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Generalized Euler-Riesz difference sequence space with a paranormed fractional ordered μ

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Abstract. In this study, we introduce the sequence space $\ell^{\mu}(p, \Delta^m)$ with a fractional order μ. Furthermore, we give some topological properties of this space. Also we introduce $\alpha-, \beta-$, and $\gamma-$duals of $\ell^{\mu}(p, \Delta^m)$ and its some matrix mappings.

§1 Introduction

Fractional analysis stands out as an important research area in mathematics day by day. Fractional sequence spaces have been an important area of interest as a generalisation of classical sequence spaces and an application of the theory of fractional analysis. Fractional analysis and sequence spaces have developed as two independent fields of mathematics and have been unified over time, leading to the concept of fractional sequence spaces. The foundations of sequence spaces were laid with the emergence of Banach and Hilbert spaces. In particular, studies on ℓp spaces and bounded, convergent sequences have formed the basic building blocks of the theory of functional analysis. The $\alpha-, \beta-$, and $\gamma-$duals have been considered to understand the basic structure of sequence spaces and to study this structure from a broader perspective. In particular, these duals have been of critical importance for studying the analytic properties of sequence spaces and understanding the behaviour of operators defined on these spaces. Matrix transformations have been a powerful tool for studying the transitions of fractional sequence spaces between different spaces and the structural effects of these transitions. These transformations have also contributed to the further development of operator theory and numerical analysis methods. Fractional analysis, on the other hand, has been a broad field of study, which has its foundations in the contributions of Leibniz and Riemann and is now recognised as a branch of mathematics. The first studies on fractional sequence spaces were made by extending classical sequence spaces with fractional derivative and integral operators. These studies are particularly noteworthy in the analysis of different behaviours of physical systems, data com-

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pression and modelling of physical systems. In this context, the following studies of fractional sequence spaces have shed light on the applications of fractional analysis in different fields.

Hardy [1] talked about the difference sequence spaces and it was followed by Kızmaz [2]: $X(\Delta) = \{x = (x_k) : \Delta x \in X\}$, where $X = \ell_\infty, c$ or c_0. After, Et and Çolak [3] generalized these spaces as $X(\Delta^m) = \{x = (x_k) : \Delta^m x \in X\}$, where $m \in N$, $\Delta^0 x_k = x_k$, $\Delta^m x_k = \Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}$ such that

$$\Delta^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+\nu}.$$

Baliarsingh and Dutta [4,5] also introduced the fractional difference operator for any proper fraction $\mu > 0$,

$$\Delta^\mu x_k = \sum_{v=0}^{\infty} (-1)^v \frac{\Gamma(\mu+1)}{\nu!(\mu-v+1)} x_{k-v},$$

and defined new sequence spaces, called difference sequence spaces of a positive fractional order μ as

$$\lambda(\Gamma, \Delta^\mu, u) = \left\{ x = (x_k) : \left(\sum_{v=0}^{k} u_\nu \Delta^\mu x_\nu \right) \in \lambda \right\},$$

where λ is any sequence space. In this definition, the gamma function of a real number μ such that $\mu \notin \{0, -1, -2, -3, ...\}$ is given by the integral,

$$\Gamma(\mu) = \int_0^\infty e^{-t} t^{\mu-1} dt. \tag{1}$$

Yaying [6] introduced paranomed difference sequence spaces of fractional order μ, $X^{\,t}(p, \Delta^\mu)$ for $X = r_0, r_c, r_\infty$.

The matrix domain of an infinite matrix A in a sequence space $\lambda \subset \omega$ is a sequence and $\lambda_A = \{x \in \omega : Ax \in \lambda\}$.

Altay and Başar [7,8] defined the sequence space

$$r^q(p) = \left\{ x \in \omega : \sum_{n\in N} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \right|^{p_n} < \infty \right\}.$$

After, Altay et al. [9] and Altay and Başar [10] defined the sequence space $e^r(p)$ by

$$e^r(p) = \left\{ x \in \omega : \sum_{n\in N} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \right|^{p} < \infty \right\},$$

where $0 < p < \infty$.

