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Suitable sets for strongly topological gyrogroups

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Abstract. A discrete subset S of a topological gyrogroup G with the identity 0 is said to be a *suitable set* for G if it generates a dense subgyrogroup of G and $S \cup \{0\}$ is closed in G. In this paper, it is proved that each countable Hausdorff topological gyrogroup has a suitable set; moreover, it is shown that each separable metrizable strongly topological gyrogroup has a suitable set.

§1 Introduction

In 1990, K H Hofmann and S A Morris in [18] introduced the concept of a suitable set for a topological group as an example of a 'thin' closed generating set. It was shown that each locally compact group has a suitable set. Fundamental results on suitable sets for topological groups were obtained by Comfort et al. in [11] and Dikranjan et al. in [12] and [13]. I Guran in [17], F Lin, A Ravsky and T Shi in [21] considered suitable sets for paratopological groups. In 2003, T Banakh and I Protasov generalized Guran's results to left topological groups in [5].

A generalization of a group, a gyrogroup (see Definition 2.2 below) was introduced by A A Ungar [30] in 2002, while studying a c-ball $\mathbb{R}_c^3 = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\}$ of relativistically admissible velocities endowed with Einstein velocity addition $\oplus_E$. Recall that for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{c^2}} \left(\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right),$$

where

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}$$

is the Lorentz factor. It turns out that $(\mathbb{R}_c^3, \oplus_E)$ is a gyrogroup, which fails to be a group, because the operation $\oplus_E$ is not associative. Recently, the topic of gyrogroups was investigated

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by many scholars, see [15, 16, 19, 20, 22, 23, 24, 25, 26, 27, 28, 30].

In 2017, W Atiponrat [2] introduced the concept of topological gyrogroups, which is a generalization of a topological group. Namely, a topological gyrogroup G is a gyrogroup $(G, \oplus)$ endowed with a topology such that the multiplication map $\oplus$ from $G \times G$ to G is jointly continuous and the inverse map $\ominus : G \to G$ is continuous. It turns out that topological gyrogroups possess nice properties. In particular, Z Cai, S Lin, and W He in [10] proved that every topological gyrogroup is a rectifiable space, so every first-countable topological gyrogroup is metrizable. Then R Shen in [25] proved that every weakly first-countable paratopological left-loop is first-countable. M Bao and F Lin introduced the concept of strongly topological gyrogroups, and proved that every feathered strongly topological gyrogroup is paracompact, every T_0 strongly topological gyrogroup is completely regular and every T_0 strongly topological gyrogroup with a countable pseudocharacter is submetrizable, see [3, 4, 6, 7, 8, 9, 31, 32] for more details.

In this paper, we mainly consider suitable sets for (strongly) topological gyrogroups. A subset S of a topological gyrogroup G is said to be a *suitable set* for G if (1) $\overline{\langle S \rangle} = G$, (2) S has the discrete topology, and (3) $S \cup \{0\}$ is closed in G. We show that each countable Hausdorff topological gyrogroup has a suitable set, and each separable metrizable strongly topological gyrogroup has a suitable set, which generalizes some results for topological groups in [11, 12].

All spaces throughout this paper are supposed to be Hausdorff, unless otherwise stated. Let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. Let X be a topological space, and let A be a subset of X. The *closure* of A in X is denoted by $\overline{A}$. For undefined notation and terminology, the reader may refer to [1, 14].

§2 Motivation and preliminaries

In this section, we provide a motivation to study suitable sets in topological gyrogroups. Also we recall and introduce notions and notation used in the paper.

Definition 2.1 ([2])**.** A *groupoid* is a pair $(G, \oplus)$, where G is a nonempty set and $\oplus$ is a binary operation on G. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is called a *groupoid homomorphism*, if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. Furthermore, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a *groupoid automorphism*. We denote the set of all automorphisms of a groupoid $(G, \oplus)$ by $\mathrm{Aut}(G, \oplus)$.

Definition 2.2 ([29])**.** A groupoid $(G, \oplus)$ is called a *gyrogroup*, if its binary operation satisfies the following conditions.

(G1) There exists a unique identity element $0 \in G$ such that $0 \oplus a = a = a \oplus 0$ for all $a \in G$.

(G2) For each $x \in G$, there exists a unique inverse element $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$.

(G3) There exists a map $\mathrm{gyr} : G \times S \to \mathrm{Aut}(G, \oplus)$, such that $x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$ for all $z \in G$.

(G4) For any $x, y \in G$, $\text{gyr}[x \oplus y, y] = \text{gyr}[x, y]$.

Definition 2.3 ([26]). A nonempty subset H of a gyrogroup $(G, \oplus)$ is called a *subgyrogroup* of G (denoted by $H \leq G$), provided the following conditions hold.

(i) The restriction $\oplus|_{H \times H}$ is a binary operation on H, i.e., $(H, \oplus|_{H \times H})$ is a groupoid.

(ii) For any $x, y \in H$, the restriction of $\text{gyr}[x, y]$ to H, $\text{gyr}[x, y]|_H : H \to \text{gyr}[x, y](H)$, is a bijective homomorphism.

(iii) $(H, \oplus|_{H \times H})$ is a gyrogroup.

A subgyrogroup H of G is said to be an *L-subgyrogroup* [26], denoted by $H \leq_L G$, if $\text{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Definition 2.4 ([2]). A triple $(G, \tau, \oplus)$ is called a *topological gyrogroup*, provided the following conditions hold.

(1) (G, τ) is a topological space.

(2) $(G, \oplus)$ is a gyrogroup.

(3) The binary operation $\oplus : G \times G \to G$ is jointly continuous, where $G \times G$ is endowed with the product topology and the inversion $\ominus : G \to G$, $x \mapsto \ominus x$, is continuous.

