

A stochastic predator-prey model with additional food for predator incorporating fear effect and a prey refuge

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Abstract. In this paper, we study a predator-prey model with additional food for predator. By using white noise to perturb the natural growth rates and introduce a jump process, we model the corresponding stochastic differential equations. The effect of fear and prey refuge on population dynamics is also considered. First, we use Itô's formula to prove the existence and uniqueness of a global positive solution and its boundedness. Next, sufficient conditions for the extinction and persistence of both species have been given. Then the stochastic permanence of our system is investigated under some conditions. Our main results demonstrate that sufficiently large white noise could drive both species to extinction. However, Lévy noise enhances the survival of both prey and predator species. Our analytical derivations are justified through numerical simulations which show the reliability of the model from the ecological point of view. In addition, we have investigated the impact of fear effect, prey refuge and the additional food biomass on this model by numerical simulation.

§1 Introduction

Population model, as an important part of ecology, has been widely studied and explored for its rich dynamic characteristics aiming to provide theoretical guidance for the conservation, development and utilization of biological resources [1]. Among the most important population models, the predator-prey model plays an important role in understanding the interactions between different species in unstable natural environments. Modeling and analysing of such systems are very interesting and active research topics for ecologists and mathematicians. Over the years, many scholars have made great contributions to this topic [7, 15, 36].

In the past few decades, most studies have only considered direct effects of predator species on prey species, as such effects are readily observable under any circumstances. However,

Received: 2022-04-13. Revised: 2022-11-05.

MR Subject Classification: 60H10, 92D25.

Keywords: fear effect, prey refuge, additional food, Lévy noise, stochastic permanence.

Digital Object Identifier(DOI): <https://doi.org/10.1007/s11766-026-4733-3>.

Supported by Guangxi Natural Science Foundation (2026GXNSFAA00641039).

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some actual data suggest that indirect effects of predator species on prey species also have a significant effect on population dynamics, in some cases even greater than direct predation [4, 20, 21]. Any prey will respond to the perceived risk of predation and exhibit different types of anti-predation behaviors, such as new selection of habitat, foraging behaviours, vigilance and several psychological changes [5, 28–30]. This predator-targeting activity can be considered to be beneficial for adult survival, but the long-term cost reduces the basic reproductive capacity of the prey. Furthermore, when some prey are fully aware of the imminent predation risk, new habitat selection can sometimes negatively impact an individual's lifetime reproductive success. Poor habitat selection, that is, poor quality of new habitats, not only affects the reproduction of adult insects, but also affects the survival of adult insects [36]. In 2011, the experiment of Zanette et al. [39] showed that the song sparrows (*Melospiza Melodia*) produced 40% fewer offspring due to fear of predators. This reduction is due to the effect of anti-predator behavior on birth rates and offspring survival. In the study of Krapivsky et al. [19], it has been found that lambs behave differently for different positions different numbers of their predator (lion). Thus, theoretical biologists and evolutionary ecologists have realized that not only direct kills or shocks need to be considered, but indirect shocks or fear costs must be factored into predator and prey population models.

On the other hand, not all prey are captured by predators because they usually have refuges to avoid predators in reality [12, 37]. In order to effectively avoid predators and increase the survival rate, the prey species often actively seek refuge. This phenomenon, known as the refuge effect [40, 42], has been scientifically demonstrated for a long time. Mukherjee [26] showed that prey sanctuary thresholds in food chain models can determine long-term survival for all species; furthermore, over-execution of predator defense strategies may be the root cause of predator extinction from the system. Khajanchi [14] demonstrates that the constant prey refuge plays an important role on the coexistence of stage structured predator-prey species. The prey's sanctuary depends on prey biomass and the number of predators. Predator systems with prey refuges are also one of the hotspots in biomathematical research, and many scholars have made great achievements in this field [18, 23, 35].

The behavioral characteristics of predation can be called functional responses, which play a dominant role in some complex dynamical behaviors, such as the steady states, bistability, periodic oscillations, chaos and bifurcation phenomena [16, 27, 33]. The functional response depends on many factors, such as different prey densities, the efficiency with which predators search for and kill prey, processing time, competition between predators, etc. Das et al. [8] constructed functional responses when provide additional food to predator, which assumed that the predator consumes both the additional food and the prey according to their availability.

For predator-prey system, let $x(t)$ be the prey density at time t , and $y(t)$ be the predator population density at time t . It is assumed that in the presence of direct predation and fear factor, the prey population follows a logical growth, and the predator preys on prey according to the functional responses of predator towards prey and additional food. Furthermore, population dynamics are inevitably affected by environmental white noise which is the im-

portant component in an ecosystem. To capture how environmental fluctuations affect the system, stochastic perturbations need to be taken into account [13, 24, 25, 32]. Applying the technique used in [9] to include stochastic effects by perturbing the growth rate with white noise $r_i \rightarrow r_i + \sigma_i \frac{dB_i(t)}{dt}$, $i = 1, 2$, a system of stochastic differential equation taking into account fear effects, prey refuge and environmental noise factors is obtained (the proof can be shown by the similar procedure as in [38])

$$\begin{cases} dx(t) = \left[r_1 x(t) e^{-\alpha y(t)} - c_1 x^2(t) - \frac{a(1-m)x(t)y(t)}{b + \beta\eta A + (1-m)x(t)} \right] dt + \sigma_1 x(t) dB_1(t), \\ dy(t) = \left[r_2 y(t) - c_2 y^2(t) + \frac{p[(1-m)x(t) + \eta A]y(t)}{b + \beta\eta A + (1-m)x(t)} \right] dt + \sigma_2 y(t) dB_2(t), \end{cases} \quad (1)$$

where $r_1, r_2, c_1, c_2, \alpha, \beta, \eta, A, a, b, p$, and m are positive constants, r_i stands for the intrinsic growth rate of species, and c_i describes the strength of competition among individual of species, $i = 1, 2$. Function $g(\alpha, y) = e^{-\alpha y}$ represents the fear function which stands for the cost of anti-predator defence of prey due to fear induced by predator and α is the level of fear. The function $g(\alpha, y)$ has some special properties, as several field data show that the effect of fear reduces the reproductive process of prey species. For more details on the fear function $g(\alpha, y)$, see [31]. β is the quality of additional food. A is the biomass of species that serve as additional food for predators and ηA represents the effectual food level. a is the maximum rate of predation and b is the half-saturation value of the predators, p is the product of the maximum predation rate and conversion factor [11, 34]. m is the strength of prey refuge and $m \in [0, 1)$. σ_i^2 ($i = 1, 2$) stands for the intensity of white noise. Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and \mathcal{F}_0 contains all \mathbb{P} -null sets), and $B_i(t) (t \geq 0) (i = 1, 2)$ be a scalar standard Brownian motion defined on this probability space.

In addition, the population system may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. However, the stochastic Lotka-Volterra model (1) cannot explain such phenomena [2, 3, 10]. To explain these phenomena, introducing a jump process into the underlying population dynamics provides a feasible and more realistic model. In this paper, we develop a stochastic predator-prey model with additional food for predator and Lévy noise as follows:

$$\begin{cases} dx(t) = x(t) \left\{ \left[r_1 e^{-\alpha y(t)} - c_1 x(t) - \frac{a(1-m)y(t)}{b + \beta\eta A + (1-m)x(t)} \right] dt + \sigma_1 dB_1(t) \right\} \\ \quad + \int_{\mathbb{Z}} \gamma_1(u) x(t^-) \tilde{N}(dt, du), \\ dy(t) = y(t) \left\{ \left[r_2 - c_2 y(t) + \frac{p[(1-m)x(t) + \eta A]}{b + \beta\eta A + (1-m)x(t)} \right] dt + \sigma_2 dB_2(t) \right\} \\ \quad + \int_{\mathbb{Z}} \gamma_2(u) y(t^-) \tilde{N}(dt, du), \end{cases} \quad (2)$$

with initial data $X_0 = (x_0, y_0) \in \mathbb{R}_+^2$, where $x(t^-)$ and $y(t^-)$ represent the left limit of $x(t)$ and $y(t)$, respectively; N is a Poisson counting measure with compensator \tilde{N} and characteristic measure λ on a measurable subset \mathbb{Z} of $(0, \infty)$ with $\lambda(\mathbb{Z}) < \infty$ and $\tilde{N}(dt, du) = N(dt, du) -$

$\lambda(du)dt$. The parameter $\gamma_i(u)$ is the effect of Lévy noise on the i th species. For biological reasons, we suppose that $1 + \gamma_i(u) > 0$, where $\gamma_i(u) > 0$ means the increasing of the species (e.g., planting) and $-1 < \gamma_i(u) < 0$ means the decreasing of the species (e.g., harvesting and epidemics), $u \in \mathbb{Z}, i = 1, 2$. The Brownian motion and Lévy jumps are mutually independent.