Fractional ordered Euler-Riesz difference sequence spaces c_0^μ, c^μ, and ℓ_∞^μ were defined by Jena and Dutta in [11]. They introduced the matrix $\tilde{B}(\Delta^\mu) = \tilde{B}^\mu = (b_{nk}^\mu)$ defined by

$$\tilde{B}^\mu = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots \\ \frac{2q_1 - \mu q_2}{2^2 Q_2} & \frac{q_2}{2^2 Q_2} & 0 & \cdots \\ \frac{3q_1 - 3\mu q_2 + \frac{\mu(\mu-1)q_3}{2!}}{2^3 Q_3} & \frac{3q_2 - \mu q_3}{2^3 Q_3} & \frac{q_3}{2^3 Q_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $b_{nk}^{\mu} = 0$ if $k > n$ and

$$b_{nk}^{\mu} = \sum_{j=k}^{n}\left[\sum_{i=j}^{n}\binom{n}{i}(-1)^n\frac{\Gamma(\mu+1)}{n!\Gamma(\mu-n+1)}\frac{q_j}{2^nQ_i}\right],$$

if $0 \le k \le n$ and hence $\tilde{B}^{\mu}$ transform of $x = (x_k) \in \omega$ is defined as the sequence $y = (y_k) \in \omega$ as follows:

$$y_n = (\tilde{B}^{\mu}x)_n = \sum_{l=0}^{n}\sum_{j=k}^{n}\left[\sum_{i=j}^{n}\binom{n}{i}(-1)^n\frac{\Gamma(\mu+1)}{n!\Gamma(\mu-n+1)}\frac{q_jx_l}{2^nQ_i}\right].$$

Bektaş and Bayram [12] introduced $\ell^{\mu}(p)$ by

$$\ell^{\mu}(p) = \left\{x \in \omega : \sum_{n=1}^{\infty}\left|(\tilde{B}^{\mu}x)_n\right|^{p_n} < \infty\right\}.$$

Tripathy made significant contributions to the fields of convergence of series of fuzzy real numbers, generalized difference sequence spaces, and Kthe–Toeplitz duals [14–16].

For any complex numbers a and x, and any $B > 0$, the following inequality holds

$$|ax| \le B(|a|^{p'}B^{-p'} + |x|^p), \tag{2}$$

where $p > 1$ and $p^{-1} + (p\prime)^{-1} = 1$ [16].

Throughout this paper, we will take C_{μ}^{n} instead of $(-1)^n\frac{\Gamma(\mu+1)}{n!\Gamma(\mu-n+1)}$.

In this study, the topological structures of fractional sequence spaces, their relations with dual spaces and coverage relations related to matrix transformations will be examined and the relations between these concepts will be investigated. The main aim of the study is to contribute to the existing literature in this field by considering fractional sequence spaces from a broad perspective.

§2 Main results

The topological structures of fractional sequence spaces are critical to understand the internal structure of these spaces and determine their applications in functional analysis. In this section, the topological and structural properties of fractional sequence spaces resulting from the combination of fractional analysis and sequence spaces will be given.

Definition 2.1 Let $m \in N$ and $\tilde{B}^{\mu}(\Delta^m)$ transform of $x = (x_k) \in \omega$ be defined the sequence $y = (y_k) \in \omega$ as follows:

$$y_n = (\tilde{B}^{\mu}\Delta^m x)_n = \sum_{l=0}^{n}\sum_{j=k}^{n}\left[\sum_{i=j}^{n}\binom{n}{i}C_{\mu}^{j-k}\frac{q_j\Delta^m x_l}{2^nQ_i}\right],$$

where $Q_n = \sum\limits_{k=0}^{n} q_k$ and (q_n) is a sequence of positive numbers. Then we define

$$\ell^{\mu}(p, \Delta^m) = \left\{x \in \omega : \sum_{n=1}^{\infty}\left|(\tilde{B}^{\mu}\Delta^m x)_n\right|^{p_n} < \infty\right\}.$$

Theorem 2.2 Suppose that $p = (p_k) \in \ell_{\infty}$ such that $0 < p_k \le H < \infty$ and $M =$

$max\{1, H\}$. Then $\ell^\mu(p, \Delta^m)$ is a complete linear metric space paranormed by

$$g_{\tilde{B}^\mu(\Delta^m)}(x) = \left(\sum_{n=0}^{\infty} \left|(\tilde{B}^\mu \Delta^m x)_n\right|^{p_n}\right)^{1/M}.$$