It is easy to see that each topological group is a topological gyrogroup $(G, \tau, \oplus)$ when we put $\text{gyr}[x, y](z) = z$ all $x, y, z \in G$. A well-known example of a topological gyrogroup, which is not a topological group, is the following *Möbius topological gyrogroup*.

Example 2.5 ([2]). *Let D be an open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Define a Möbius addition $\oplus_M : D \times D \to D$ by putting*

$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b} \text{ for all } a, b \in D.$$

Then $(D, \oplus_M)$ is a gyrogroup with

$$\text{gyr}[a, b](c) = \frac{1 + a\bar{b}}{1 + \bar{a}b} c \text{ for any } a, b, c \in D.$$

But $(D, \oplus_M)$ is not a group, because the operation $\oplus_M$ is not associative. Indeed, it is easy to check that $(1/2 \oplus_M i/2) \oplus_M (-1/2) \neq 1/2 \oplus_M (i/2 \oplus_M (-1/2))$. If τ is the usual topology on D then $(D, \tau, \oplus_M)$ is a topological gyrogroup.

Definition 2.6 ([6]). A topological gyrogroup G is a *strongly topological gyrogroup* if there exists a neighborhood base $\mathscr{U}$ of 0 such that $\text{gyr}[x, y](U) = U$ for each $x, y \in G$ and $U \in \mathscr{U}$. In this case we shall say that G is a strongly topological gyrogroup with a neighborhood base $\mathscr{U}$ of 0. Clearly, we may assume that U is symmetric for each $U \in \mathscr{U}$.

We claim that $(D, \tau, \oplus_M)$ in Example 2.5 is a strongly topological gyrogroup [6]. Indeed, for any $n \in \omega$, let $U_n = \{x \in D : |x| \leq \frac{1}{n}\}$. Then, $\mathscr{U} = \{U_n : n \in \omega\}$ is a neighborhood base of 0. Moreover, since $\overline{1 + a\bar{b}} = 1 + \bar{a}b$ for each $a, b \in D$, we have $|\frac{1+a\bar{b}}{1+\bar{a}b}| = 1$. Therefore, we see that $\text{gyr}[x, y](U) \subset U$, for any $x, y \in D$ and each $U \in \mathscr{U}$. By [26, Proposition 2.6], it follows that $\text{gyr}[x, y](U) = U$.

Moreover, Möbius gyrogroups, Einstein gyrogroups, and Proper velocity gyrogroups tudied in [15, 16, 29], are all strongly topological gyrogroups, see [6].

Definition 2.7 ([18])**.** Let G be a topological gyrogroup and S a subset of G. Then S is said to be a *suitable set* for G if S is discrete in itself, generates a dense subgyrogroup of G, and $S \cup \{0\}$ is closed in G.

By the same notations of [12], let $\mathcal{S}$ (resp., $\mathcal{S}_c$) be the class of topological gyrogroups having a suitable (resp., closed suitable) set. The subset S of the group G often has the stronger property to generate G, instead of generating just a dense subgroup of G. We denote by $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ the corresponding subclasses of $\mathcal{S}$ and $\mathcal{S}_c$, respectively.

The following proposition generalizes [11, Proposition 1.4].

Proposition 2.8. If a topological gyrogroup $(G, \oplus)$ has a suitable set, then G is Hausdorff or $|G| \leq 2$.

Proof. Assume that G is not Hausdorff and $|G| \geq 3$. Let S be the suitable set for G. Since G is not Hausdorff and T_0 and T_3 are equivalent in topological gyrogroups by [2], for every $g \in G$ a set $\overline{\{g\}}$ contains a point $h \neq g$. By the assumption of $|G| \geq 3$, it follows that $S \cup \{0\}$ has at least two points. Take an arbitrary point $s \in S \setminus \{0\}$. Since S is discrete in itself, we have $S \cap \overline{\{s\}} = \{s\}$. Furthermore, $S \cup \{0\}$ is closed in G; thus $\overline{\{s\}} = \{s, 0\}$. Therefore, $\overline{\{0\}} = \{s, 0\}$. It follows that S has at most two points, s and 0. Moreover, since $\overline{\{0\}}$ is a gyrogroup, it is clear that $s \oplus s = 0$. Then, $s = \ominus s = 0$, that is, $G = S$, this is a contradiction. □

Recall that given a space X, a *pseudocharacter* $\psi(X)$ of x is the smallest infinite cardinal κ such that any point of X is an intersection of at most κ open subsets of X and *extent* $e(X)$ is the supremum of cardinalities of closed discrete subspaces of X. Similarly to the proof of [12, Lemma 2.3], we can show the following.

Proposition 2.9. A Hausdorff topological gyrogroup G which has a suitable set satisfies $d(G) \leq e(G) \cdot \psi(G)$.

Proof. We assume that A is a suitable set for G. If U is an open neighborhood of 0 in G, then $A \setminus U$ is discrete and closed in G, which implies $|A \setminus U| \leq e(G)$. Pick a family γ of open sets in G such that $\bigcap \gamma = \{0\}$ and $|\gamma| = \psi(G)$. Since $A \setminus \{0\} \subset \bigcup\{A \setminus U : U \in \gamma\}$, it follows that $|A| \leq e(G) \cdot \psi(G)$. The subgyrogroup $H = \langle A \rangle$ of G satisfies $|H| \leq |A| \cdot \aleph_0$. Since A is a suitable set and H is dense in G, we can conclude that

$$d(G) \leq |H| \leq |A| \cdot \aleph_0 \leq e(G) \cdot \psi(G).$$

□

Therefore, the following result is obtained immediately.