Motivated by these arguments, we first propose a deterministic predator-prey model with fear effect and prey refuge, then we incorporate Lévy noise into the model. To the best of the author's knowledge, no work has been done on the stochastic model where the prey is affected by fear and provide additional food into the system.

Based on the above discussion, this paper intends to study the dynamical properties of a stochastic predator-prey model with additional food for predator and Lévy noise. The organization of this paper is as follows. In the next section, we present some preliminary results of model (2). In Section 3, we prove the existence and boundedness of global positive unique solutions. And in Section 4, we establish some sufficient conditions for the extinction and persistence of both species. We further investigate the stochastic permanence of our system in Section 5. Some numerical simulations are carried out in Section 6. We close the paper with a conclusion in Section 7.

§2 Preliminaries

In this section, we begin with the introduction of the generalized Itô formula with jumps (more details see, e.g., Reference [41]). Let $x(t) \in \mathbb{R}^n$ be a solution of the following stochastic differential equation with Lévy jumps

$$dx(t) = F(x(t^-), t^-)dt + G(x(t^-), t^-)dB(t) + \int_{\mathbb{Z}} H(x(t^-), t^-, u)\tilde{N}(dt, du), \quad (3)$$

where $F : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{Z} \rightarrow \mathbb{R}^n$ are measurable functions. Given $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, we define the operator LV by

$$\begin{aligned} LV(x(t), t) &= V_t(x(t), t) + V_x(x(t), t)F(x(t), t) + \frac{1}{2}\text{trace}[G^T(x(t), t)V_{xx}(x(t), t)G(x(t), t)] \\ &\quad + \int_{\mathbb{Z}} \{V(x(t) + H(x(t), t, u), t) - V(x(t), t) - V_x(x(t), t)H(x(t), t, u)\} \lambda(du), \end{aligned} \quad (4)$$

where $V_t(x(t), t) = \frac{\partial V(x(t), t)}{\partial t}$, $V_x(x(t), t) = \left(\frac{\partial V(x(t), t)}{\partial x_1}, \dots, \frac{\partial V(x(t), t)}{\partial x_n} \right)$, $V_{xx}(x(t), t) = \left(\frac{\partial^2 V(x(t), t)}{\partial x_i \partial x_j} \right)_{n \times n}$.

Then the generalized Itô formula with jumps is as follows

$$\begin{aligned} dV(x(t), t) &= LV(x(t), t)dt + V_x(x(t), t)G(x(t), t)dB(t) \\ &\quad + \int_{\mathbb{Z}} \{V(x(t) + H(x(t), t, u), t) - V(x(t), t)\} \tilde{N}(dt, du). \end{aligned} \quad (5)$$

From now on, we make the following fundamental assumptions on the jump-diffusion coefficients of model (2).

Assumption 1. For any $p > 0$ and $i = 1, 2$, there exist constants $K_j > 0$ ($j = 1, 2, 3$) such that

$$\int_{\mathbb{Z}} \{|\gamma_i(u)|^2 \vee [\ln(1 + \gamma_i(u))]^2\} \lambda(du) \leq K_1 < \infty,$$

$$\int_{\mathbb{Z}} \{\gamma_i(u) - \ln(1 + \gamma_i(u))\} \lambda(du) \leq K_2 < \infty,$$

$$\int_{\mathbb{Z}} \{(1 + \gamma_i(u))^p - 1 - p\gamma_i(u)\} \lambda(du) \leq K_3 < \infty, \quad p \geq 1,$$

where $i = 1, 2$, which implies that the intensity of Lévy noise cannot be too strong, otherwise, the solution of system (2) may explode in some finite time.

§3 Existence and boundedness of solutions

In this section, under Assumption 1, we show the solution of system (2) is globally positive and bounded. First of all, we introduce the following theorem.

Theorem 3.1. *Let Assumption 1 hold. Then for any given initial value $(x_0, y_0) \in \mathbb{R}_+^2$, the system (2) will possess a unique solution $(x(t), y(t))$, for all $t \geq 0$, and the solution will remain in \mathbb{R}_+^2 with probability 1.*

Proof. Since the coefficients of (2) are locally Lipschitz continuous, by the stochastic differential equation theory, for any given initial condition $(x_0, y_0) \in \mathbb{R}_+^2$, there is a unique local solution $(x(t), y(t))$ for $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that the solution is global, we need to prove $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large such that x_0 and y_0 remain within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, we define the stopping time by

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k \right) \text{ or } y(t) \notin \left(\frac{1}{k}, k \right) \right\},$$

where $\inf \emptyset = \infty$ (as usual ∞ denotes the empty set). Denote $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$ and since τ_k is nondecreasing as $k \rightarrow \infty$, then $\tau_\infty \leq \tau_e$ a.s. Now we show that $\tau_\infty = \infty$ a.s. If not, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Thus, by denoting $\Omega_k = \{\tau_k \leq T\}$, there exists $k_1 \geq k_0$ such that

$$\mathbb{P}(\Omega_k) \geq \varepsilon \quad \text{for all } k \geq k_1. \quad (6)$$

Define a C^2 -function $V: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ by $V(x, y) = x - 1 - \ln x + y - 1 - \ln y$ which is non-negative (more applications of Lyapunov function to global stability, see [17]). If $(x(t), y(t)) \in \mathbb{R}_+^2$, by using Itô formula, we get

$$dV(x, y) = LV(x, y)dt + \sigma_1(x - 1)dB_1(t) + \sigma_2(y - 1)dB_2(t) \\ + \int_{\mathbb{Z}} \{[\gamma_1(u)x - \ln(1 + \gamma_1(u))] + [\gamma_2(u)y - \ln(1 + \gamma_2(u))]\} \tilde{N}(dt, du), \quad (7)$$

where

$$LV(x, y) = (x - 1) \left[r_1 e^{-\alpha y} - c_1 x - \frac{a(1 - m)y}{b + \beta\eta A + (1 - m)x} \right] + \left(1 - \frac{1}{y} \right) \frac{p[(1 - m)x + \eta A]y}{b + \beta\eta A + (1 - m)x} \\ + (y - 1)(r_2 - c_2 y) + \frac{\sigma_1^2 + \sigma_2^2}{2} \\ + \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) + \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du) \\ \leq x \left[r_1 e^{-\alpha y} - c_1 x - \frac{a(1 - m)y}{b + \beta\eta A + (1 - m)x} \right] - \left[r_1 e^{-\alpha y} - c_1 x - \frac{a(1 - m)y}{b + \beta\eta A + (1 - m)x} \right]$$

$$\begin{aligned}
& + y(r_2 - c_2y) - (r_2 - c_2y) + \frac{p[(1-m)x + \eta A]y}{b + \beta\eta A + (1-m)x} + \frac{\sigma_1^2 + \sigma_2^2}{2} \\
& + \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) + \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du) \\
& \leq -c_1x^2 + (r_1 + c_1)x - c_2y^2 + \left[c_2 + r_2 + p + \frac{a(1-m) + p\eta A}{b + \beta\eta A} \right] y + \frac{\sigma_1^2 + \sigma_2^2}{2} + 2K_2 \\
& \leq \frac{(r_1 + c_1)^2}{4c_1} + \frac{\left[c_2 + r_2 + p + \frac{a(1-m) + p\eta A}{b + \beta\eta A} \right]^2}{4c_2} + \frac{\sigma_1^2 + \sigma_2^2}{2} + 2K_2 \\
& := K > 0,
\end{aligned} \tag{8}$$

where K is a positive constant. Thus

$$\begin{aligned}
dV(x, y) & \leq Kdt + \sigma_1(x-1)dB_1(t) + \sigma_2(y-1)dB_2(t) + \int_{\mathbb{Z}} [\gamma_1(u)x - \ln(1 + \gamma_1(u))] \tilde{N}(dt, du) \\
& + \int_{\mathbb{Z}} [\gamma_2(u)y - \ln(1 + \gamma_2(u))] \tilde{N}(dt, du).
\end{aligned} \tag{9}$$

