3.9}$$

$$\begin{cases} \text{(a) } B:\Omega\times\mathbb{S}^d\to\mathbb{S}^d; \\ \text{(b) There exists } L_B>0 \text{ such that} \\ \qquad \|B(x,\varepsilon_1)-B(x,\varepsilon_2)\|\le L_B\|\varepsilon_1-\varepsilon_2\| \quad ,\forall\varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega; \\ \text{(c) There exists } m_B>0 \text{ such that} \\ \qquad (B(x,\varepsilon_1)-B(x,\varepsilon_2))\cdot(\varepsilon_1-\varepsilon_2)\ge m_B\|\varepsilon_1-\varepsilon_2\|^2 \quad ,\forall\varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \text{ a.e. } x\in\Omega; \\ \text{(d) } x\mapsto B(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } B(x,0)=0 \text{ a.e. } x\in\Omega. \end{cases} \tag{3.10}$$

$$\begin{cases} \text{(a) } p:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}; \\ \text{(b) There exists a constant } L_p \text{ such that} \\ \qquad |p(x,r_1)-p(x,r_2)|\le L_p|r_1-r_2| \quad ,\forall r_1,r_2\in\mathbb{R} \text{ a.e. } x\in\Gamma_3 \ ; \\ \text{(c) } (p(x,r_1)-p(x,r_2))(r_1-r_2)\ge 0 \quad ,\forall r_1,r_2\in\mathbb{R} \text{ a.e. } x\in\Gamma_3; \\ \text{(d) The mapping } x\mapsto p(x,r) \text{ is measurable} \quad ,\forall r\in\mathbb{R}; \\ \text{(e) } p(x,r)=0 \quad \forall r\le 0 \quad \text{a.e. } x\in\Gamma_3. \end{cases} \tag{3.11}$$

Also, we suppose that the forces and the tractions verify

$$f_0\in W^{1,1}(\mathbb{R}_+;\ H),\ f_2\in W^{1,1}\left(\mathbb{R}_+;\ L^2(\Gamma_2)^d\right). \tag{3.12}$$

Let $f:\mathbb{R}_+\to V'$ be the mapping defined by

$$(f(t),v)_{V'\times V}=\int_\Omega f_0(t).v dx+\int_{\Gamma_2} f_2(t).v da, \tag{3.13}$$

for all $v\in V$, t a.e. $\in\mathbb{R}_+$.

We note that the assumption (3.12) implies that

$$f\in W^{1,1}(\mathbb{R}_+;\ V'). \tag{3.14}$$

§4 Well-posedness

In this section, we will show the well-posedness result of the above system. To this end, define the auxiliary variable η as in [9]

$$\eta(x,s,t)=u(x,t)-u(x,t-s),\quad x\in\Omega,\ t,s\ge 0. \tag{4.1}$$

From (4.1), we obtain

$$\int_0^\infty g(s)B\varepsilon(u(t-s))ds=\int_0^\infty g(s)B\varepsilon(u(t)-\eta(s))ds,$$

then, the considered system becomes

$$\begin{array}{lll}
\rho u_{tt} - Div\sigma = f_0 & \text{in} & \Omega \times (0,\infty), \\
\sigma = A\varepsilon(u_t) + G\varepsilon(u) - \int_0^{\infty} g(s)B\varepsilon(u-\eta(s))ds & \text{in} & \Omega \times (0,\infty), \\
\eta_t = u_t - \eta_s & \text{in} & \Omega \times (0,\infty) \times (0,\infty), \\
u(x,t) = 0 & \text{on} & \Gamma_1 \times (0,\infty), \\
\eta(x,s,t) = 0 & \text{on} & \Gamma_1 \times (0,\infty) \times (0,\infty), \\
\sigma\nu = f_2 & \text{on} & \Gamma_2 \times (0,\infty), \\
-\sigma_\nu = p\left(u_\nu\right) & \text{on} & \Gamma_3 \times (0,\infty), \\
\sigma_\tau = 0 & \text{on} & \Gamma_3 \times (0,\infty), \\
u(-t) = u_0(t),\ u_t(0) = u_1 & in & \Omega \times [0,\infty), \\
\eta(s,0) = u_0(0) - u_0(s) = \eta_0(s) & \text{for} & s \in (0,\infty).
\end{array} \tag{4.2}$$

Let us consider the following additional assumption.
(A1) The kernel function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is class C^1, nonincreasing function satisfying

$$g(0) > 0,\ g_0 = \int_0^{\infty} g(s)ds < \frac{m_G}{L_B}. \tag{4.3}$$

Introducing the state vector $U = (u, v, \eta)^T$ with $v = u_t$, we can give a formulation as a first order evolution system of (4.2).
We consider the Hilbert space $\mathscr{H}$

$$\mathscr{H} = V \times H \times \mathcal{M}, \tag{4.4}$$

endowed with the following inner product

$$\langle U, \widetilde{U} \rangle_{\mathscr{H}} = \langle u, \widetilde{u} \rangle_V + \langle v, \widetilde{v} \rangle_H + \langle \eta, \widetilde{\eta} \rangle_{\mathcal{M}}, \tag{4.5}$$

for every $U = (u, v, \eta)^T$ and $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{\eta})^T$ in $\mathscr{H}$.
The set $\mathcal{M}$ is defined by

$$\mathcal{M} = \left\{ \chi : \mathbb{R}_+ \to V, \int_0^{+\infty} g(s) \|\chi(s)\|_V^2 ds < +\infty \right\}, \tag{4.6}$$

equipped with the inner product

$$\langle \chi, \phi \rangle_{\mathcal{M}} = \int_0^{+\infty} g(s) \left\langle \varepsilon(\chi(s)), \varepsilon(\phi(s)) \right\rangle_{\mathcal{H}} ds. \tag{4.7}$$

Assuming that σ is a regular function $\left(\text{say } \sigma \in C^1\left(\overline{\Omega}; \mathbb{S}^d\right)\right)$, we view the system (4.2) as

$$\left\{ \begin{array}{l} U_t(t) + \mathcal{A}U(t) = F(t), \quad \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0)^T, \end{array} \right. \tag{4.8}$$

where the operator $\mathcal{A}$, the domain $\mathcal{D}(\mathcal{A})$ and $F(t)$ are respectively given by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} -v \\ -\frac{1}{\rho} Div(A\varepsilon(v) + G\varepsilon(u)) + \frac{1}{\rho} Div(\int_0^{\infty} g(s)B\varepsilon(u-\eta(s))ds \\ \eta_s - v \end{pmatrix}, \tag{4.9}$$

$$F(t) = \begin{pmatrix} 0 \\ \frac{f_0(t)}{\rho} \\ 0 \end{pmatrix}, \tag{4.10}$$

and

$$\mathcal{D}(\mathcal{A})=\left\{\begin{array}{c}(u,v,\eta)^T\in\mathscr{H};\ v\in V;\ \eta_s\in\mathcal{M};\\ \sigma\nu=f_2 \text{ on } \Gamma_2;\\ -\sigma_\nu=p\left(u_\nu\right)\text{on } \Gamma_3;\\ \sigma_\tau=0 \text{ on } \Gamma_3\end{array}\right\}. \tag{4.11}$$