Proof. Obviously $g_{\tilde{B}^\mu(\Delta^m)} = (0, 0, \ldots 0)$ and $g_{\tilde{B}^\mu(\Delta^m)}(-x) = g_{\tilde{B}^\mu(\Delta^m)}(x)$ for all $x \in \ell^\mu(p, \Delta^m)$. Now, let $x, y \in \ell^\mu(p, \Delta^m)$ and $\alpha_1, \alpha_2 \in R$. Since Δ^m is linear, we obtain that

$$\begin{aligned} g_{\tilde{B}^\mu(\Delta^m)}(\alpha_1 x + \alpha_2 y) &= \left(\sum_{n=0}^{\infty} \left|(\tilde{B}^\mu \Delta^m(\alpha_1 x + \alpha_2 y))_n\right|^{p_n}\right)^{1/M} \\ &= \left(\sum_{n=0}^{\infty} \left|(\alpha_1 \tilde{B}^\mu \Delta^m x + \alpha_2 \tilde{B}^\mu \Delta^m y))_n\right|^{p_n}\right)^{1/M} \\ &\leq \left(H^* \sum_{n=0}^{\infty} \left|(\tilde{B}^\mu \Delta^m x)_n\right|^{p_n}\right)^{1/M} + \left(H^* \sum_{n=0}^{\infty} \left|(\tilde{B}^\mu \Delta^m y)_n\right|^{p_n}\right)^{1/M} \\ &\leq \alpha_1 g_{\tilde{B}^\mu(\Delta^m)}(x) + \alpha_2 g_{\tilde{B}^\mu(\Delta^m)}(y), \end{aligned}$$

where $H^* = max\{1, |\alpha_1|^M, |\alpha_2|^M\}$. This implies that $g_{\tilde{B}^\mu(\Delta^m)}$ is subadditive. Also, the linearity of $\ell^\mu(p, \Delta^m)$ with respect to the coordinate-wise addition and scalar multiplication can be easily seen. Suppose that $\{x^n\}$ is a sequence in $\ell^\mu(p, \Delta^m)$ such that $g_{\tilde{B}^\mu(\Delta^m)}(x^n - x) \to 0$ and also (a_n) is any sequence of scalars such that $a_n \to a$ as $n \to \infty$. $\{g_{\tilde{B}^\mu(\Delta^m)}(x^n)\}$ is bounded because of the subadditivity of $g_{\tilde{B}^\mu(\Delta^m)}$. Then we have

$$\begin{aligned} g_{\tilde{B}^\mu(\Delta^m)}(\alpha_s x^s - \alpha x) &= \left(\sum_{n=0}^{\infty} \left|(\tilde{B}^\mu(\alpha_s \Delta^m x^s - \alpha \Delta^m x)_n\right|^{p_n}\right)^{1/M} \\ &\leq \left(\sum_{n=0}^{\infty} \left|\sum_{l=0}^{n} \sum_{j=k}^{n} \left[\sum_{i=j}^{n} \binom{n}{i} C_\mu^{j-k} \frac{q_j(\alpha_s \Delta^m x_l^s - \alpha \Delta^m x_l)}{2^n Q_i}\right]\right|^{p_k}\right)^{1/M} \\ &\leq |\alpha_s - \alpha| g_{\tilde{B}^\mu(\Delta^m)}(x^s) + |\alpha| g_{\tilde{B}^\mu(\Delta^m)}(x^s - x). \end{aligned}$$

Accordingly, $g_{\tilde{B}^\mu(\Delta^m)}(\alpha_s x^s - \alpha x) \to 0$ as $s \to \infty$, shows that scalar multiplication is continuous. Consequently, $g_{\tilde{B}^\mu(\Delta^m)}$ is a paranorm on $\ell^\mu(p, \Delta^m)$.

