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Exploring growth models with multiplicative counting: Connections with the geometric and bigeometric calculi

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Abstract. With a perspective of interest in the modeling of dynamic processes, here we investigate various types of basic growth equations, which in their formulation quantify the change of the variables, the state, or the independent one, using balance equations in which the counts (aggregation-reduction) are of the multiplicative type. We enter the context of the "differential" equations typical of non-Newtonian calculations, such as geometric calculus, bi-geometric calculus, or the lesser-known logarithmic calculus, when we take the step to the limit. In these new possibilities of dynamic laws, we highlight the interpretive aspects. A particular case is to review the equivalents of the logistic equation of the standard calculation in the new accounting calculations, where we make graphical and semantic comparisons. Finally, the construction of a geometric type equation is exemplified, with applications inherent to the financial mathematics.

§1 Introduction

Every basic dynamic law of instantaneous type, including the simplest ones (*e.g.*, with a scalar state variable), is a relation between a way of measuring some variation of the dependent variable resulting from a slight change in the independent one (*e.g.*, usually time), and the values that these variables have. This is, for a function $y = f(x)$ and two pairs (x_i, y_i), such that $y_i = f(x_i)$, $i \in \{0, 1\}$, firstly, we have ways to measure variation: first $x_0 \to x_1$ as $V(x_0, x_1)$, and then $y_0 \to y_1$ as $W(y_0, y_1)$; and secondly, there exists a number $\mathcal{L}[W, V]$ that compares numerically (in some way) $W(y_0, y_1)$ with $V(x_0, x_1)$. Thirdly, having as the reference point $(x_0, f(x_0))$, *i.e.*, (x_1, y_1) "near" (x_0, y_0), this is, resorting to metric structures in the domain and image set of the function, there exists an operator $f \to \mathcal{L}_f$ where

$$\mathcal{L}_f(x_0) := \lim_{x \to x_0} \mathcal{L}[W(x_0, x), V(y_0, f(x))]. \tag{1}$$

So a dynamic law (or an $\mathcal{L}$-equation) is a rule to determine the variation in the initial condition (x_0, y_0). Alternatively, it is the problem of explicitly finding a function $f(\cdot)$, $y = f(x)$,

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such that

$$\mathcal{L}_y(x) = F(x, y(x)) \quad \text{and} \quad y_0 = f(x_0), \tag{2}$$

where the function $F(\cdot)$ is given *a priori*. Although, if we are interested in representing a specific phenomenon, the establishment of $F(\cdot,\cdot)$ is the first and essential part of the modeling cycle. So we need to know very well what the operator $\mathcal{L}$ is measuring as a rate.

When we are in the case $V(a,b) = W(a,b) = b - a$ and $\mathcal{L}[V,W] = W/V$, then (1) defines the usual derivative of f in x_0, which will be denoted by $f'(\cdot)$ or $\mathcal{D}_f(x_0)$, and (2) corresponds to the first order differential equation $y' = F(x,y)$. However, there are other ways to define the variations V and W, and to introduce a comparison of them to determine different types of derivatives $\mathcal{L}$ in a specific numerical context.

In 1972, R Katz and M Grossman introduced a version of non-Newtonian calculus [1,2], where the derivative operator $\mathcal{L}$ is a non-dimensional number (see [2]) and linear in a multiplicative sense (see [3,4]), this is,

$$\mathcal{L}_{f(\cdot)^{\alpha} g(\cdot)^{\beta}} = (\mathcal{L}_{f(\cdot)})^{\alpha} \cdot (\mathcal{L}_{g(\cdot)})^{\beta}, \tag{3}$$

for a pair of positive functions $f(\cdot)$ and $g(\cdot)$ and real numbers α and β. Specifically, they work the case $V(a,b) = W(a,b) = b/a$, with a and b being positive real numbers. For a complete context in terms of the arithmetic and the metric space involved, as well as the calculus behind the introduction of these derivatives, see [5,6].

In classical calculus, changes in a function are measured by differences and accumulations by additions. Still, in bigeometric calculus, changes and accumulations in the arguments and values of a function are measured by quotients and products, respectively [7].

The bigeometric calculation can be considered a by-product when choosing to replace differences with quotients (and additions with by-products) for the estimation of deviations [7,8]. In this sense, although the bigeometric calculus (reserved for functions between sets of positive real numbers) is, in a certain sense, isomorphic to the standard, its value lies in the fact that it is situated within the latter, thereby playing an essential auxiliary role. Some applications of bigeometric calculus are evidenced in the fractal dynamics of biological systems [9], the fractal dynamics of materials [10], numerical systems [11], among others. This article explores types of basic dynamic laws for the three derivatives (or ways of measuring change).

From the modeling perspective, the three derivatives that we will explore can be thought of as other ways of expressing a growth rate of a variable state concerning another (the independent). These rates, intrinsically involve or determine a counting model. The simplest cases would be that they do not vary, *i.e.*, this variation is constant, and the case when it depends directly on the variables. There is always the possibility of considering that some processes increase or others make them decrease in the latter case. In particular, we have the direct cases and the compensated ones (increase-decrease), *i.e.*, the logistic formulations.

In section 2, starting from the possibilities of measurement and comparison of variations of the dependent variable concerning the independent variable, we briefly present the geometric and bigeometric derivatives, together with their relations. Section 3 explores the possibilities of equations with unknown $x(\cdot)$ of the type $\mathcal{L}_x(t) = a$ with a as a constant, as well as $\mathcal{L}_x(t) = \alpha x(t)$

and α as constants. Section 4 is reserved for “analogs” in non-Newtonian derivatives of the ordinary logistic equation. Finally, in section 5, we present the deduction of a geometric equation that could be applied to model capital growth in investment and tax burden scenarios.