The well-posedness of the problem (4.8) is ensured by the following theorem

Theorem 4.1. *Under the assumptions (A1) and (3.8)-(3.11), for an initial datum $U_0\in\overline{\mathcal{D}(\mathcal{A})}$and $F\in L^1(\mathbb{R}_+;\ \mathscr{H})$, the system (4.8) has a unique mild solution*

$$U\in C(\mathbb{R}_+;\mathscr{H}). \tag{4.12}$$

Proof. To prove Theorem 4.1, it suffices to show that the operator $\mathcal{A}+\omega I$ is an m-accretive operator for some positive constant ω. An operator $\mathcal{A}:D(\mathcal{A})\subset\mathscr{H}\to\mathscr{H}$ is said to be accretive if $(\mathcal{A}z_1-\mathcal{A}z_2,z_1-z_2)_{\mathscr{H}}\geq 0,\forall z_1,z_2\in D(\mathcal{A})$. The operator $\mathcal{A}$ is said to be m-accretive if $\mathcal{A}$ is accretive and $R(\mathcal{A}+I)$ the range of $\mathcal{A}+I$ is all of $\mathscr{H}$.

First, we show that $\mathcal{A}+\omega I$ is accretive operator for some positive constant ω.

Let $U_i=(u_i,v_i,\eta_i)\in\mathcal{D}(\mathcal{A}),\ i=1,\ 2$. Then, we have

$$\begin{aligned}&\langle(\mathcal{A}+\omega I)\,U_1-(\mathcal{A}+\omega I)\,U_2,U_1-U_2\rangle_{\mathscr{H}}\\ =&-\langle v_1-v_2,u_1-u_2\rangle_V\\ &-\langle\frac{1}{\rho}Div\left(A\varepsilon(v_1)\right)-A\varepsilon(v_2)\right),v_1-v_2\rangle_H\\ &-\langle\frac{1}{\rho}Div\left(G\varepsilon(u_1)-G\varepsilon(u_2)\right),v_1-v_2\rangle_H\\ &+\left\langle\frac{1}{\rho}Div\int_0^\infty g(s)B\varepsilon(u_1-\eta_1(s))ds,v_1-v_2\right\rangle_H\\ &-\left\langle\frac{1}{\rho}Div\int_0^\infty g(s)B\varepsilon(u_2-\eta_2(s))ds,v_1-v_2\right\rangle_H\\ &-\langle-\eta_{1s}+\eta_{2s}+v_1-v_2,\eta_1-\eta_2\rangle_{\mathcal{M}}\\ &+\omega\langle U_1-U_2,U_1-U_2\rangle_{\mathscr{H}}.\end{aligned}$$

The use of Green's formula and boundary conditions leads to

$$\begin{aligned}&\langle(\mathcal{A}+\omega I)\,U_1-(\mathcal{A}+\omega I)\,U_2,U_1-U_2\rangle_{\mathscr{H}}\\ =&-\langle v_1-v_2,u_1-u_2\rangle_V\\ &+\langle A\varepsilon(v_1)-A\varepsilon(v_2),\varepsilon(v_1)-\varepsilon(v_2)\rangle_{\mathcal{H}}\\ &+\langle G\varepsilon(u_1)-G\varepsilon(u_2),\varepsilon(v_1)-\varepsilon(v_2)\rangle_{\mathcal{H}}\\ &-\left\langle\int_0^\infty g(s)B\varepsilon\left(u_1-\eta_1(s)\right)ds,\varepsilon(v_1)-\varepsilon(v_2)\right\rangle_{\mathcal{H}}\\ &+\left\langle\int_0^\infty g(s)B\varepsilon\left(u_2-\eta_2(s)\right)ds,\varepsilon(v_1)-\varepsilon(v_2)\right\rangle_{\mathcal{H}}\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Gamma_3} \left(p(u_{1\nu})-p(u_{2\nu})\right)\left(v_{1\nu}-v_{2\nu}\right)da \\
&-\frac{1}{2}\int_0^\infty g'(s)\|\eta_1(s)-\eta_2(s)\|_V^2 ds \\
&-\langle v_1-v_2,\eta_1-\eta_2\rangle_{\mathcal{M}}+\omega\|U_1-U_2\|_{\mathscr{H}}^2 .
\end{aligned}
$$

Also using Cauchy- Schwarz's and Young's inequalities, (A1), (b) of (3.9), (3.11) and (3.10), (c) of (3.8) and (2.3), we get

$$
\begin{aligned}
&\langle (\mathcal{A}+\omega I)\, U_1-(\mathcal{A}+\omega I)\, U_2, U_1-U_2\rangle_{\mathscr{H}} \\
\geq &-\left(\frac{3}{2\mu}-m_A\right)\|v_1-v_2\|_V^2 \\
&-\mu\left(1+L_G^2+g_0^2L_B^2+L_p^2C_0^4\right)\|u_1-u_2\|_V^2 \\
&-\mu\left(L_B^2 g_0+g_0\right)\int_0^\infty g(s)\|\eta_1(s)-\eta_2(s)\|_V^2 ds \\
&+\omega\|U_1-U_2\|_{\mathscr{H}}^2 .
\end{aligned}
$$

Therefore, by taking $\mu > \frac{3}{2m_A}$, we obtain

$$
\langle (\mathcal{A}+\omega I)\, U_1-(\mathcal{A}+\omega I)\, U_2, U_1-U_2\rangle_{\mathscr{H}} \geq (\omega-L)\,\|U_1-U_2\|_{\mathscr{H}}^2,
$$

where $L = max\left(\mu\left(1+L_G^2+g_0^2L_B^2+L_p^2C_0^4\right),\ \mu\left(L_B^2g_0+g_0\right)\right)$. Thus, $\mathcal{A}+\omega I$ is accretive when $\omega > L$.

