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Anti-pancyclic arcs in strong tournaments

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Abstract. A *tournament* is an orientation of the edges of a complete graph. An arc in a digraph D is *pancyclic* if it is contained in a cycle of length k for every $3 \le k \le |V(D)|$. An arc uv in a digraph D is k*-anticyclic* if there is a path from u to v of length $k-1$ in D. If for every $3 \le k \le |V(D)|$, an arc uv is k-anticyclic, then we say that uv is *anti-pancyclic* in D. It has been proved in Discrete Appl. Math. 79 (1997) 127-135 that every arc of a 3-strong and arc-3-cyclic tournament T is k-anticyclic for each $k \ge 4$, unless T is isomorphic to two tournaments, each of which has exactly 8 vertices. In J. Combin. Inform. System Sci. 19 (1994) 207-214, Moon showed that every strong tournament contains at least three pancyclic arcs and characterized the tournaments that attain this lower bound. In this paper we investigate the number of anti-pancyclic arcs in strong tournaments and show that every strong tournament with order $n \ge 6$ contains at least four anti-pancyclic arcs unless it is isomorphic to five tournaments, each of which has exactly 6 vertices. Consequently, every strong tournament with order $n \ge 7$ contains at least four anti-pancyclic arcs.

§1 Terminology and introduction

Our main source of terminology and notation is [3]. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. Denote $|V(D)|$ the *order* of D. If xy is an arc of D, then we say that x *dominates* y, y is *dominated* by x, y is an *out-neighbour* of x and xy is an *out-arc* of x. More generally, if X and Y are two disjoint subdigraphs of D (or subsets of $V(D)$) such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y and denote it by $X \to Y$. In addition, we use $X \rightsquigarrow Y$ to denote there is at least one arc from X to Y and use $A(X, Y)$ to denote the set of arcs going from X to Y. Let $W \subseteq V(D)$. Then $D\langle W\rangle$ is a subdigraph of D induced by W and $D - W = D\langle V(D) \setminus W\rangle$. Since we do not distinguish

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between a single vertex element set $\{x\}$ and the element x itself, we will often write $D - x$ rather than $D - \{x\}$.

For any vertex x of D, the set $N_D^+(x)$ consists of all vertices dominated by x in D, and more generally, for a subdigraph X of D, we define $N_D^+(X) = \bigcup_{x\in V(X)} N_D^+(x) - V(X)$ (if the digraph D is clear from the context, we write $N^+(X)$ instead of $N_D^+(X)$).

Paths and cycles in a digraph are always assumed to be directed. A *bypath* of an arc uv is a path from u to v. A cycle (resp. bypath) of length k is said to be a *k-cycle* (resp. *k-bypath*). A cycle (resp. path) in a digraph D is *hamiltonian* if it includes all the vertices of D. The directed path on cycle C from a vertex x to a vertex y is denoted by $C[x, y]$.

An arc in a digraph D is called *pancyclic* if it is contained in a k-cycle for every $3 \le k \le |V(D)|$. A digraph D is *arc-k-cyclic* if every arc of D is contained in a k-cycle. An arc is *k-anticyclic* if it has a bypath of length $k - 1$. Moreover, an arc in a digraph D is called *anti-pancyclic* if it is k-anticyclic for each $3 \le k \le |V(D)|$. If every arc of a digraph D is pancyclic and anti-pancyclic, then we say that D is *completely strong path-connected*.

A digraph D is *strong*, if for any two distinct vertices $x, y \in V(D)$, D contains a path from x to y and a path from y to x. A *strong component* H of a digraph D is a maximal strong subdigraph of D. A digraph D is *k-strong* if $|V(D)| \ge k+1$ and the subdigraph $D-S$ is strong for any set S of at most $k - 1$ vertices. The number $\kappa(D) = \max\{\ k \mid D \text{ is } k\text{-strong}\}$ is called the *strong connectivity number* of D.

If D is strong and x is a vertex of D such that $D - x$ is not strong, then we say that x is a *cut-vertex* of D. If D is a digraph with $\kappa(D) = k \ge 1$, then a *reductor* of D is a subdigraph X such that $|V(X)| = k$ and $D - V(X)$ is not strong. Note that when $k = 1$, the unique vertex in X is a cut-vertex.

If we replace every arc xy of D by yx, then we call the resulting digraph (denoted by D^{-1}) the *converse digraph* of D.

A *tournament* is an orientation of the edges of a complete graph. For a non-strong tournament T, the unique sequence $T_1, T_2, \dots, T_k$ $(k \ge 2)$ of the strong components of T satisfying $T_i \to T_j$ for every $1 \le i < j \le k$, is called the *strong decomposition* of T.

In 1967, Alspach [1] proved that every arc of a regular tournament is pancyclic. Since then many mathematicians have studied the path-connectivity in tournaments, and specially, k-anticyclic arcs. Most of them used the following two types of conditions.

The first one is in terms of degree. For example, Alspach, Reid and Roselle proved the following result.

Theorem 1.1 ([2]). *Every arc of a regular tournament with order $n \ge 7$ is k-anticyclic for each $k \ge 4$.*

The second one is the arc-3-cyclic condition. For example, it has been proved in [5] that every arc of a 3-strong and arc-3-cyclic tournament T is k-anticyclic for each $k \ge 4$, unless T is isomorphic to two tournaments, each of which has exactly 8 vertices. This extends the above mentioned result in [2].

In 1994, Moon [6] showed that every non-trivial strong tournament contains at least three pancyclic arcs and characterized the tournaments that attain this lower bound. In this paper we investigate the anti-pancyclic arcs in strong tournaments and obtain a similar result to [6].

Theorem 1.2 (Main result)**.** *Let T be a strong tournament with order $n \geq 6$. Then T contains at least four anti-pancyclic arcs unless it is isomorphic to T_i^* for $1 \leq i \leq 5$, which are shown in Figure 1.*

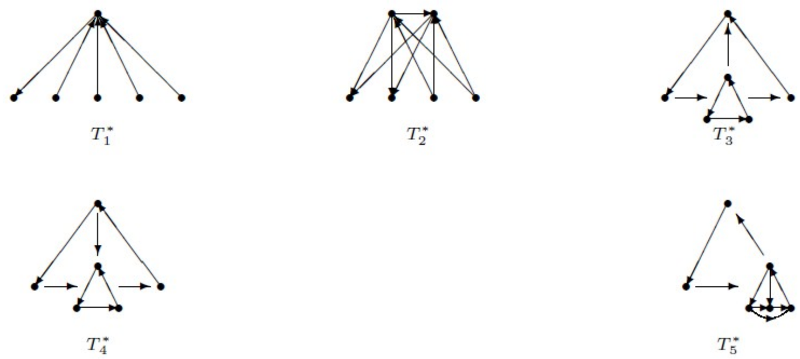

Figure 1. The digraph T_i^* for each $1 \leq i \leq 5$ is a strong tournament. The arcs which are not shown are oriented from left to right. The strong tournament T_i^* for each $1 \leq i \leq 4$ contains exactly three anti-pancyclic arcs, and T_5^* contains exactly two anti-pancyclic arcs (The details can be seen in Lemma 2.17, Lemma 2.19 and Lemma 2.29).

Since each T_i^* for $1 \leq i \leq 5$ consists of 6 vertices, we can immediately obtain the following result.

Corollary 1.3. *Every strong tournament with order $n \geq 7$ contains at least four anti-pancyclic arcs.*

§2 Preliminaries

In this section we always assume that T is a strong tournament with order $n \geq 6$ and X is a reductor of T. Let $X_1, X_2, \ldots, X_\alpha$ ($\alpha \geq 1$) be the strong decomposition of X and $T_1, T_2, \ldots, T_\beta$ ($\beta \geq 2$) be the strong decomposition of $T - V(X)$.

Since every strong tournament has a hamiltonian cycle (Camion' theorem [4]), we may assume that $C_x^i = x_1^i x_2^i \ldots x_{p_i}^i x_1^i$ is a hamiltonian cycle of the strong component X_i for each $1 \leq i \leq \alpha$ and $C_t^j = t_1^j t_2^j \ldots t_{q_j}^j t_1^j$ is a hamiltonian cycle of the strong component T_j for each $1 \leq j \leq \beta$. Note that when $|V(X_i)| = 1$ for some $1 \leq i \leq \alpha$, C_x^i denotes the single vertex x_1^i of X_i. Similarly, when $|V(T_j)| = 1$ for some $1 \leq j \leq \beta$, C_t^j denotes the single vertex t_1^j of T_j.

Since X is reductor of T, every vertex of X dominates some vertex of T_1 and is dominated by some vertex of T_β. So when $|V(T_1)| = 1$, we have $X \to T_1$, and similarly, when $|V(T_\beta)| = 1$, we know $T_\beta \to X$.

2.1 Sufficient conditions for an arc to be anti-pancyclic in T

Below we consider what arcs are anti-pancyclic in T.