Corollary 2.10. *A non-separable Lindelöf Hausdorff topological gyrogroup of countable pseudocharacter does not have a suitable set.*

Example 2.11. *There exists a non-separable Lindelöf Hausdorff topological gyrogroup G of countable pseudocharacter such that G does not have a suitable set and G is not a topological group.*

Proof. Let D be the topological gyrogroup in Example 2.5, and let H be the Lindelöf non-separable topological group with countable pseudocharacter in (a) of [12, Theorem 2.4]. Then D has a suitable set by in the following Corollary 4.15 and H does not have any suitable set. Moreover, the product $G = D \times H$ is a Lindelöf non-separable topological gyrogroup with countable pseudocharacter, hence it does not have any suitable set by Corollary 2.10. Clearly, G is not a topological group. □

In this paper we mainly consider the following question.

Question 2.12. *If G belongs to some class $\mathcal{C}$ of Hausdorff topological gyrogroups, does G have a suitable set?*

§3 Countable topological gyrogroup with a suitable set

In this section, we study the suitable sets in the class $\mathcal{C}$ of Hausdorff countable topological gyrogroups. We prove that every Hausdorff countable topological gyrogroup G has a closed discrete subset S such that $\langle S\rangle = G$. First, we need some lemmas.

Let G be a gyrogroup. Fix an $n \in \mathbb{N}$. For any $x_1, \dots, x_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$, denote by $R[\varepsilon_1 x_1, \dots, \varepsilon_n x_n]$ the set of all elements which is added some brackets in the summand $\varepsilon_1 x_1 \oplus \dots \oplus \varepsilon_n x_n$ such that the summand belongs to G, where

$$\varepsilon_i x_i = \begin{cases} x_i, & \varepsilon_i = 1, \\ \ominus x_i, & \varepsilon_i = -1. \end{cases}$$

Clearly, $R[\varepsilon_1 x_1, \dots, \varepsilon_n x_n]$ is a countable set, so enumerate $R[\varepsilon_1 x_1, \dots, \varepsilon_n x_n]$ as

$$\{f_m(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) : m \in \mathbb{N}\}.$$

If $A_1, \dots, A_n \subset G$, then we denote $R[\varepsilon_1 A_1, \dots, \varepsilon_n A_n]$ and $f_m(\varepsilon_1 A_1, \dots, \varepsilon_n A_n)$ as the sets

$$\bigcup_{x_1 \in A_1, \dots, x_n \in A_n} R[\varepsilon_1 x_1, \dots, \varepsilon_n x_n] \text{ and } \bigcup_{x_1 \in A_1, \dots, x_n \in A_n} f_m(\varepsilon_1 x_1, \dots, \varepsilon_n x_n),$$

respectively.

In the class of topological gyrogroups, since the multiplication is jointly continuous and the inverse is continuous, it is easy to prove the following lemma.

Lemma 3.1. *Let $a_1, a_2, \dots, a_n$ be points of a topological gyrogroup G, and let V be a neighborhood of the point $f_m(\varepsilon_1 a_1, \dots, \varepsilon_n a_n)$. Then there exists neighborhoods $U_1, \dots, U_n$ of $a_1, \dots, a_n$ in G respectively such that $f_m(\varepsilon_1 U_1, \dots, \varepsilon_n U_n) \subset V$.*

A topological space X is *zero-dimensional* if it has a base consisting of clopen subsets.

Lemma 3.2. *Let G be a nondiscrete Hausdorff topological gyrogroup and U a nonempty open subset which generates G. Then every point $x \in U$ has an open neighborhood V_x of x such that*

$V_x \subset U$ and $\langle U \setminus \overline{V_x} \rangle = G$. In particular, if G is zero-dimensional, then V_x can be chosen to be clopen in G.

Proof. Let U be a nonempty open subset which generates G. Take an arbitrary point $x \in U$. Since G is not discrete, it is obvious that $U \setminus \{x\}$ is dense in U, then it follows that $\langle U \setminus \{x\} \rangle$ is dense in $\langle U \rangle = G$. Moreover, since $U \setminus \{x\}$ is open in G and every open subgyrogroup is closed in G by [2, Proposition 7], we can conclude that $\langle U \setminus \{x\} \rangle$ is open and closed in G. Therefore, $\langle U \setminus \{x\} \rangle = G$.

Since $x \in \langle U \setminus \{x\} \rangle$, there exist $y_1, y_2, \ldots, y_n \in U \setminus \{x\}$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ and $m \in \mathbb{N}$ such that $x = f_m(\varepsilon_1 y_1, \ldots, \varepsilon_n y_n)$, where

$$\varepsilon_i y_i = \begin{cases} y_i, & \varepsilon_i = 1, \\ \ominus y_i, & \varepsilon_i = -1. \end{cases}$$

Because each $y_i \neq x$, we can find an open neighborhood O of x such that $y_i \notin \overline{O} \subset U$, for $i = 1, \ldots, n$. Then for each $i \in \{1, \ldots, n\}$, there is an open neighborhood O_i of y_i such that $O_i \subset U$, $O \cap O_i = \emptyset$, and $f_m(\varepsilon_1 O_1, \ldots, \varepsilon_n O_n) \subset O$ by Lemma 3.1. Put $W = f_m(\varepsilon_1 O_1, \ldots, \varepsilon_n O_n)$. Then W is an open neighborhood of x. By the regularity of G, there exists an open neighborhood V_x of x such that $\overline{V_x} \subset W \subset O$. Therefore, $O_i \subset U \setminus O \subset U \setminus \overline{V_x}$, for $i = 1, 2, \ldots, n$. So, $\overline{V_x} \subset W \subset \langle U \setminus \overline{V_x} \rangle$. Thus $\langle U \setminus \overline{V_x} \rangle = \langle U \rangle = G$.

The last statement of this lemma is obvious. □

Now we can prove our main theorem in this section.