Integrating both sides of (9) from 0 to $\tau_k \wedge T$, we obtain

$$\begin{aligned}
\int_0^{\tau_k \wedge T} dV(x(t), y(t)) & \leq \int_0^{\tau_k \wedge T} Kdt + \int_0^{\tau_k \wedge T} \sigma_1(x-1)dB_1(t) + \int_0^{\tau_k \wedge T} \sigma_2(y-1)dB_2(t) \\
& + \int_0^{\tau_k \wedge T} \int_{\mathbb{Z}} [\gamma_1(u)x - \ln(1 + \gamma_1(u))] \tilde{N}(dt, du) \\
& + \int_0^{\tau_k \wedge T} \int_{\mathbb{Z}} [\gamma_2(u)y - \ln(1 + \gamma_2(u))] \tilde{N}(dt, du).
\end{aligned}$$

Taking expectations of the above inequality and using the definition of the stopping times τ_k leads to

$$EV(x_{\tau_k \wedge T}, y_{\tau_k \wedge T}) \leq V(x_0, y_0) + KE(\tau_k \wedge T) \leq V(x_0, y_0) + KT.$$

On the other hand, by (6), we get $\mathbb{P}(\Omega_k) \geq \varepsilon$. Noting that for every $\omega \in \Omega_k$, either $x_{\tau_k}(\omega)$ or $y_{\tau_k}(\omega)$ equals either k or $1/k$, we have

$$V(x_{\tau_k \wedge T}(\omega), y_{\tau_k \wedge T}(\omega)) \geq \varepsilon(k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k \right).$$

Setting $k \rightarrow \infty$ leads to the contradiction

$$\infty > V(x_0, y_0) + KT = \infty.$$

Therefore, we have $\tau_\infty = \infty$ a.s. The proof is complete. \square

Now we prove the boundedness of the moments of $x(t)$ and $y(t)$.

Theorem 3.2. *Let $X(t) = (x(t), y(t))$ be a solution of system (2). For any initial value $X_0 = (x_0, y_0) \in \mathbb{R}_+^2$, there exists $M_j(n) > 0$ ($j = 1, 2$) such that*

$$\begin{cases} \limsup_{t \rightarrow \infty} E(x^n(t)) \leq M_1(n), \\ \limsup_{t \rightarrow \infty} E(y^n(t)) \leq M_2(n), \end{cases} \quad \text{for any } n \geq 1. \tag{10}$$

Proof. Applying Itô's formula to the first equation of (2), we can easily obtain

$$d(e^t x^n(t)) = ne^t x^n(t) \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + r_1 e^{-\alpha y(t)} - c_1 x(t) - \frac{a(1-m)y(t)}{b + \beta\eta A + (1-m)x(t)} + \frac{n-1}{2} \sigma_1^2 \right] dt$$

$$+ ne^t x^n(t) \sigma_1 dB_1(t) + e^t x^n(t) \int_{\mathbb{Z}} [(1 + \gamma_1(u))^n - 1] \tilde{N}(dt, du), \tag{11}$$

where $\tilde{\gamma}_{1n} = \int_{\mathbb{Z}} \{(1 + \gamma_1(u))^n - 1 - n\gamma_1(u)\} \lambda(du)$. Integrating two sides of (11) and taking expectations leads to

$$\begin{aligned} E(e^t x^n(t)) &= x_0^n + n \int_0^t \left\{ E(e^s x^n(s)) \left[\frac{1 + \tilde{\gamma}_{1n}}{n} + r_1 e^{-\alpha y(t)} - \frac{a(1-m)y(t)}{b + \beta\eta A + (1-m)x(t)} \right. \right. \\ &\quad \left. \left. + \frac{n-1}{2} \sigma_1^2 \right] - c_1 E(e^s x^{n+1}(s)) \right\} ds \\ &\leq x_0^n + n \int_0^t \left[E(e^s x^n(s)) \left(\frac{1 + \tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2} \sigma_1^2 \right) - c_1 E(e^s x^{n+1}(s)) \right] ds. \end{aligned}$$

Now let $h(x) = x^n \left[\left(\frac{1 + \tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2} \sigma_1^2 \right) - c_1 x \right]$. In order to find the maximum value of $h(x)$, we first calculate $h'(x)$ and obtain

$$\begin{aligned} h'(x) &= nx^{n-1} \left[\left(\frac{1 + \tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2} \sigma_1^2 \right) - c_1 x \right] + x^n (-c_1) \\ &= x^{n-1} \left[n \left(\frac{1 + \tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2} \sigma_1^2 \right) - c_1(n+1)x \right]. \end{aligned}$$

When $x = \frac{n(\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2)}{c_1(n+1)}$, we get the critical point by getting $h'(x) = 0$. Further, we notice that $h''(x) < 0$ at the critical point and the maximum value at the critical point is given by

$$h_{\max} = \left(\frac{n}{c_1} \right)^n \left(\frac{\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2}{n+1} \right)^{n+1}.$$

Therefore,

$$\begin{aligned} E(e^t x^n(t)) &\leq x_0^n + nE \int_0^t e^s \left(\frac{n}{c_1} \right)^n \left(\frac{\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2}{n+1} \right)^{n+1} ds \\ &\leq x_0^n + \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2 \right)^{n+1}}{c_1^n} (e^t - 1), \end{aligned}$$

i.e.,

$$\begin{aligned} E(x^n(t)) &\leq \left[x_0^n - \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2 \right)^{n+1}}{c_1^n} \right] e^{-t} \\ &\quad + \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2 \right)^{n+1}}{c_1^n}. \end{aligned}$$

One can observe that for $t = 0$, $E(x^n(t)) = x_0^n$. And when $t \rightarrow \infty$,

$$\limsup_{t \rightarrow \infty} E(x^n(t)) \leq \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1+\tilde{\gamma}_{1n}}{n} + r_1 + \frac{n-1}{2}\sigma_1^2 \right)^{n+1}}{c_1^n} := M_1(n).$$

Therefore, we conclude that $\limsup_{t \rightarrow \infty} E(x^n(t)) \leq M_1(n)$ for $n \geq 1$.

Similarly for predator species, we have

$$d(e^t y^n(t)) = ne^t y^n(t) \left\{ \frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 - c_2 y(t) + \frac{p[(1-m)x(t) + \eta A]}{b + \beta\eta A + (1-m)x(t)} + \frac{n-1}{2} \sigma_2^2 \right\} dt$$

$$+ ne^t y^n(t) \sigma_2 dB_2(t) + e^t y^n(t) \int_{\mathbb{Z}} [(1 + \gamma_2(u))^n - 1] \tilde{N}(dt, du), \quad (12)$$

where $\tilde{\gamma}_{2n} = \int_{\mathbb{Z}} \{(1 + \gamma_2(u))^n - 1 - n\gamma_2(u)\} \lambda(du)$. Integrating both sides of (12) from 0 to t and taking expectation, we have

$$\begin{aligned} E(e^t y^n(t)) &= y_0^n + n \int_0^t E(e^s y^n(s)) \left[\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 - c_2 y(t) + \frac{p[(1-m)x(t) + \eta A]}{b + \beta \eta A + (1-m)x(t)} + \frac{n-1}{2} \sigma_2^2 \right] ds \\ &\leq y_0^n + n \int_0^t E(e^s y^n(s)) \left[\left(\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 + p + \frac{\eta A}{b + \beta \eta A} + \frac{n-1}{2} \sigma_2^2 \right) - c_2 y(t) \right] ds \\ &\leq y_0^n + n E \int_0^t e^s \left(\frac{n}{c_2} \right)^n \left(\frac{\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 + p + \frac{\eta A}{b + \beta \eta A} + \frac{n-1}{2} \sigma_2^2}{n+1} \right)^{n+1} ds \\ &\leq y_0^n + \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 + p + \frac{\eta A}{b + \beta \eta A} + \frac{n-1}{2} \sigma_2^2 \right)^{n+1}}{c_2^n} (e^t - 1). \end{aligned}$$

