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Nonisospectral Botie-Pempinelli-Tu hierarchy and its integrable coupling

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Abstract. An efficient scheme is applied to generate a nonisospectral Botie-Pempinelli-Tu (BP-T) integrable hierarchy under the case where $\lambda_t = \sum_{j=0}^{n} k_j(t)\lambda^{-j}$. Based on an expanding higher-dimensional Lie algebra, we obtain a nonisospectral BPT integrable coupling hieararchy. It follws that some new nonisospectral nonlinear systems are obtained by reducing these two non-isospectral BPT hierarchies. Actually, these nonisospectral integrable models that we obtained can enrich the existing integrable models and possibly describe new nonlinear phenomena.

§1 Introduction

In the fields of mathematical physics and theoretical physics, integrable nonlinear systems have attracted wide attention due to the fact that they successfully describe and explain nonlinear phenomena in natural science [1]. Among them, the nonisospectral equations are meaningful and indispensable. A large number of nonisospectral deformations of classic integrable systems are considered [1,2]. Some solitary waves can be described by some nonisospectral equations in a certain type of nonuniform media [3,4]. In [5], the authors presented the nonisospectral linear representations which can help find some geometric properties of some important soliton systems. Additionally, there are many excellent research results of nonisospectral equations related to the inverse scattering transform [6,7]. Therefore, how to generate nonisospectral equations and their extended systems is a crucial research topic [8,9]. A method for generating integrable systems was proposed by Magri [10], which was called the Lax-pair method. Based on it, Tu proposed a method for generating integrable Hamiltonian hierarchies by making use of a trace identity in [11]. Some integrable systems and the corresponding Hamiltonian structures as well as other properties were obtained by using the Tu scheme, such as the work in [12-18]. And Zhang et al deduced an integrable couplings of the well-known BPT hierarchy by constructing a new loop algebra in [19]. There are many different methods for generating isospectral integrable equations. However, as far as we know, fewer works has been done on the nonisospectral integrable equations. In [20,21], Ma proposed a method of constructing its

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corresponding nonisospearal $\lambda_t = \lambda^n (n \geq 0)$ hierarchy of evolution equations by starting from Lax equation $L\psi = \lambda\psi, \lambda_t = \lambda^n$. If $\psi_t = B_n\psi$, then the compatibility condition admits a nonisospectral hierarchy

$$u_t = \sigma_n = \Phi^n g_0, n \geq 0,$$

where the operator Φ is a hereditary operator which satisfying $u_t = K_m(u) = \Phi^m f(\lambda_t = 0)$ [22]. In [23,24], Qiao adopted the Lenard series method to obtain some nonisospectral integrable hierarchies under the case $\lambda_t = \lambda^{m+1}M$.

BPT spatial spectral problem was first introduced by Boiti, Pempinelli and Tu in [25], where they considered a particular case of the more general spectral problem proposed by Boiti and Tu in [26] and they showed that the soliton equations of the related hierarchy are Hamiltonian systems with commuting flows on a symplectic Kahler manifold. In this article, an efficient scheme is applied to generate a nonisospectral BPT integrable hierarchy under the case where $\lambda_t = \sum\limits_{j=0}^{n} k_j(t)\lambda^{-j}$ [27]. One can find that the case is a generalized expression for the case $\lambda_t = \lambda^n$. Based on it, we deduce an expanding nonisospectral integrable hierarchies by using the knowledge of integrable coupling [28]. In [29,30], Ma presented the integrable couplings can be derived based on the corresponding non-semisimple Lie algebra $\overline{g}$ which has the following triangular block matrix form

$$M(A,B) = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}, \tag{1}$$

where A and B are two arbitrary square matrices of the same order. By reducing the nonisospectral BPT integrable hierarchy and its integrable coupling hierarchy, we obtain some new nonisospectral nonlinear systems. Ma proposed an effective method to calculate symmetry algebras and successfully applied it to coupled KdV systems [31]. Then, the types of algebraic structures of the spaces of the isospectral Lax operators were established, and it means that the theoretical basis of the Lax operator method has been essentially formed [32]. Under obtaining the expanding nonisospectral integrable systems, some properties including symmetry algebras, Bäcklund transformations [34], analytic solutions [35], and so on, could be studied [36]. The research object of this paper is the generation of the nonisospectral integrable hierarchy and its extended hierarchy, which is different from the related research on the isospectral hierarchy. As far as we know, the nonisospectral BPT hierarchy and its extended integrable hierarchy obtained in this paper are new and different from the existing literature.