Theorem 3.3. *Every countable Hausdorff topological gyrogroup G belongs to $\mathcal{S}_{cg}$.*

Proof. If G is finitely generated or discrete, then the theorem is clear. Therefore, we may suppose that G is neither finitely generated nor discrete. Enumerate G as $\{g_n : n < \omega\}$. It suffice to find a subset S in G and an open neighborhood U_n of g_n, for each $n < \omega$ satisfying $\langle S \rangle = G$ and $U_n \cap S$ is finite.

Next we will, by induction, find a clopen set V_n in G and a finite set $S_n \subset G$ for each $n < \omega$ so that the following conditions hold:

(i) $g_n \in \bigcup_{i=0}^{n} V_i$ for each $n \in \omega$;

(ii) $G = \langle G \setminus (\bigcup_{i=0}^{n} V_i) \rangle$ for each $n \in \omega$;

(iii) for each $n > 0$, $V_n \subset G \setminus (\bigcup_{i=0}^{n-1} V_i)$;

(iv) $V_i \cap S_n = \emptyset$, for $i < n$; and

(v) $g_n \in \langle \bigcup_{i=0}^{n} S_i \rangle$ for each $n \in \omega$.

Then set $U_n = \bigcup_{i=0}^{n} V_i$ for each $n \in \omega$, and put $S = \bigcup_{n<\omega} S_n$. Clearly, $\langle S \rangle = G$ and $U_n \cap S$ is finite for each $n \in \omega$.

Therefore, it suffices to construct S_n and V_n inductively as follows.

Set $S_0 = \{g_0\}$. Since every Hausdorff topological gyrogroup is regular and every countable non-empty regular space is zero-dimensional [14, Corollary 6.2.8], it follows that the countable topological gyrogroup G is zero-dimensional. By Lemma 3.2, there exists a clopen neighborhood V_0 of g_0 such that $G = \langle G \setminus V_0 \rangle$.

Assume that the finite sets $S_0, S_1, \ldots, S_k$ and clopen sets $V_0, V_1, \ldots, V_k$ have been defined satisfying the above properties (i)-(v). Clearly, if $g_{k+1} \in \langle \bigcup_{i=0}^{k} S_i \rangle$, then set $S_{k+1} = \emptyset$. If $g_{k+1} \notin \langle \bigcup_{i=0}^{k} S_i \rangle$, then it follows from (ii) that there exist

$$y_1, y_2, \ldots, y_m \in G \setminus (\bigcup_{i=0}^{k} V_i), \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{1, -1\}$$

and $n \in \mathbb{N}$ such that

$$g_{k+1} = f_n(\varepsilon_1 y_1, \ldots, \varepsilon_m y_m).$$

Set $S_{k+1} = \{y_1, y_2, \ldots, y_m\}$. Thus both (iv) and (v) are satisfied.

Obviously, if $g_{k+1} \in \bigcup_{i=0}^{k} V_i$, then put $V_{k+1} = \emptyset$. If $g_{k+1} \notin \bigcup_{i=0}^{k} V_i$, then it follows from Lemma 3.2 that there exists a clopen neighborhood V_{k+1} of g_{k+1} such that $V_{k+1} \subset G \setminus (\bigcup_{i=0}^{k} V_i)$ and $G = \langle G \setminus (\bigcup_{i=0}^{k+1} S_i) \rangle$. Then (i)-(iii) are all satisfied.

Therefore, the sets S_n and V_n are defined for all n with the required properties. □

Corollary 3.4 ([11])**.** *Every countable Hausdorff topological group G belongs to $\mathcal{S}_{cg}$.*

§4 A strongly topological gyrogroup with a suitable set

In this section, we mainly prove that every separable metrizable strongly topological gyrogroup has a suitable set. First, we need some lemmas.

Lemma 4.1. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup with a symmetric neighborhood base $\mathscr{U}$ at 0. Suppose further that U, V, and W are all open neighborhoods of 0 such that $V \oplus V \subset W$, $W \oplus W \subset U$ and $V, W \in \mathscr{U}$. If a subset A of G is U-disjoint (that is, if $b \notin a \oplus U$, for any distinct $a, b \in A$), then for each $x \in G$ the set $x \oplus V$ intersects at most one of the element of the family $\{a \oplus V : a \in A\}$. In particular, the family of open sets $\{a \oplus V : a \in A\}$ is discrete in G.*

Proof. We need to show that, for every $x \in G$, the open neighborhood $x \oplus V$ of x intersects at most one element of the family $\{a \oplus V : a \in A\}$. We assume the contrary that, for some $x \in G$, there exist distinct elements $a, b \in A$ such that $(x \oplus V) \cap (a \oplus V) \neq \emptyset$ and $(x \oplus V) \cap (b \oplus V) \neq \emptyset$. We show that $b \in a \oplus U$ as follows.

Since $(x \oplus V) \cap (a \oplus V) \neq \emptyset$, there exist $v_1, v_2 \in V$ such that $x \oplus v_1 = a \oplus v_2$. Then, $a = (a \oplus v_2) \oplus \text{gyr}[a, v_2](\ominus v_2) = (x \oplus v_1) \oplus \text{gyr}[a, v_2](\ominus v_2)$. Therefore,

$$\begin{aligned} a &\in (x \oplus v_1) \oplus \text{gyr}[a, v_2](V) \\ &= (x \oplus v_1) \oplus V \\ &= x \oplus (v_1 \oplus \text{gyr}[v_1, x](V)) \\ &= x \oplus (v_1 \oplus V) \\ &\subset x \oplus (V \oplus V) \\ &\subset x \oplus W. \end{aligned}$$

Thus, $a \in x \oplus W$. By the same reasoning, we also have $b \in x \oplus W$.