Next, we prove that $\lambda I+\mathcal{A}$ is surjective for some $\lambda > 0$. For this , we consider for $(h_1,h_2,h_3)^T \in \mathscr{H}$,

$$
(\lambda I+\mathcal{A})\,(u,v,\eta)^T = (h_1,h_2,h_3)^T .
$$

Then, the previous equation reads

$$
\begin{aligned}
\lambda u - v =&\ h_1, \\
\lambda v-\frac{1}{\rho}Div\left(G\varepsilon(u)+A\varepsilon(v)\right)+\frac{1}{\rho}Div\left(\int_0^\infty g(s)B\varepsilon\left(u-\eta(s)\right)ds\right) =&\ h_2, \qquad (4.13)\\
\lambda\eta+\eta_s-v =&\ h_3.
\end{aligned}
$$

Suppose that we have found $v \in V$. Thus, the first equation in (4.13) yields

$$
u=\frac{h_1+v}{\lambda}. \qquad (4.14)
$$

Then,

$$
u \in V.
$$

Integrating the third equation in (4.13) with $\eta(0)=0$, we find

$$
\eta(s)=e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right)d\tau, \quad s\in\mathbb{R}_+. \qquad (4.15)
$$

Note that (4.13) is equivalent to

$$
\begin{aligned}
&\lambda v-\frac{1}{\rho}Div\left(G\varepsilon\left(\frac{h_1+v}{\lambda}\right)+A\varepsilon(v)\right) \\
+&\ \frac{1}{\rho}Div\left(\int_0^\infty g(s)B\varepsilon\left(\frac{h_1+v}{\lambda}-e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right)d\tau\right)ds\right) \\
=&\ h_2. \qquad (4.16)
\end{aligned}
$$

We define the operator $T : V \to V'$ by

$$\begin{aligned}\langle T(v), v\rangle_{V'\times V} =&\langle \lambda v, v\rangle_H + \langle G\varepsilon\left(\frac{h_1+v}{\lambda}\right) + A\varepsilon(v), \varepsilon(v)\rangle_{\mathcal{H}}\\ &-\left\langle \int_0^\infty g(s)B\varepsilon\left(\frac{h_1+v}{\lambda} - e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right)d\tau\right)ds, \varepsilon(v)\right\rangle_{\mathcal{H}}\\ &+\int_{\Gamma_3} p\left(\frac{h_1+v_\nu}{\lambda}\right)v_\nu da - \int_{\Gamma_2} f_2 v da. \end{aligned}\tag{4.17}$$

To show that the mapping $T : V \to V'$ is surjective. We will prove that T is maximal monotone and coercive (see for instance Corollary 1.2 page 45 in [2]).

Let show that $T : V \to V'$ is maximal monotone. Accordingly to Theorem 1.3 in ([2], p.45), it's enough to prove that T is monotone and hemicontinuous.

First, we will show that T is monotone. Let $v,\ \widetilde{v} \in V$. Then, we have

$$\begin{aligned}&\langle Tv - T\widetilde{v}, v-\widetilde{v}\rangle_{V'\times V}\\ =&\langle \lambda v - \lambda\widetilde{v}, v-\widetilde{v}\rangle_H + \langle G\varepsilon\left(\frac{h_1+v}{\lambda}\right) - G\varepsilon\left(\frac{h_1+\widetilde{v}}{\lambda}\right), \varepsilon(v)-\varepsilon(\widetilde{v})\rangle_{\mathcal{H}}\\ &+\langle A\varepsilon(v) - A\varepsilon(\widetilde{v}), \varepsilon(v)-\varepsilon(\widetilde{v})\rangle_{\mathcal{H}}\\ &-\left\langle \int_0^\infty g(s)B\varepsilon\left(\frac{h_1+v}{\lambda} - e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right)d\tau\right)ds, \varepsilon(v)-\varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\\ &+\left\langle \int_0^\infty g(s)B\varepsilon\left(\frac{h_1+\widetilde{v}}{\lambda} - e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+\widetilde{v}\right)d\tau\right)ds, \varepsilon(v)-\varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\\ &+\int_{\Gamma_3}\left(p\left(\frac{h_1+v_\nu}{\lambda}\right) - p\left(\frac{h_1+\widetilde{v}_\nu}{\lambda}\right)\right)(v_\nu - \widetilde{v}_\nu)\,da.\end{aligned}$$

From (c) of (3.9), we have

$$\left\langle G\varepsilon\left(\frac{h_1+v}{\lambda}\right) - G\varepsilon\left(\frac{h_1+\widetilde{v}}{\lambda}\right), \varepsilon(v)-\varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}} \geq \frac{1}{\lambda}m_G\|v-\widetilde{v}\|_V^2. \tag{4.18}$$

Using (c) of (3.8) yields

$$\langle A\varepsilon(v) - A\varepsilon(\widetilde{v}), \varepsilon(v)-\varepsilon(\widetilde{v})\rangle_{\mathcal{H}} \geq m_A\|v-\widetilde{v}\|_V^2. \tag{4.19}$$

The use of Cauchy-Schwarz's inequality, (b) of (3.10) and (A1) leads to

$$\begin{aligned}&-\left\langle \int_0^\infty g(s)B\varepsilon\left(\frac{h_1+v}{\lambda} - e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right)d\tau\right)ds, \varepsilon(v)-\varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\\ &+\left\langle \int_0^\infty g(s)B\varepsilon\left(\frac{h_1+\widetilde{v}}{\lambda} - e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+\widetilde{v}\right)d\tau\right)ds, \varepsilon(v)-\varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\\ \geq& -\frac{g_0L_B}{\lambda}\|v-\widetilde{v}\|_V^2.\end{aligned}\tag{4.20}$$

From (c) of (3.11) , we have

$$\int_{\Gamma_3}\left(p\left(\frac{h_{1\nu}+v_\nu}{\lambda}\right) - p\left(\frac{h_{1\nu}+\widetilde{v}_\nu}{\lambda}\right)\right)(v_\nu - \widetilde{v}_\nu)\,da \geq 0 \tag{4.21}$$

Combining (4.18)-(4.21), we get

$$\langle Tv - T\widetilde{v}, v-\widetilde{v}\rangle_{V\times V'} \geq \left(\frac{m_G - g_0L_B}{\lambda}\right)\|v-\widetilde{v}\|_V^2.$$

Using (4.3), we conclude that T is strongly monotone for some $\lambda > 0$ which also implies that

T is monotone and coercive.

Next, we will prove that T is hemicontinuous (see Definition 2.2 in [3]). Indeed, we will show that

$$\mathrm{w}-\lim_{\zeta\to 0} T\left(v+\zeta v_1\right)=T(v), \quad \forall v, v_1 \in V.$$

$\mathrm{w}-\lim\limits_{\zeta\to 0} T\left(v+\zeta v_1\right)$ denotes the weak limit of $T\left(v+\zeta v_1\right)$ when $\zeta \to 0$.