§2 A nonisospectral BPT hierarchy

A basis of the loop algebras $\overline{A}_1$ is given by

$$\begin{cases} h(n) = \frac{1}{2}\begin{pmatrix} \lambda^n & 0 \\ 0 & -\lambda^n \end{pmatrix}, \quad e(n) = \frac{1}{2}\begin{pmatrix} 0 & \lambda^n \\ \lambda^n & 0 \end{pmatrix}, \quad f(n) = \frac{1}{2}\begin{pmatrix} 0 & \lambda^n \\ -\lambda^n & 0 \end{pmatrix}, \\ \deg h(n) = \deg e(n) = \deg f(n) = n, \end{cases} \tag{2}$$

equipped with

$$[h(n), e(m)] = f(m+n), \quad [h(n), f(m)] = e(m+n), \quad [e(n), f(m)] = -h(m+n).$$

Based on the loop algebra $\overline{A}_1$, we consider the corresponding nonisospectral problems

$$\begin{cases}\psi_x = U\psi, \quad U = 2h(1) + qe(0) + rf(-1) + sh(-1) = \begin{pmatrix} \lambda + \frac{1}{2}s\lambda^{-1} & \frac{1}{2}q + \frac{1}{2}r\lambda^{-1} \\ \frac{1}{2}q - \frac{1}{2}r\lambda^{-1} & -\lambda - \frac{1}{2}s\lambda^{-1}\end{pmatrix}, \\ \psi_t = V\psi, \quad V = \sum\limits_{m\geq 0}(a_m h(-m) + b_m e(-m) + c_m f(-m)), \\ \lambda_t = \sum\limits_{m\geq 0} k_m(t)\lambda^{1-m}. \end{cases} \tag{3}$$

In order to deduce the nonisospectral integrable hierarchy, one need solve the nonisospectral zero curvature equation

$$\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda_t^{(n)} - V_x^{(n)} + [U, V^{(n)}] = 0. \tag{4}$$

From (3), we have

$$\begin{aligned}\frac{\partial U}{\partial \lambda}\lambda_t &= \begin{pmatrix} 1 - \frac{1}{2}s\lambda^{-2} & -\frac{1}{2}r\lambda^{-2} \\ \frac{1}{2}r\lambda^{-2} & -1 + \frac{1}{2}s\lambda^{-2}\end{pmatrix}\sum_{m\geq 0} k_m(t)\lambda^{1-m} \\ &= \sum_{m\geq 0} k_m(t)[2h(1-m) - rf(-1-m) - sh(-1-m)].\end{aligned}$$

By solving the stationary zero curvature equation

$$V_x = \frac{\partial U}{\partial \lambda}\lambda_t + [U, V], \tag{5}$$

we obtain the recursion relations

$$\begin{cases} a_{nx} = 2k_{n+1}(t) - sk_{n-1}(t) + rb_{n-1} - qc_n, \\ b_{nx} = 2c_{n+1} + sc_{n-1} - ra_{n-1}, \\ c_{nx} = -rk_{n-1}(t) + 2b_{n+1} + sb_{n-1} - qa_n. \end{cases} \tag{6}$$

Take $a_0 = 2\beta,\ b_0 = c_0 = 0$, and then one has

$$\begin{aligned} &b_1 = \beta q,\ c_2 = \frac{\beta q_x}{2} + \beta r,\ a_2 = -\frac{\beta q^2}{4} + 2k_3(t)x - sk_1(t)x + \beta_1, \cdots, \\ &a_{2m+1} = b_{2m} = c_{2m+1} = k_{2m}(t) = 0. \end{aligned}$$