Therefore, there exists $w_1 \in W$ such that $a = x \oplus w_1$. Then,

$$x = a \oplus \text{gyr}[x, w_1](\ominus w_1) \in a \oplus \text{gyr}[x, w_1](W) = a \oplus W.$$

Hence,

$$\begin{aligned} b &\in (a \oplus W) \oplus W \\ &= a \oplus (W \oplus \text{gyr}[W, a](W)) \\ &= a \oplus (W \oplus W) \\ &\subset a \oplus U. \end{aligned}$$

□

Let G be a topological gyrogroup. For κ an infinite cardinal, the topological gyrogroup G is said to be *left κ-totally bounded* if for every nonempty open subset U of G there is a subset $F \subset G$ such that $|F| < \kappa$ and $G = F \oplus U$. We denote $lb(G)$ by the least cardinal $\kappa \geq \omega$ such that G is left κ-totally bounded. Each left ω-totally bounded topological gyrogroup is also called *left precompact*.

Lemma 4.2. *Let G be a strongly topological gyrogroup with $lb(G) = \kappa$. If $\tau < \kappa$, then there exists an open neighborhood V of 0 and a subset $\{p_\alpha : \alpha < \tau\}$ such that for each $p \in G$ the set $p \oplus V$ intersects at most one of the elements of the family $\{p_\alpha \oplus V : \alpha < \tau\}$.*

Proof. Since $lb(G) = \kappa$ and $\tau < \kappa$, it follows that there exists a nonempty open neighborhood U of 0 in G such that no $F \subset G$ with $|F| \leq \tau$ satisfies $G = F \oplus U$. By induction, it is easy to find a set $\{p_\alpha : \alpha < \tau\}$ such that each p_α satisfies $p_\alpha \notin \bigcup_{\beta<\alpha}(p_\beta \oplus U)$. Then, from Lemma 4.1 we can find a nonempty open neighborhood V of 0 in G such that for each $p \in G$ the set $p \oplus V$ intersects at most one of the elements of the family $\{p_\alpha \oplus V : \alpha < \tau\}$. □

The strongly topological gyrogroup G in Example 2.5 is left precompact and non-pseudocompact. However, the following result shows that each pseudocompact strongly topological gyrogroup is left precompact.

Theorem 4.3. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup with a symmetric open neighborhood base $\mathscr{U}$ at 0. If G is pseudocompact, then it is left precompact.*

Proof. Let U be an arbitrary symmetric open neighborhood of 0 in G and $V, W \in \mathscr{U}$ such that $V \oplus V \subset W$ and $W \oplus W \subset U$. Let

$$\mathscr{F} = \{A \subset G : (b \oplus V) \cap (a \oplus V) = \emptyset, \text{ for any distinct } a, b \in A\}.$$

Define $\leq$ in G such that $A_1 \leq A_2$ if and only if $A_1 \subset A_2$, for any $A_1, A_2 \in \mathscr{F}$. Then, $(\mathscr{F}, \leq)$ is a poset and the union of any chain of V-disjoint sets is again a V-disjoint set. Therefore, it follows from Zorn's Lemma that there exists a maximal element A in $\mathscr{F}$ so that $\{a \oplus V : a \in A\}$ is a disjoint family of non-empty open sets in G. By Lemma 4.1, the family of open sets $\{a \oplus V : a \in A\}$ is discrete in G. Furthermore, G is pseudocompact, and we have that A is finite. Finally, we show that $A \oplus U = G$ as follows.

Take an arbitrary $x \in G$. If $x \notin A$, then it follows from the maximality of A that there exists an element $a \in A$ such that $(x \oplus V) \cap (a \oplus V) \neq \emptyset$. Then, there exist $v_1, v_2 \in V$ such that $x \oplus v_1 = a \oplus v_2$. By the right cancellation law, we have

$$\begin{aligned} x &= (x \oplus v_1) \oplus \mathrm{gyr}[x, v_1](\ominus v_1) \\ &= (a \oplus v_2) \oplus \mathrm{gyr}[x, v_1](\ominus v_1) \\ &\in (a \oplus v_2) \oplus \mathrm{gyr}[x, v_1](V) \\ &= (a \oplus v_2) \oplus V \\ &= a \oplus (v_2 \oplus \mathrm{gyr}[v_2, a](V)) \\ &= a \oplus (v_2 \oplus V) \\ &\subset a \oplus (V \oplus V) \\ &\subset a \oplus U. \end{aligned}$$

Therefore, $A \oplus U = G$. □

Theorem 4.4. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup with a symmetric open neighborhood base $\mathscr{U}$ at 0, and H is an open L-subgyrogroup of G. If H has a suitable set, then G has a suitable set. If H has a closed suitable set, then G has a closed suitable set.*

Proof. Let S be a suitable set for H. Since H is a L-subgyrogroup, two distinct cosets of H are disjoint. Then let A select one point from each coset of H in G such that $0 \notin A$ and $|A \cap (g \oplus H)| = 1$ for each $x \in G$. We claim that $S \cup A$ is suitable for G.

Indeed, $S \cup \{0\}$ and H are all closed in G, thus $S \cup A$ is discrete in $G \setminus \{0\}$, then there is at most an accumulation point 0 since $S \cup A \cup \{0\}$ is closed in G. Now it suffices to prove that $\langle S \cup A \rangle$ is dense in G. Since $\langle S \rangle$ is dense in H, the subgyrogroup $\langle S \cup A \rangle$ is dense in G. If S is closed in H, then $S \cup A$ is closed in G. □

However, the following question is open.

Question 4.5. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup, and that H is an open subgyrogroup of G with a suitable set. Does H have a suitable set?*

The following lemma gives a partial answer to Question 4.5.