Let $\widetilde{v} \in V$ and $\lambda>0$, then

$$\begin{aligned}
&\langle T\left(v+\zeta v_1\right), \widetilde{v}\rangle_{V'\times V}-\langle T(v), \widetilde{v}\rangle_{V'\times V}\\
=&\zeta\langle\lambda v_1, \widetilde{v}\rangle_H+\langle G\varepsilon\left(\frac{h_1+v+\zeta v_1}{\lambda}\right)-G\varepsilon\left(\frac{h_1+v}{\lambda}\right), \varepsilon\left(\widetilde{v}\right)\rangle_{\mathcal{H}}\\
&+\langle A\varepsilon\left(v+\zeta v_1\right)-A\varepsilon\left(v\right), \varepsilon(\widetilde{v})\rangle_{\mathcal{H}}\\
&-\left\langle\int_0^{\infty} g(s) B\varepsilon\left(\frac{h_1+v+\zeta v_1}{\lambda}-e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v+\zeta v_1\right) d\tau\right) ds, \varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\\
&+\left\langle\int_0^{\infty} g(s) B\varepsilon\left(\frac{h_1+v}{\lambda}-e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right) d\tau\right) ds, \varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\\
&+\int_{\Gamma_3}\left(p\left(\frac{h_{1\nu}+v_\nu+\zeta v_{1\nu}}{\lambda}\right)-p\left(\frac{h_{1\nu}+v_\nu}{\lambda}\right)\right)\widetilde{v}_\nu da. \qquad (4.22)
\end{aligned}$$

To show that each term in the right-hand side of (4.22) converges to zero as $\zeta\to 0$, we estimate each term in the right-hand side of (4.22) as follows.

Exploiting Cauchy-Schwarz's inequality and (2.6), we obtain

$$\begin{aligned}
\left|\zeta\langle\lambda v_1, \widetilde{v}\rangle_H\right| &\leq |\zeta|\lambda\|v_1\|_H\|\widetilde{v}\|_H\\
&\leq |\zeta|\lambda d^2\|v_1\|_V\|\widetilde{v}\|_V \to 0. \qquad (4.23)
\end{aligned}$$

The use of Cauchy-Schwarz's inequality and (b) of (3.9) leads to

$$\begin{aligned}
\left|\left\langle G\varepsilon\left(\frac{h_1+v+\zeta v_1}{\lambda}\right)-G\varepsilon\left(\frac{h_1+v}{\lambda}\right), \varepsilon\left(\widetilde{v}\right)\right\rangle_{\mathcal{H}}\right| &\leq \frac{|\zeta|}{\lambda} L_G\|v_1\|_V\|\widetilde{v}\|_V\\
&\to 0. \qquad (4.24)
\end{aligned}$$

Similar to (4.24) and by using (b) of (3.8) instead of (3.9) , we get

$$\left|\langle A\varepsilon\left(v+\zeta v_1\right)-A\varepsilon\left(v\right), \varepsilon(\widetilde{v})\rangle_{\mathcal{H}}\right| \leq |\zeta|\, L_A\|v_1\|_V\|\widetilde{v}\|_V \to 0. \qquad (4.25)$$

Exploiting Cauchy-Schwarz's inequality, (A1) and (b) of (3.10), we obtain

$$\begin{aligned}
&\left|-\left\langle\int_0^{\infty} g(s) B\varepsilon\left(\frac{h_1+v+\zeta v_1}{\lambda}-e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v+\zeta v_1\right) d\tau\right) ds, \varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\right.\\
&\left.+\left\langle\int_0^{\infty} g(s) B\varepsilon\left(\frac{h_1+v}{\lambda}-e^{-\lambda s}\int_0^s e^{\lambda\tau}\left(h_3(\tau)+v\right) d\tau\right) ds, \varepsilon(\widetilde{v})\right\rangle_{\mathcal{H}}\right|\\
&\leq \frac{|\zeta| L_B g_0}{\lambda}\|v_1\|_V\|\widetilde{v}\|_V \to 0. \qquad (4.26)
\end{aligned}$$

Applying again Cauchy-Schwarz's inequality and using (b) of (3.11) and (2.3), one has

$$\left|\int_{\Gamma_3}\left(p\left(\frac{h_{1\nu}+v_\nu+\zeta v_{1\nu}}{\lambda}\right)-p\left(\frac{h_{1\nu}+v_\nu}{\lambda}\right)\right)\widetilde{v}_\nu da\right|$$

$$\begin{aligned}
&\leq \|p\left(\frac{h_{1\nu}+v_\nu+\zeta v_{1\nu}}{\lambda}\right)-p\left(\frac{h_{1\nu}+v_\nu}{\lambda}\right)\|_{L^2(\Gamma_3)}\|\widetilde{v}_\nu\|_{L^2(\Gamma_3)}\\
&\leq \frac{L_p|\zeta|}{\lambda}\|v_1\|_{L^2(\Gamma_3)^d}\|\widetilde{v}\|_{L^2(\Gamma_3)^d}\\
&\leq \frac{L_p c_0^2|\zeta|}{\lambda}\|v_1\|_V\|\widetilde{v}\|_V \to 0. \qquad (4.27)
\end{aligned}$$

Therefore, combining (4.23)-(4.27) yields

$$|\langle T\left(v+\zeta v_1\right),\widetilde{v}\rangle_{V'\times V}-\langle T(v),\widetilde{v}\rangle_{V'\times V}| \to 0, \qquad (4.28)$$

as $\zeta \to 0$. It follows that $T : V \to V'$ is hemicontinuous.

Due to the coercivity, the monotonicity and the hemicontinuity of T, we conclude that T is maximal monotone and coercive which implies that this operator is surjective. Therefore, given any $(h_1,h_2,h_3)^T \in \mathscr{H}$, there exists $v \in V$ that satisfies Equation (4.16).

From (4.15), we deduce that there exists $C > 0$ such that

$$\|\eta\|_{\mathcal{M}}^2 \leq C\left(\|h_3\|_{\mathcal{M}}^2+\|v\|_V^2\right).$$

Then, we get $\eta \in \mathcal{M}$.