Note that

$$V_+^{(n)} = \sum_{m=0}^{n}(a_m h(-m) + b_m e(-m) + c_m f(-m))\lambda^n = \lambda^n V - V_-^{(n)},$$

$$\lambda_{t,+}^{(n)} = \lambda^n \lambda_t - \lambda_{t,-}^{(n)} = \sum_{m=0}^{n} k_m(t)\lambda^{n-m+1},$$

then (5) can be broken down into

$$-V_{+,x}^{(n)} + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(n)} + [U, V_+^{(n)}] = V_{-,x}^{(n)} - \frac{\partial U}{\partial \lambda}\lambda_{t,-}^{(n)} - [U, V_-^{(n)}]. \tag{7}$$

It follows that

$$\deg V_{+,x}^{(n)} =: (0,0,0),\ \deg \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(n)} =: (1,-1,-1),\ \ \deg([U, V_+^{(n)}]) =: (1,0,-1,-1;0,0,0),$$

$$\deg V_{-,x}^{(n)} =: (-1,-1,-1),\ \ \deg \frac{\partial U}{\partial \lambda}\lambda_{t,-}^{(n)} =: (0,-2,-2),\ \deg([U, V_-^{(n)}]) =: (1,0,-1,-1;-1,-1,-1).$$

We find that the gradation of the left-hand side of (7) is more than -1, while the right-hand side is less than 0. Therefore, take the gradations 0 and -1 in (7), one has

$$\begin{aligned}&-V_{+,x}^{(n)}+\frac{\partial U}{\partial\lambda}\lambda_{t,+}^{(n)}+[U,V_{+}^{(n)}]=-2c_{n+1}e(0)+(-b_{n+1,x}+2c_{n+2})e(-1)+2b_{n+1}f(0)\\&+(c_{n+1,x}-2b_{n+2}+qa_{n+1})f(-1)+2k_{n+1}(t)h(0)+(a_{n+1,x}+qc_{n+1}-2k_{n+2}(t))e(-1).\end{aligned}$$

Choose $n=2m-1$, one can find $\triangle_n=0$ so that $V^{(n)}=V_{+}^{(n)}$.

Theorem 1. By means of (4) and (6), we obtain the nonisospectral BPT hierarchy as follows:

$$\begin{aligned}u_t=\begin{pmatrix}q\\r\\s\end{pmatrix}_t&=\begin{pmatrix}2c_{2m}\\-sb_{2m-1}+rk_{2m-1}(t)\\sk_{2m-1}(t)-rb_{2m-1}\end{pmatrix}=J_1\begin{pmatrix}b_{2m+1}\\-c_{2m}\\a_{2m}\end{pmatrix}+\begin{pmatrix}0\\0\\2k_{2m+1}(t)\end{pmatrix}\\&=J_1L_1^m\begin{pmatrix}\beta q\\0\\2\beta\end{pmatrix}+J_1\sum_{i=0}^{m-1}(L_1^iK_{2m+1-2i}(t)xQ_1)+J_1\sum_{i=0}^{m-1}(L_1^iK_{2m-1-2i}(t)Q_2)\\&\quad+J_1\sum_{i=0}^{m-1}(L_1^i\alpha_{2m-2i}R_1)+\begin{pmatrix}0\\0\\2k_{2m+1}(t)\end{pmatrix}\\&=\Phi^mJ_1\begin{pmatrix}\beta q\\0\\2\beta\end{pmatrix}+\sum_{i=0}^{m-1}(\Phi^iJ_1K_{2m+1-2i}(t)xQ_1)+\sum_{i=0}^{m-1}(\Phi^iJ_1K_{2m-1-2i}(t)Q_2)\\&\quad+\sum_{i=0}^{m-1}(\Phi^iJ_1\alpha_{2m-2i}R_1)+\begin{pmatrix}0\\0\\2k_{2m+1}(t)\end{pmatrix},\end{aligned}\tag{8}$$