§2 Preliminaries

Considering that given an operation in a set $\mathcal{A}$, its inverse operation (if it exists) provides a means of comparison or measurement of change, from an element x toward others x_*, both in $\mathcal{A}$, we have, for instance, the following well known possibilities $\Delta(x_*, x) := x_* - x \quad (\text{with} \quad \mathcal{A} = \mathbb{R})$ and $\nabla(x_*, x) := x_*/x \quad (\text{with} \quad \mathcal{A} = \mathbb{R}^+)$, the *difference* and the *ratio* associated with addition and multiplication on the real numbers.

Note that some families of elementary functions implicitly relate these variations. In this regard, all the functions $y = f(x)$ in Table 1 carry $x_0 \to y_0$ and represent four possible associations of Δ with ∇.

Table 1. Elementary function families, their variation relations, and slope definitions.

Family	$x \to y$	Function	Relation	Slope
Linear	$\mathbb{R} \to \mathbb{R}$	$y = y_0 + m(x - x_0)$	$\Delta y = m \cdot \Delta x$	$m = \Delta y / \Delta x$
Potential	$\mathbb{R}^+ \to \mathbb{R}^+$	$y = y_0 \cdot (x/x_0)^{\ln(m)}$	$\nabla y = (\nabla x)^m$	$m = (\nabla y)^{1/\ln(\nabla X)}$
Exponential	$\mathbb{R} \to \mathbb{R}^+$	$y = y_0 \cdot m^{x - x_0}$	$\nabla y = m^{\Delta x}$	$m = (\nabla y)^{1/\Delta x}$
Logarithmic	$\mathbb{R}^+ \to \mathbb{R}$	$y = y_0 + \log_m(x/x_0)$	$\Delta y = \log_m(\nabla x)$	$m = (\nabla x)^{1/\Delta y}$

Note that the graphs of these four families of functions can be visualized in Figure 1 for different values of m, which we will call the slope. This denomination is justified in that given any two points (x_i, y_i), $i \in \{1, 2\}$, in the space $X \times Y$, defined by $X \to Y$, in the second column of Table 1, there is a single respective linear, potential, exponential, or logarithmic function that passes through them, in which $(x_0, y_0) \in \{(x_1, y_1), (x_2, y_2)\}$ and m are given by the last column, where it is understood that $\Delta s = s_2 - s_1$ and $\nabla s = s_2/s_1$, $s \in \{x, y\}$. In addition, the linear, potential, exponential, and logarithmic families in the third column of Table 1 become constant functions when $m = 0$, $m = 1$, $m = 1$, or $m \in \{0, \infty\}$, respectively.

Another lesser-known case is considered in the complete and ordered field $(\mathbb{R}^+, \cdot . \odot)$; here $a \odot b = a^{\ln(b)}$, the comparison (through the inverse operation of $\odot$) is

$$\sqcap(x_*, x) := x_*^{1/\ln(x)}.$$

In fact, notice the regularities:

(i) $x + \Delta(x_*, x) = x + (x^* - x) = x^*$,

(ii) $x \cdot \nabla(x_*, x) = x \cdot (x_*/x) = x_*$, and

(iii) $x \odot \sqcap(x_*, x) = x \odot x_*^{1/\ln(x)} = x^{\ln(x_*)/\ln(x)} = \left(x^{1/\ln(x)}\right)^{\ln(x_*)} = e^{\ln(x_*)} = x_*$.

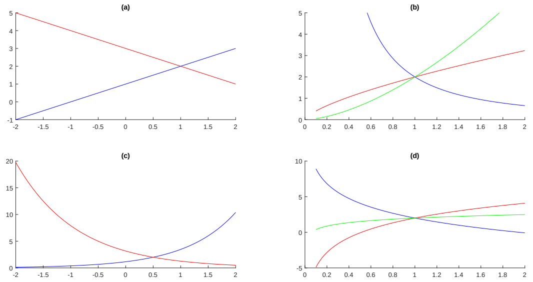

Figure 1. Graph of the functions of the third column of Table 1 for different values of the slope m. In (a) linear-family $y = m(x - x_0) + y_0$, $(x_0, y_0) = (1, 2)$, with $m = 1$ in blue, and $m = -1$ in red. In (b) potential-family $y = y_0 \cdot (x/x_0)^{\ln(m)}$, $(x_0, y_0) = (1, 2)$, with $m = 0.2$ in blue, $m = 2$ in red and $m = 5$ in green. In (c) exponential-family $y = y_0 \cdot m^{x-x_0}$, $(x_0, y_0) = (0.5, 2)$, with $m = 3$ in blue and $m = 0.4$ in red, and (d) logarithmic-family $y = y_0 + \log_m(x/x_0)$, $(x_0, y_0) = (1, 2)$, with $m = 3$ in blue, $m = -3$ in red, and $m = 0.7$ in green.

Now, considering a relation $x \to y$, to compare some variation of y as a response to a variation in x, notice that the slopes in the last column of Table 1 (except for the respective limit) potentially define a derivative. So, we have some possibilities

- Use ∇ (quotient) to compare Δ_y with Δ_x. This is Δ_y/Δ_x; an expression that, when passing to the limit $\Delta_x \to 0$, defines the usual derivative in the field $(\mathbb{R}, +, \cdot)$, which for a function $f : A \subset \mathbb{R} \to \mathbb{R}$ with $y = f(x)$ is given by
$$\mathcal{D}_f(x) = \lim_{\Delta_x \to 0} \nabla(\Delta(y_*, y), \Delta(x_*, x)) = \lim_{\Delta_x \to 0} \frac{\Delta_y}{\Delta_x}, \tag{4}$$
when the limit exists.