On the other hand, from the third equation of (4.13), it results in that there is a constant $C_1 > 0$ such that

$$\|\eta_s\|_{\mathcal{M}}^2 \leq C_1\left(\|h_3\|_{\mathcal{M}}^2+\|v\|_V^2+\|\eta\|_{\mathcal{M}}^2\right).$$

Thus, we deduce that $\eta_s \in \mathcal{M}$. Therefore $U = (u,v,\eta)^T \in \mathcal{D}(\mathcal{A})$ and thus $\lambda I + \mathcal{A}$ is surjective for some positive λ.

Using Theorem A.29 in ([1], p.236) (see also [5], page 322) and the fact that $F \in L^1(\mathbb{R}_+;\ \mathscr{H})$, we conclude that the system (4.8) has a unique mild solution $U \in C(\mathbb{R}_+;\ \mathscr{H})$ if $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ completing the proof of Theorem 4.1. ☐

§5 Exponential stability

In this section, we will show an exponential stability result of the energy solution of (4.8). For this purpose, we establish some technical lemmas that we need to prove our main results. The stability results hold under the following additional assumptions.

(A2) There exists a positive constant α such that

$$g'(t) \leq -\alpha g(t), \quad \forall t \geq 0. \qquad (5.1)$$

(A3) There exists a constant $\beta > 0$ such that

$$\|f_t(t)\|_{V'} \leq \beta\|f(t)\|_{V'}, \quad \forall t \geq 0, \quad \forall x \in \Omega, \qquad (5.2)$$

where f_t is the derivative of f with respect to t.

Remark 5.1. As a simple example of function f of $W^{1,1}(\mathbb{R}_+;V')$ which satisfies (A3), we can take

$$f(t,x) = ce^{-\beta t}h(x),$$

where c is a constant and h is a function of V'.

Now, we recall that the energy functional E is defined by

$$E(t)=\frac{1}{2}\left(\|u_t(t)\|_H^2+\|u(t)\|_V^2+\int_0^{+\infty}g(s)\,\|\eta(s,t)\|_V^2\,ds+\|f(t)\|_{V'}^2\right),\quad \forall t\in\mathbb{R}_+. \tag{5.3}$$

Lemma 5.2. *Assume that (A1), (A3) and (3.8)-(3.12) hold and let $U_0\in\mathcal{D}(\mathcal{A})$. Then, the energy functional defined by (5.3) satisfies for $\mu>0$, for all $t\geq 0$*

$$\begin{aligned}E'(t)\leq &-(m_A-C_1)\,\|u_t(t)\|_V^2+\mu C_2\|u(t)\|_V^2+\mu C_3\int_0^{+\infty}g(s)\,\|\eta(s,t)\|_V^2\,ds\\&+\frac{1}{2}\int_0^{+\infty}g'(s)\,\|\eta(s,t)\|_V^2\,ds+(\mu+\beta)\,\|f(t)\|_{V'}^2,\end{aligned}\tag{5.4}$$

where $C_1=\frac{7}{4\mu}, C_2=L_G^2+g_0^2L_B^2+1+L_p^2C_0^4, C_3=g_0+g_0L_B^2$.

Proof. Assuming that σ is regular, we differentiate the equation of (5.3) with respect to t, use the first and the third equation of (4.2) and integrate over Ω using Green's formula and boundary conditions, and obtain

$$\begin{aligned}E'(t)=&\langle u_{tt}(t),u_t(t)\rangle_H+\langle u_t(t),u(t)\rangle_V+\int_0^{+\infty}g(s)\langle\eta_t(s,t),\eta(s,t)\rangle_V ds+\langle f_t(t),f(t)\rangle_{V'}\\=&-\langle A\varepsilon(u_t(t)),\varepsilon(u_t(t))\rangle_{\mathcal{H}}-\langle G\varepsilon(u(t)),\varepsilon(u_t(t))\rangle_{\mathcal{H}}\\&+\left\langle\int_0^{\infty}g(s)B\varepsilon\left(u(t)-\eta(s,t)\right)ds,\varepsilon(u_t(t))\right\rangle_{\mathcal{H}}-\int_{\Gamma_3}p(u_\nu(t))u_{t\nu}(t)da\\&+\langle f(t),u_t(t)\rangle_{V'\times V}+\langle u_t(t),u(t)\rangle_V+\int_0^{\infty}g(s)\langle u_t(t),\eta(s,t)\rangle_V ds\\&+\frac{1}{2}\int_0^{+\infty}g'(s)\,\|\eta(s,t)\|_V^2\,ds+\langle f_t(t),f(t)\rangle_{V'}.\end{aligned}$$

By using (A1), (A3), (3.8) – (3.11), (2.3), Cauchy -Schwarz's and Young's inequalities, estimate (5.4) is established. ☐

Remark 5.3. We deduce that the energy functional is not decreasing in general as the second, the third and the last term in the right-hand side of (5.4) are not negative.

Lemma 5.4. *Assume that (A1), (A3) and (3.8)-(3.12) hold and let $U_0\in\mathcal{D}(\mathcal{A})$. Then the functional*

$$I_1(t)=\langle u_t(t),u(t)\rangle_H-\left(\int_0^t g(s)ds\right)\|f(t)\|_{V'}^2, \tag{5.5}$$

satisfies for $\varepsilon>0$, for all $t\geq 0$

$$\begin{aligned}I_1'(t)\leq&(-l+3\varepsilon)\,\|u(t)\|_V^2+C_4\int_0^{+\infty}g(s)\,\|\eta(s,t)\|_V^2\,ds+C_5\,\|u_t(t)\|_V^2\\&+\left(\frac{1}{4\varepsilon}-\lambda_0\right)\|f(t)\|_{V'}^2,\end{aligned}\tag{5.6}$$

where $C_4=\frac{g_0L_B^2}{4\varepsilon}, C_5=\frac{L_A^2}{4\varepsilon}+d^2$.