Lemma 4.6. *Suppose that $(G, \tau, \oplus)$ is a separable strongly topological gyrogroup with a symmetric open neighborhood base $\mathscr{U}$ at 0, and H is an open subgyrogroup of G. If H has a suitable set, then G has a suitable set. If H has a closed suitable set, then G has a closed suitable set.*

Proof. Let S be a suitable set for H. Since G is separable, there exists a countable subset $A = \{g_n : n \in \omega\}$ of G such that $g_0 = 0$ and $\overline{A} = G$, then $A \oplus H = G$ since H is open in G. Then, by induction on n, we can choose a subset B of A satisfies the following conditions:

(i) B is closed discrete;

(ii) $\overline{\langle B \cup S \rangle} = G$;

(iii) $B \cap (g \oplus H) = \{g\}$.

Indeed, take $g_0 = \{0\}$. If $H = G$, then let $B = \{0\}$; otherwise, $G \setminus H \neq \emptyset$, since $G \setminus H$ is open, there exists a minimum $n_1 \in \mathbb{N}$ such that $g_{n_1} \in (g_{n_1} \oplus H) \setminus H$ and $g_i \in H$ for any $i < n_1$. Assume the points defined $g_0, g_{n_1}, \ldots, g_{n_k}$ such that $g_{n_i} \in (g_{n_i} \oplus H) \setminus \bigcup_{j<i}(g_{n_j} \oplus H)$ for each $i \leq k$ and $g_j \in \bigcup_{i=0}^{k-1}(g_{n_i} \oplus H)$ for any $n_m \leq j < n_{m+1}$ and $m \leq k-1$. If $\bigcup_{i=0}^{k}(g_{n_i} \oplus H) = G$, let $B = \{g_{n_i} : i \leq k\}$; otherwise, the set $G \setminus \bigcup_{i\leq k}(g_{n_i} \oplus H)$ is a nonempty open subset of G, then there exists a minimum $n_{k+1} \in \mathbb{N}$ such that $g_{n_{k+1}} \in (g_{n_{k+1}} \oplus H) \setminus \bigcup_{i\leq k}(g_{n_i} \oplus H)$ and $g_j \in \bigcup_{i\leq k}(g_{n_i} \oplus H)$ for each $j \leq n_{k+1}$. If there exists $N \in \mathbb{N}$ such that $\bigcup_{i\leq N}(g_{n_i} \oplus H) = G$, then $B = \{g_{n_i} : i \leq N\}$ is a finite set; otherwise, put $B = \{g_{n_i} : i \in \omega\}$. By our construction of B, it is easy to see that B satisfies the conditions (i)-(iii).

By (ii), $\langle S \cup B \rangle$ is dense in G. Moreover, $S \cup \{0\}$ and H are all closed in G, thus $(S \cup A) \setminus \{0\}$ is discrete in $G \setminus \{0\}$, then there is at most an accumulation point 0 since $S \cup A \cup \{0\}$ is closed in G. If S is closed in H, then $S \cup A$ is closed in G. □

Lemma 4.7. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup with a symmetric open neighborhood base $\mathscr{U}$ at 0, B is a left precompact subset of G and S is dense in B. Then, for every neighborhood U of 0 in G, there is a finite set $K \subset S$ such that $B \subset K \oplus U$.*

Proof. We assume that U is an arbitrary neighborhood of 0 in G and $V \in \mathscr{U}$ such that $V \oplus V \subset U$. Since B is left precompact in G, there exists a finite set F in G such that $B \subset F \oplus V$. Take an arbitrary $x \in F$ such that $B \cap (x \oplus V) \neq \emptyset$. Then $S \cap (x \oplus V) \neq \emptyset$ and we pick a point $y_x \in S \cap (x \oplus V)$. Then the finite set

$$K_1 = \{y_x : x \in F \text{ and } B \cap (x \oplus V) \neq \emptyset\}$$

is contained in S. We claim $B \subset K_1 \oplus U$.

Indeed, if $b \in B$, then there exists an element $x \in F$ such that $b \in x \oplus V$, so $b \in B \cap (x \oplus V) \neq \emptyset$. Therefore, $y_x \in x \oplus V$. We can find $v_1 \in V$ such that $y_x = x \oplus v_1$. Then

$$\begin{aligned} x &= (x \oplus v_1) \oplus \mathrm{gyr}[x, v_1](\ominus v_1) \\ &= y_x \oplus \mathrm{gyr}[x, v_1](\ominus v_1) \\ &\in y_x \oplus \mathrm{gyr}[x, v_1](V) \\ &= y_x \oplus V. \end{aligned}$$

Thus,

$$\begin{aligned} b &\in x \oplus V \\ &\subset (y_x \oplus V) \oplus V \\ &= y_x \oplus (V \oplus \mathrm{gyr}[V, y_x](V)) \\ &= y_x \oplus (V \oplus V) \\ &\subset y_x \oplus U \\ &\subset K_1 \oplus U. \end{aligned}$$

□

Lemma 4.8. *Every subgyrogroup H of a left precompact strongly topological gyrogroup G is left precompact.*

Proof. Take an arbitrary open neighborhood U of 0 in H, then there is an open neighborhood V of 0 in G such that $V \cap H = U$. Since H is a left precompact subset of G, by Lemma 4.7, we can find a finite set $F \subset H$ such that $H \subset F \oplus V$. Therefore, for every $h \in H$, there exist $f \in F$ and $v \in V$ such that $h = f \oplus v$. Thus, $v = (\ominus f) \oplus h \in H \oplus H \subset H$. Then, $v \in V \cap H = U$. It follows that $H \subset F \oplus U$, that is, $H = F \oplus U$. □

By Theorem 4.3 and Lemma 4.8, we have the following corollary.