- Use $\sqcap$ (*i.e.*, $\oslash$, the inverse of $\odot$) to compare ∇_y with ∇_x. This is $\nabla_y \oslash \nabla_x$; an expression that, when passing to the limit $\nabla_x \to 1$, defines the *multiplicative derivative*
$$\mathcal{Q}_f(x) = \lim_{\nabla_x \to 1} \sqcap(\nabla(y_*, y), \nabla(x_*, x)) = \lim_{\nabla_x \to 1} \nabla_y^{1/\ln(\nabla_x)}, \tag{5}$$
when this limit, defined according to $(\mathbb{R}, +, \cdot)$ as a metric space (see [5]), exists.

From the identities $\Delta_y = (\Delta_y/\Delta_x) \cdot \Delta_x$ and $\nabla_y = (\nabla_y \oslash \nabla_x) \odot \nabla_x$, the approximation relationship is clear.
$$\Delta_y \sim \mathcal{D}_f(x) \cdot \Delta_x \quad \text{and} \quad \nabla_y \sim \mathcal{Q}_f(x) \odot \nabla_x = \mathcal{Q}_f(x)^{\ln(\nabla_x)} = \nabla_x^{\mathcal{Q}_f(x)}. \tag{6}$$
The calculi $((\mathbb{R}, +, \cdot), \mathcal{D}))$ (differential) and $((\mathbb{R}^+, \cdot, \odot), \mathcal{Q}))$ (multiplicative), denoted respectively by $\mathcal{C}_\mathcal{D}$ and $\mathcal{C}_\mathcal{Q}$, are isomorphic. This means, the function $\ln : \mathbb{R}^+ \to \mathbb{R}$ is an isomorphism between the fields, and for $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R} \to \mathbb{R}$ with $\ln(f(\cdot)) = g(\ln(\cdot))$, we have $\ln(\mathcal{Q}_f(x_0)) = \mathcal{D}_g(\ln(x_0))$ when one of the derivatives exists. Then, we have $\mathcal{Q}_f(x) = \exp\left(\mathcal{D}_{\ln \circ f \circ \exp}(y_0)\right)$ with $y_0 = \ln(x_0)$. Therefore, using the chain rule, it gets
$$\mathcal{Q}_f(x_0) = \exp\left(\frac{\mathcal{D}_{f \circ \exp}(y_0)}{(f \circ \exp)(y_0)}\right) = \exp\left(\frac{\mathcal{D}_f(e^{y_0}) \cdot e^{y_0}}{f(e^{y_0})}\right) = \exp\left(\mathcal{D}_f(x_0) \cdot x_0/f(x_0)\right), \tag{7}$$

equivalently,

$$\mathcal{D}_f(x_0) = \frac{f(x_0)}{x_0} \ln(\mathcal{Q}_f(x_0)). \tag{8}$$

For instance, using (8) and (3), we have

$$\mathcal{D}_{f^\alpha g^\beta}(x) = \frac{f^\alpha g^\beta}{x} \ln(\mathcal{Q}_f^\alpha \cdot \mathcal{Q}_g^\beta) = f^\alpha g^\beta \left\{ \frac{\alpha}{f} \left[\frac{f}{x} \ln(\mathcal{Q}_f) \right] + \frac{\beta}{g} \left[\frac{g}{x} \ln(\mathcal{Q}_g) \right] \right\},$$

then

$$\mathcal{D}_{f^\alpha g^\beta}(x) = f^\alpha g^\beta \left\{ \alpha \mathcal{D}_f / f + \beta \mathcal{D}_g / g \right\}. \tag{9}$$

Notice that with (α, β) equal to $(1,1)$ (or $(1,-1)$), we obtain the product (or quotient) differentiation rule.

The importance of $\mathcal{C}_\mathcal{Q}$ is that it is contained in $\mathcal{C}_\mathcal{D}$ because $\mathbb{R}^+ \subset \mathbb{R}$, and the operation $\odot$ derives from the field of real numbers. So that, $(\mathbb{R}, +, \cdot)$, as a field and as a calculus, has a copy of itself in its interior.

In the formulas given in (6), variation of the same type between the dependent and independent variables are related. Nevertheless, we have the following identities

$$\nabla_y^{1/\Delta_x} = \left[\nabla_y^{1/\ln(\nabla_x)} \right]^{\Delta_{\ln(x)}/\Delta_x} \quad \text{and} \quad \nabla_x^{1/\Delta_y} = \exp\left(\frac{\Delta \ln(x)}{\Delta x} / \frac{\Delta y}{\Delta x} \right), \tag{10}$$

which insinuates defining

$$\mathcal{G}_f^+(x) := \lim_{\Delta_x \to 0} \nabla_y^{1/\Delta_x} = \mathcal{Q}_f(x)^{1/x} \quad \text{and} \quad \mathcal{G}_f^-(x) := \lim_{\Delta_x \to 0} \nabla_x^{1/\Delta_y} = e^{1/(x\,\mathcal{D}_f(x))}. \tag{11}$$

The number $\mathcal{G}_f^+$, when it exists, is named the *geometric derivative* , which also satisfies (3); *i.e.*, $\mathcal{G}_f^+$ is of multiplicative type. Concerning $\mathcal{G}_f^-(x)$, to the extent of our searches, it is not recorded in the literature, and we will refer to it *as the inverse geometric derivative.* This is a justified name considering the equality $\mathcal{G}_f(x) = \mathcal{G}_{f^{-1}}(y)$ for an invertible $x \to y = f(x)$.

The equality relations between these derivatives for a function $y = f(x)$, defined in the appropriate domain and having the necessary properties to exist, are summarized in Table 2.

Table 2. Comparison and conversion identities for different derivative operators.