Lemma 2.1. *If $|V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_j)| \geq 4$ for some $1 < i < j < \beta$, then every arc from T_i to T_j is anti-pancyclic in T.*

Proof. Without loss of generality we consider the arc $t_1^i t_1^j$, and suppose $t_1^\beta \to x_1^1 \to t_1^1$. Since $|V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_j)| \geq 4$ and $T_r \to T_s$ for each $1 \leq r < s \leq \beta$, we can easily find a 2-bypath and a 3-bypath of $t_1^i t_1^j$ in $V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_j)$. Moreover, $t_1^i t_1^j$ has a 4-bypath $t_1^i t_1^\beta x_1^1 t_1^1 t_1^j$.

Since $X_\mu \to X_\tau$ for each $1 \leq \mu < \tau \leq \alpha$ and $T_r \to T_s$ for each $1 \leq r < s \leq \beta$, we can insert every vertex of $C_t^i, C_t^{i+1}, \ldots, C_t^{j-1}, C_t^{j+1}, \ldots, C_t^\beta, C_x^1, \ldots, C_x^\alpha, C_t^1, \ldots, C_t^{i-1}, C_t^j$ one by one into the 4-bypath $t_1^i t_1^\beta x_1^1 t_1^1 t_1^j$ to obtain an ℓ-bypath of $t_1^i t_1^j$ for each $5 \leq \ell \leq n-1$. That is to say the 4-bypath can be extended to an ℓ-bypath of $t_1^i t_1^j$ for each $5 \leq \ell \leq n-1$. This extension technique is applied similarly in the following lemmas. So $t_1^i t_1^j$ is anti-pancyclic in T. □

Lemma 2.2. *If $|V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_j)| = 3$ for some $1 < i < j < \beta$, and $T_i \rightsquigarrow X$ or $X \rightsquigarrow T_j$, then every arc from T_i to T_j is anti-pancyclic in T.*

Proof. Comparing with Lemma 2.1, we only need to show that every arc from T_i to T_j has a 3-bypath. Below we consider the case $T_i \rightsquigarrow X$ and the other case $X \rightsquigarrow T_j$ can be proved just by considering T^{-1}.

Since $|V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_j)| = 3$, we have $j = i+2$ and $|V(T_i)| = |V(T_{i+1})| = |V(T_{i+2})| = 1$. Suppose without loss of generality that $t_1^i \to x \to t_1^1$ for some $x \in V(X)$. Then the unique arc $t_1^i t_1^j$ from T_i to T_j has a 3-bypath $t_1^i x t_1^1 t_1^j$ and the rest proof is similar to Lemma 2.1. □

Lemma 2.3. *If $|V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_\beta)| \geq 3$ for some $1 < i < \beta$, and there exists an arc from T_i to X_1, say $t_1^i x_1^1$, then every arc from t_1^i to T_β is anti-pancyclic in T.*

Proof. Without loss of generality we consider the arc $t_1^i t_1^\beta$ and assume $x_1^1 \to t_1^1$. Since $|V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_\beta)| \geq 3$, the arc $t_1^i t_1^\beta$ has a 2-bypath in $T\langle V(T_i) \cup V(T_{i+1}) \cup \ldots \cup V(T_\beta)\rangle$. Moreover, $t_1^i x_1^1 t_1^1 t_1^\beta$ is a 3-bypath of $t_1^i t_1^\beta$. Along with $C_x^1, \ldots, C_x^\alpha, C_t^1, \ldots, C_t^{i-1}, C_t^i[t_2^i, t_{q_i}^i], C_t^{i+1}, \ldots, C_t^\beta$ the above 3-bypath can be successively extended to an ℓ-bypath of $t_1^i t_1^\beta$ for each $4 \leq \ell \leq n-1$. So the arc $t_1^i t_1^\beta$ is anti-pancyclic in T. □

By Lemma 2.3 and considering the converse digraph T^{-1} of T, we can immediately obtain the following lemmas.

Lemma 2.4. *If $|V(T_1) \cup V(T_2) \cup \ldots \cup V(T_i)| \geq 3$ for some $1 < i < \beta$, and there exists an arc from X_α to T_i, say $x_1^\alpha t_1^i$, then every arc from T_1 to t_1^i is anti-pancyclic in T.*

Lemma 2.5. *If $|V(T_j) \cup V(T_{j+1}) \cup \ldots \cup V(T_\beta)| \geq 4$ for some $2 \leq j \leq \beta - 2$, and $T_i \rightsquigarrow X_1$ for some $j < i < \beta$, then every arc from T_j to T_β is anti-pancyclic in T.*

Proof. Without loss of generality we consider the arc $t_1^j t_1^\beta$. Since $|V(T_j) \cup V(T_{j+1}) \cup \ldots \cup V(T_\beta)| \geq 4$, the arc $t_1^j t_1^\beta$ has an ℓ-bypath in $T\langle V(T_j) \cup \ldots \cup V(T_\beta)\rangle$ for every $2 \leq \ell \leq 3$. Since $T_i \rightsquigarrow X_1$, we may assume without loss of generality that $t_1^i \to x_1^1 \to t_1^1$. Then $t_1^j t_1^\beta$ has a 4-bypath $t_1^j t_1^i x_1^1 t_1^1 t_1^\beta$. Along with the cycles $C_t^j, \ldots, C_t^i, C_x^1, \ldots, C_x^\alpha, C_t^1, \ldots, C_t^{j-1}, C_t^{i+1}, \ldots, C_t^\beta$ the above 4-bypath can be successively extended to an ℓ-bypath of $t_1^j t_1^\beta$ for each $5 \leq \ell \leq n-1$. So the arc $t_1^j t_1^\beta$ is anti-pancyclic in T. □

By Lemma 2.5 and considering the converse digraph T^{-1} of T, we can immediately obtain the following lemmas.

Lemma 2.6. *If $|V(T_1) \cup V(T_2) \cup \ldots \cup V(T_j)| \geq 4$ for some $3 \leq j \leq \beta - 1$, and $X_\alpha \rightsquigarrow T_i$ for some $1 < i < j$, then every arc from T_1 to T_j is anti-pancyclic in T.*

Lemma 2.7. *If $\alpha \geq 2$ and $|V(X)| \geq 3$, then every arc from X_1 to X_α is anti-pancyclic in T.*

Proof. Without loss of generality we consider the arc $x_1^1 x_1^\alpha$, and suppose $x_1^1 \to t_1^1$ and $t_1^\beta \to x_1^\alpha$. Since $|V(X)| \geq 3$, the arc $x_1^1 x_1^\alpha$ has a 2-bypath in X. Moreover, $x_1^1 t_1^1 t_1^\beta x_1^\alpha$ is a 3-bypath of the arc $x_1^1 x_1^\alpha$. Along with $C_x^1, \ldots, C_x^{\alpha-1}, C_x^\alpha[x_2^\alpha, x_{p_\alpha}^\alpha], C_t^1, \ldots, C_t^\beta$ the above 3-bypath can be successively extended to an ℓ-bypath of $x_1^1 x_1^\alpha$ for each $4 \leq \ell \leq n-1$. So $x_1^1 x_1^\alpha$ is anti-pancyclic in T. □

Lemma 2.8. *If $\alpha \geq 2$ and $x_1^1 \to y \to x_1^\alpha$ for some $y \in V(T) \setminus V(X)$, then the arc $x_1^1 x_1^\alpha$ is anti-pancyclic in T.*

Proof. By Lemma 2.7, we may assume that $\alpha = 2$, $|V(X_1)| = |V(X_2)| = 1$ and only need to find a 2-bypath of $x_1^1 x_1^\alpha$. Clearly, $x_1^1 y x_1^\alpha$ is the desired 2-bypath. So $x_1^1 x_1^\alpha$ is anti-pancyclic in T. □

Lemma 2.9. *If $\beta \geq 3$, $|V(T_1)| \geq 3$ and $t_1^\beta \to x_1^\alpha \to t_1^1$, then every arc from $T_1 - t_1^1$ to T_j is anti-pancyclic for each $2 \leq j \leq \beta - 1$.*

Proof. Without loss of generality we consider the arc $t_i^1 t_1^j$ for $i \neq 1$. Since $|V(T_1)| \geq 3$, the arc $t_i^1 t_1^j$ has a 2-bypath and a 3-bypath in $T\langle V(T_1) \cup V(T_2) \cup \ldots \cup V(T_j)\rangle$. Moreover, $t_i^1 t_1^\beta x_1^\alpha t_1^1 t_1^j$ is a 4-bypath of $t_i^1 t_1^j$. Let P be a hamiltonian path of X ending at x_1^α. Since every vertex of P is dominated by some vertex of T_β, along with $C_t^1[t_i^1, t_{q_1}^1], C_t^2, \ldots, C_t^{j-1}, C_t^{j+1}, \ldots, C_t^\beta, P, C_t^1[t_1^1, t_{i-1}^1], C_t^j$ the arc $t_i^1 t_1^j$ can be successively extended to an ℓ-bypath of $t_i^1 t_1^j$ for each $5 \leq \ell \leq n-1$. So $t_i^1 t_1^j$ is anti-pancyclic in T. □

By Lemma 2.9 and considering T^{-1}, we can immediately obtain the following lemmas.