where

$$J_1=\begin{pmatrix}0&-2&0\\2&\partial&-q\\0&q&-\partial\end{pmatrix},\ Q_1=\begin{pmatrix}q\\0\\2\end{pmatrix},\ Q_2=\frac{1}{2}\begin{pmatrix}-qsx+r\\0\\-2sx\end{pmatrix},\ R_1=\frac{1}{4}\begin{pmatrix}q\\0\\1\end{pmatrix},$$

$$L_1=\frac{1}{4}\begin{pmatrix}\partial^2-2s-q^2+2q\partial^{-1}r+q\partial^{-1}q_x&\partial s-q\partial^{-1}(qs)&\partial r-q\partial^{-1}(qr)\\-2\partial&-2s&-2r\\-2q+\partial^{-1}(2q_x+4r)&-2\partial^{-1}(qs)&-2\partial^{-1}(qr)\end{pmatrix},$$

with

$$\Phi=J_1L_1J_1^{-1}=\frac{1}{4}\begin{pmatrix}-q_x\partial^{-1}q-q^2+\partial^2-2s-2r\partial^{-1}q&0&-2q_x\partial^{-1}-2q-2r\partial^{-1}\\qs\partial^{-1}q-s\partial&-2s&2qs\partial^{-1}\\qr\partial^{-1}q-r\partial&-2r&2qr\partial^{-1}\end{pmatrix}.\tag{9}$$

To the nonisospectral BPT hierarchy (8), we show the first two nonlinear systems as follows:

$$\begin{cases}q_t=\beta q_x+2\beta r,\\r_t=-s\beta q+rk_1(t),\\s_t=-r\beta q+sk_1(t),\end{cases}\tag{10}$$

and

$$\begin{cases} q_t = \frac{\beta}{4}(q_{xxx} + 2r_{xx} - 4q_x s - 2qs_x - \frac{3}{2}q^2 q_x + 4rs - q^2 r) + \frac{\beta_1}{2}(q_x + 2r) \\ \qquad + \frac{k_1(t)}{2}(r_x - qs - xq_x s - xqs_x - 2xrs) + k_3(t)(q + xq_x + 2xr), \\ r_t = -\frac{\beta}{4}(q_{xx}s + 2r_x s - 2qs^2 - \frac{1}{2}q^3 s) - \frac{\beta_1}{2}qs - \frac{k_1(t)}{2}(rs - xqs^2) + k_3(t)(xqs + r), \\ s_t = -\frac{\beta}{4}(q_{xx}s + 2r_x r - 2qsr - \frac{1}{2}q^3 r) - \frac{\beta_1}{2}qr - \frac{k_1(t)}{2}(r^2 - xqsr) + k_3(t)(xqr + s). \end{cases} \tag{11}$$

§3 An expanding nonisospectral BPT hierarchy

The Lie algebras A_1=span$\{h, e, f\}$ can be expanded into the Lie algebras A_{11}=span$\{f_i\}_{i=1}^5$, where

$$f_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ f_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ f_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ f_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[f_1, f_2] = f_3,\ \ [f_1, f_3] = f_2,\ \ [f_2, f_3] = -f_1,\ \ [f_1, f_4] = \frac{1}{2}f_5,\ \ [f_1, f_5] = \frac{1}{2}f_4,$$

$$[f_2, f_4] = \frac{1}{2}f_4,\ \ [f_2, f_5] = -\frac{1}{2}f_5,\ \ [f_3, f_4] = \frac{1}{2}f_5,\ \ [f_3, f_5] = -\frac{1}{2}f_4,\ \ [f_4, f_5] = 0.$$

To the Lie algebra A_{11}, the following loop algebra can be introduced
$\widetilde{A}_{11} = span\{f_i(n)\}_{i=1}^5,\ \ f_i(n) = f_i\lambda^n,\ \ [f_i(n), f_j(m)] = [f_i, f_j]\lambda^{m+n},\ \ i, j = 1, \ldots, 5,\ m, n \in Z.$
From $\widetilde{A}_{11}$, we consider a nonisospectral problem