$y = f(x)$	$\mathcal{D}_f(x)$	$\mathcal{Q}_f(x)$	$\mathcal{G}_f^+(x)$	$\mathcal{G}_f^-(x)$
$\mathcal{D}_f(x)$	$-$	$(y/x)\ \ln(\mathcal{Q}_f)$	$y\ \ln(\mathcal{G}_f^+)$	$1/(x\ln(\mathcal{G}_f^-))$
$\mathcal{Q}_f(x)$	$\exp\{x\,\mathcal{D}_f/y\}$	$-$	$(\mathcal{G}_f^+)^x$	$\exp\{1/(y\ \ln(\mathcal{G}_f^-))\}$
$\mathcal{G}_f^+(x)$	$\exp\{\mathcal{D}_f/y\}$	$\mathcal{Q}_f^{1/x}$	$-$	$\exp\{1/(x \cdot y\ \ln(\mathcal{G}_f^-))\}$
$\mathcal{G}_f^-(x)$	$\exp\{1/(x\,\mathcal{D}_f)\}$	$\exp\{1/(y\ln(\mathcal{Q}_f))\}$	$\exp\{1/(x \cdot y\ \ln(\mathcal{G}_f^+))\}$	$-$

§3 Elementary equations to model primary growths

In what follows, we reserve the notation $x(t)$ to represent a function of time. In the first subsection, we consider the cases in which one of the three derivatives of $x(\cdot)$ is a constant and the consequences for the others. Similarly, in the second subsection, the analysis is similar, but one of the three derivatives of the unknown function is the function except for a multiplicative constant. Let us observe that the theory of "differential" equations of geometric and bigeometric type exists, see [12,13,14,15,16]. Furthermore, this is supported by the respective integral operators' construction (see [17,18,19]). Our interest here is to highlight the comparative aspects that may be useful for modeling purposes.

3.1 $\mathcal{L}_x = a$

Knowing the functions with constant derivatives is important since, for modeling purposes, these derivatives represent growth rates or rhythms; therefore, they characterize the functions with constant growth. Furthermore, due to the first-order Taylor expansion, they are those that, given a point (t_0, x_0) on the graph of the function $x(\cdot)$, allow us to approximate the function in a neighborhood of the said point. In this regard, Table 3 shows the recursion that derives from the equation in the first column and the last column the respective family by assuming (on the diagonal for each of the derivatives) that $\mathcal{D}$, $\mathcal{Q}$, $\mathcal{G}^+$, and $\mathcal{G}^-$ are constant. The equivalent equations for the other derivatives are shown in each row.

Table 3. Solutions to basic growth equations assuming constant derivatives.

	Recursion	$\mathcal{D}_x(t)$	$\mathcal{Q}_x(t)$	$\mathcal{G}_x^+(t)$	$\mathcal{G}_x^-(t)$	$x(t)$
(a)	$x(t+\Delta_t) = x(t) + a \cdot \Delta_t$	a	$e^{\frac{a \cdot t}{x}}$	$e^{\frac{a}{x}}$	$e^{\frac{1}{t \cdot a}}$	$x_0 + a(t - t_0)$
(b)	$x(t \cdot \nabla_t) = x(t) \cdot \nabla_t^{\ln(a)}$	$x \cdot \frac{\ln(a)}{t}$	a	$a^{\frac{1}{t}}$	$e^{\frac{1}{\ln(a^x)}}$	$x_0 \cdot \left(\frac{t}{t_0}\right)^{\ln(a)}$
(c)	$x(t+\Delta_t) = x(t) \cdot a^{\Delta_t}$	$x \cdot \ln(a)$	a^t	a	$e^{\frac{1}{\ln(a^{t \cdot x})}}$	$x_0 \cdot a^{t-t_0}$
(d)	$x(t \cdot \nabla_t) = x(t) + \log_a(\nabla_t)$	$\frac{1}{\ln(a^t)}$	$e^{\frac{1}{\ln(a^x)}}$	$e^{\frac{1}{\ln(a^{t \cdot x})}}$	a	$x_0 + \log_a\left(\frac{t}{t_0}\right)$

Then, tangent functions, therefore approximations, $t \to y(t)$ to any function $t \to x(t)$ (it is understood that the function is differentiable) at the point (t_0, x_0) are:

$$\text{(A)} \qquad y(t) = x_0 + \mathcal{D}_x(t_0)\,(t - t_0), \quad \text{(B)} \quad y(t) = x_0 \cdot (t/t_0)^{\ln(\mathcal{Q}_x(t_0))},$$

$$\text{(C)} \qquad y(t) = x_0 \cdot \mathcal{G}_x(t_0)^{t-t_0}, \quad \text{and} \quad \text{(D)} \quad y(t) = x_0 + \ln(t/t_0)/\ln(\mathcal{G}_x^-(t_0)).$$

Notice that if dimensionally the state variable $x_f(\cdot)$ is [units] and those of t are [time], then the units of parameter a (the slope) are respectively (A) $a^{[\text{time}^{-1}]}$, (B) a is adimensional, and (C) a[units/time].

3.2 $\mathcal{L}_x = a(x) = \alpha \cdot x$

Let us now consider that in Cases $(a) - (d)$ from Table 3, where the parameter a in the first column is replaced by the state-dependent function $a(x) = \alpha \cdot x$.

Table 4. Solutions to growth equations with linear state-dependent derivatives.