Lemma 2.10. *If $\beta \geq 3$, $|V(T_\beta)| \geq 3$ and $t_1^\beta \to x_1^1 \to t_1^1$, then every arc from T_j to $T_\beta - t_1^\beta$ is anti-pancyclic in T for each $2 \leq j \leq \beta - 1$.*

Lemma 2.11. *Every arc from X_1 to T_β is anti-pancyclic in T.*

Proof. Without loss of generality we consider the arc $x_1^1 t_1^\beta$ and suppose $x_1^1 \to t_1^1$. Then $x_1^1 t_1^1 t_1^\beta$ is a 2-bypath of the arc $x_1^1 t_1^\beta$. Along with the cycles $C_x^1, \ldots, C_x^\alpha, C_t^1, C_t^2, \ldots, C_t^\beta$ the above 2-bypath can be successively extended to an ℓ-bypath of $x_1^1 t_1^\beta$ for each $3 \le \ell \le n-1$. So the arc $x_1^1 t_1^\beta$ is anti-pancyclic in T. □

Lemma 2.12. *If $\alpha \ge 2$, $|V(T_1)| \ge 3$, $|V(T_\beta)| \ge 3$, $t_1^\beta \to x_1^1$ and $x_1^\alpha \to t_1^1$, then the arc $t_i^1 t_j^\beta$ is anti-pancyclic for each $i \ne 1$ and $j \ne 1$.*

Proof. Since $|V(T_1)| \ge 3$ and $|V(T_\beta)| \ge 3$, the arc $t_i^1 t_j^\beta$ has an ℓ-bypath in $T - V(X)$ for each $2 \le \ell \le 4$. Furthermore, $t_i^1 t_1^\beta x_1^1 x_1^\alpha t_1^1 t_j^\beta$ is a 5-bypath of $t_i^1 t_j^\beta$. Along with $C_t^1[t_i^1, t_{q_1}^1], C_t^2, \ldots, C_t^\beta[t_{j+1}^\beta, t_1^\beta], C_x^1, \ldots, C_x^\alpha, C_t^1[t_1^1, t_{i-1}^1], C_t^\beta[t_2^\beta, t_j^\beta]$ the above 5-bypath can be successively extended to an ℓ-bypath of $t_i^1 t_j^\beta$ for each $6 \le \ell \le n-1$. So $t_i^1 t_j^\beta$ is anti-pancyclic in T. □

Lemma 2.13. *If $\alpha = 1$, $|V(T_1)| \ge 3$, $|V(T_\beta)| \ge 3$, $x_1^1 \to t_1^1$ and there exists a vertex in T_β, say t_1^β, dominating X, then the arc $t_i^1 t_j^\beta$ is anti-pancyclic in T for each $i \ne 1$ and $j \ne 1$.*

Proof. Since $|V(T_1)| \ge 3$ and $|V(T_\beta)| \ge 3$, the arc $t_i^1 t_j^\beta$ has an ℓ-bypath in $T - V(X)$ for each $2 \le \ell \le 3$. Furthermore, $t_i^1 t_1^\beta x_1^1 t_1^1 t_j^\beta$ is a 4-bypath of $t_i^1 t_j^\beta$. Since $t_1^\beta \to X$, we can successively along with $C_t^1[t_i^1, t_{q_1}^1], C_t^2, \ldots,$
$C_t^\beta[t_{j+1}^\beta, t_1^\beta], C_x^1, C_t^1[t_1^1, t_{i-1}^1], C_t^\beta[t_2^\beta, t_j^\beta]$ extend the above 4-bypath to an ℓ-bypath of $t_i^1 t_j^\beta$ for each $5 \le \ell \le n-1$. So $t_i^1 t_j^\beta$ is anti-pancyclic in T. □

Lemma 2.14. *If $\beta = 2$, $|V(T_1)| = 1$, $t_1^2 \to x_1^1$ and T_2 is 2-strong, then at least one out-arc of t_1^2 in T_2 is anti-pancyclic in T.*

Proof. Since T_2 is 2-strong, there are at least two out-neighbours of t_1^2 in T_2, say u and v. Assume without loss of generality that $u \to v$. Then $t_1^2 uv$ is a 2-bypath of the arc $t_1^2 v$. Since $T_2 - t_1^2$ is strong, there is a hamiltonian path P of $T_2 - t_1^2$ ending at v. It follows from $X \to t_1^1 \to P$ that the 3-bypath $t_1^2 x_1^1 t_1^1 v$ of $t_1^2 v$ can be successively extended to an ℓ-bypath of $t_1^2 v$ for each $4 \le \ell \le n-1$ along with $C_x^1, C_x^2, \ldots, C_x^\alpha$ and P. So $t_1^2 v$ is anti-pancyclic in T. □

2.2 Lower bound of the number of anti-pancyclic arcs in T

Lemma 2.15. *If $\beta \ge 7$, then T contains at least four anti-pancyclic arcs.*

Proof. According to Lemma 2.1, it suffices to consider the case that $\beta = 7$ and $|V(T_i)| = 1$ for $i = 2, 3, 4, 5, 6$. Moreover, $t_1^2 t_1^5$, $t_1^2 t_1^6$ and $t_1^3 t_1^6$ are three anti-pancyclic arcs in T.

If $T_4 \rightsquigarrow X_1$, then by Lemma 2.3 $t_1^4 t_1^7$ is the fourth anti-pancyclic arc. Assume in the following that $X_1 \to T_4$.

If $X_\alpha \rightsquigarrow T_4$, then by Lemma 2.4 $t_1^1 t_1^4$ is the fourth anti-pancyclic arc. So assume now that $T_4 \to X_\alpha$.

Combining $X_1 \to T_4$ with $T_4 \to X_\alpha$, we have $\alpha \ge 2$ and $x_1^1 x_1^\alpha$ is the fourth anti-pancyclic arc by Lemma 2.8. □

Lemma 2.16. *If $\beta = 6$, then T contains at least four anti-pancyclic arcs.*

Proof. According to Lemma 2.1, it suffices to consider the case that $|V(T_i)| = 1$ for $i = 2, 3, 4, 5$, and moreover, $t_1^2t_1^5$ is an anti-pancyclic arc in T.

If $T_3 \rightsquigarrow X_1$, then $t_1^3t_1^5$, $t_1^3t_1^6$ and $t_1^2t_1^6$ are another three anti-pancyclic arcs by Lemma 2.2, Lemma 2.3 and Lemma 2.5, respectively. So assume in the following that $X_1 \to T_3$. By considering T^{-1}, we may assume that $T_4 \to X_\alpha$.

If $T_4 \rightsquigarrow X_1$, then $t_1^2t_1^6$ and $t_1^3t_1^6$ are another two desired anti-pancyclic arcs by Lemma 2.5, and $t_1^4t_1^6$ is the fourth anti-pancyclic arc by Lemma 2.3. Assume in the following that $X_1 \to T_4$. This yields $\alpha \geq 2$ and by considering T^{-1}, we may assume that $T_3 \to X_\alpha$.

Now $x_1^1x_1^\alpha$ (by Lemma 2.8) and $t_1^2t_1^4, t_1^3t_1^5$ (by Lemma 2.2) are all anti-pancyclic arcs, and we are done. □

Lemma 2.17. *If $\beta = 5$, then T contains at least four anti-pancyclic arcs unless it is isomorphic to T_1^*.*

Proof. According to Lemma 2.1, we only need to consider the case that $|V(T_i)| = 1$ for $i = 2, 3, 4$. If $|V(T_1)| \geq 3$, then by Lemma 2.9 at least two arcs from T_1 to T_i are anti-pancyclic for each $2 \leq i \leq 4$, and we are done. So assume in the following that $|V(T_1)| = 1$. By a similar argument and Lemma 2.10 we may also assume that $|V(T_5)| = 1$. Hence, $T_5 \to X \to T_1$.

Case 1. $T_3 \rightsquigarrow X_1$.

In this case the arcs $t_1^3t_1^5$ and $t_1^2t_1^5$ are anti-pancyclic by Lemma 2.3 and Lemma 2.5, respectively, and we need to find another two anti-pancyclic arcs.

If $X_\alpha \rightsquigarrow T_4$, then $t_1^1t_1^4$ and $t_1^2t_1^4$ are the desired anti-pancyclic arcs by Lemma 2.4 and Lemma 2.2, respectively. So assume below that $T_4 \to X_\alpha$.

If $X_\alpha \rightsquigarrow T_3$, then $t_1^1t_1^3$ and $t_1^1t_1^4$ are another two anti-pancyclic arcs by Lemma 2.4 and Lemma 2.6, respectively. Assume in the following that $T_3 \to X_\alpha$.

If $X_\alpha \rightsquigarrow T_2$, then $t_1^1t_1^4$ is the third anti-pancyclic arc by Lemma 2.6. Moreover, it is not difficult to check that $t_1^1t_1^5$ is the fourth anti-pancyclic arc. So assume that $T_2 \to X_\alpha$.

Now $|N^+(X_\alpha)| = 1$, and thus, $|V(X)| = 1$. So T is isomorphic to T_1^* which contains exactly three anti-pancyclic arcs $t_1^3t_1^5$, $t_1^2t_1^5$ and $t_1^2t_1^4$.