That is,

$$\begin{aligned} E(y^n(t)) &\leq \left[y_0^n - \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 + p + \frac{\eta A}{b + \beta \eta A} + \frac{n-1}{2} \sigma_2^2 \right)^{n+1}}{c_2^n} \right] e^{-t} \\ &\quad + \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 + p + \frac{\eta A}{b + \beta \eta A} + \frac{n-1}{2} \sigma_2^2 \right)^{n+1}}{c_2^n}. \end{aligned}$$

By the similar reason as above, we have

$$\limsup_{t \rightarrow \infty} E(y^n(t)) \leq \left(\frac{n}{n+1} \right)^{n+1} \frac{\left(\frac{1 + \tilde{\gamma}_{2n}}{n} + r_2 + p + \frac{\eta A}{b + \beta \eta A} + \frac{n-1}{2} \sigma_2^2 \right)^{n+1}}{c_2^n} := M_2(n), n \geq 1.$$

The proof is now complete. \square

§4 Extinction and persistence

Extinction and persistence are two important issues in population dynamics. In this section, our goal is to find the conditions for which the prey population as well as the predator population go to extinction and persistence. Let us first define the local extinction of a population.

Definition 4.1. For system (2), the population $x(t)$ is said to be extinct with probability 1 if

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ a.s.}$$

For later applications, we cite a strong law of large numbers for local martingales as the first lemma below [22].

Lemma 4.1. Let $J(t) (t \geq 0)$ be a local martingale vanishing at time 0 and define

$$\rho_J(t) := \int_0^t \frac{d\langle J \rangle(s)}{(1+s)^2}, \quad t \geq 0,$$

where $\langle J \rangle(t) := \langle J, J \rangle(t)$ is the Meyer's angle bracket process. Then

$$\lim_{t \rightarrow \infty} \frac{J(t)}{t} = 0 \text{ a.s. provided that } \lim_{t \rightarrow \infty} \rho_J(t) < \infty \text{ a.s.}$$

Theorem 4.1. Let $(x(t), y(t))$ be the solution with any initial value $(x_0, y_0) \in \mathbb{R}_+^2$ of system (2). Then the prey and predator population goes to extinction with probability 1, i.e., $\lim_{t \rightarrow \infty} x(t) = 0$ and $\lim_{t \rightarrow \infty} y(t) = 0$ a.s., if $r_1 < \frac{\sigma_1^2}{2}$ and $r_2 + \frac{p\eta A}{b + \beta\eta A} < \frac{\sigma_2^2}{2}$.

Proof. We recall the first equation of system (2) and apply Itô formula on it by taking $V_1(x, t) = \ln x$, then we have

$$d \ln x(t) = \left\{ r_1 e^{-\alpha y(t)} - c_1 x(t) - \frac{a(1-m)y(t)}{b + \beta\eta A + (1-m)x(t)} - \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_1(u)) - \gamma_1(u)] \lambda(du) \right\} dt + \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(dt, du). \tag{13}$$

Integrating both sides of (13) from 0 to t leads to

$$\begin{aligned} \ln x(t) &\leq \ln x_0 + \int_0^t \left\{ r_1 - \frac{\sigma_1^2}{2} - c_1 x(s) - \frac{a(1-m)y(s)}{b + \beta\eta A + (1-m)x(s)} - \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) \right\} ds + P_1(t) \\ &\leq \ln x_0 + (r_1 - \frac{\sigma_1^2}{2})t + P_1(t), \end{aligned} \tag{14}$$

where $P_1(t) = \int_0^t \sigma_1 dB_1(s) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(ds, du)$. By Lemma 4.1 and the strong law of large numbers, we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma_1 dB_1(s) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(ds, du) = 0 \quad \text{a.s.,}$$

i.e.,

$$\lim_{t \rightarrow \infty} \frac{P_1(t)}{t} = 0 \quad \text{a.s.} \tag{15}$$

Thus from (14), we get

$$\limsup_{t \rightarrow \infty} \frac{\ln x(t)}{t} \leq \limsup_{t \rightarrow \infty} \left[\frac{\ln x_0}{t} + \frac{\left(r_1 - \frac{\sigma_1^2}{2}\right)t}{t} + \frac{P_1(t)}{t} \right] = r_1 - \frac{\sigma_1^2}{2}.$$

Using the condition $r_1 \leq \frac{\sigma_1^2}{2}$, we have $\limsup_{t \rightarrow \infty} \frac{\ln x(t)}{t} \leq 0$, i.e.,

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \text{a.s.}$$

Hence, for every $\epsilon > 0$ (no matter however small), there exists t_1 such that $\frac{p[(1-m)x + \eta A]}{b + \beta\eta A + (1-m)x} < \epsilon'$ for every $t \geq t_1$, where $\epsilon' = \epsilon + \frac{p\eta A}{b + \beta\eta A}$. From the second equation of (2), apply Itô formula on it by taking $V_2(y, t) = \ln y$, then we have, for $t \geq t_1$

$$\begin{aligned} d \ln y(t) &= \left\{ r_2 - c_2 y(t) + \frac{p[(1-m)x(t) + \eta A]}{b + \beta\eta A + (1-m)x(t)} - \frac{\sigma_2^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \right\} dt \\ &\quad + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du) \\ &\leq \left\{ r_2 - c_2 y(t) + \epsilon' - \frac{\sigma_2^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \right\} dt \end{aligned}$$

$$+ \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du). \quad (16)$$

Integrating both sides of (16) from 0 to t leads to

$$\begin{aligned} \ln y(t) &\leq \ln y_0 + \int_0^t \left\{ r_2 - c_2 y(s) + \epsilon' - \frac{\sigma_2^2}{2} - \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du) \right\} ds + P_2(t) \\ &\leq \ln y_0 + (r_2 + \epsilon' - \frac{\sigma_2^2}{2})t + P_2(t), \end{aligned} \quad (17)$$

where $P_2(t) = \int_0^t \sigma_2 dB_2(s) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(ds, du)$, and then

$$\limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} \leq \limsup_{t \rightarrow \infty} \left[\frac{\ln y_0}{t} + \frac{(r_2 + \epsilon' - \frac{\sigma_2^2}{2})t}{t} + \frac{P_2(t)}{t} \right] = r_2 + \epsilon + \frac{p\eta A}{b + \beta\eta A} - \frac{\sigma_2^2}{2}.$$

Since $\epsilon > 0$ is arbitrarily small and using the given condition, we can conclude that $r_2 + \epsilon + \frac{p\eta A}{b + \beta\eta A} - \frac{\sigma_2^2}{2} \leq 0$, therefore $\limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} \leq 0$. It leads to

$$\lim_{t \rightarrow \infty} y(t) = 0 \text{ a.s.}$$

The proof is complete. \square

Next we will show the persistence of the system (2) under some conditions.

Definition 4.2. The dynamical system (2) is said to be persistent in the mean if

$$\liminf_{t \rightarrow \infty} \langle y(t) \rangle > 0, \text{ where } \langle y(t) \rangle = \frac{1}{t} \int_0^t y(r) dr.$$

Lemma 4.2. (See Lemma 5.1 in [7]) Suppose $Z(t) \in \mathbb{C}(\Omega \times [0, \infty), \mathbb{R}_+)$.