Recursion	$\mathcal{D}_x(t)$	$\mathcal{Q}_x(t)$	$\mathcal{G}_x^+(t)$	$\mathcal{G}_x^-(t)$	$x(t)$
(a)	$\alpha \cdot x$	$e^{\alpha \cdot t}$	e^{α}	$e^{1/(t \cdot \alpha \cdot x)}$	$x_0 e^{\alpha(t-t_0)}$
(b)	$\left(\frac{x}{t}\right) \cdot \ln(\alpha \cdot x)$	$\alpha \cdot x$	$(\alpha \cdot x)^{1/t}$	$e^{1/\ln((\alpha \cdot x)^x)}$	$(\alpha \cdot x_0)^{t/t_0}/\alpha$
(c)	$x \cdot \ln(\alpha \cdot x)$	$\alpha^t \cdot x^t$	$\alpha \cdot x$	$e^{1/\ln((\alpha \cdot x)^{t \cdot x})}$	$(\alpha \cdot x_0)^{e^{t-t_0}}/\alpha$
(d)	$\frac{1}{\ln(\alpha \cdot x)^t}$	$e^{1/(\ln(\alpha \cdot x)^x}$	$e^{1/\ln((\alpha \cdot x)^{t \cdot x})}$	$\alpha \cdot x$	$\frac{(\alpha x/e)^x}{t} = \frac{(\alpha x_0/e)^{x_0}}{t_0}$

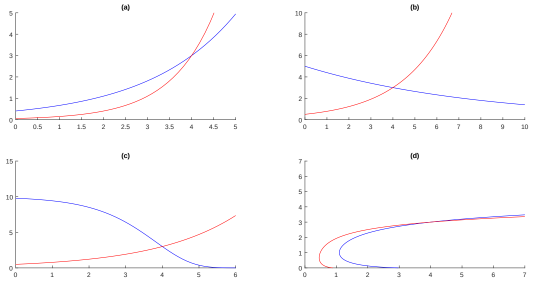

Figure 2. Graph of the functions in the last column of Table 4 for different values of the slope α. In (a) $x(t) = x_0\, e^{\alpha(t-t_0)}$, with $(t_0, x_0) = (4, 3)$, $\alpha = 1$ in red, and $\alpha = 0.5$ in blue. In (b) $x(t) = (\alpha x_0)^{t/t_0}/\alpha$, with $(t_0, x_0) = (4, 3)$, $\alpha = 2$ in red, and $\alpha = 0.2$ in blue. In (c) $x(t) = (\alpha\, x_0)^{\exp(t-t_0)}/\alpha$, with $(t_0, x_0) = (4, 3)$, $\alpha = 1.5$ in red, and $\alpha = 0.1$ in blue. In (d) the family $(\alpha x/e)^x/t = (\alpha x_0/e)^{x_0}/t_0$, with $(t_0, x_0) = (4, 3)$, with $\alpha = 1$ in blue, and $\alpha = 1.5$ in red.

§4 Logistic type equations

The relationships between the dependent variables x and y, $x \to y$, which follow an S-shaped curve in their graph, also called logistic, are common in various disciplinary contexts, standing out in studies of population growth, the spread of epidemic diseases, or disseminating information on social networks. In the first subsection, we review the classical (that is, interpreted) derivation of the standard logistic equation (derivative $\mathcal{D}$) that is carried out in population ecology. Then, in the subsections that follow, and by analogy, we obtain their versions in the derivatives $\mathcal{Q}$, $\mathcal{G}^+$, and $\mathcal{G}^-$, highlighting their usefulness.

4.1 $\mathcal{D}$–logistic equation

In ecology [20], representing the abundance of a population at an instance t by a variable $x(t)$, the widely applied logistic equation is introduced, given by

$$\mathcal{D}_x(t) = x'(t) = r \cdot \{1 - x(t)/K\} \cdot x(t), \quad r \in \mathbb{R} \quad \text{and} \quad K > 0, \tag{12}$$

where the parameters r and K are called intrinsic per capita growth rates and carrying capacity, respectively.

Equation (12) is deduced (going to the limit $\Delta_t \to 0$) from a birth-death balance relation over a time interval $[t, t + \Delta_t]$, of type

$$x(t + \Delta_t) = x(t) + b(x) \cdot x(t) \cdot \Delta_t - d(x) \cdot x(t) \cdot \Delta_t,$$

where $b(x)$ and $d(x)$ are respectively the density-dependent per capita birth and death rates. This equation determines $x(t+\Delta_t)$, the abundance at the end of the interval $[t, t+\Delta_t]$, from its value at the beginning, $x(t)$. Note that, considering that per capita birth (resp. Mortality) correlates negatively (resp. Positively) with abundance, $b(x)$ and $d(x)$ are modeled by linear functions

$$b(x) = b_0 - b_1 \cdot x \quad \text{and} \quad d(x) = d_0 + d_1 \cdot x,$$

where b_0, b_1, d_0 and d_1, with $b_0 > d_0$, are positive constants. These parameters are related to the parameters of (12) by $r = b_0 - d_0$ and $K = r/(b_1 + d_1)$. Thus, in the difference that determines the change, the minuend flow decreases with the abundance while the subtrahend flow increases with that abundance. Observe that $b(\cdot)$ and $d(\cdot)$ are functions with constant Newtonian derivatives and

$$[\Delta_x/\Delta_t]\,/\,x \sim \mathcal{D}_x/x = \Delta(b(x), d(x)). \tag{13}$$

Then, we have that the rate of change, per unit of x, is the comparison, through the delta operator, between an inflow and an outflow. Let us explore the structure given by (13) in the following subsections.

4.2 $\mathcal{Q}$–logistic equation

We will consider a process characterized by a time variable $x(\cdot)$ with the property that given an instant t when twice the time passes, it is amplified or decreased when multiplied by a factor $a(x)$ according to $a(x) > 1$ (or $a(x) < 1$). Therefore, its new value $x(2t)$ is $a(x) \cdot x(t)$. A condition satisfied by the functions that satisfy the relation

$$x_2/x_1 = (t_2/t_1)^{\log_2(a(x_1))}, \quad x_i = x(t_i), \quad i \in \{1, 2\}.$$

A condition that is equivalent to saying that $\nabla x^{1/\ln(\nabla t)} = a(x)^{\ln(2)}$. In other words, passing to the limit when $\nabla t \to 1$, satisfy the $\mathcal{Q}$-equation

$$\mathcal{Q}_x(t) = a(x)^{1/\ln(2)}. \tag{14}$$

Let us now assume the logistic character of equation (14), that is, the function $a(x)$ is the balance between a component that tends to decrease with respect to x and another that tends to increase. If, in logistic case, factors corresponded to lines (derivative $\mathcal{D}$ constant), now they are functions with derivative $\mathcal{Q}$ constant.