Now, we have to show that $\ell^\mu(p, \Delta^m)$ is complete. Let $x^s = \{x_n^s\}$ be any Cauchy sequence in $\ell^\mu(p, \Delta^m)$. So for each $\varepsilon > 0$, there exists $N(\varepsilon) \in N^+$ such that

$$\begin{aligned} \left|(\tilde{B}^\mu(\Delta^m x^s - \Delta^m x^t))_n\right| &\leq g_{\tilde{B}^\mu(\Delta^m)}(x^s - x^t) \\ &= \left(\sum_{n=0}^{\infty} \left|(\tilde{B}^\mu(\Delta^m x^s - \Delta^m x^t))_n\right|^{p_n}\right)^{1/M} < \varepsilon, \end{aligned}$$

for all $s, t \geq N(\varepsilon)$.

Hence, for each fixed $n \in N$, $\{(\tilde{B}^\mu \Delta^m x^0)_n, (\tilde{B}^\mu \Delta^m x^1)_n, \ldots\}$ forms a Cauchy sequence of scalars and the completeness of the scalar field implies its converges. Assume that for each $n \in N$, $(\tilde{B}^\mu \Delta^m x^s)_n \to (\tilde{B}^\mu \Delta^m x)_n$ as $s \to \infty$ and let us define the sequence

$$x = \{(\tilde{B}^\mu \Delta^m x)_0, (\tilde{B}^\mu \Delta^m x)_1, \ldots\},$$

using these limits. For each $r \in N$ and $s, t \geq N(\varepsilon)$ we obtain

$$\sum_{n=0}^{r} \left|(\tilde{B}^{\mu}(\Delta^m x^s - \Delta^m x^t))_n\right|^{p_n} \leq g_{\tilde{B}^{\mu}(\Delta^m)}(x^s - x^t) < \varepsilon^M.$$

For any $t \geq N(\varepsilon)$ and as $t \to \infty$, we obtain from the last inequality that

$$\sum_{n=0}^{r} \left|(\tilde{B}^{\mu}(\Delta^m x^s - \Delta^m x))_n\right|^{p_n} < \varepsilon^M,$$

for each $r \in N$. Then

$$\sum_{n=0}^{r} \left|(\tilde{B}^{\mu}(\Delta^m x^s - \Delta^m x))_n\right|^{p_n} \leq g_{\tilde{B}^{\mu}(\Delta^m)}(x^s - x) < \varepsilon, \tag{3}$$

holds. Taking $\varepsilon = 1$, $s \geq N(1)$ and using Minkowskii's inequality, we have for each $r \in N$ that

$$\begin{aligned}\left(\sum_{n=0}^{r} \left|(\tilde{B}^{\mu}(\Delta^m x))_n\right|^{p_n}\right)^{1/M} &\leq g_{\tilde{B}^{\mu}(\Delta^m)}(x^s - x) + g_{\tilde{B}^{\mu}(\Delta^m)}(x^s) \qquad (4)\\ &< 1 + g_{\tilde{B}^{\mu}(\Delta^m)}(x^s),\end{aligned}$$

which implies that $x \in \ell^{\mu}(p, \Delta^m)$. Also, from (4), $g_{\tilde{B}^{\mu}(\Delta^m)}(x^s - x) < \varepsilon$ holds for all $s \geq N(\varepsilon)$ and see that $x^s \to x \in \ell^{\mu}(p, \Delta^m)$ as $s \to \infty$. This proves the completeness of $\ell^{\mu}(p, \Delta^m)$.

Theorem 2.3 The Riesz sequence space $\ell^{\mu}(p, \Delta^m)$ is linearly isomorphic to $\ell(p)$, where $0 < p_k \leq H < \infty$.

Proof. Let us define the transform as follows

$$\varphi : \ell^{\mu}(p, \Delta^m) \to \ell(p), \varphi(x) = \tilde{B}^{\mu}(\Delta^m x).$$

Clearly, the linearity of $\tilde{B}^{\mu}(\Delta^m)$ implies the linearity of φ and $x = 0$ whenever $\varphi(x) = 0$, that is, φ is injective. From (2.2) in [11], for the sequence $y = (y_n) \in \ell(p)$, the sequence $x = (x_n)$ is defined by

$$x_n = n^m \sum_{l=0}^{n} \sum_{j=k}^{n} C_{-\mu}^{j-k} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i y_i. \tag{5}$$