(I) If T, δ and δ_0 are positive constants such that

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_{i=1}^n \alpha_i B_i(t) \quad \text{a.s. } \forall t \geq T,$$

where α_i are constants for $i = 1, 2, \dots, n$, then

$$\liminf_{t \rightarrow \infty} \langle Z(t) \rangle \geq \frac{\delta}{\delta_0} \quad \text{a.s.}$$

(II) If T, δ and δ_0 are positive constants such that

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_{i=1}^n \alpha_i B_i(t) \quad \text{a.s. } \forall t \geq T,$$

where α_i are constants for $i = 1, 2, \dots, n$, then

$$\begin{cases} \limsup_{t \rightarrow \infty} \langle Z(t) \rangle \leq \frac{\delta}{\delta_0} \quad \text{a.s.} & \text{if } \delta > 0, \\ \lim_{t \rightarrow \infty} \langle Z(t) \rangle = 0, \quad \text{a.s.} & \text{if } \delta < 0. \end{cases}$$

Lemma 4.3. Let's consider the following one dimensional stochastic system

$$dz(t) = z(t) (r_1 e^{-\alpha y} - c_1 z(t)) dt + \sigma_1 z(t) dB_1(t) + \int_{\mathbb{Z}} \gamma_1(u) z(t^-) \tilde{N}(dt, du), \text{ with } z(0) = x(0).$$

(I) If $r_1 - \frac{\sigma_1^2}{2} < 0$, then $\lim_{t \rightarrow \infty} z(t) = 0$ a.s.

(II) If $r_1 - \frac{\sigma_1^2}{2} > 0$, then $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t z(s) ds = r_1 - \frac{\sigma_1^2}{2}$.

(III) For $n \geq 1$, $\limsup_{t \rightarrow \infty} E(z^n(t)) \leq M_1(n)$, where $M_1(n) = \left(\frac{n}{n+1}\right)^{n+1} \frac{\left(\frac{1+\bar{\gamma}_1 n}{n} + r_1 + \frac{n-1}{2} \sigma_1^2\right)^{n+1}}{c_1^n}$.

Proof. The above conclusion can be obtained by the similar calculation as Theorem 3.2 and Theorem 4.1. So we omit it here. □

Theorem 4.2. Let $(x(t), y(t))$ be the solution of system (2) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$, then $E(x(t)) \geq Q$, where $Q = \min \left\{ x_0, \frac{(\bar{a}M_2(1))^2}{4c_1} \right\}$ and $\bar{a} = \frac{a(1-m)}{b+\beta\eta A}$.

Proof. Recalling the first equation of system (2), we get

$$dx(t) = x(t) \left\{ \left[r_1 e^{-\alpha y(t)} - c_1 x(t) - \frac{a(1-m)y(t)}{b+\beta\eta A + (1-m)x(t)} \right] dt + \sigma_1 dB_1(t) \right\} + \int_{\mathbb{Z}} \gamma_1(u) x(t^-) \tilde{N}(dt, du).$$

Let us consider $f(x(t), t) = x(t)e^t$. After using Itô formula it leads to

$$\begin{aligned} d(f(x(t), t)) &= x(t)e^t \left[1 + r_1 e^{-\alpha y(t)} - c_1 x(t) - \frac{a(1-m)y(t)}{b+\beta\eta A + (1-m)x(t)} \right] dt \\ &\quad + e^t x(t) \sigma_1 dB_1(t) + e^t \int_{\mathbb{Z}} \gamma_1(u) x(t) \tilde{N}(dt, du). \end{aligned}$$

Taking expectation on both sides and using Theorem 3.2, we have

$$\begin{aligned} E(f(x, t)) &= x_0 + \int_0^t e^s E \left[x \left(1 + r_1 e^{-\alpha y} - c_1 x - \frac{a(1-m)y}{b+\beta\eta A + (1-m)x} \right) \right] ds \\ &\geq x_0 + \int_0^t e^s E [x(-c_1 x - \bar{a}M_2(1))] ds, \end{aligned} \tag{18}$$

where $\bar{a} = \frac{a(1-m)}{b+\beta\eta A}$. Now let $\zeta(x) = x(-c_1 x - \bar{a}M_2(1))$. In order to find the maximum value of $\zeta(x)$, we first calculate $\zeta'(x)$ and obtain $\zeta'(x) = 0$ when $x = -\frac{\bar{a}M_2(1)}{2c_1}$. Also, $\zeta''(x) = -2c_1 < 0$. Hence $\zeta(x)$ has the supremum value at the point $x = -\frac{\bar{a}M_2(1)}{2c_1}$, i.e., $\sup(\zeta(x)) = \frac{(\bar{a}M_2(1))^2}{4c_1}$ on $[0, \infty)$.

Now applying supremum property on (18), then

$$E(f(x, t)) \geq x_0 + \frac{(\bar{a}M_2(1))^2}{4c_1} \int_0^t e^s ds = x_0 + \frac{(\bar{a}M_2(1))^2}{4c_1} (e^t - 1).$$

Thus,

$$E(x(t)) \geq \left(x_0 - \frac{(\bar{a}M_2(1))^2}{4c_1} \right) e^{-t} + \frac{(\bar{a}M_2(1))^2}{4c_1}.$$

For $t = 0$, $E(x(t)) \geq x_0$. Also, $E(x(t)) \geq \frac{(\bar{a}M_2(1))^2}{4c_1}$ as $t \rightarrow \infty$.

Let $Q = \min \left\{ x_0, \frac{(\bar{a}M_2(1))^2}{4c_1} \right\}$, then $E(x(t)) \geq Q$. The proof is now complete. □

Theorem 4.3. Let $(x(t), y(t))$ be the solution of system (2) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$. We assume that $r_1 - \frac{\sigma_1^2}{2} > 0$, then system (2) will be persistent in the mean if $r_2 + \frac{p(1-m)Q}{b+\beta\eta A+(1-m)M_1(1)} > \frac{\sigma_2^2}{2}$.

Proof. Using stochastic integration process of (13) and dividing both sides by t , we get

$$\begin{aligned} \frac{\ln x(t) - \ln x_0}{t} &= \frac{1}{t} \int_0^t r_1 e^{-\alpha y(r)} dr - \frac{\sigma_1^2}{2} - \frac{c_1}{t} \int_0^t x(r) dr - \frac{a(1-m)}{t} \int_0^t \frac{y(r)}{b+\beta\eta A + (1-m)x(r)} dr \\ &\quad - \frac{1}{t} \int_0^t \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) dr - \frac{P_1(t)}{t}, \end{aligned} \tag{19}$$

where $P_1(t) = \int_0^t \sigma_1 dB_1(r) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(dr, du)$.

Let us consider $V_3(z, t) = \ln z(t)$ and apply Itô formula, then we have

$$\begin{aligned} d \ln z(t) = & \left\{ r_1 e^{-\alpha y(t)} - c_1 z(t) - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \right\} dt \\ & + \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \tilde{N}(dt, du). \end{aligned}$$

Again using stochastic integration process of above equation and dividing both sides by t , we get

$$\begin{aligned} \frac{\ln z(t) - \ln z_0}{t} = & \frac{1}{t} \int_0^t r_1 e^{-\alpha y(r)} dr - \frac{\sigma_1^2}{2} - \frac{c_1}{t} \int_0^t z(r) dr - \frac{P_1(t)}{t} \\ & - \frac{1}{t} \int_0^t \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du) dr. \end{aligned} \quad (20)$$

From (19) and (20), we have

$$0 \geq \frac{\ln x(t) - \ln z(t)}{t} = \frac{c_1}{t} \int_0^t (z(r) - x(r)) dr - \frac{a(1-m)}{t} \int_0^t \frac{y(r)}{b + \beta\eta A + (1-m)x(r)} dr,$$

which implies $x(t) \leq z(t)$ and

$$\frac{1}{t} \int_0^t (z(r) - x(r)) dr \leq \frac{a(1-m)}{c_1 t} \int_0^t \frac{y(r)}{b + \beta\eta A + (1-m)x(r)} dr \leq \frac{a(1-m)}{c_1 t(b + \beta\eta A)} \int_0^t y(r) dr. \quad (21)$$