Proof. We assume that σ is regular and we differentiate (5.5) with respect to t to obtain

$$I_1'(t)=\|u_t(t)\|_H^2+\langle u_{tt}(t),u(t)\rangle_H-g(t)\|f(t)\|_{V'}^2-2\left(\int_0^t g(t)dt\right)\langle f_t(t),f(t)\rangle_{V'}.$$

Then, we multiply the first equation of (4.2) by $u(t)$, we integrate over Ω using Green's formula, boundary conditions and (3.9), and arrive at

$$\begin{aligned} I_1'(t) \leq& \|u_t(t)\|_H^2 - \langle A\varepsilon(u_t(t)), \varepsilon(u(t))\rangle_{\mathcal{H}} - m_G\|u(t)\|_V^2 \\ &+ \left\langle \int_0^\infty g(s)B\varepsilon(u(t)-\eta(s,t))ds, \varepsilon(u(t))\right\rangle_{\mathcal{H}} - \int_{\Gamma_3} p(u_\nu(t))u_\nu(t)da \\ &+ \langle f(t), u(t)\rangle_{V'\times V} - g(t)\|f(t)\|_{V'}^2 - 2\left(\int_0^t g(t)dt\right)\langle f_t(t), f(t)\rangle_{V'}. \end{aligned} \tag{5.7}$$

Using Cauchy-Schwarz's, Young's inequalities and (3.8), we obtain

$$\begin{aligned} \langle A\varepsilon(u_t(t)), \varepsilon(u(t))\rangle_{\mathcal{H}} &\leq \|A\varepsilon(u_t(t))\|_{\mathcal{H}}\|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq \frac{L_A^2}{4\varepsilon}\|u_t(t)\|_V^2 + \varepsilon\|u(t)\|_V^2. \end{aligned} \tag{5.8}$$

Thanks to Cauchy-Schwarz's, Young's inequalities, (3.10) and (A1), we get

$$\begin{aligned} &\left\langle \int_0^\infty g(s)B\varepsilon(u(t)-\eta(s,t))ds, \varepsilon(u(t))\right\rangle_{\mathcal{H}} \\ &\leq \| \int_0^\infty g(s)B\varepsilon(u(t)-\eta(s,t))ds\|_{\mathcal{H}}\|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq L_B\int_0^\infty g(s)\left(\|u(t)\|_V + \|\eta(s,t)\|_V\right)ds\|u(t)\|_V \\ &\leq (g_0L_B+\varepsilon)\|u(t)\|_V^2 + \frac{L_B^2 g_0}{4\varepsilon}\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \tag{5.9}$$

The use of Cauchy-Schwarz's and Young's inequality leads to

$$\langle f(t), u(t)\rangle_{V'\times V} \leq \frac{1}{4\varepsilon}\|f(t)\|_{V'}^2 + \varepsilon\|u(t)\|_V^2. \tag{5.10}$$

By exploiting Cauchy-Schwarz's and (A3), we obtain

$$\begin{aligned} -2\left(\int_0^t g(s)ds\right)\langle f_t(t), f(t)\rangle_{V'} &\leq 2\int_0^t g(s)ds\|f_t(t)\|_{V'}\|f(t)\|_{V'} \\ &\leq 2\beta g_0\|f(t)\|_{V'}^2. \end{aligned} \tag{5.11}$$

Combining(5.8)-(5.11), using (2.6) and (c) of (3.11), we get

$$\begin{aligned} I_1'(t) \leq& (-l+3\varepsilon)\|u(t)\|_V^2 + \frac{g_0L_B^2}{4\varepsilon}\int_0^{+\infty} g(s)\|\eta(s,t)\|_V^2 ds \\ &+ \left(\frac{L_A^2}{4\varepsilon} + d^2\right)\|u_t(t)\|_V^2 + \left(\frac{1}{4\varepsilon} - \lambda_0\right)\|f(t)\|_{V'}^2, \end{aligned}$$

where $l = m_G - g_0L_B$ and $\lambda_0 = min\ g(t) - 2\beta g_0$. □

Lemma 5.5. *Assume that (A1) and (3.8)-(3.12) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional*

$$I_2(t) = -\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(s,t)ds\right\rangle_H, \tag{5.12}$$

satisfies for $\varepsilon > 0$, for all $t \geq 0$

$$\begin{aligned} I_2'(t) \leq& -g_0\|u_t(t)\|_H^2 + C_6\varepsilon\|u_t(t)\|_V^2 + C_7\varepsilon\|u(t)\|_V^2 + C_8\int_0^{+\infty} g(s)\|\eta(s,t)\|_V^2 ds \\ &- C_9\int_0^{+\infty} g'(s)\|\eta(s,t)\|_V^2 ds + \varepsilon\|f(t)\|_{V'}^2, \end{aligned} \tag{5.13}$$

where $C_6 = L_A^2 + d^2, C_7 = L_G^2 + L_B^2 + L_p^2 C_0^4, C_8 = \frac{4g_0}{4\varepsilon} + \frac{g_0^3}{4\varepsilon} + L_B g_0, C_9 = \frac{g(0)d^2}{4\varepsilon}$.

Proof. We assume that σ is regular and we differentiate (5.12) with respect to t and use the third equation of (4.2) to obtain

$$I_2'(t) = -\langle u_{tt}(t), \int_0^\infty g(s)\eta(s,t)ds\rangle_H - g_0\|u_t(t)\|_H^2 + \langle u_t(t), \int_0^\infty g(s)\eta_s(s,t)ds\rangle_H.$$

We integrate with respect to s the last term on the right-hand side of the above equality and use the fact that $\lim_{s\to+\infty} g(s) = 0,\ \eta(0,t) = 0$ to find

$$I_2'(t) = -\langle u_{tt}(t), \int_0^\infty g(s)\eta(s,t)ds\rangle_H - g_0\|u_t(t)\|_H^2 - \langle u_t(t), \int_0^\infty g'(s)\eta(s,t)ds\rangle_H.$$

Taking into account the first equation of (4.2), integrating over Ω and using Green's formula, boundary conditions, we get

$$\begin{aligned} I_2'(t) =& \langle A\varepsilon(u_t(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\rangle_{\mathcal{H}} + \langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\rangle_{\mathcal{H}} \\ &- \left\langle \int_0^\infty g(s)B\varepsilon(u(t)-\eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\ &+ \int_{\Gamma_3} p(u_\nu(t)) \int_0^\infty g(s)\eta_\nu(s,t)dsda - \langle f(t), \int_0^\infty g(s)\eta(s,t)ds\rangle_{V'\times V} \\ &- g_0\|u_t(t)\|_H^2 - \langle u_t(t), \int_0^\infty g'(s)\eta(s,t)ds\rangle_H. \end{aligned} \tag{5.14}$$

Thanks to Cauchy Schwarz's and Young's inequalities, (A1) and (3.8), it follows that

$$\begin{aligned} &\langle A\varepsilon(u_t(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\rangle_{\mathcal{H}} \\ \le& \|A\varepsilon(u_t(t))\|_{\mathcal{H}} \| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\|_{\mathcal{H}} \\ \le& \varepsilon L_A^2\|u_t(t)\|_V^2 + \frac{g_0}{4\varepsilon}\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \tag{5.15}$$