Case 2. $X_1 \to T_3$.

Suppose first that $\alpha = 1$. Then by considering T^{-1} and Case 1, T contains at least four anti-pancyclic arcs unless it is isomorphic to $(T_1^*)^{-1} \cong T_1^*$.

Suppose now that $\alpha \geq 2$. Then T is 2-strong. By considering T^{-1} we may assume that $T_3 \to X_\alpha$. It follows from Lemma 2.8 that $x_1^1x_1^\alpha$ is the first anti-pancyclic in T. In the rest proof we look for another three anti-pancyclic arcs.

If $X_\alpha \rightsquigarrow T_4$, then $t_1^1t_1^4$ and $t_1^2t_1^4$ are the second and third anti-pancyclic arcs by Lemma 2.4 and Lemma 2.2, respectively. When $t_1^2 \rightsquigarrow X_1$, we know that $t_1^2t_1^5$ is the fourth anti-pancyclic arc by Lemma 2.3. When $X_1 \to t_1^2$, since T is 2-strong, we have $|N^-(X_1)| = 2$ and $t_1^4 \rightsquigarrow X_1$. Hence, $t_1^2t_1^5$ is the fourth anti-pancyclic arc by Lemma 2.5.

Assume in the following that $T_4 \to X_\alpha$. By considering T^{-1} we may assume that $X_1 \to T_2$. Since T is 2-strong, we have $T_4 \leadsto X_1$ and $X_\alpha \leadsto T_2$. Then $t_1^2t_1^5$ and $t_1^1t_1^4$ are the second and third anti-pancyclic arcs by Lemmas 2.5 and 2.6, respectively. Moreover, it is easy to check that $t_1^1t_1^5$ is fourth anti-pancyclic arc in T, and we are done. □

Lemma 2.18. *If $\beta = 4$, then T contains at least four anti-pancyclic arcs unless it is isomorphic to T_2^*.*

Proof. Suppose $|V(T_2)| \geq 3$. Then every arc from T_2 to T_3 is anti-pancyclic by Lemma 2.1. So there are already at least three anti-pancyclic arcs in T. If $T_2 \leadsto X_1$, say $t_1^2x_1^1$, then $t_1^2t_1^4$ is the fourth anti-pancyclic arc by Lemma 2.3, and we are done. So assume $X_1 \to T_2$. If $X_\alpha \leadsto T_2$, say $x_1^\alpha t_1^2$, then $t_1^1t_1^2$ is the fourth anti-pancyclic arc in T by Lemma 2.4. So assume $T_2 \to X_\alpha$. Hence, $\alpha \geq 2$ and $x_1^1x_1^\alpha$ is the fourth anti-pancyclic arc by Lemma 2.8. Therefore, we may assume in the following that $|V(T_2)| = 1$. Considering T^{-1} we may also assume that $|V(T_3)| = 1$.

By Lemmas 2.9 and 2.10 we may assume $|V(T_1)| = 1$ and $|V(T_4)| = 1$, respectively. Hence, $T_4 \to X \to T_1$.

Since $T - \{t_1^2, t_1^3, t_1^4\}$ is not strong and X is a reductor of T, we have $|V(X)| \leq 3$. Combining with $n \geq 6$ we get $2 \leq |V(X)| \leq 3$. Note that when $|V(X)| = 3$, we know that T is a 3-strong tournament with order 7, so T is 3-regular.

Case 1. $|V(X)| = 3$ and $\alpha = 1$.

Since T is 3-regular, there is a vertex x in X such that $t_1^2 \to x \to t_1^3$. So by Theorem 1.1 $t_1^2t_1^4$, $t_1^1t_1^3$, $t_1^1t_1^4$ and $t_1^2t_1^3$ are the desired four anti-pancyclic arcs in T.

Case 2. $|V(X)| = 3$ and $\alpha = 3$.

By Theorem 1.1 the four arcs $t_1^1t_1^3$, $t_1^2t_1^4$, $x_1^1x_1^3$, $t_1^1t_1^4$ are the desired anti-pancyclic arcs in T.

Case 3. $|V(X)| = 2$.

In this case $\alpha = 2$ and T is a 2-strong tournament with order 6.

Suppose first that $t_1^2 \to x_1^1$. Then $x_1^2 \to t_1^2$ and $t_1^2t_1^4$ is the first anti-pancyclic arc by Lemma 2.3. If $x_1^2 \to t_1^3$, then $t_1^3 \to x_1^1$ and it is easy to check that $t_1^1t_1^3$, $t_1^2x_1^1$ and $x_1^2t_1^3$ are another three anti-pancyclic arcs. If $t_1^3 \to x_1^2$ and $t_1^3 \to x_1^1$, then $t_1^1t_1^4$, $t_1^3x_1^1$ and $t_1^3x_1^2$ are the desired three anti-pancyclic arcs. If $t_1^3 \to x_1^2$ and $x_1^1 \to t_1^3$, then $t_1^1t_1^3$, $t_1^2t_1^3$ and $x_1^1x_1^2$ are another three anti-pancyclic arcs in T.

Suppose now that $x_1^1 \to t_1^2$. By considering T^{-1} we may assume that $t_1^3 \to x_1^2$. This yields that $t_1^3 \to x_1^1$ and $x_1^2 \to t_1^2$. Now T is isomorphic to T_2^* and contains exactly three anti-pancyclic arcs $t_1^1t_1^4$, $x_1^1t_1^2$, and $t_1^3x_1^2$. □

Lemma 2.19. *If $\beta = 3$, then T contains at least four anti-pancyclic arcs unless it is isomorphic to T_i^* for $3 \leq i \leq 4$.*

Proof. In view of the order of T_2, we distinguish the following two cases.

Case 1. $|V(T_2)| \geq 3$.

By Lemmas 2.9 and 2.10 we may assume that $|V(T_1)| = |V(T_3)| = 1$. Then $T_3 \to X \to T_1$.

Suppose first that $T_2 \rightsquigarrow X_1$, say $t_1^2 x_1^1$. Then $t_1^2 t_1^3$ and $t_{q_2}^2 t_1^3$ are two anti-pancyclic arcs.

If $X_\alpha \rightsquigarrow T_2$, then by considering T^{-1} at least two arcs from T_1 to T_2 are also anti-pancyclic in T and we are done. Assume below that $T_2 \to X_\alpha$.

If $\alpha \geq 2$, then it is easy to check that $t_1^2 x_1^\alpha$ and $t_{q_2}^2 x_1^\alpha$ are another two anti-pancyclic arcs, and we are done.

If $\alpha = 1$, then $|N^+(X_1)| = 1$, and thus, $|V(X)| = 1$. By Lemma 2.3 every arc from T_2 to T_3 is anti-pancyclic in T. So when $|V(T_2)| \geq 4$, we are done; when $|V(T_2)| = 3$, T is isomorphic to T_3^* which contains exactly three anti-pancyclic arcs $t_1^2 t_1^3$, $t_2^2 t_1^3$ and $t_3^2 t_1^3$.

Suppose now that $X_1 \to T_2$. If $\alpha \geq 2$, then T is 2-strong and by considering T^{-1} we may assume that $T_2 \to X_\alpha$. This yields that $|N^+(X_\alpha)| = 1$ which contradicts that T is 2-strong. So $\alpha = 1$, and consequently, $|N^-(X)| = 1$ which means $|V(X)| = 1$. By Lemma 2.4, every arc from T_1 to T_2 is anti-pancyclic. So when $|V(T_2)| \geq 4$, we are done; when $|V(T_2)| = 3$, T is isomorphic to T_4^*, which contains exactly three anti-pancyclic arcs $t_1^1 t_1^2$, $t_1^1 t_2^2$, $t_1^1 t_3^2$.

Case 2. $|V(T_2)| = 1$.

If $|V(T_1)| = |V(T_3)| = 1$, then $|V(X)| \leq 2$, and consequently, $n \leq 5$, a contradiction. If $|V(T_1)| \geq 3$ and $|V(T_3)| \geq 3$, then by Lemmas 2.9 and 2.10, at least two arcs from T_1 to T_2 and at least two arcs from T_2 to T_3 are anti-pancyclic. So we only need to consider the case $|V(T_1)| = 1$ and $|V(T_3)| \geq 3$ (the other case $|V(T_1)| \geq 3$ and $|V(T_3)| = 1$ can be proved by considering T^{-1}).

Note that $T_3 \rightsquigarrow X_1$. So by Lemma 2.10 at least two arcs from T_2 to T_3 are anti-pancyclic in T. If $|A(X_1, T_3)| \geq 2$, then we are done by Lemma 2.11. If $|A(X_1, T_3)| = 1$, then at least two vertices of T_3 dominate some vertex of X_1. Consequently, every arc from T_2 to T_3 is anti-pancyclic by Lemma 2.10. Adding the arc from X_1 to T_3 there are at least four anti-pancyclic arcs in all. If $|A(X_1, T_3)| = 0$, i.e., $T_3 \to X_1$, then every arc from T_2 to T_3 is anti-pancyclic by Lemma 2.10. When $|V(T_3)| \geq 4$, we are done. When $|V(T_3)| = 3$, we look for the fourth anti-pancyclic arc as follows.