In this sense, $a(x)$ is assumed to be the product of two factors $B(x)$ and $1/D(x)$, respectively decreasing and increasing with x. More precisely, $B(1) = B_0 > 1$, $D(1) = D_0 > 1$, $B_0 > D_0$, and with constants elasticities (that is, $x \cdot \mathcal{D}_f(x)/f(x)$, in continuous terms $(\Delta_f/f)/(\Delta_x/x)$) constants $B_1 > 0$ and $D_1 > 0$. So,

$$B(x) = B_0\, x^{-B_1} \quad \text{and} \quad D(x) = D_0\, x^{D_1}.$$

Functions determine $a(x) = B(x)/D(x)$ in (14), and an equation takes the form

$$\mathcal{Q}_x = \left[\frac{B_0/D_0}{x^{B_1+D_1}}\right]^{1/\ln(2)} = \mathcal{R}_0^{\{1-\ln(x)/\ln(\mathcal{K})\}}, \tag{15}$$

where $\mathcal{R}_0 := (B_0/D_0)^{1/\ln(2)}$ is the $\mathcal{Q}$-derivative when $x \sim 1$ and $\mathcal{K} := (B_0/D_0)^{1/(B_1+D_1)}$ is the unique equilibrium.

The logistic equation (15) in terms of the usual derivative ($x' = \mathcal{D}_x$) turns out to be non-autonomous of time, but a separable equation. In fact,

$$x'(t) = r(t)\left\{1 - \frac{\ln(x)}{\ln(\mathcal{K})}\right\} x(t), \quad \text{where} \quad r(t) := \mathcal{R}_0/t. \tag{16}$$

If this equation is a population model, we would be in the case of a per capita rate $x'(\cdot)/x(\cdot)$ whose intrinsic value (low density) decreases over time and a dense dependent factor that makes it reduce logarithmically. In addition, it determines as equilibria the zero ($x'/x \to 0$ as $x \to 0$) or a $\mathcal{K}$ levels population.

To find the solution, we have

$$\frac{x'(t)}{x(t)\,\{1 - \ln(x(t))/\ln(\mathcal{K})\}} = \frac{\ln(\mathcal{R}_0)}{t}.$$

It is clear, directly studying the sign of $x'(\cdot)$, that assuming $B_0 > D_0$, while $x(t) < \mathcal{K}$ (*resp.* $>$), the solution will be increasing (*resp.* decreasing).

Therefore, integrating on $[t_0, t]$, by the following sequence of calculation

$$\int_{\ln(x_0)}^{\ln(x)} \frac{du}{1 - u/\ln(\mathcal{K})} = \ln(\mathcal{R}_0)\ \ln(t/t_0),$$

$$-\ln(\mathcal{K}) \cdot \ln\left\{\frac{\ln(\mathcal{K}) - \ln(x)}{\ln(\mathcal{K}) - \ln(x_0)}\right\} = \ln(\mathcal{R}_0)\ \ln(t/t_0),$$

and

$$\ln\left\{\frac{\ln(\mathcal{K}/x_0)}{\ln(\mathcal{K}/x)}\right\} = \ln(t/t_0)^{\ln(\mathcal{R}_0)/\ln(\mathcal{K})},$$

we get

$$x(t) = \mathcal{K}\left(\frac{x_0}{\mathcal{K}}\right)^{1/(t/t_0)^{\lambda}}, \quad \text{where} \quad \lambda = \ln(\mathcal{R}_0)/\ln(\mathcal{K}) = (B_1 + D_1)/\ln(2).$$

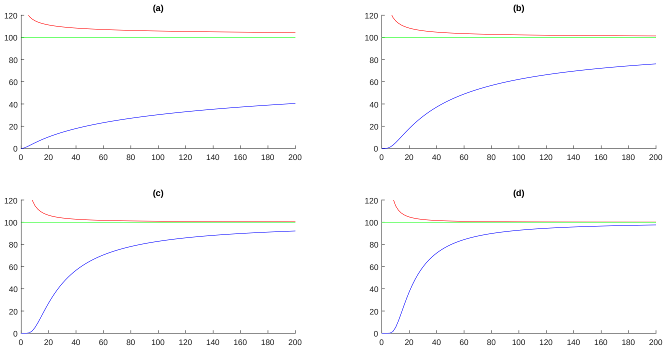

Figure 3. Graph of function $x(t) = \mathcal{K}(x_0/\mathcal{K})^{1/(t/t_0)^{\lambda}}$, where $\mathcal{K} = 100$, $t_0 = 10$ and $x_0 = 5$ (in blue) and $x_0 = 115$ (in red). In (a), (b), (c) and (d) the parameter $\lambda = (B_1 + D_1)/\ln(2)$ is respectively 0.4, 0.8, 1.2 and 1.6. Note that in (a), (b), (c) and (d), since $B_0/D_0 = \mathcal{K}^{\lambda \ln(2)} = 100^{\lambda \ln(2)}$, the value of B_0 is approximately 3.6, 12.9, 46.1 and 165.2 times the value of D_0, respectively.