Then

$$\begin{aligned} g_{\tilde{B}^{\mu}(\Delta^m)}(x) &= \left(\sum_{n=1}^{\infty} \left|\sum_{l=0}^{n} \sum_{j=k}^{n} \left[\sum_{i=j}^{n} \binom{n}{i} C_{\mu}^{j-k} \frac{q_j \Delta^m x_l}{2^n Q_i}\right]\right|^{p_n}\right)^{1/M} \\ &= \left(\sum_{n=1}^{\infty} \left|\sum_{j=0}^{n} \delta_{n\ j}\ y_j\right|^{p_n}\right)^{1/M} \\ &= \left(\sum_{n=1}^{\infty} |y_n|^{p_n}\right)^{1/M} < \infty, \end{aligned}$$

holds where $\delta_{n\ j}$ is Kronecker delta. This implies that $x \in \ell^{\mu}(p, \Delta^m)$. Consequently, φ is a linear bijection and preserves the paranorm. Therefore $\ell^{\mu}(p, \Delta^m)$ is linearly isomorphic to $\ell(p)$.

Theorem 2.4 The Riesz sequence space $\ell^{\mu}(p, \Delta^m)$ is linearly isomorphic to $\ell(p, \Delta^m)$ where $0 < p_k \leq H < \infty$.

Proof is easy.

§3 $\alpha-, \beta-$ and $\gamma-$duals of $\ell^{\mu}(p, \Delta^m)$

For the sequence X and Y, the multiplier space is defined as follows:

$$S(X, Y) = \{z = (z_k) \in \omega : xz \in Y \text{ forall } x \in X\}.$$

The $\alpha-$, $\beta-$, and $\gamma-$duals of a sequence space X are defined as $X^{\alpha} = S(X, \ell_1)$, $X^{\beta} = S(X, cs)$, $X^{\gamma} = S(X, bs)$, respectively.

Here we will show the space of all bounded, convergent and absolutely convergent series bs, cs and ℓ_1, respectively.

Theorem 3.1 Let us define the following sets in the case where $1 < p_k \leq H < \infty$ for every $k \in N$ and B an integer

$$M_1(p) = \bigcup_{B>1} \left\{ a = (a_k) : \sup_{K \in F} \sum_{k \in N} \left| \sum_{n \in K} \left[\tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) Q_k \right] B^{-1} \right|^{p_k'} < \infty \right\},$$

$$M_2(p) = \bigcup_{B>1} \{ a = (a_k) : \sup_{n \in N} \sum_{k \in N} \left| Q_k \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) B^{-1} \right|^{p_k'} < \infty,$$

$$\text{and } \left(\left(\frac{2^k}{q_k} Q_k k^m a_k \right)^{p_k} \right) \in c_0 \},$$

$$M_3(p) = \bigcup_{B>1} \{ a = (a_k) : \sup_{n \in N} \sum_{k \in N} \left| Q_k \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) B^{-1} \right|^{p_k'} < \infty$$

$$\text{and } \left(\left(Q_k \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) \right)^{p_k} \right) \in \ell_{\infty} \},$$

where $p_k' = \frac{p_k}{p_k - 1}$ and

$$\tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) = \sum_{j=k}^{n} C_{-\mu}^{n-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} k^m a_k.$$

Then $[\ell^{\mu}(p, \Delta^m)]^{\alpha} = M_1(p)$, $[\ell^{\mu}(p, \Delta^m)]^{\beta} = M_2(p)$ and $[\ell^{\mu}(p, \Delta^m)]^{\gamma} = M_3(p)$.

Proof. Let us take any $a = (a_k) \in \omega$ and consider $x = (x_k)$ as defined in (5) where

$$x_n = n^m \sum_{l=0}^{n} \sum_{j=k}^{n} C_{-\mu}^{n-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i y_i.$$

Then we obtain that for each $n \in N$

$$a_n x_n = \sum_{l=0}^{n} \sum_{j=k}^{n} C_{-\mu}^{n-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i n^m a_n y_l.$$

On the other hand, if we define the matrix $G = (g_{nk})$ as $g_{nk} = 0$ if $k > n$, $g_{nk} = \frac{2^n}{q_n} Q_n n^m a_n$ if $k = n$ and $g_{nk} = \sum_{j=k}^{n} C_{-\mu}^{n-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i n^m a_n$, if $0 \leq k < n$.