If $\alpha \geq 2$ and $|V(X)| \geq 3$, then by Lemma 2.7 $x_1^1 x_1^\alpha$ is the fourth anti-pancyclic arc.

If $|V(X)| = 2$ and $X_\alpha \rightsquigarrow T_3$, say $x_1^\alpha t_1^3$, then it is easy to see that $t_1^1 t_1^3$ is the fourth anti-pancyclic arc.

If $|V(X)| = 2$ and $T_3 \to X_\alpha$, then it is not difficult to check that every arc from T_3 to X_1 is anti-pancyclic in T.

If $\alpha = 1$ and $X_1 \rightsquigarrow T_2$, then every arc from T_1 to T_3 is anti-pancyclic in T.

If $\alpha = 1$ and $T_2 \to X_1$, then $|N^+(X_1)| = 1$ and thus $|V(X)| = 1$. Now T is isomorphic to T_4^* which contains exactly three anti-pancyclic arcs. □

Lemma 2.20. *If $\beta = 2$, $|V(T_1)| \geq 3$ and $|V(T_2)| \geq 3$, then T contains at least four anti-pancyclic arcs.*

Proof. If $\alpha \geq 2$, then we are done by Lemma 2.12. Assume now that $\alpha = 1$. Then $X = X_1 = X_\alpha$. If there is a vertex of T_2 such that it dominates X, then by Lemma 2.13, there are at least four anti-pancyclic arcs from T_1 to T_2, and we are done. Assume below that for any vertex of T_2, there is a vertex of X dominating it. Hence, there are at least three anti-pancyclic arcs from X to T_2 by Lemma 2.11.

By considering T^{-1} and a similar argument as above, we can deduce that either there are at least four anti-pancyclic arcs from T_1 to T_2, or there are at least three anti-pancyclic arcs from T_1 to X. In either case, there are at least four anti-pancyclic arcs in all, and we are done. □

Lemma 2.21. *If $\beta = 2$, $|V(T_1)| = 1$, and T_2 is 2-strong, then T contains at least four anti-pancyclic arcs.*

Proof. Since T_2 is 2-strong, we have $|V(T_2)| \geq 5$. By Lemma 2.14 and Lemma 2.11 there are at least four anti-pancyclic arcs. □

Lemma 2.22. *If $\beta = 2$, $|V(T_1)| = 1$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ ($k \geq 2$) of $T_2 - z$ satisfies $|V(H_1)| \geq 3$, then there are at least four anti-pancyclic arcs in T.*

Proof. Since $|V(T_1)| = 1$, we have $X \to T_1$. Let $C_h^1 = z_1^1 z_2^1 \ldots z_{h_1}^1 z_1^1$ be a hamiltonian cycle of H_1. If there are at least four arcs from X_1 to T_2, then by Lemma 2.11 we are done. So assume in the following that $|A(X_1, T_2)| \leq 3$. Suppose without loss of generality that $z_k z$ is an arc from H_k to z.

Case 1. $|A(X_1, T_2)| = 0$.

In this case $T_2 \to X_1$. If $H_1 \rightsquigarrow z$, say $z_1^1 z$, then it is not difficult to check that the arc $z_1^1 z$ and every arc from H_1 to H_k are all anti-pancyclic in T. If $z \to H_1$, then it is not difficult to check that every arc from z to H_1 and every arc from H_1 to H_k are all anti-pancyclic in T.

Case 2. $1 \leq |A(X_1, T_2)| \leq 2$.

In this case, there is already at least one anti-pancyclic arc which is from X_1 to T_2 by Lemma 2.11, and we need to find another three anti-pancyclic arcs in T. Since $|V(H_1)| \geq 3$, we have $|A(H_1, X_1)| \geq 1$. Consider the following three subcases.

Subcase 2.1. There are at least two consecutive vertices on C_h^1, say z_1^1 and z_2^1, such that $z_1^1 \rightsquigarrow X_1$ and $z_2^1 \rightsquigarrow X_1$.

If $z \to z_2^1$, then every arc $z_i^1 z_k$ for each $i \neq 2$ is anti-pancyclic in T. So when $|V(H_1)| \geq 4$, we are done. Assume in the following that $|V(H_1)| = 3$. When $z \to z_3^1$, we can deduce that $z_2^1 z_k$ is the fourth anti-pancyclic arc. When $z_3^1 \to z$, we know that $z_3^1 z$ is the fourth anti-pancyclic arc. So assume in the following that $z_2^1 \to z$, and $z_2^1 z$ is the second anti-pancyclic arc.

If $z \to z_3^1$, then $z_1^1 z_k$ and $z_2^1 z_k$ are the desired another two anti-pancyclic arcs. So assume in the following that $z_3^1 \to z$, and $z_3^1 z$ is the third anti-pancyclic arc.

Proceeding in this way we may assume that $z_i^1 \to z$ and $z_i^1 z$ is anti-pancyclic arc for each $2 \leq i \leq h_1$. Since T_2 is strong, we have $z \to z_1^1$. When $h_1 \geq 4$, we are done. When $h_1 = 3$, $z_1^1 z$ is the fourth anti-pancyclic arc in T.

Subcase 2.2. There are at least two vertices of H_1 dominating some vertex of X_1, and they are all not consecutive on C_h^1.

Since $|A(X_1,T_2)| \le 2$, we have $h_1 = 4$ and $|V(X_1)| = 1$. Assume without loss of generality that $\{z_1^1, z_3^1\} \to X_1 \to \{z_2^1, z_4^1\}$. Hence, the two arcs from X_1 to H_1 are anti-pancyclic by Lemma 2.11.

If $z \to z_2^1$, then $z_1^1 z_k$ and $z_4^1 z_k$ are another two desired anti-pancyclic arcs in T. If $z_2^1 \to z$, then $z_2^1 z$ is the third anti-pancyclic arc, and $z_1^1 z_k$ (when $z \to z_3^1$) or $z_3^1 z$ (when $z_3^1 \to z$) is the fourth anti-pancyclic arc.

Subcase 2.3. There is only one vertex of H_1, say z_1^1, such that $z_1^1 \rightsquigarrow X_1$.

Since $|A(X_1,T_2)| \le 2$, we have $h_1 = 3$, $|V(X_1)| = 1$, and $\{H_2, \dots, H_k, z\} \to X_1 \to \{z_2^1, z_3^1\}$. By Lemma 2.11 the two arcs $x_1^1 z_2^1$ and $x_1^1 z_3^1$ are anti-pancyclic in T.

If $z \to z_2^1$, then $z_1^1 z_k$ and $z_3^1 z_k$ are another two desired anti-pancyclic arcs. If $z_2^1 \to z$, then $z_2^1 z$ is the third anti-pancyclic arc and $z_3^1 z$ (when $z_3^1 \to z$) or $z_1^1 z_k$ (when $z \to z_3^1$) is the fourth anti-pancyclic arc.

Case 3. $|A(X_1,T_2)| = 3$.

In this case, the arcs from X_1 to T_2 are three of our desired anti-pancyclic arcs in T by Lemma 2.11, and we need to find the fourth one.

Suppose first that $H_1 \rightsquigarrow X_1$, say $z_1^1 x_1^1$. Then $z_1^1 z_k$ (when $z \to z_2^1$) or $z_2^1 z$ (when $z_2^1 \to z$) is the fourth anti-pancyclic arc.

Suppose now that $X_1 \to H_1$. Then $|V(X_1)| = 1$, $|V(H_1)| = 3$ and $\{H_2, \dots, H_k, z\} \to X_1$. If $z_i^1 \to z$ for some $1 \le i \le 3$, then $z_i^1 z$ is the fourth anti-pancyclic arc. If $z \to H_1$, then $z z_i^1$ is the fourth anti-pancyclic arc for each $i \in \{1, 2, 3\}$. □

Lemma 2.23. *If $\beta = 2$, $|V(T_1)| = 1$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \dots, H_k$ ($k \ge 2$) of $T_2 - z$ satisfies $|V(H_1)| = 1$ and $|V(H_k)| \ge 3$, then there are at least four anti-pancyclic arcs in T.*

Proof. Since $|V(T_1)| = 1$, we have $X \to T_1$. Let $V(H_1) = \{z_1\}$ and $C_h^k = z_1^k z_2^k \dots z_{h_k}^k z_1^k$ be a hamiltonian cycle of H_k. If there are at least four arcs from X_1 to T_2, then by Lemma 2.11 we are done. So assume in the following that $|A(X_1,T_2)| \le 3$.

Case 1. $|A(X_1,T_2)| = 0$.

In this case $T_2 \to X_1$. If $z \rightsquigarrow H_k$, say $z z_1^k$, then it is not difficult to check that the arc $z z_1^k$ and every arc from H_1 to H_k are all anti-pancyclic in T. If $H_k \to z$, then it is not difficult to check that every arc from H_k to z and every arc from H_1 to H_k are all anti-pancyclic in T.