We can estimate the second term of the right hand side of (5.14) similar as (5.15) by using (3.9) instead of (3.8)

$$\begin{aligned} &\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\rangle_{\mathcal{H}} \\ \le& \|G\varepsilon(u(t))\|_{\mathcal{H}} \| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\|_{\mathcal{H}} \\ \le& \varepsilon L_G^2\|u(t)\|_V^2 + \frac{g_0}{4\varepsilon}\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \end{aligned} \tag{5.16}$$

By using Cauchy Schwarz's, Young's inequalities, (A1) and (3.10), we obtain

$$\begin{aligned} &- \left\langle \int_0^\infty g(s)B\varepsilon(u(t)-\eta(s))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\ \le& \| \int_0^\infty g(s)B\varepsilon(u(t)-\eta(s,t))ds\|_{\mathcal{H}} \| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds\|_{\mathcal{H}} \\ \le& L_B \int_0^\infty g(s)\|u(t)-\eta(s,t)\|_V ds \int_0^\infty g(s)\|\eta(s,t)\|_V ds \end{aligned}$$

$$\le \varepsilon L_B^2 \|u(t)\|_V^2 + \left(\frac{g_0^3}{4\varepsilon} + L_B g_0\right) \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds. \tag{5.17}$$

Applying again Cauchy Schwarz's, Young's inequalities and using (A1), (2.3), we obtain

$$\begin{aligned}
&\int_{\Gamma_3} p(u_\nu(t)) \left(\int_0^\infty g(s)\eta_\nu(s,t)ds\right) da \\
\le& \left(\int_{\Gamma_3} p^2(u_\nu(t))da\right)^{\frac{1}{2}} \left(\int_{\Gamma_3} \left(\int_0^\infty g(s)\eta_\nu(s,t)ds\right)^2 da\right)^{\frac{1}{2}} \\
\le& \varepsilon L_p^2 C_0^4 \|u(t)\|_V^2 + \frac{g_0}{4\varepsilon} \int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{5.18}$$

The fifth term of the right-hand side of (5.14) can be handled by

$$-\langle f(t), \int_0^\infty g(s)\eta(s,t)ds\rangle_{V'\times V} \le \varepsilon \|f(t)\|_{V'}^2 + \frac{g_0}{4\varepsilon}\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds. \tag{5.19}$$

The use of Cauchy Schwarz's, Young's inequalities, (A1) and (2.6) leads to

$$\begin{aligned}
&-\left\langle u_t(t), \int_0^\infty g'(s)\eta(s,t)ds\right\rangle_H \\
\le& \|u_t(t)\|_H \|\int_0^\infty g'(s)\eta(s,t)ds\|_H \\
\le& \varepsilon\|u_t(t)\|_H^2 - \frac{g(0)}{4\varepsilon}\left(\int_0^\infty g'(s)\|\eta(s,t)\|_H^2 ds\right) \\
\le& d^2\varepsilon\|u_t(t)\|_V^2 - \frac{g(0)d^2}{4\varepsilon}\left(\int_0^\infty g'(s)\|\eta(s,t)\|_V^2 ds\right).
\end{aligned} \tag{5.20}$$

Combining the estimates (5.15)-(5.20) and (5.14), we obtain (5.13). ☐

We define a Lyapunov functional L as follows

$$L(t) = E(t) + \epsilon(NI_1(t) + I_2(t)), \tag{5.21}$$

where ϵ, N are positive constants to be chosen later.

Theorem 5.6. *Assume that (A1)-(A3) and (3.8)-(3.12) hold. Assume that m_G satisfies*

$$m_G > \frac{3}{4\lambda_0} + g_0 L_B, \tag{5.22}$$

and there exists a positive constant δ_0 independent of m_A such that

$$m_A > \delta_0, \tag{5.23}$$

where $\lambda_0 > 0$ satisfies (5.31) and δ_0 is given in (5.33), and the kernal of the memory term $g(t)$ satisfies

$$g(t) > 2\beta g_0 \tag{5.24}$$

then, for any $U_0 \in \mathcal{D}(\mathcal{A})$, there exist two positive constants δ_1, δ_2 such that the solution of the problem (4.2) satisfies

$$\|U(t)\|_{\mathscr{H}}^2 \le \delta_2 e^{-\delta_1 t}. \tag{5.25}$$

Proof. We differentiate (5.21) with respect to t and use (5.4), (5.6), (5.13) to obtain for all $t \ge 0$

$$L'(t) = E'(t) + \epsilon(NI_1'(t) + I_2'(t))$$

$$\begin{aligned}
&\leq -\epsilon\left(g_0\|u_t(t)\|_H^2+\left(N(l-3\varepsilon)-\varepsilon C_7-\frac{C_2\mu}{\epsilon}\right)\|u(t)\|_V^2\right)\\
&-\epsilon\left(C_3\left(N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon\right)-\frac{C_3\mu}{\epsilon}\right)\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds\\
&+\left(C_1-m_A+\epsilon(C_5N+\varepsilon C_6)\right)\|u_t(t)\|_V^2+\left(\frac{1}{2}-\epsilon C_9\right)\int_0^\infty g'(s)\|\eta(s,t)\|_V^2 ds\\
&+\epsilon C_{10}\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds-\epsilon\left(N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon-\frac{\mu+\beta}{\epsilon}\right)\|f(t)\|_{V'}^2 \qquad (5.26)
\end{aligned}$$

where $C_{10}=NC_4+C_8+C_3\left(N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon\right)$,
From (A2), we get

$$\begin{aligned}
&\left(\frac{1}{2}-\epsilon C_9\right)\int_0^\infty g'(s)\|\eta(t,s)\|_V^2 ds+\epsilon C_{10}\int_0^\infty g(s)\|\eta(t,s)\|_V^2 ds\\
&\leq\left(\frac{1}{2}-\epsilon C_{11}\right)\int_0^\infty g'(s)\|\eta(t,s)\|_V^2 ds,
\end{aligned}$$

where $C_{11}=C_9+\frac{C_{10}}{\alpha}$.
Inserting the above inequality in (5.26), we get

$$\begin{aligned}
L'(t)=&E'(t)+\epsilon(NI_1'(t)+I_2'(t))\\
\leq&-\epsilon\left(g_0\|u_t(t)\|_H^2+\left(N(l-3\varepsilon)-\varepsilon C_7-\frac{C_2\mu}{\epsilon}\right)\|u(t)\|_V^2\right)\\
&-\epsilon\left(C_3\left(N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon\right)-\frac{C_3\mu}{\epsilon}\right)\int_0^\infty g(s)\|\eta(s,t)\|_V^2 ds\\
&+\left(C_1-m_A+\epsilon(C_5N+\varepsilon C_6)\right)\|u_t(t)\|_V^2+\left(\frac{1}{2}-\epsilon C_{11}\right)\int_0^\infty g'(s)\|\eta(s,t)\|_V^2 ds\\
&-\epsilon\left(N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon-\frac{\mu+\beta}{\epsilon}\right)\|f(t)\|_{V'}^2. \qquad (5.27)
\end{aligned}$$