4.3 $\mathcal{G}^+$–logistic equation

We will suppose that a time variable $x(\cdot)$ has the property that given an instant t when passing a unit of time, this is amplified or diminished when multiplied by a factor $a(x)$ according to $a(x) > 1$ (or

$a(x) < 1$), which corresponds to a condition of type

$$x_2/x_1 = a(x_1)^{t_2-t_1}, \quad x_i = x(t_i), \quad i \in \{1,2\}.$$

This condition is equivalent to $\nabla x^{\,1/\Delta t} = a(x)$, which by making $\Delta t \to 0$, we obtain

$$\mathcal{G}_x^+(t) = a(x).$$

Now, what we have called the logistic characteristic, is obtained by assuming that $a(x)$ at balance (which we will assume by quotient) between constant functions of $\mathcal{G}^+$, that is, an exponential $\beta(x) = \beta_0 \cdot (1/\beta_1)^x$, which with initial value β_0 then decreases with x (*i.e.*, $\beta_1 > 1$) and another exponential $\delta(x) = \delta_0 \cdot \delta_1^x$, initially δ_0 and increasing with x (*i.e.*, $\delta_1 > 1$). So, since $a(x) = \beta(x)/\delta(x)$, we have the equation

$$\mathcal{G}_x^+ = \frac{\beta_0/\delta_0}{(\beta_1 \cdot \delta_1)^x} = {\rho_0}^{1-x/\kappa},$$

where $\rho_0 := \beta_0/\delta_0$ is the $\mathcal{G}^+$-derivative when $x \sim 0$ and the carrying capacity parameter is given by $\kappa = \ln(\beta_0/\delta_0)/\ln(\beta_1\delta_1)$, by a certain equilibrium.

In terms of the usual derivative ($x' = \mathcal{D}_x$), this equation is

$$x'(t) = \ln(\rho_0) \cdot \{1 - x(t)/\kappa\},$$

which is none other than the standard logistic equation.

4.4 $\mathcal{G}^-$–logistic equation

If we consider a process in which the state variable x duplicates when the elapsed time is duplicated, this causality is expressible in the relation $x_2 - x_1 = x_1 \cdot \log_2(t_2/t_1)$, since $t_2 = 2\,t_1$ implies $x_2 = 2\,x_1$. Now, if we suppose that the proportion of time a that must pass for duplicating the value of the state variable is state-dependent, that is, $a(x)$, we have the extension

$$x_2 - x_1 = x_1 \cdot \log_{a(x_1)}(t_2/t_1), \quad x_i = x(t_i), \quad i \in \{1,2\},$$

so that if $t_2 = a(x_1) \cdot t_1$, we have $x_2 = 2 \cdot x_1$. In terms of variations, this expression is expressed by $\Delta x = x \cdot \log_{a(x)}(\nabla t)$, which, when rearranged, remains $\nabla t^{\,1/\Delta x} = a(x)^{\,1/x}$ and, in the limit when $\nabla t \to 1$, leads us to the $\mathcal{G}^-$-equation

$$\mathcal{G}_x^-(t) = a(x)^{\,1/x}.$$

Note that this time amplifying factor, $a(x)$, which produces the duplication of the state, must be dimensionless and satisfy the logistic assumption, that is, be the balance between two functions, one that increases and the other subtracts. Namely,

$$\mathbf{b}(x) = \mathbf{b}_0 + \log_{1/\mathbf{b}_1}(x/1) \quad \text{and} \quad \mathbf{d}(x) = \mathbf{d}_0 + \log_{\mathbf{d}_1}(x/1), \quad \text{with} \quad \mathbf{b}_1, \mathbf{d}_1 > 1,$$

and where the number 1 in the argument of the of logarithms have the units of measure of x. These functions satisfy the following incremental type properties $\mathbf{b}(\mathbf{b}_1 \cdot x) = \mathbf{b}_0 - 1$ and $\mathbf{d}(\mathbf{d}_1 \cdot x) = \mathbf{d}_0 + 1$.

Then,

$$a(x) = \mathbf{b}(x) - \mathbf{d}(x) = (\mathbf{b}_0 - \mathbf{d}_0) - \frac{\ln(\mathbf{b}_1 \cdot \mathbf{d}_1)}{\ln(\mathbf{b}_1) \cdot \ln(\mathbf{d}_1)} \ln(x),$$

so that the $\mathcal{G}^-$-equation

$$\mathcal{G}_x^- = \mathbf{r}_0^{1/x} \cdot \{1 - \ln(x)/\ln(\mathbf{K})\}^{1/x},$$

is obtained, where $\mathbf{r}_0 := \mathbf{b}_0 - \mathbf{d}_0$ is the $\mathcal{G}^-$-derivative when $x \sim 1$ and the parameter that analogs the carrying capacity is given by $\mathbf{K} := \exp(\mathbf{r}_0 \cdot \ln(\mathbf{b}_1)\ln(\mathbf{d}_1)/\ln(\mathbf{b}_1 \cdot \mathbf{d}_1))$, which is an equilibrium.

The corresponding growth differential equation is

$$x' = \mathbf{r}(t,x) \cdot x, \quad \text{with} \quad \mathbf{r}(t,x) = 1/\{t \ln(a(x))\},$$

where this unitary growth rate satisfies $\partial_t \mathbf{r} < 0$ and $\lim_{x\to\mathbf{K}} r(t,x) = 0$.