Case 2. $1 \le |A(X_1,T_2)| \le 2$.

In this case, at least one arc which is from X_1 to T_2 is anti-pancyclic and at least one vertex of H_k, say z_1^k, dominates X_1.

If $z \to z_2^k$, then it is not difficult to check that every arc $z_1 z_i^k$ for $i \ne 1$ is anti-pancyclic in T. So when $h_k \ge 4$ or $|A(X_1,T_2)| = 2$, we are done. Assume in the following that $h_k = 3$ and $|A(X_1,T_2)| = 1$. Then $z_1 z_2^k$ and $z_1 z_3^k$ are another two anti-pancyclic arcs, and at least one of $z \rightsquigarrow X_1$ and $z_1 \rightsquigarrow X_1$ holds. In any case, we know that $z z_2^k$ is the fourth anti-pancyclic arc. Below we assume that $z_2^k \to z$.

If $z \to z_3^k$, then it is not difficult to check $z_1 z_3^k$ and $z z_3^k$ are another two anti-pancyclic arcs. In the case when $z_1 \rightsquigarrow X_1$ or $z \rightsquigarrow X_1$, we know that $z_1 z_1^k$ is the fourth anti-pancyclic arc. In the case when $X_1 \to \{z, z_1\}$, the two arcs from X_1 to T_2 together with $z_1 z_3^k$ and $z z_3^k$ are the desired anti-pancyclic arcs, and we are done. Assume in the following $z_3^k \to z$.

Proceeding in this manner, we may assume that $z_j^k \to z$ for each $2 \leq j \leq h_k$.

If $z_1^k \to z$, then $H_k \to z$ and every arc $z_i^k z$ for $i \neq 2$ is anti-pancyclic in T. So when $h_k \geq 4$ or $|A(X_1, T_2)| = 2$, we are done. Assume now that $h_k = 3$ and $|A(X_1, T_2)| = 1$. Then either $z \rightsquigarrow X_1$ or $z_2^k \rightsquigarrow X_1$ holds, which implies either $z_1 z_2^k$ or $z_2^k z$ is the fourth anti-pancyclic arc. Assume in the following $z \to z_1^k$.

If $z \rightsquigarrow X_1$ or $z_1 \rightsquigarrow X_1$, then $z z_1^k$, $z_1 z_1^k$ and $z_1 z_2^k$ are another three anti-pancyclic arcs in T. If $X_1 \to \{z, z_1\}$, then $H_k \to X_1$. Now the two arcs from X_1 to $\{z, z_1\}$ together with $z_1 z_1^k$ and $z_1 z_2^k$ are the desired four anti-pancyclic arcs in T.

Case 3. $|A(X_1, T_2)| = 3$.

In this case, the arcs from X_1 to T_2 are three of our desired anti-pancyclic arcs by Lemma 2.11, and we need to find the fourth one.

Suppose first that $H_k \rightsquigarrow X_1$, say $z_1^k x_1^1$. If $z \to z_2^k$, then $z_1 z_2^k$ is the fourth anti-pancyclic arc. Assume that $z_2^k \to z$. If $z \to z_3^k$, then $z_1 z_3^k$ is the fourth anti-pancyclic arc. So assume that $z_3^k \to z$. Proceeding in this manner, we may assume that $z_i^k \to z$ for each $2 \leq i \leq h_k$. Then $z_1^k z$ (when $z_1^k \to z$) or $z_1 z_2^k$ (when $z \to z_1^k$) is the fourth anti-pancyclic arc.

Suppose now that $X_1 \to H_k$. Then $|V(X_1)| = 1$, $|V(H_k)| = 3$ and $\{z, z_1\} \to X_1$. Since T_2 is strong, we may assume without loss of generality that $z_1^k \to z$. Then $z_1 z_3^k$ is the fourth anti-pancyclic arc, and we are done. □

Lemma 2.24. *If $\beta = 2$, $|V(T_1)| = 1$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ ($k \geq 3$) of $T_2 - z$ satisfies $|V(H_1)| = |V(H_k)| = 1$ and $|V(H_i)| \geq 3$ for some $2 \leq i \leq k-1$, then there are at least four anti-pancyclic arcs in T.*

Proof. Let $V(H_1) = \{z_1\}$, $V(H_k) = \{z_k\}$, and w be an arbitrary vertex of H_i. If $w \rightsquigarrow X_1$, then it is easy to see that $w z_k$ is an anti-pancyclic arc in T. Otherwise, if $X_1 \to w$, then every arc from X_1 to w is anti-pancyclic by Lemma 2.11. Since $|V(H_i)| \geq 3$, there are already at least three anti-pancyclic arcs in T.

If $X_1 \rightsquigarrow \{z_1, z, z_k\}$, then by Lemma 2.11 it is the fourth anti-pancyclic arc. If $\{z_1, z_k, z\} \to X_1$, then $z_1 w$ is the fourth anti-pancyclic arc. □

Lemma 2.25. *If $\beta = 2$, $|V(T_1)| = 1$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ of $T_2 - z$ satisfies $k \geq 6$ and $|V(H_i)| = 1$ for each $1 \leq i \leq k$, then there are at least four anti-pancyclic arcs in T.*

Proof. Let $V(H_i) = \{z_i\}$ for $i = 1, 2, \ldots, k$. For any $i \in \{2, 3, \ldots, k-2\}$, if $z_i \rightsquigarrow X_1$, then $z_i z_k$ is an anti-pancyclic arc; if $X_1 \to z_i$, then every arc from X_1 to z_i is anti-pancyclic. Since $k \geq 6$, there are already at least three anti-pancyclic arcs in T.

If $X_1 \rightsquigarrow z$, say $x_1^1 z$, then $x_1^1 z$ is the fourth anti-pancyclic arc by Lemma 2.11. If $z \to X_1$, then $z_1 z_{k-1}$ is the fourth anti-pancyclic arc. □

Lemma 2.26. *If $\beta = 2$, $|V(T_1)| = 1$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ of $T_2 - z$ satisfies $k = 5$ and $|V(H_i)| = 1$ for each $1 \le i \le 5$, then there are at least four anti-pancyclic arcs in T.*

Proof. Let $V(H_i) = \{z_i\}$ for $i = 1, 2, \ldots, 5$. Suppose first that $z_3 \rightsquigarrow X_1$. Then $z_3 z_5$ and $z_2 z_5$ are two anti-pancyclic arcs.

If $z_2 \rightsquigarrow X_1$, then $z_2 z_4$ is the third anti-pancyclic arc. Otherwise, if $X_1 \to z_2$, then the arc from X_1 to z_2 is the third anti-pancyclic arc by Lemma 2.11.

If $X_1 \rightsquigarrow \{z, z_1\}$, then the arc from X_1 to z or to z_1 is the fourth anti-pancyclic arc by Lemma 2.11. Otherwise, if $\{z, z_1\} \to X_1$, then $z_1 z_4$ is the fourth anti-pancyclic arc.

Suppose now that $X_1 \to z_3$. Then the arc from X_1 to z_3 is the first anti-pancyclic arc.

If $z_2 \rightsquigarrow X_1$, then $z_2 z_5$ is the second anti-pancyclic arc. In the case when $z \rightsquigarrow X_1$, $z_2 z_3$ and $z_2 z_4$ are another two desired anti-pancyclic arcs. In the case when $X_1 \to z$ and $X_1 \rightsquigarrow z_5$, they are just another two desired anti-pancyclic arcs by Lemma 2.11. In the case when $z_5 \to X_1 \to z$, the arc from X_1 to z and $z_2 z_3$ are another two desired anti-pancyclic arcs.

If $X_1 \to z_2$, then the arc from X_1 to z_2 is the second anti-pancyclic arc. When $z_4 \rightsquigarrow X_1$, $z_2 z_5$ is the third anti-pancyclic arc. When $X_1 \to z_4$, the arc from X_1 to z_4 is the third anti-pancyclic arc.

In the case when $X_1 \rightsquigarrow \{z_1, z\}$, the arc from X_1 to z_1 or to z is the fourth anti-pancyclic arc. In the case when $\{z_1, z\} \to X_1$, $z_1 z_3$ is the fourth anti-pancyclic arc. □

Lemma 2.27. *If $\beta = 2$, $|V(T_1)| = 1$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ of $T_2 - z$ satisfies $k = 4$ and $|V(H_i)| = 1$ for each $1 \le i \le 4$, then there are at least four anti-pancyclic arcs in T.*

Proof. Let $V(H_i) = \{z_i\}$ for $i = 1, 2, 3, 4$. If $|A(X_1, T_2)| \ge 4$, then by Lemma 2.11 we are done. Below we consider the case $|A(X_1, T_2)| \le 3$. Note that $X \to T_1$.

Case 1. $|A(X_1, T_2)| = 0$.

In this case $T_2 \to X_1$, and clearly, $z_2 z_4$, $z_1 z_3$ are two anti-pancyclic arcs in T. Moreover, it is not difficult to check that $z z_2$ (when $z \to z_2$) or $z_2 z$ (when $z_2 \to z$) is the third anti-pancyclic arc, and $z z_3$ (when $z \to z_3$) or $z_3 z$ (when $z_3 \to z$) is the fourth anti-pancyclic arc.