By definition of E, I_1 and I_2 and using Cauchy Schwarz's, Young's inequalities, (A1) and (2.6), we have

$$\begin{aligned}
|I_1(t)|=&\left|\langle u_t(t),u(t)\rangle_H-\left(\int_0^t g(s)ds\right)\|f(t)\|_{V'}^2\right|\leq\frac{1}{2}\|u_t(t)\|_H^2+\frac{1}{2}\|u(t)\|_H^2+\int_0^t g(t)\|f(t)\|_{V'}^2\\
&\leq max(1,d^2,2g_0)E(t), \qquad (5.28)
\end{aligned}$$

and

$$\begin{aligned}
|I_2(t)|=&\left|\left\langle u_t(t),\int_0^{+\infty}g(s)\eta(s,t)ds\right\rangle_H\right|\leq\frac{1}{2}\left(\|u_t(t)\|_H^2+g_0d^2\int_0^{+\infty}g(s)\|\eta(t,s)\|_V^2 ds\right)\\
&\leq max(1,g_0d^2)E(t). \qquad (5.29)
\end{aligned}$$

Combining (5.3), (5.28), (5.29), we get

$$|L(t)-E(t)|\leq\epsilon\gamma_3E(t),$$

where γ_3 is a positive constant such that

$$\gamma_3=Nmax(1,d^2,2g_0)+max(1,g_0d^2).$$

Using (5.21), we get

$$(1-\epsilon\gamma_3)E(t)\leq L(t)\leq(1+\epsilon\gamma_3)E(t), \qquad (5.30)$$

then by choosing $\epsilon < \frac{1}{\gamma_3}$, we have $L \sim E$.
Now, using (5.22), we can pick ε such that

$$\frac{1}{4\lambda_0} < \varepsilon < \frac{l}{3}, \tag{5.31}$$

and therefore N large enough such that

$$\begin{aligned} N(l-3\varepsilon) - \varepsilon C_7 &> 0, \\ N\left(\lambda_0 - \frac{1}{4\varepsilon}\right) - \varepsilon &> 0, \\ N\left(\lambda_0 - \frac{1}{4\varepsilon}\right) - \varepsilon - 2C_{11}\beta &> 0, \\ N\left(\lambda_0 - \frac{1}{4\varepsilon}\right) - \varepsilon - \gamma_3\beta &> 0. \end{aligned} \tag{5.32}$$

Next, we assume that m_A satisfies (5.23) under the following choice of δ_0

$$\delta_0 = max\left\{\frac{C_2\mu\left(C_5N + \varepsilon C_6\right)}{N\left(l-3\varepsilon\right) - \varepsilon C_7}, \frac{\left(\mu+\beta\right)\left(C_5N+\varepsilon C_6\right)}{N\left(\lambda_0 - \frac{1}{4\varepsilon}\right) - \varepsilon}\right\} + C_1, \tag{5.33}$$

Now, we assume that μ satisfies

$$\mu < min$$
$$\left\{\frac{N(l-3\varepsilon)-\varepsilon C_7}{C_2\gamma_3}, \frac{N(l-3\varepsilon)-\varepsilon C_7}{2C_2C_{11}}, \frac{N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon-\gamma_3\beta}{\gamma_3}, \frac{N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon-2C_{11}\beta}{2C_{11}}\right\}. \tag{5.34}$$

From (5.34), we have

$$\frac{C_2\mu}{N\left(l-3\varepsilon\right)-\varepsilon C_7} < min\left\{\frac{1}{\gamma_3}, \frac{1}{2C_{11}}\right\}, \tag{5.35}$$

and

$$\frac{\mu+\beta}{N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon} < min\left\{\frac{1}{\gamma_3}, \frac{1}{2C_{11}}\right\}. \tag{5.36}$$

It follows from (5.33) that

$$\frac{C_2\mu}{N\left(l-3\varepsilon\right)-\varepsilon C_7} < \frac{m_A - C_1}{C_5N+\varepsilon C_6}, \tag{5.37}$$

and

$$\frac{\mu+\beta}{N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon} < \frac{m_A - C_1}{C_5N+\varepsilon C_6}. \tag{5.38}$$

Therefore, combining (5.35)-(5.38) we can fix ϵ such that

$$max\left\{\frac{C_2\mu}{N\left(l-3\varepsilon\right)-\varepsilon C_7}, \frac{\mu+\beta}{N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon}\right\} < \epsilon < min\left\{\frac{m_A-C_1}{C_5N+\varepsilon C_6}, \frac{1}{2C_{11}}, \frac{1}{\gamma_3}\right\}. \tag{5.39}$$

Consequently, we arrive at

$$L'(t) \le -\epsilon C_{12}E(t) + (\frac{1}{2} - \epsilon C_{11}) \int_0^\infty g'(s)\|\eta(s,t)\|_V^2 ds,$$

where

$$C_{12} = 2min\times$$

$$\left(g_0,\ N(l-3\varepsilon)-\varepsilon C_7-\frac{C_2\mu}{\epsilon},\ C_3\left(N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon\right)-\frac{C_3\mu}{\epsilon},\ N\left(\lambda_0-\frac{1}{4\varepsilon}\right)-\varepsilon-\frac{\mu+\beta}{\varepsilon}\right).$$

Using (A1), (5.39) and (5.30), we obtain

$$L'(t)\le\frac{-\epsilon C_{12}}{1+\epsilon\gamma_3}L(t). \tag{5.40}$$

We integrate (5.40) from 0 to t to obtain

$$L(t)\le L(0)e^{-\delta_1 t},\quad \forall t\in\mathbb{R}_+, \tag{5.41}$$

where $\delta_1=\frac{\epsilon C_{12}}{1+\epsilon\gamma_3}$

From (5.30) and (5.41), we have

$$\|U(t)\|^2_{\mathscr{H}}\le 2E(t)\le\frac{2}{1-\epsilon\gamma_3}L(t)\le\frac{2}{1-\epsilon\gamma_3}L(0)e^{-\delta_1 t},$$

which yields (5.25) with $\delta_2=\frac{1}{1-\epsilon\gamma_3}L(0)$.

Thus the proof of Theorem 5.6 is completed. ☐

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Declarations

Conflict of interest The authors declare no conflict of interest.

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