Corollary 4.9. *Every subgyrogroup H of a pseudocompact strongly topological gyrogroup G is left precompact.*

Lemma 4.10. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup with a symmetric open neighborhood base $\mathscr{U}$ at 0, and suppose that G is non-pseudocompact left precompact with a countable dense subgyrogroup P. Then there exists a subset $L \subset P$ such that L is closed discrete in G and $\langle L \rangle = P$. In particular, L is suitable for G.*

Proof. Since G is not pseudocompact, we can fix a sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of G such that $U_n \in \mathscr{U}$, $\overline{U_{n+1}} \subset U_n$ for each $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} = \emptyset$. Let $\{x_n : n \in \omega\}$ be an enumeration of elements of P.

We construct an increasing sequence $\{L_k : k \in \omega\}$ of finite subsets of P by induction which satisfies the following conditions:

(1) $x_k \in \langle L_k \rangle$;

(2) $L_{k+1} \setminus L_k \subset U_k$;

(3) $G = \langle L_k \rangle \oplus U_k$.

Since the subgyrogroup P is dense in G, it follows from [2, Lemma 9] that $G = \overline{P} \subset P \oplus U_0$. So $G = P \oplus U_0$. Since G is left precompact, it follows from Lemma 4.7 that we can find a finite subset K_0 of P such that $K_0 \oplus U_0 = G$. Therefore, for any $x_0 \in G$, there exist $a_0 \in K_0, u_0 \in U_0$ such that $x_0 = a_0 \oplus u_0$. Then $u_0 = (\ominus a_0) \oplus x_0 \in P$ and let $L_0 = K_0 \cup \{u_0\}$.

We assume that for some $n \in \omega$ we have defined an increasing sequence $L_0, \ldots, L_n$ of finite subsets of P which satisfies (1) and (3) for each $k \le n$ and (2) for every $k < n$. Since P is dense in G, it is clear that $\langle U_n \cap P \rangle$ is dense in the gyrogroup $G_n = \langle U_n \rangle$. Thus, $\langle U_n \cap P \rangle \oplus U_{n+1} = G_n$. It follows from Lemma 4.8 that G_n is left precompact, hence we can find a finite subset F_{n+1} of $\langle U_n \cap P \rangle$ such that $F_{n+1} \oplus U_{n+1} = G_n$. Clearly, we can find a finite subset K_{n+1} of $U_n \cap P$ with $F_{n+1} \subset \langle K_{n+1} \rangle$, so $\langle K_{n+1} \rangle \oplus U_{n+1} = G_n$. Let $L'_{n+1} = L_n \cup K_{n+1}$. By (3), we have

$$\begin{aligned} G &= \langle L_n \rangle \oplus U_n \\ &\subset \langle L'_{n+1} \rangle \oplus G_n \\ &= \langle L'_{n+1} \rangle \oplus (\langle K_{n+1} \rangle \oplus U_{n+1}) \\ &= (\langle L'_{n+1} \rangle \oplus \langle K_{n+1} \rangle) \oplus \text{gyr}[\langle L'_{n+1} \rangle, \langle K_{n+1} \rangle](U_{n+1}) \end{aligned}$$

$$\begin{aligned} &= (\langle L'_{n+1}\rangle \oplus \langle K_{n+1}\rangle) \oplus U_{n+1} \\ &= \langle L'_{n+1}\rangle \oplus U_{n+1}. \end{aligned}$$

Therefore, there exist $a_{n+1} \in \langle L'_{n+1}\rangle, u_{n+1} \in U_{n+1}$ such that $x_{n+1} = a_{n+1} \oplus u_{n+1}$. Since $a_{n+1} \in \langle L'_{n+1}\rangle \subset P$ and $x_{n+1} \in P$, it follows that $u_{n+1} = (\ominus a_{n+1}) \oplus x_{n+1} \in P$. Then let $L_{n+1} = L'_{n+1} \cup \{u_{n+1}\}$. It is clear that L_{n+1} is a finite subset of P and $L_n \subset L_{n+1}$. At the same time, $\langle L_{n+1}\rangle \oplus U_{n+1} = G$. Moreover, $L_{n+1} \setminus L_n \subset K_{n+1} \cup \{u_{n+1}\} \subset U_n$. Therefore, we complete the construction.

Finally, set $L = \bigcup\{L_n : n \in \omega\}$. It follows from (2) that $L \setminus \overline{U_k} \subset L_k$ is a finite set for each $k \in \omega$. Then, $\bigcap\{\overline{U_n} : n \in \omega\}$ implies that L is a closed discrete subset of G. Moreover, (1) guarantees that $\langle L\rangle = P$. □

Now we can prove one of main results in this section.

Theorem 4.11. *Suppose that $(G, \tau, \oplus)$ is a non-pseudocompact strongly topological gyrogroup with a symmetric open neighborhood base $\mathscr{U}$ at 0. If G is separable, then $G \in \mathcal{S}_c$.*

Proof. It follows from [7] that G is Tychonoff. We divide the proof into two cases.

Case 1: G is not left precompact.

Then G has a neighborhood U of 0 such that $F \oplus U \neq G$ for any finite $F \subset G$. By Lemma 4.1, we need to take $V, W \in \mathscr{U}$ such that $V \oplus V \subset W$ and $W \oplus W \subset U$, then there exists a subset $A = \{a_n : n \in \omega\} \subset G$ with $a_i \neq a_j$ if $i \neq j$ and the family $\gamma = \{a_n \oplus V : n \in \omega\}$ is discrete in G. Let B be a countable dense subset of G, and set $B_V = B \cap V = \{b_n : n \in \omega\}$. We prove that $S = A \cup (\bigcup_{n\in\mathbb{N}} a_n \oplus b_n)$ is a suitable set of open subgyrogroup $G_1 = \langle V \cup A\rangle$.