Then

$$(x'/x) \cdot \ln[\mathbf{r}_0 \cdot (1 - \ln(x)/\ln(\mathbf{K}))] = 1/t.$$

The change of variables $u(x) = \mathbf{r}_0 \cdot (1 - \ln(x)/\ln(\mathbf{K}))$ implies $du = -(\mathbf{r}_0/\ln(K))(x'/x)\,dt$, therefore

$$-\frac{\ln(\mathbf{K})}{\mathbf{r}_0} \int_{u(t_0)}^{u(t)} \ln(u)\,du = \ln(t/t_0).$$

Integrating it we get

$$-\ln(\mathbf{K}) \cdot u[\ln(u) - 1]|_{u(t_0)}^{u(t)} = \ln(t/t_0)^{\mathbf{r}_0},$$

which leads us to the implicit solution

$$t^{\mathbf{r}_0} K^{a(x)\ln(a(x)/e)} = t_0^{\mathbf{r}_0} K^{a(x_0)\ln(a(x_0)/e)}.$$

§5 Examples

5.1 Capital growth

Now, let us consider the case of a variable $C(t)$ (*e.g.*, a capital) that per unit of time is affected simultaneously by two processes:

(i) One that makes it increase by a fraction $\alpha(C)$ that negatively correlated to the variable ($\alpha'(C) < 0$). That is, at each instant t during a unit of time, consecutively passes from $C(t)$ to $C(t)(1+\alpha(C))$, in a compound way (*e.g.*, recapitalizations with diminishing returns). So, when passing Δ_t units of time, we have the transition

$$C(t) \to C(t)\{1 + \alpha(C(t))\}^{\Delta_t}.$$

(ii) Another process that, in each unit of time, causes C to decrease by a fraction $\beta(C)$ that now correlates positively with the variable ($\beta'(C) > 0$). That is, it goes from $C(t)$ to $C(t)(1- beta(C))$, also in compound form (*for example*, consecutive progressive taxes). So, passing Δ_t units of time, we have the transition

$$C(t) \to C(t)\{1 - \beta(C(t))\}^{\Delta_t}.$$

Superimposing these processes, we have $C(t+\Delta_t) = C(t)\{(1+\alpha(C(t)))(1-\beta(C(t)))\}^{\Delta_t}$. Then, when considering particular cases

$$\alpha(C) = \alpha_0/C \quad \text{and} \quad \beta(C) = C/(\beta_0 + C) < 1, \quad \text{with} \quad \alpha_0, \beta_0 > 0,$$

we have the recursive case $\nabla_C = a(C)^{\Delta_t}$, but with positive function $a(x)$, $x > 0$, the product of two factors is

$$a(x) = \Big(1 + \frac{\alpha_0}{x}\Big) \cdot \Big(1 - \frac{x}{\beta + x}\Big) = \frac{\beta(x + \alpha)}{x(x + \beta)}.$$

The parameter α_0 indicates the level of C when we have duplication in the increasing factor. Inversely, in the second factor, the decreasing one, parameter β_0 is the level of C where we reduce to half.

So that, by doing $\Delta_t \to 0$ in $\nabla_C^{1/\Delta_t} = a(C)$, we obtain the geometric equation

$$\mathcal{G}_C^{+}(t) = a(C(t)),$$

that as a standard differential one is

$$C' = C \cdot \ln(a(C)). \tag{17}$$

An equation that is contextualized in the positive real numbers has an equilibrium point C_e (*e.g.* , a long-term capital result) given by the relation $a(C_e) = 1$, then

$$C_e = \sqrt{\alpha_0\,\beta_0},$$

which corresponds to the geometric mean of the parameters α_0 and β_0.

Defining $F(x) = x\,\ln(a(x))$, notice that $F'(x) = \ln(a(x)) + x\,a'(x)/a(x)$ and $a'(x) = -\beta_0\,\{x^2 + 2\alpha_0 x + \alpha_0\beta_0\}/\{x^2(x+\beta_0)^2\}$. Then, using $a(C_e) = 1$, we have $F'(C_e) = C_e \cdot a'(x_e) = -2\{\sqrt{\alpha_0}\}\{\sqrt{\alpha_0} + \sqrt{\beta_0}\} < 0$, that is, C_e is a globally stable equilibrium.

5.2 Cauchy-Euler equation

Let us consider the case of a second-order differential equation, the Cauchy-Euler equation, which, given real numbers a, b, and c, is given by $a\,t^2\,x'' + b\,t\,x' + c\,x = 0$. A fundamental identity obtained by differentiating $x' = (x/t)\ln(\mathcal{Q}_x)$ is

$$x''(t) = \frac{x}{t^2}\{\ln^2(\mathcal{Q}_x) - \ln(\mathcal{Q}_x) + \ln(\mathcal{Q}_x^{(2)})\},$$

with $Q_x^{(2)}$ the second order derivative of $x(\cdot)$.

Then by substituting in the Cauchy-Euler equation, we have

$$a\,x\{\ln^2(\mathcal{Q}_x) - \ln(\mathcal{Q}_x) + \ln(\mathcal{Q}_x^2)\} + b\,x\,\ln(\mathcal{Q}_x) + c\,x = 0.$$

Then, when looking for non-zero solutions, we obtain the $\mathcal{Q}$-equation of second order

$$a\,\{\ln^2(\mathcal{Q}_x) - \ln(\mathcal{Q}_x) + \ln(\mathcal{Q}_x^2)\} + b\,\ln(\mathcal{Q}_x) + c = 0,$$

which is autonomous of time and equivalent to

$$\Big\{Q_x^{(2)} \cdot Q_x^{\ln(Q_x)}\Big\}^a = Q_x^{a-b}/e^c.$$

Now, let us find conditions to have a solution of the form $x(t) = t \odot A = t^{\ln(A)}$ with $A > 0$. In this regard, let us note that $Q_x(t) = (Q_t \odot A) \cdot (t \odot Q_A) = (e \odot A) \cdot (t \odot 1) = e^{\ln(A)} \cdot t^{\ln(1)} = A$ and also

$Q_x^{(2)}(t) = Q_A = 1$. Therefore,

$$\left\{1 \cdot A^{\ln(A)}\right\}^a = A^{a-b}/e^c.$$

A condition that, when represented by λ, a real number such that $\lambda = \ln(A)$ leads us to the quadratic polynomial of the standard resolution method, $a\lambda^2 + (a-b)\lambda + c = 0$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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