From (16), we get

$$\begin{aligned} d \ln y(t) = & \left\{ r_2 - c_2 y(t) + \frac{p[(1-m)x(t) + \eta A]}{b + \beta\eta A + (1-m)x(t)} - \frac{\sigma_2^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \right\} dt \\ & + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du) \\ \geq & \left\{ \frac{p(1-m)z(t)}{b + \beta\eta A + (1-m)z(t)} - \left[\frac{p(1-m)z(t)}{b + \beta\eta A + (1-m)z(t)} - \frac{p(1-m)x(t)}{b + \beta\eta A + (1-m)x(t)} \right] \right. \\ & + r_2 - c_2 y(t) - \frac{\sigma_2^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \left. \right\} dt \\ & + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du) \\ = & \left\{ \frac{p(1-m)z(t)}{b + \beta\eta A + (1-m)z(t)} - \frac{p(1-m)(b + \beta\eta A)(z(t) - x(t))}{(b + \beta\eta A + (1-m)z(t))(b + \beta\eta A + (1-m)x(t))} \right. \\ & + r_2 - c_2 y(t) - \frac{\sigma_2^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \left. \right\} dt \\ & + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du) \\ \geq & \left\{ r_2 - c_2 y(t) - \frac{\sigma_2^2}{2} + \frac{p(1-m)z(t)}{b + \beta\eta A + (1-m)z(t)} - \frac{p(1-m)}{b + \beta\eta A} (z(t) - x(t)) \right. \\ & + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \left. \right\} dt + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du). \end{aligned}$$

Using (21) and the Lemma 4.3, we have

$$d \ln y(t) \geq \left\{ r_2 - c_2 y(t) - \frac{\sigma_2^2}{2} + \frac{p(1-m)x(t)}{b + \beta\eta A + (1-m)M_1(1)} - \frac{p(1-m)}{b + \beta\eta A} (z(t) - x(t)) \right\}$$

$$\begin{aligned} & + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \Big\} dt + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du) \\ \geq & \left\{ r_2 - c_2 y(t) - \frac{\sigma_2^2}{2} + \frac{p(1-m)Q}{b + \beta\eta A + (1-m)M_1(1)} - \frac{p(1-m)}{b + \beta\eta A} (z(t) - x(t)) \right. \\ & \left. + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du) \Big\} dt + \sigma_2 dB_2(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dt, du). \end{aligned}$$

Using stochastic integration process of above equation and dividing both sides by t and with the help of (21), we get

$$\begin{aligned} \frac{\ln y(t) - \ln y_0}{t} & \geq r_2 - \frac{\sigma_2^2}{2} + \frac{p(1-m)Q}{b + \beta\eta A + (1-m)M_1(1)} - \frac{c_2}{t} \int_0^t y(r) dr - \frac{pa(1-m)^2}{tc_1(b + \beta\eta A)^2} \int_0^t y(r) dr - \frac{P_2(t)}{t} \\ & = r_2 - \frac{\sigma_2^2}{2} + \frac{p(1-m)Q}{b + \beta\eta A + (1-m)M_1(1)} - \frac{c_1 c_2 (b + \beta\eta A)^2 + pa(1-m)^2}{tc_1(b + \beta\eta A)^2} \int_0^t y(r) dr - \frac{P_2(t)}{t}, \end{aligned}$$

where $P_2(t) = \int_0^t \sigma_2 dB_2(r) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \tilde{N}(dr, du)$.

Applying (I) of Lemma 4.2, we have

$$\liminf_{t \rightarrow \infty} \langle y(t) \rangle \geq \frac{c_1(b + \beta\eta A)^2 \left(r_2 + \frac{p(1-m)Q}{b + \beta\eta A + (1-m)M_1(1)} - \frac{\sigma_2^2}{2} \right)}{c_1 c_2 (b + \beta\eta A)^2 + pa(1-m)^2} > 0,$$

since $r_2 + \frac{p(1-m)Q}{b + \beta\eta A + (1-m)M_1(1)} > \frac{\sigma_2^2}{2}$. Hence the system will be persistent in the mean. □

§5 Stochastic permanence

In the research of population dynamics, stochastic permanence is one of the most important properties. In this section, we shall discuss this property.

Definition 5.1. If for arbitrary $\varepsilon \in (0, 1)$, there are two positive constants χ_1 and χ_2 such that for positive initial value $X_0 = (x_0, y_0)$, the solution $X(t)$ of system (2) has the property that

$$\liminf_{t \rightarrow \infty} P\{|X(t)| \geq \chi_1\} \geq 1 - \varepsilon, \quad \liminf_{t \rightarrow \infty} P\{|X(t)| \leq \chi_2\} \geq 1 - \varepsilon,$$

then system (2) is said to be stochastically permanent.

Theorem 5.1. Under Assumptions 1, system (2) is stochastically permanent if $\min\{G_1, G_2\} > 0$, where

$$\begin{aligned} G_1 & = r_1 e^{-\alpha M_2(1)} - \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_1(u)) - \gamma_1(u)] \lambda(du), \\ G_2 & = r_2 - \frac{\sigma_2^2}{2} + \int_{\mathbb{Z}} [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \lambda(du). \end{aligned}$$

Proof. First, it is easy to derive

$$\lim_{\phi \rightarrow 0^+} \left\{ \frac{\phi}{2} \sigma_1^2 + \int_{\mathbb{Z}} \left[\frac{1}{\phi(1 + \gamma_1(u))^\phi} - \frac{1}{\phi} \right] \lambda(du) \right\} = \int_{\mathbb{Z}} \ln \left(\frac{1}{1 + \gamma_1(u)} \right) \lambda(du) = - \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \lambda(du).$$

If $G_1 > 0$, then we can choose a sufficiently small $\phi > 0$ such that

$$r_1 e^{-\alpha M_2(1)} - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} \gamma_1(u) \lambda(du) - \left\{ \frac{\phi}{2} \sigma_1^2 + \int_{\mathbb{Z}} \left[\frac{1}{\phi(1 + \gamma_1(u))^\phi} - \frac{1}{\phi} \right] \lambda(du) \right\} > 0. \quad (22)$$

Define $g(t) = 1/x(t)$ for $x(t) > 0$. Making use of Itô formula to $g(t)$ yields

$$\begin{aligned} dg(t) &= -\frac{1}{x} \left[\left(r_1 e^{-\alpha y} - c_1 x - \frac{a(1-m)y}{b + \beta\eta A + (1-m)x} \right) dt + \sigma_1 dB_1(t) \right] + \frac{\sigma_1^2}{x} dt \\ &\quad + \int_{\mathbb{Z}} \left(\frac{1}{x(1 + \gamma_1(u))} - \frac{1}{x} + \frac{\gamma_1(u)}{x} \right) \lambda(du) dt + \int_{\mathbb{Z}} \left(\frac{1}{x(1 + \gamma_1(u))} - \frac{1}{x} \right) \tilde{N}(dt, du) \\ &= -g \left[r_1 e^{-\alpha y} - \frac{a(1-m)y}{b + \beta\eta A + (1-m)x} - \sigma_1^2 - \int_{\mathbb{Z}} \left(\frac{1}{1 + \gamma_1(u)} - 1 + \gamma_1(u) \right) \lambda(du) \right] dt \\ &\quad + c_1 dt + g \int_{\mathbb{Z}} \left(\frac{1}{1 + \gamma_1(u)} - 1 \right) \tilde{N}(dt, du). \end{aligned}$$