Since $\langle B_V\rangle$ is dense in $\langle V\rangle$ and $B_V \subset \langle S\rangle$, we have that $\langle S\rangle$ is dense in G_1. For every $g \in G$, there exists a neighborhood O of 0 in G such that $(g \oplus O) \cap S \subset \{a_n, a_n \oplus b_n\}$. Therefore, S is closed and discrete and hence it is a suitable set of G_1. Then since G_1 is an open subgyrogroup of G, it follows from Lemma 4.6 that G has a closed suitable set.

Case 2: G is left precompact.

Since G is non-pseudocompact, we can choose a discrete family $\gamma = \{U_n : n \in \omega\}$ of non-empty open subsets of G. Let $B = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of G.

Since G is precompact, for every $n \in \omega$, there exists a finite subset $A_n = \{a(n,i) : 1 \leq i \leq m_n\}$ of G such that $A_n \oplus U_n = G$. Fix an $n \in \omega$ and define $H_n^i = \{d_n\} \cap (a(n,i) \oplus U_n)$ for each $i \leq m_n$. Then $H_n = \bigcup\{H_n^i : 1 \leq i \leq m_n\}$.

The set $T_n = \bigcup\{(\ominus a(n,i)) \oplus H_n^i : 1 \leq i \leq m_n\}$ is closed and discrete in G and lies in U_n. Since the family γ is discrete, the set $T = \bigcup\{T_n : n \in \omega\}$ is closed and discrete in G. Let $A = \bigcup\{A_n : n \in \omega\}$. For every $n \in \omega$, choose a point $y_n \in U_n$ such that $d_n \in A_n \oplus y_n$ and denote by G_2 the closure of $P = \langle A \cup \{y_n : n \in \omega\}\rangle$ in G. The gyrogroup G_2 is closed and left precompact by Lemma 4.8. Moreover, for each $n \in \omega$, $G_2 \cap U_n \neq \emptyset$, so G_2 is not pseudocompact.

It follows from Lemma 4.10 that there is a closed discrete subset L of G_2 such that $\langle L\rangle = P$. We find that $T \cup L$ is closed and discrete in G and $P \subset \langle T \cup L\rangle \supset \langle T \cup A\rangle \supset B$. Hence $\langle T \cup L\rangle$ is dense in G. Therefore, $T \cup L$ is a closed suitable set for G. □

Lemma 4.12. *Let G be a compact metrizable strongly topological gyrogroup. Then G has a closed suitable set.*

Proof. Since G is compact metrizable, it is separable, hence there exists a countable dense subgroup P. Let $P = \{x_n : n \in \omega\}$ be a enumeration of P. Moreover, we can choose a decreasing sequence $\{U_n : n \in \omega\}$ of open neighborhoods of the identity 0 in G satisfying the following conditions:

(1) $U_{n+1} \oplus U_{n+1} \subset U_n$ for each $n \in \omega$;

(2) $\bigcap_{n\in\omega} U_n = \{0\}$.

By the same construction of Lemma 4.10, we can find an increasing sequence $\{L_k : k \in \omega\}$ of finite subsets of P by induction which satisfies the following conditions:

(1) $x_k \in \langle L_k\rangle$;

(2) $L_{k+1} \setminus L_k \subset U_k$;

(3) $G = \langle L_k\rangle \oplus U_k$.

By a similar proof of Lemma 4.10, we can find a closed discrete subset L for P. Then L is a closed suitable set for G since P is dense in G. □

A space X is *paracompact*, if each its open cover has a locally finite open refinement. A space X is *submetrizable*, if there exists a continuous injective map of X to a metrizable space.

Corollary 4.13. *Suppose that $(G, \tau, \oplus)$ is a separable left precompact Hausdorff strongly topological gyrogroup of countable pseudocharacter with a symmetric open neighborhood base $\mathscr{U}$ at 0. If P is a countable dense subgyrogroup of G, then there exists a discrete subset L of P such that L is closed in $G \setminus \{0\}$ and $P = \langle L\rangle$. So L is a suitable set for both P and G.*

Proof. If G is not pseudocompact, then it follows from Lemma 4.10 that the conclusion holds. Assume that G is pseudocompact, then from [6, 7] that each strongly topological gyrogroup of countable pseudocharacter is paracompact and submetrizable, hence it is compact and metrizable, thus G has a closed suitable set by Lemma 4.12. □

A space X is a *σ-space* if it has a σ-locally finite network.

Corollary 4.14. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup. If G is a separable σ-space then G has a suitable set.*

Proof. Since each σ-space has a countable pseudocharacter, G has countable pseudocharacter. If G is not pseudocompact, the conclusion holds from Theorem 4.11. From [6, 7], each strongly topological gyrogroup of countable pseudocharacter is paracompact and submetrizable, hence it is compact and metrizable, thus separable precompact, so we can apply Lemma 4.13 to conclude that G has a suitable set. □

Corollary 4.15. *Suppose that $(G, \tau, \oplus)$ is a strongly topological gyrogroup. If G is a separable metrizable space, then G has a suitable set.*

We now close our paper with the following three questions.

Question 4.16. *Suppose that* $(G, \tau, \oplus)$ *is a metrizable strongly topological gyrogroup, does* G *have a suitable set?*

Question 4.17. *Does each locally compact (strongly) topological gyrogroup have a suitable set? What if the space is compact?*

Question 4.18. *Suppose that* $(G, \tau, \oplus)$ *is a Hausdorff strongly topological gyrogroup with a symmetric open neighborhood base* $\mathscr{U}$ *at* 0 *which satisfies* $d(G) < b(G)$*, does* G *have a closed suitable set?*

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Conflict of interest The authors declare no conflict of interest.

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