$$\begin{cases} \psi_x = M\psi, \ \ M = 2f_1(1) + u_1 f_2(0) + u_2 f_3(-1) + u_3 f_1(-1) + u_4 f_4(-1) + u_5 f_5(0), \\ \psi_t = N\psi, \ \ N = \sum\limits_{m\geq 0}(a_m f_1(-m) + b_m f_2(-m) + c_m f_3(-m) + d_m f_4(-m) + e_m f_5(-m)), \\ \lambda_t = \sum\limits_{m\geq 0} k_m(t)\lambda^{-m}, \end{cases} \tag{12}$$

then the stationary zero curvature equation admits that

$$\begin{cases} a_{nx} = 2k_{n+1}(t) - u_3 k_{n-1}(t) + u_2 b_{n-1} - u_1 c_n, \\ b_{nx} = 2c_{n+1} - u_2 a_{n-1} + u_3 c_{n-1}, \\ c_{nx} = -u_2 k_{n-1}(t) + 2b_{n+1} - u_1 a_n + u_3 b_{n-1}, \\ d_{nx} = -u_4 k_{n-1}(t) + e_{n+1} + \frac{1}{2}u_1 d_n - \frac{1}{2}u_2 e_{n-1} + \frac{1}{2}u_3 e_{n-1} - \frac{1}{2}u_4 b_{n-1} - \frac{1}{2}u_5 a_n + \frac{1}{2}u_5 c_n, \\ e_{nx} = d_{n+1} - \frac{1}{2}u_1 e_n + \frac{1}{2}u_2 d_{n-1} + \frac{1}{2}u_3 d_{n-1} - \frac{1}{2}u_4 a_{n-1} - \frac{1}{2}u_4 c_{n-1} + \frac{1}{2}u_5 b_n. \end{cases} \tag{13}$$

Take $a_0 = 2\beta,\ b_0 = c_0 = d_0 = e_0 = 0$, then

$$b_1 = \beta u_1,\ e_1 = \beta u_5,\ c_2 = \frac{\beta u_{1x}}{2} + \beta u_2,\ a_2 = -\frac{\beta u_1^2}{4} + 2k_3(t)x - u_3 k_1(t)x + \beta_1,$$

$$d_2 = \beta u_{5x} + 2\beta u_4,\ \cdots,\ a_{2m+1} = b_{2m} = c_{2m+1} = e_{2m} = d_{2m+1} = k_{2m}(t) = 0.$$

When $n = 2m-1$, $N^{(n)} = N_+^{(n)} = \sum\limits_{m=0}^{n}(a_m f_1(n-m) + b_m f_2(n-m) + c_m f_3(n-m) + d_m f_4(n-m) + e_m f_5(n-m))$.

Theorem 2. From (12) and (13), we obtain the expanding nonisospectral BPT hierarchy

$$
\begin{aligned}
u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{pmatrix}_t &= \begin{pmatrix}2c_{2m}\\-u_3b_{2m-1}+u_2k_{2m-1}(t)\\-u_2b_{2m-1}+u_3k_{2m-1}(t)\\\frac{1}{2}(u_2-u_3)e_{2m-1}+\frac{1}{2}u_4b_{2m-1}+u_4k_{2m-1}(t)\\d_{2m}\end{pmatrix}=J_2\begin{pmatrix}b_{2m+1}\\-c_{2m}\\a_{2m}\\e_{2m+1}\\d_{2m}\end{pmatrix}+\begin{pmatrix}0\\0\\2k_{2m+1}(t)\\0\\0\end{pmatrix}\\
&=J_2L_2^m\begin{pmatrix}\beta u_1\\0\\2\beta\\\beta u_5\\0\end{pmatrix}+J_2\sum_{i=0}^{m-1}(L_2^iK_{2m+1-2i}(t)xQ_3)+J_2\sum_{i=0}^{m-1}(L_2^iK_{2m-1-2i}(t)Q_4)\\
&\quad+J_2\sum_{i=0}^{m-1}(L_2^i\alpha_{2m-2i}R_2)+\begin{pmatrix}0\\0\\2k_{2m+1}(t)\\0\\0\end{pmatrix} \qquad (14)\\
&=\Phi_2^mJ_2\begin{pmatrix}\beta u_1\\0\\2\beta\\\beta u_5\\0\end{pmatrix}+\sum_{i=0}^{m-1}(\Phi_2^iJ_2K_{2m+1-2i}(t)xQ_3)+\sum_{i=0}^{m-1}(\Phi_2^iJ_2K_{2m-1-2i}(t)Q_4)\\
&\quad+\sum_{i=0}^{m-1}(\Phi_2^iJ_2\alpha_{2m-2i}R_2)+\begin{pmatrix}0\\0\\2k_{2m+1}(t)\\0\\0\end{pmatrix},
\end{aligned}
$$