Case 2. $|A(X_1, T_2)| = 1$.

In this case the unique arc from X_1 to T_2 is the first anti-pancyclic arc by Lemma 2.11.

Suppose first that $z_2 \rightsquigarrow X_1$. Then $z_2 z_4$ is the second anti-pancyclic arc. In the case when $X_1 \to z_3$, we have $\{z_1, z_2, z_4, z\} \to X_1$, and $z_2 z_3$, $z_1 z_3$ are another two anti-pancyclic arcs. In the case when $z_3 \rightsquigarrow X_1$, we look for the rest two anti-pancyclic arcs.

It is not difficult to check that $z_1 z_4$ (when $z \to z_2$) or $z_2 z$ (when $z_2 \to z$) is the third anti-pancyclic arc, and $z_3 z$ (when $z_3 \to z$) or $z_1 z_3$ (when $z \to z_3$) is the fourth anti-pancyclic arc.

Suppose now that $X_1 \to z_2$. Then $|V(X_1)| = 1$ and $\{z_1, z_3, z_4, z\} \to X_1$. It is not difficult to check that zz_3 (when $z \to z_3$) or z_3z (when $z_3 \to z$) is the second anti-pancyclic arc. Moreover, we can further deduce that zz_2 and z_1z_4 (when $z \to z_2$) or z_2z and z_1z_4 (when $z_2 \to z$) are the rest two anti-pancyclic arcs.

Case 3. $|A(X_1, T_2)| = 2$.

In this case the two arcs from X_1 to T_2 are anti-pancyclic. Suppose first that $z_2 \rightsquigarrow X_1$. Then z_2z_4 is the third anti-pancyclic arc. If $z_3 \rightsquigarrow X_1$ and $z_3 \to z$, then z_3z is the fourth anti-pancyclic arc in T. If $z_3 \rightsquigarrow X_1$ and $z \to z_3$, then z_1z_3 is the fourth anti-pancyclic arc. If $X_1 \to z_3$, then z_2z_3 is the last desired anti-pancyclic arc.

Suppose now that $X_1 \to z_2$, which yields $|V(X_1)| = 1$, and the arc from X_1 to z_2 is the first anti-pancyclic arc.

If $X_1 \to z$, then $\{z_1, z_3, z_4\} \to X_1$, and the arc from X_1 to z is the second anti-pancyclic arc. In the case when $z \to z_3$, zz_3 and z_1z_3 are another two anti-pancyclic arcs. In the case when $z_3 \to z$, z_3z and z_4z are another two desired anti-pancyclic arcs.

If $X_1 \to z_1$, then $\{z, z_3, z_4\} \to X_1$, and the arc from X_1 to z_1 is the second anti-pancyclic arc. In the case when $z \to z_3$, zz_3 and z_1z_3 are another two anti-pancyclic arcs. In the case when $z_3 \to z$, z_3z and z_1z_4 are another two anti-pancyclic arcs.

If $X_1 \to z_3$, then $\{z_1, z_4, z\} \to X_1$, and the arc from X_1 to z_3 is the second anti-pancyclic arc. Moreover, we can deduce that z_1z_2 and z_1z_3 are another two anti-pancyclic arcs.

If $X_1 \to z_4$, then $\{z, z_1, z_3\} \to X_1$, and the arc from X_1 to z_4 is the second anti-pancyclic arc. In the case when $z \to z_2$, z_1z_2 and z_1z_4 are another two anti-pancyclic arcs. In the case when $z_2 \to z$, z_2z and z_1z_4 are another two desired anti-pancyclic arcs.

Case 4. $|A(X_1, T_2)| = 3$.

In this case, the arcs from X_1 to T_2 are three of our desired anti-pancyclic arcs by Lemma 2.11, and we need to find the fourth one. If $z_2 \rightsquigarrow X_1$, then z_2z_4 is the fourth anti-pancyclic arc. So assume in the following that $X_1 \to z_2$.

Suppose first that $z_3 \rightsquigarrow X_1$. If $z_3 \to z$, then z_3z is the fourth anti-pancyclic arc. Otherwise, if $z \to z_3$, then z_1z_4 is the fourth anti-pancyclic arc whenever $z \to z_2$ or $z_2 \to z$.

Suppose now that $X_1 \to z_3$. Then $|V(X_1)| = 1$. If $X_1 \to z_1$, then $\{z, z_4\} \to X_1$ and z_1z_3 is the fourth anti-pancyclic arc. If $X_1 \to z_4$, then $\{z_1, z\} \to X_1$. In the case when $z \to z_3$, z_1z_3 is the fourth anti-pancyclic arc. In the case when $z_3 \to z$, z_1z_4 is the fourth anti-pancyclic arc. If $X_1 \to z$, then $\{z_1, z_4\} \to X_1$. In the case when $z \to z_2$ or $z_2 \to z \to z_3$, z_1z_4 is the fourth anti-pancyclic arc. In the case when $\{z_2, z_3\} \to z$, z_3z is the fourth anti-pancyclic arc. □

Lemma 2.28. *If $\beta = 2$, $|V(T_1)| = 1$, $X_\alpha \rightsquigarrow T_2$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ of $T_2 - z$ satisfies $k = 3$ and $|V(H_i)| = 1$ for each $1 \le i \le 3$, then there are at least four anti-pancyclic arcs in T.*

Proof. Let $V(H_i) = \{z_i\}$ for $i = 1, 2, 3$. Since $k = 3$, T_2 is the unique strong tournament with order 4. So we may assume without loss of generality that $z \to z_2$. If $|A(X_1, T_2)| \ge 4$, then by Lemma 2.11 we are done. Below we consider the case $|A(X_1, T_2)| \le 3$.

Case 1. $|A(X_1, T_2)| = 0$.

In this case $T_2 \to X_1$, and it is easy to check that zz_2 and z_1z_3 are two anti-pancyclic arcs in T.

Since $X_\alpha \rightsquigarrow T_2$, we have $\alpha \geq 2$ and assume without loss of generality that $X_\alpha \rightsquigarrow z$. Then t_1^1z and $t_1^1z_1$ are another two anti-pancyclic arcs.

Case 2. $|A(X_1, T_2)| = 1$.

In this case the unique arc from X_1 to T_2 is the first anti-pancyclic arc in T.

Firstly, suppose that $X_1 \rightsquigarrow z_1$. Then $\{z_2, z_3, z\} \to X_1$ and zz_2, zz_1 are another two anti-pancyclic arcs in T.

If $\alpha = 1$, then $t_1^1z_1$ is the fourth anti-pancyclic arc. If $\alpha \geq 2$ and $|V(X)| \geq 3$, then $x_1^1x_1^\alpha$ is the fourth anti-pancyclic arc. If $|V(X)| = 2$, since $X_\alpha \rightsquigarrow T_2$, say $x_1^\alpha w$, where $w \in V(T_2)$, then t_1^1w is the fourth anti-pancyclic arc.

Secondly, suppose that $X_1 \rightsquigarrow z_2$. Then $\{z_1, z_3, z\} \to X_1$ and z_1z_3, zz_2 and z_1z_2 are another three anti-pancyclic arcs in T.

Thirdly, suppose that $X_1 \rightsquigarrow z_3$. Then $\{z_1, z_2, z\} \to X_1$ and z_1z_3, z_2z_3 are another two anti-pancyclic arcs in T.

If $\alpha = 1$, then $t_1^1z_3$ is the fourth anti-pancyclic arc. If $\alpha \geq 2$ and $|V(X)| \geq 3$, then $x_1^1x_1^\alpha$ is the fourth anti-pancyclic arc. If $|V(X)| = 2$, since $X_\alpha \rightsquigarrow T_2$, say $x_1^\alpha w$, where $w \in V(T_2)$, then $x_1^\alpha w$ is the fourth anti-pancyclic arc.

Finally, suppose that $X_1 \rightsquigarrow z$. Then $\{z_1, z_2, z_3\} \to X_1$ and z_1z_3 is the second anti-pancyclic arc in T. Since $X_\alpha \rightsquigarrow T_2$, we may assume without loss of generality that $x_1^\alpha \to w$, where $w \in V(T_2)$. Then t_1^1w is the third anti-pancyclic arc.

If $\alpha = 1$, then $w = z$ and $t_1^1z_1$ is the fourth anti-pancyclic arc. If $\alpha \geq 2$ and $|V(X)| \geq 3$, then $x_1^1x_1^\alpha$ is the fourth anti-pancyclic arc. If $|V(X)| = 2$, then $x_1^\alpha w$ is the fourth anti-pancyclic arc.

Case 3. $|A(X_1, T_2)| = 2$.

In this case the two arcs from X_1 to T_2 are anti-pancyclic, and we need to find another two anti-pancyclic arcs.

Suppose first that $z_1 \rightsquigarrow X_1$. Then z_1z_3 is the third anti-pancyclic arc.