We have $a_n x_n = (Gy)_n$. Thus $ax = (a_k x_k) \in \ell_1$ whenever $x \in \ell^{\mu}(p, \Delta^m)$ if and only if $Gy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This means that $a \in [\ell^{\mu}(p, \Delta^m)]^{\alpha}$ if and only if $G \in (\ell(p), \ell_1)$. Consequently, from Lemma 3.1(i) in [17], $[\ell^{\mu}(p, \Delta^m)]^{\alpha} = M_1(p)$.

Consider the following equation for the $\beta-$, and $\gamma-$duals

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left[k^m \sum_{l=0}^{k} \sum_{j=s}^{k} C_{-\mu}^{k-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i y_l \right]$$

$$= \sum_{k=0}^{n} y_k Q_k y_i \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right). \tag{6}$$

If we define the matrix $H = (h_{nk})$ as $h_{nk} = 0$ if $k > n$, $h_{nk} = \frac{2^n}{q_n} Q_n n^m a_n$ if $k = n$ and $h_{nk} = Q_k \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right)$ if $0 \leq k < n$, we have $\sum\limits_{k=1}^{n} a_k x_k = (Hy)_n$ for each $n \in N$. Thus $ax = (a_k x_k) \in cs$ whenever $x \in \ell^{\mu}(p, \Delta^m)$ if and only if $Hy \in c$ whenever $y = (y_k) \in \ell(p)$. This yields that $a \in [\ell^{\mu}(p, \Delta^m)]^{\beta}$ if and only if $H \in (\ell(p), c)$. Consequently, from Lemma 3.2 (iii) in [11], we conclude there exists an integer $B > 1$ and

$$\sup_{n \in N} \sum_{k \in N} \left| Q_k \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) B^{-1} \right|^{p_k'} < \infty,$$

$$\left\{ \left(\frac{2^k}{q_k} Q_k k^m a_k \right)^{p_k} \right\} \in \ell_{\infty}.$$

This means that $[\ell^{\mu}(p, \Delta^m)]^{\beta} = M_2(p)$.

Similarly, $ax = (a_k\ x_k) \in bs$ whenever $x \in \ell^{\mu}(p, \Delta^m)$ if and only if $Hy \in \ell_{\infty}$ whenever $y = (y_k) \in \ell(p)$. This implies that $a \in [\ell^{\mu}(p, \Delta^m)]^{\gamma}$ if and only if $H \in (\ell(p), \ell_{\infty})$. Hence $[\ell^{\mu}(p, \Delta^m)]^{\gamma} = M_3(p)$.

Theorem 3.2 Let us define the following sets in the case where $0 < p_k \leq 1$ for every $k \in N$ and B an integer

$$M_4(p) = \left\{ a \in \omega : \sup_{K \in F} \sup_{k \in N} \left| \sum_{n \in K} \left[\tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) Q_k \right] \right|^{p_k} < \infty \right\},$$

$$M_5(p) = \left\{ a \in \omega : \sup_{k \in N} \left| \tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) Q_k \right|^{p_k} < \infty \text{ and } \sup_{k \in N} \left| \frac{2^k}{q_k} Q_k k^m a_k \right|^{p_k} < \infty \right\},$$

where

$$\tilde{B}^{\mu} \left(\frac{k^m a_k}{q_k} \right) = \sum_{j=k}^{n} C_{-\mu}^{n-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} k^m a_k.$$

Then $[\ell^{\mu}(p, \Delta^m)]^{\alpha} = M_4(p)$ and $[\ell^{\mu}(p, \Delta^m)]^{\beta} = [\ell^{\mu}(p, \Delta^m)]^{\gamma} = M_5(p)$.

Proof. Similar to the proof of the previous theorem, using the same notations and calculations, we obtain the proposition that $a \in [\ell^{\mu}(p, \Delta^m)]^{\alpha}$ if and only if $G \in (\ell(p), \ell_1)$. From (6) and Lemma 3.1 (ii) in [17], $a \in [\ell^{\mu}(p, \Delta^m)]^{\alpha}$ if and only if

$$\sup_{K \in F} \sup_{k \in N} \left| \sum_{n \in K} \left[\sum_{j=k}^{n} C_{-\mu}^{n-j} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i n^m a_n \right] \right|^{p_k} < \infty.$$

Consequently, we also observe the equality $[\ell^{\mu}(p, \Delta^m)]^{\alpha} = M_4(p)$.