where

$$
J_2=\begin{pmatrix}0&-2&0&0&0\\2&\partial&-u_1&0&0\\0&u_1&-\partial&0&0\\0&-\frac{u_5}{2}&-\frac{u_5}{2}&1&\frac{u_1}{2}-\partial\\0&0&0&0&1\end{pmatrix},\ Q_3=\begin{pmatrix}u_1\\0\\2\\u_5\\0\end{pmatrix},\ Q_4=\frac{1}{2}\begin{pmatrix}-u_1u_3x+u_2\\0\\-2u_3x\\-u_3u_5x+2u_4\\0\end{pmatrix},\ R_2=\frac{1}{4}\begin{pmatrix}2u_1\\0\\2\\u_5\\0\end{pmatrix},
$$

$\Phi_2=J_2L_2J_2^{-1}$,

$$
L_2=\frac{1}{4}\begin{pmatrix}A&\partial u_3-u_1\partial^{-1}(u_1u_3)&\partial u_2-u_1\partial^{-1}(u_1u_2)&0&0\\-2\partial&-2u_3&-2u_2&0&0\\-2u_1+2\partial^{-1}(u_{1x}+2u_2)&-2\partial^{-1}(u_1u_3)&-2\partial^{-1}(u_1u_2)&0&0\\B&C&D&E&F\\-2u_5&-2u_4&2u_4&4\partial+2u_1&-2(u_2+u_3)\end{pmatrix},
$$

with

$$
\begin{cases}A=\partial^2-2u_3-u_1^2+2u_1\partial^{-1}u_2+u_1\partial^{-1}u_{1x}, & B=-2\partial u_5+2u_4-u_5\partial+u_5\partial^{-1}u_{1x}+2u_5\partial^{-1}u_2,\\ C=-2\partial u_4+u_1u_4+u_3u_5+u_5\partial^{-1}u_1u_3, & D=2\partial u_4-u_1u_4-u_2u_5-u_5\partial^{-1}u_1u_2,\\ E=4\partial^2+2\partial u_1-2u_1\partial-u_1^2+2u_2-2u_3, & F=-2\partial(u_2+u_3)+u_1u_2+u_1u_3.\end{cases}
$$

To the extended nonisospectral BPT hierarchy (14), the first nonlinear system is

$$\begin{cases} u_{1t} = \beta u_{1x} + 2\beta u_2, \\ u_{2t} = -\beta u_1 u_3 + k_1(t) u_2, \\ u_{3t} = -\beta u_1 u_2 + k_1(t) u_3, \\ u_{4t} = \frac{\beta}{2}(u_2 u_5 + u_1 u_4 - u_3 u_5) + k_1(t) u_4, \\ u_{5t} = \beta u_{5x} + 2\beta u_4, \end{cases} \tag{15}$$

which is an extended system of the system (10). Obviously, the hierarchy (14) is an integrable coupling of the nonisospectral BPT hierarchy (8).

§4 Conclusions and discussions

This paper introduced the nonisospectral problems (3) based on the loop algebras $\overline{A}_1$, and thus deduced the nonisospectral BPT hierarchy (8). The nonisospectral problem (12) was introduced by using the extended Lie algebras A_{11}, and then obtained the expanded nonisospectral BPT hierarchy (14). By reducing these two nonisospectral BPT hierarchies, some new nonisospectral nonlinear systems were obtained. Actually, these nonisospectral integrable model that we obtained can enrich the existing integrable models and possibly describe new nonlinear phenomena [37-44].

Declarations

Conflict of interest The authors declare no conflict of interest.

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