If $z \rightsquigarrow X_1$ and $z_3 \rightsquigarrow X_1$, then zz_2 is the fourth anti-pancyclic arc. If $z \rightsquigarrow X_1 \to z_3$, then $|V(X_1)| = 1$, $X_1 \to z_2$ and z_1z_2 is the fourth anti-pancyclic arc. If $X_1 \to z$ and $z_3 \rightsquigarrow X_1$, then $|V(X_1)| = 1$, $X_1 \to z_2$ and z_1z_2 is still the fourth anti-pancyclic arc. If $X_1 \to \{z, z_3\}$, then $|V(X_1)| = 1$, $z_2 \to X_1$ and z_2z_3 is the fourth anti-pancyclic arc.

Suppose now that $X_1 \to z_1$. Then $|V(X_1)| = 1$. If $X_1 \to z_2$, then $\{z, z_3\} \to X_1$ and zz_2, zz_1 are another two anti-pancyclic arc.

If $X_1 \to z_3$, then $\{z, z_2\} \to X_1$ and z_2z_3 is the third anti-pancyclic arc. In the case when $\alpha = 1$, $t_1^1z_3$ is the fourth anti-pancyclic arc. In the case when $\alpha \geq 2$ and $|V(X)| \geq 3$, $x_1^1x_1^\alpha$ is the fourth anti-pancyclic arc. In the case when $|V(X)| = 2$, since $X_\alpha \rightsquigarrow T_2$, say $x_1^\alpha w$, where $w \in V(T_2)$, then $x_1^\alpha w$ is the fourth anti-pancyclic arc.

If $X_1 \to z$, then $\{z_2, z_3\} \to X_1$. Since $X_\alpha \rightsquigarrow T_2$, say $x_1^\alpha w$, where $w \in V(T_2)$, then $t_1^1 w$ is the third anti-pancyclic arc. In the case when $\alpha = 1$, $t_1^1 z$ and $t_1^1 z_1$ are another two anti-pancyclic arcs. In the case when $\alpha \geq 2$ and $|V(X)| \geq 3$, $x_1^1 x_1^\alpha$ is the fourth anti-pancyclic arc. In the case when $|V(X)| = 2$, $x_1^\alpha w$ is the fourth anti-pancyclic arc.

Case 4. $|A(X_1, T_2)| = 3$.

In this case, the arcs from X_1 to T_2 are three of our desired anti-pancyclic arcs by Lemma 2.11, and we need to find the fourth one. If $z_1 \rightsquigarrow X_1$, then $z_1 z_3$ is the fourth anti-pancyclic arc. So assume in the following that $X_1 \to z_1$, then $|V(X_1)| = 3$ or $|V(X_1)| = 1$.

If $|V(X_1)| = 3$, then $\{z, z_2, z_3\} \to X_1$ and zz_1 is the fourth anti-pancyclic arc.

If $|V(X_1) = 1$, since $|A(X_1, T_2)| = 3$, we have $|N^-(X_1)| = 1$ and thus $|V(X)| = 1$. Whenever $z \to X_1$, $z_2 \to X_1$ or $z_3 \to X_1$, we can deduce that $t_1^1 z_1$ is the fourth anti-pancyclic arc. □

Lemma 2.29. *If $\beta = 2$, $|V(T_1)| = 1$, $T_2 \to X_\alpha$, and there is a cut-vertex z of T_2 such that the strong decomposition $H_1, H_2, \ldots, H_k$ of $T_2 - z$ satisfies $k = 3$ and $|V(H_i)| = 1$ for each $1 \leq i \leq 3$, then T is isomorphic to T_5^* and has exactly two anti-pancyclic arcs.*

Proof. Let $V(H_i) = \{z_i\}$ for $i = 1, 2, 3$. Since $T_2 \to X_\alpha$, we have $|N^+(X_\alpha)| = 1$. Because X is a reductor of T, we know that $|V(X)| = 1$.

Since $k = 3$, T_2 is the unique strong tournament with 4 vertices. So we may assume without loss of generality that $z \to z_2$. Now T is isomorphic to T_5^* and only the two arcs zz_2 and $z_1 z_3$ are anti-pancyclic in T. □

Lemma 2.30. *If $\beta = 2$, $|V(T_1)| = 1$ and T_2 is 3-cycle, then there are at least four anti-pancyclic arcs in T.*

Proof. Since $n \geq 6$ and $T - T_2$ is not strong, we have $2 \leq |V(X)| \leq 3$. Note that when $|V(X)| = 2$, T is 2-strong and then $|N^+(X_1) \cap V(T_2)| \leq 1$; when $|V(X)| = 3$, T is 3-strong with order 7, and then T is 3-regular.

Case 1. $|V(X)| = 3$ and $\alpha = 3$.

By Theorem 1.1 it is not difficult to check that $x_1^1 x_1^3$, $t_1^1 t_1^2$, $t_1^1 t_2^2$ and $t_1^1 t_3^2$ are the desired anti-pancyclic arcs.

Case 2. $|V(X)| = 3$ and $\alpha = 1$.

By Theorem 1.1 it is not difficult to check that $x_i^1 t_1^1$, $t_1^1 t_i^2$ are the desired anti-pancyclic arcs in T for $i = 1, 2, 3$.

Case 3. $|V(X)| = 2$ and $|N^+(X_1) \cap V(T_2)| = 0$.

In this case T is 2-strong and $T_2 \to X_1$. By the symmetry of T_2 we may assume that $x_1^2 \to t_1^2$. Then $t_1^1 t_1^2$ is the first anti-pancyclic arc.

If $x_1^2 \to t_2^2$, then $t_3^2 \to x_1^2$. It is not difficult to check that $x_1^2 t_1^2$, $t_3^2 t_1^2$ and $t_3^2 x_1^1$ are another three anti-pancyclic arcs in T.

If $t_2^2 \to x_1^2$, then $t_2^2 x_1^1$ is the second anti-pancyclic arc in T. In the case when $x_1^2 \to t_3^2$, we know that $x_1^2 t_3^2$ and $t_2^2 t_3^2$ are another two desired anti-pancyclic arcs. In the case when $t_3^2 \to x_1^2$, then $t_3^2 x_1^2$ and $t_3^2 t_1^2$ are another two anti-pancyclic arcs.

Case 4. $|V(X)| = 2$ and $|N^+(X_1) \cap V(T_2)| = 1$.

By the symmetry of T_2 we may assume that $x_1^1 \to t_1^2$. It follows that $\{t_2^2, t_3^2\} \to X_1$ and then $t_1^2 \to x_1^2$. So $x_1^1 t_1^2$ and $x_1^1 x_1^2$ are two anti-pancyclic arcs by Lemmas 2.11 and 2.8, respectively.

If $x_1^2 \to t_2^2$, then $x_1^2 t_2^2$ and $t_1^1 t_2^2$ are another two anti-pancyclic arcs. If $t_2^2 \to x_1^2$, then $x_1^2 \to t_3^2$, and thus $t_2^2 x_1^2$ and $t_1^1 t_3^2$ are another two anti-pancyclic arcs. □

§3 Proof of the main result

Now we show that Lemmas 2.15-2.30 cover all possible cases. Let X be a reductor of T, $X_1, X_2, \ldots, X_\alpha$ ($\alpha \geq 1$) be the strong decomposition of X, and $T_1, T_2, \ldots, T_\beta$ ($\beta \geq 2$) be the strong decomposition of $T - V(X)$.

If $\beta \geq 3$, then by Lemmas 2.15-2.19 there are at least four anti-pancyclic arcs unless it is isomorphic to T_i^* for $1 \leq i \leq 4$. Assume in the following that $\beta = 2$.

If $|V(T_1)| \geq 3$ and $|V(T_2)| \geq 3$, then by Lemma 2.20 T has at least four anti-pancyclic arcs and we are done. If $|V(T_1)| = |V(T_2)| = 1$, then $|V(X)| = 1$, which contradicts the assumption $n \geq 6$. So we only need to consider the case $|V(T_1)| = 1$ and $|V(T_2)| \geq 3$ (the other case $|V(T_1)| \geq 3$ and $|V(T_2)| = 1$ can be similarly proved by considering T^{-1}).

If T_2 is 2-strong, then there are at least four anti-pancyclic arcs in T by Lemma 2.21. Assume now that T_2 is not 2-strong. Then there is a cut-vertex of T_2, say z, such that $H_1, H_2, \ldots, H_k$ ($k \geq 2$) is the strong decomposition of $T_2 - z$.

If $|V(H_i)| \geq 3$ for some $i \in \{1, 2, \ldots, k\}$, then we are done by Lemmas 2.22-2.24. Assume now that $|V(H_i)| = 1$ for each $1 \leq i \leq k$.

If $k = 2$ or $k \geq 4$, then there are at least four anti-pancyclic arcs by Lemma 2.30 or Lemmas 2.25-2.27. If $k = 3$ and $X_\alpha \rightsquigarrow T_2$, then there are at least four anti-pancyclic arcs by Lemma 2.28. If $k = 3$ and $T_2 \to X_\alpha$, then by Lemma 2.29 T is isomorphic to T_5^* which contains exactly two anti-pancyclic arcs. Now the proof of Theorem 1.1 is complete. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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