Then define $w(t) = g(t)^\phi$. By Itô formula and Theorem 3.2, we have

$$\begin{aligned} dw(t) &= \phi g^{\phi-1} \left\{ -g \left[r_1 e^{-\alpha y} - \frac{a(1-m)y}{b + \beta\eta A + (1-m)x} - \sigma_1^2 - \int_{\mathbb{Z}} \left(\frac{1}{1 + \gamma_1(u)} - 1 + \gamma_1(u) \right) \lambda(du) \right] + c_1 \right\} dt \\ &\quad + \int_{\mathbb{Z}} \left\{ \left[g + g \left(\frac{1}{1 + \gamma_1(u)} - 1 \right) \right]^\phi - g^\phi - \phi g^\phi \left(\frac{1}{1 + \gamma_1(u)} - 1 \right) \right\} \lambda(du) dt + \frac{\phi(\phi-1)}{2} g^\phi \sigma_1^2 dt \\ &\quad - \phi g^\phi \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \left\{ \left[g + g \left(\frac{1}{1 + \gamma_1(u)} - 1 \right) \right]^\phi \right\} \tilde{N}(dt, du) \\ &= -\phi g^\phi \left[r_1 e^{-\alpha y} - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} \left(\frac{1}{1 + \gamma_1(u)} - 1 + \gamma_1(u) \right) \lambda(du) - \frac{\phi \sigma_1^2}{2} \right] dt \\ &\quad + \phi g^{\phi-1} \left[c_1 + \frac{a(1-m)y}{b + \beta\eta A + (1-m)x} \right] dt + g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 - \phi \left(\frac{1}{1 + \gamma_1(u)} - 1 \right) \right] \lambda(du) dt \\ &\quad - \phi g^\phi \sigma_1 dB_1(t) + g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du) \\ &= -\phi g^\phi \left[r_1 e^{-\alpha y} - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} \gamma_1(u) \lambda(du) - \frac{\phi \sigma_1^2}{2} - \int_{\mathbb{Z}} \left(\frac{1}{\phi(1 + \gamma_1(u))^\phi} - \frac{1}{\phi} \right) \lambda(du) \right] dt \\ &\quad - \phi g^\phi \sigma_1 dB_1(t) + g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du) \\ &\leq -\phi g^\phi \left[r_1 e^{-\alpha M_2(1)} - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} \gamma_1(u) \lambda(du) - \frac{\phi \sigma_1^2}{2} - \int_{\mathbb{Z}} \left(\frac{1}{\phi(1 + \gamma_1(u))^\phi} - \frac{1}{\phi} \right) \lambda(du) \right] dt \\ &\quad - \phi g^\phi \sigma_1 dB_1(t) + g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du). \end{aligned} \tag{23}$$

According to (22), there exists a constant $\xi > 0$ such that

$$R =: r_1 e^{-\alpha M_2(1)} - \frac{\sigma_1^2}{2} - \int_{\mathbb{Z}} \gamma_1(u) \lambda(du) - \frac{\phi \sigma_1^2}{2} - \int_{\mathbb{Z}} \left(\frac{1}{\phi(1 + \gamma_1(u))^\phi} - \frac{1}{\phi} \right) \lambda(du) > \frac{\xi}{\phi} > 0. \tag{24}$$

Define $V(t) = e^{\xi t} w(t) = e^{\xi t} g(t)^\phi$. Making use of Itô formula to $V(t)$ leads to

$$\begin{aligned} dV(t) &\leq \xi e^{\xi t} g^\phi dt - \phi e^{\xi t} g^\phi R dt - \phi e^{\xi t} g^\phi \sigma_1 dB_1(t) + e^{\xi t} g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du) \\ &= -\phi e^{\xi t} g^\phi \left(R - \frac{\xi}{\phi} \right) dt - \phi e^{\xi t} g^\phi \sigma_1 dB_1(t) + e^{\xi t} g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du) \\ &=: e^{\xi t} \mathbb{G}(g) dt - \phi e^{\xi t} g^\phi \sigma_1 dB_1(t) + e^{\xi t} g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du). \end{aligned}$$

It follows from (24) that $\mathbb{G}(g) = -\phi g^\phi (R - \xi/\phi)$ is upper bounded in $(0, +\infty)$, namely $\mathbb{G} :=$

$\sup_{g>0} \mathbb{G}(g) < \infty$. Thus, we obtain

$$dV(t) \leq e^{\xi t} \mathbb{G} dt - \phi e^{\xi t} g^\phi \sigma_1 dB_1(t) + e^{\xi t} g^\phi \int_{\mathbb{Z}} \left[\frac{1}{(1 + \gamma_1(u))^\phi} - 1 \right] \tilde{N}(dt, du).$$

Integrating the above inequality from 0 to t and then taking the expectation on both sides gives

$$E [e^{\xi t} g^\phi(t)] \leq g^\phi(0) + \frac{\mathbb{G}(e^{\xi t} - 1)}{\xi}.$$

Thus,

$$\limsup_{t \rightarrow \infty} E [x^{-\phi}(t)] \leq \frac{\mathbb{G}}{\xi}.$$

Similarly, if $G_2 > 0$, then we can get

$$\limsup_{t \rightarrow \infty} E [y^{-\phi}(t)] \leq \frac{\mathbb{G}}{\xi}.$$

For any given $\varepsilon \in (0, 1)$, set $\chi_1 = \left(\frac{\xi \varepsilon}{\mathbb{G}} \right)^{\frac{1}{\phi}}$. An application of Chebyshev's inequality gives

$$P\{x(t) < \chi_1\} = P\{x^{-\phi}(t) > \chi_1^{-\phi}\} \leq \frac{E [x^{-\phi}(t)]}{\chi_1^{-\phi}} = \chi_1^\phi E [x^{-\phi}(t)],$$

and

$$P\{y(t) < \chi_1\} = P\{y^{-\phi}(t) > \chi_1^{-\phi}\} \leq \frac{E [y^{-\phi}(t)]}{\chi_1^{-\phi}} = \chi_1^\phi E [y^{-\phi}(t)].$$

Then we have

$$\limsup_{t \rightarrow \infty} P \{|X(t)| < \chi_1\} \leq \chi_1^\phi \frac{\mathbb{G}}{\xi} = \varepsilon,$$

i.e.,

$$\liminf_{t \rightarrow \infty} P \{|X(t)| \geq \chi_1\} \geq 1 - \varepsilon.$$

Combining Chebyshev's inequality with (10), we can show that for any $\varepsilon \in (0, 1)$, there exists a positive constant χ_2 such that

$$\liminf_{t \rightarrow \infty} P \{|X(t)| \leq \chi_2\} \geq 1 - \varepsilon.$$

The proof is complete. □

§6 Numerical simulations

In this section, we will perform some numerical simulations using MATLAB R2013a to illustrate the effect of white noise, Lévy noise, fear effect and a prey refuge on the dynamics of model (2). For parameter setting, we refer to Reference [8] and make some changes to our model for biological feasibility reasons. In this section, we always take the following parameters:

$$\begin{aligned} r_1 = 1.5, r_2 = 0.8, c_1 = 0.5, c_2 = 2, \alpha = 1, a = 1.4, b = 0.9, m = 0.2, \\ p = 1, \beta = 0.5, \eta = 0.2, A = 0.35, \sigma_1 = \sigma_2 = 0.1, \gamma_1 = \gamma_2 = 0.15, \end{aligned} \tag{25}$$

and $\mathbb{Z} = (0, +\infty)$, $\lambda(\mathbb{Z}) = 1$ with initial value $(x_0, y_0) = (0.6, 0.6)$.

First, we can compute that $r_1 - \frac{\sigma_1^2}{2} = 1.4950 > 0$ and $r_2 + \frac{p(1-m)Q}{b+\beta\eta A+(1-m)M_1(1)} - \frac{\sigma_2^2}{2} = 0.9733 > 0$, the condition of Theorem 4.3 holds here. Hence the system (2) will be persistent in the mean. At the same time, we can easily check that these parameters satisfy Theorem 5.1. So we obtain that the system (2) is also stochastically permanent. Figure 1 depicts the fact.

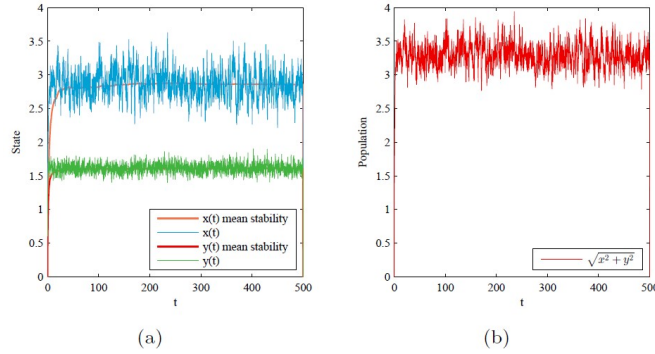


Figure 1. System (2) is persistent in the mean and stochastically permanent.