For the $\beta-$, and $\gamma-$duals, again referring the last theorem, we find that $a \in [\ell^{\mu}(p, \Delta^m)]^{\beta}$ (respectively, $a \in [\ell^{\mu}(p, \Delta^m)]^{\gamma}$) if and only if $H \in (\ell(p), c)$ (respectively, $H \in (\ell(p), \ell_{\infty})$). From (6) and Lemma 3.1 (iii) in [13], we conclude that $a \in [\ell^{\mu}(p, \Delta^m)]^{\beta}$ (respectively, $a \in$

$[\ell^\mu(p,\Delta^m)]^\gamma)$ if and only if

$$\sup_{n,k\in N}\left|\sum_{j=k}^{n} C_{-\mu}^{n-j}\frac{2^k}{q_j}\sum_{i=j-1}^{j}\binom{i}{k}Q_i n^m a_n\right|^{p_k}<\infty,$$

$$\sup_{k\in N}\left|\frac{2^k}{q_k}Q_k k^m a_k\right|^{p_k}<\infty.$$

Therefore we have $[\ell^\mu(p,\Delta^m)]^\beta=[\ell^\mu(p,\Delta^m)]^\gamma=M_5(p)$.

§4 Matrix transformations

In this section we will give characterizations of certain matrix mappings defined from $\ell^\mu(p,\Delta^m)$ to X where $X=\ell_\infty(q), bs, c(q)$ or $c_0(q)$. We define the matrix class (X,Y) by stating that $A=(a_{nk})\in(X,Y)$ if and only if for every $x=(x_k)\in X$,

$$(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k,$$

converges in Y for each $n\in N$ and the resulting sequence called the A-transform of x satisfies

$$Ax=((Ax)_n)_{n\in N}=\left(\sum_{k=1}^{\infty}a_{nk}x_k\right)_{n\in N},$$

belongs to Y. Moreover, $A\in(X,Y)$ if and only if n^{th}-row of A, denoted $(a_{nk})_{k\in N}$ belongs to X^β for all $n\in N$ and $Ax\in Y$ for every $x=(x_k)\in X$.

Now, we assume that $q=(q_n)$ is a non-decreasing bounded sequence of strictly positive real numbers and $p'_k=\frac{p_k}{p_k-1}$ for each $k\in N$. For simplicity, we adopt the following notations

$$\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)=\sum_{j=k}^{n}C_{-\mu}^{n-j}\frac{2^k}{q_j}\sum_{i=j-1}^{j}\binom{i}{k}n^m a_{n\,j},$$

for all $n,k\in N$.

Suppose $A=(a_{nk})\in(\ell^\mu(p,\Delta^m),Y)$ where $Y=\ell_\infty(q), bs, c(q)$ or $c_0(q)$. Then $(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k$ converges for each $n\in N$ and $x\in\ell^\mu(p,\Delta^m)$. From (5), for the sequence $y\in\ell(p)$, we define the sequence $x\in\ell^\mu(p,\Delta^m)$ as follows:

$$x_n=n^m\sum_{l=0}^{n}\sum_{j=k}^{n}C_{-\mu}^{n-j}\frac{2^k}{q_j}\sum_{i=j-1}^{j}\binom{i}{k}Q_i y_l.$$

From (6), we obtain that for each $s,n\in N$

$$\begin{aligned}\sum_{k=1}^{s}a_{nk}x_k &= \sum_{k=1}^{s}a_{nk}\left[k^m\sum_{l=0}^{n}\sum_{j=k}^{n}C_{-\mu}^{n-j}\frac{2^k}{q_j}\sum_{i=j-1}^{j}\binom{i}{k}Q_i y_l\right]\\ &= \sum_{k=1}^{s}y_k Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right).\end{aligned}$$

As $s\to\infty$ we have

$$(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k=\sum_{k=1}^{\infty}y_k Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)=(Hy)_n.$$