Secondly, we consider system (2) with the intensity of white noise as $\sigma_1 = 1.75$, $\sigma_2 = 1.4$ (high intensity) and keep the rest of the parameters consistent with (25). Then we get $r_1 - \frac{\sigma_1^2}{2} = -0.0313 < 0$, and $r_2 + \frac{p\eta A}{b + \beta\eta A} - \frac{\sigma_2^2}{2} = -0.1051 < 0$, which means that the conditions of extinction for both species of Theorem 4.1 are satisfied. We exhibit the fact by Figure 2.

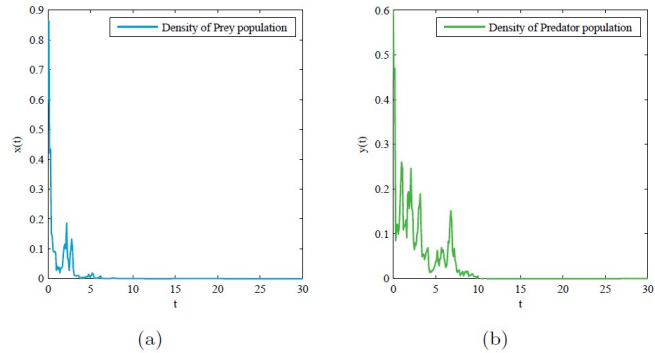


Figure 2. Both populations tend to extinction.

Now we consider some other cases. In order to obtain deep insights of the influences of Lévy noise, we keep the model parameter values the same but chose different values of γ_1 and γ_2 , say $\gamma_1 = \gamma_2 = 0$, $\gamma_1 = \gamma_2 = 0.3$ and $\gamma_1 = \gamma_2 = 0.5$. We can obtain that system (2) is persistent in the mean and find that the Lévy jumps promote the survival of both prey and predator populations to a certain extent. In the absence of Lévy noise, population numbers of the two species remain at a low level with little difference and the volatility is smaller (see Figure 3).

Next, in order to illustrate the influence of fear effect to model (2) through numerical simulation, we chose different values of α , say $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$. For the remaining parameter values, we keep them the same as in (25). We can check that these parameters satisfy the condition of population persistence in the mean, which is depicted in Figure 4. From Figure 4 we find that the increase in fear effect reduces population densities in both species and has a greater impact on prey.

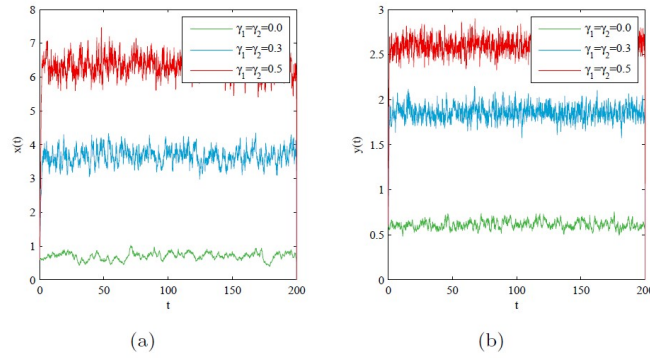


Figure 3. The Lévy jumps promote the survival of both prey and predator populations to a certain extent.

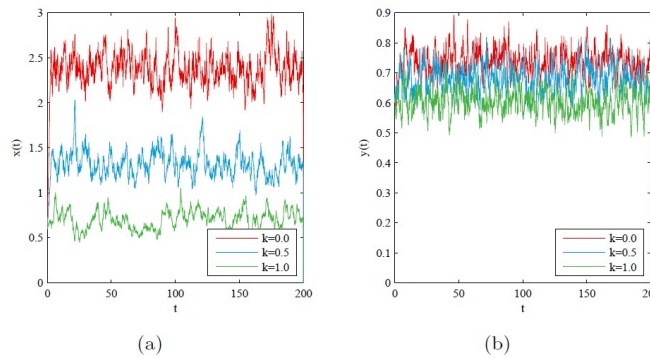


Figure 4. The presence of the fear effect will reduce the population of both species.

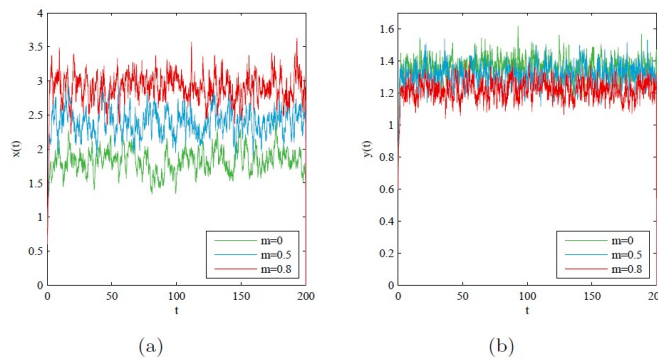


Figure 5. As the prey refuge strength increases, the prey population density increases and the predator population density decreases.

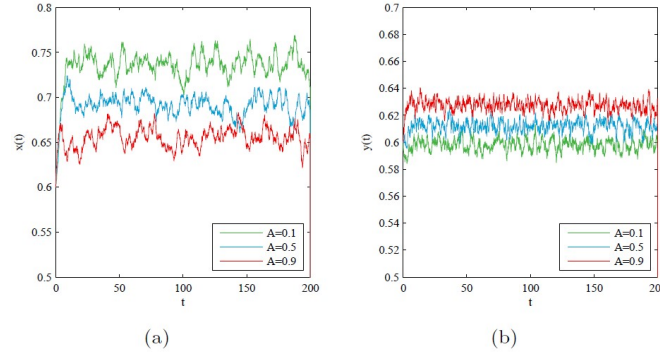


Figure 6. As the additional food biomass increases, the predator population density increases and the prey population density decreases.

Subsequently, we numerically simulate the impact of a prey refuge to model (2) and choose different values of m , say $m = 0$, $m = 0.5$ and $m = 0.8$. For the remaining parameter values, we keep them the same as in (25), shown in Figure 5. From Figure 5, it can be seen that as the prey refuge strength increases, the prey population density increases and the predator population density decreases. The prey refuge strength affects the prey more than the predators populations.

Finally, we explore the effect of different additional food biomass on model (2). Let $A = 0.1$, $A = 0.5$ and $A = 0.9$, respectively and keep the rest of the parameters consistent with (25). Obviously, the increase in additional food biomass will result in a larger population density of the predator and decreases the population density of the prey (see Figure 6).

§7 Conclusion

In this paper we focus on a stochastic prey-predator model with additional food for predator and Lévy noise. Mathematically, we have shown that the solution of the stochastic system will not explode at a finite time. Without any parametric restriction the system will possess a unique global solution starting from any interior of the positive quadrant. We also establish some sufficient conditions for the extinction and persistence in the mean of both species. The sufficient criteria for the stochastic permanence of the model have been obtained.

The study of predator-prey model can provide rich and biologically meaningful dynamics to understand the real ecological episode. Our analysis reveals that the random disturbance may change the dynamical behaviours of the population. Especially when the noise is large, it may lead to the extinction of the prey and predator populations, but the Lévy jumps promote the survival of species. We also found that the level of fear and prey refuge has significant effects in the dynamics of system (2) (see Figures 4 and 5). Besides, we have found in numerical simulation that if the effectual food level of additional food which is provided to the predator is high, then the predator dominates the prey population (see Figure 6).

There are some interesting themes worthy of further research. On the one hand, we can consider some other functional response into model (2), such as ratio-dependent functional response. On the other hand, the Lévy jump process is very useful in other scenarios as well (e.g., foragers have maximized abundances when individuals perform scale-free Lévy flights [6], which is similar to our conclusion). We leave these questions for further investigation.

Acknowledgement

The authors are grateful to the anonymous reviewers for their helpful, valuable comments and suggestions in the improvement of this manuscript.

Declarations

Conflict of interest The authors declare no conflict of interest.

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