Hence, we obtain

$$A \in ((\ell^\mu(p,\Delta^m),Y) \Leftrightarrow H \in (\ell(p),Y). \tag{7}$$

Theorem 4.1 i) If $1 < p_k \le H < \infty$, then $A \in (\ell^\mu(p,\Delta^m),\ell_\infty(q)) \Leftrightarrow$ There exists an integer $B > 0$ such that

$$\sup_{n\in N}\sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_k}{q_k}\right)B^{-1}\right|^{p_k'} < \infty, \tag{8}$$

$$\sup_{n\in N}\sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_k}{q_k}\right)B^{-1/q_n}\right|^{p_k'} < \infty. \tag{9}$$

ii) If $0 < p_k \le 1$, then $A \in (\ell^\mu(p,\Delta^m),\ell_\infty(q)) \Leftrightarrow$ There exists an integer $B > 0$ such that

$$\sup_{k\in N}\left|\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)Q_k\right|^{p_k} < \infty, \tag{10}$$

$$\sup_{n\in N}\sup_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)B^{-1/q_n}\right|^{p_k} < \infty. \tag{11}$$

Proof. i) Since Ax exists from Theorem 3.1, (9) holds. Also from (7) we obtain that $A \in (\ell^\mu(p,\Delta^m),\ell_\infty(q)) \Leftrightarrow H \in (\ell(p),\ell_\infty(q))$, i.e.,

$$\sup_{n\in N}\sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)B^{-1/q_n}\right|^{p_k'} < \infty.$$

Conversely, if the conditions (9) and (10) hold, then Ax exists and from (8) we obtain that

$$\begin{aligned} |(Ax)_n| &= \left|\sum_{k\in N} a_{nk}x_k\right| \le \sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)y_k\right| \\ &\le B^{1/q_n}\left(\sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)\right|^{p_k'}\left|B^{1/q_n}\right|^{-p_k'} + \sum_{k\in N}|y_k|^{p_k}\right). \end{aligned}$$

Thus we have $\sup_{n\in N}|(Ax)_n|^{p_n} < \infty$. This implies that $A \in (\ell^\mu(p,\Delta^m),\ell_\infty(q))$.

ii) Similar to i).

Theorem 4.2 If $0 < p_k \le H < \infty$, then $A \in (\ell^\mu(p,\Delta^m),c_0(q)) \Leftrightarrow$ (9), (11) hold and for all $B \in N$,

$$\lim_n\left(\sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)B^{\frac{1}{p_k}}\right|\right)^{q_n} = 0.$$

Proof is omitted.

Theorem 4.3 If $0 < p_k \le H < \infty$, then $A \in (\ell^\mu(p,\Delta^m),\ell_\infty(q)) \Leftrightarrow$ There exists an integer $B > 0$ and a sequence $(a_k) \subset R$ such that (9), (10), and (11) hold, along with the following statements

$$\sup_{n,k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right)\right|^{p_k} < \infty,$$

$$\sup_{n\in N}\sum_{k\in N}\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_k}{q_k}\right)B^{-1}\right|^{p_k'} < \infty,$$

$$\lim_n\left(\left|Q_k\tilde{B}^\mu\left(\frac{k^m a_{nk}}{q_k}\right) - \alpha_k\right|\right)^{q_n} = 0 \text{ for all } k \in N,$$

$$\sup_{n\in N}\sum_{k\in N}\left(\left|Q_k\tilde{B}^{\mu}\left(\frac{k^m a_{nk}}{q_k}\right)-\alpha_k\right|L^{1/q_n}B^{-1}\right)^{p'_k}<\infty \text{ for all } L\in R^+.$$

§5 Conclusion

This study offers a new perspective with studies on this subject in the literature by addressing the binary structures of the topological properties of fractional sequence spaces and the coverage relations of matrix transformations, which emerge by combining fractional analysis and sequence spaces. Dual spaces of fractional sequence spaces, that is, spaces in which continuous linear functionals of these spaces are constructed, play a central role in operator technology and functional analysis. Current studies on these topics expand and further consolidate their place in comprehensive analysis of fractional sequence propagations. A detailed analysis of topological properties and binary structures in particular is envisaged in the literature on fractional sequence spaces.

Declarations

Conflict of interest The authors declare no conflict of interest.

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