Appl. Math. J. Chinese Univ.
2026, 41(1): 194-213

Balaban index of bicyclic chain graphs

LV Xue-zheng[1,*] MA Meng-yu[2] WANG Cheng-rui[3]

Abstract. Chain graphs are $\{2K_2, C_3, C_5\}$-free graphs. Balaban index and sum-Balaban index are two important topological indices. In this paper, we concentrate on the subclass of bicyclic connected chain graphs, identifying the extremal graphs that exhibit the minimum or maximum Balaban index and sum-Balaban index within this class. Moreover, we provide a systematic ordering of all bicyclic connected chain graphs according to the magnitude of their Balaban index and sum-Balaban index.

§1 Introduction

Unless otherwise specified, we adhere to the terminology and notations as defined in [9, 10]. All graphs discussed in this paper are simple and undirected. We denote a path, a cycle, a complete graph on n vertices by P_n, C_n and K_n respectively. A graph G is termed a chain graph if it does not contain $2K_2$, C_3 and C_5 as induced subgraphs. In some literature (e.g., [7]), chain graphs are also known as double nested graphs.

Let G be a connected chain graph. The vertex set V of G can be partitioned into two disjoint parts, each of which can be further partitioned into k non-empty subsets $U_1, U_2, \ldots, U_k$ and $V_1, V_2, \ldots, V_k$. Here U_i and V_i (for $i = 1, 2, \ldots, k$) are independent sets, and for any $1 \le i \le k$, all vertices in U_i are adjacent and only adjacent to all vertices in $\cup_{j=1}^{k+1-i} V_j$. Consequently, if $u' \in U_{i+1}$ and $u'' \in U_i$ (or $v' \in V_{j+1}$ and $v'' \in V_j$), then $N_G(u') \subseteq N_G(u'')$ ($N_G(v') \subseteq N_G(v'')$). Let $|U_i| = s_i$ and $|V_i| = t_i$ for $1 \le i \le k$. The structure of G is completely determined by these parameters, and thus the chain graph G can be denoted as $G(s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_k)$. The number of edges in G is then given by

$$m = \sum_{i+j \le k+1} s_i t_j = \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} s_i t_j = \frac{1}{2}\left[\sum_{j=1}^{k} t_j \sum_{i=1}^{k+1-j} s_i + \sum_{i=1}^{k} s_i \sum_{j=1}^{k+1-i} t_j\right].$$

In 2008, Bell et al. proved that among all connected bipartite graphs with a given number of vertices and edges, chain graphs possess the largest spectral radius [8]. This finding has

Received: 2024-08-16. Revised: 2025-01-20.
MR Subject Classification: 05C07.
Keywords: Balaban index, sum-Balaban index, chain graph.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-026-5273-6.
*Corresponding author.

established the significance of chain graphs within the realm of graph spectral theory. Research on chain graphs remains a burgeoning field, with substantial advancements in the analysis of eigenvalues, energy, and other properties. For instance, Xiong and Hou [28] examined the eigenvalues and Siedel eigenvalues of chain graphs by constructing infinite pairs of non-isomorphic cospectral chain graphs and characterizing those with distinct Siedel eigenvalues. In another study [13], the authors investigated the energy and Laplacian energy of connected chain graphs, establishing upper and lower bounds for the energy and lower bounds for the Laplacian energy. It was also proven that star graph had the minimum energy among all connected chain graphs, and the unicyclic connected chain graph with the maximum Laplacian energy was also identified. In 2023, Alazemi etc. [2] explored certain properties of the Laplacian spectrum of chain graphs that can be inferred from its degree sequence. For more results on chain graphs, readers are directed to [1, 3–5, 24].

Hundreds of topological indices have been extensively studied and can be broadly categorized into three types: degree-based indices, spectral-based indices and distance-based indices. For further results on topological indices, refer to [6, 12, 18, 20–23]. The Balaban index that will be investigated in this paper is a distance-based topological index proposed by Balaban [19]. This index has been extended to weighted graphs [19] and has found successful applications in QSPR/QSAR modeling [17]. For a given connected graph $G = (V, E)$, and for any $u \in V$, let $\sigma_G(u)$ denote the sum of distances from vertex u to all the other vertices in graph G, i.e.,

$$\sigma_G(u) = \sum_{v\in V, v\neq u} d_G(u, v),$$

where $d_G(u, v)$ represents the number of edges in the shortest path between vertices u and v in G. The Balaban index (denoted as J index) of a connected graph G is defined as

$$J(G) = \frac{m}{\mu+1} \sum_{(u,v)\in E} \frac{1}{\sqrt{\sigma_G(u)\sigma_G(v)}},$$

and the sum-Balaban index (denoted as SJ index) is defined as

$$SJ(G) = \frac{m}{\mu+1} \sum_{(u,v)\in E} \frac{1}{\sqrt{\sigma_G(u)+\sigma_G(v)}},$$

where m is the number of edges in G, n is the number of vertices in G and $\mu = m - n + 1$ is the number of cycles in G (see [16, 25–27, 29]).

Significant results have been obtained in the study of J and SJ indices. Zhou et al. [29] provided upper and lower bounds for the J index among all trees, while Deng et al. [14] offered a new proof that a star has the maximum and a path has the minimum J index among all trees, respectively. The basic properties of the SJ indices and its extremal values among all trees have been studied in [15]. Additionally, Lei et al. [20] provided upper and lower bounds for the SJ index of regular graphs. Das investigated the J and SJ indices of double star graphs $DS(p, q)$ with n ($n = p + q + 2$) vertices in [11], ranking the double star graphs based on the size of the indices and identifying the extremal graphs that achieve the maximum or minimum values. A double star graph $DS(p, q)$ is a tree with diameter 3 and $p+q+2$ vertices, in which two adjacent vertices are of degree $p+1$ and $q+1$ respectively, and the remaining $p+q$ vertices are pendant vertices.

Theorem 1.1. *([11]) Let $DS(p, q)$ be a double star graph with n ($n = p+q+2, p \geq q$) vertices, then*

$F(DS(n-3,1)) > F(DS(n-4,2)) > \ldots > F(DS(\lceil\frac{n}{2}\rceil, \lfloor\frac{n}{2}\rfloor - 2)) > F(DS(\lceil\frac{n}{2}\rceil - 1, \lfloor\frac{n}{2}\rfloor - 1)),$

hence

$$F(DS(\lceil\frac{n}{2}\rceil - 1, \lfloor\frac{n}{2}\rfloor - 1)) \leq F(DS(p, q)) \leq F(DS(n-3,1)),$$

where the equality on the left holds if and only if $G \cong DS(\lceil\frac{n}{2}\rceil - 1, \lfloor\frac{n}{2}\rfloor - 1)$, and the equality on the right side holds if and only if $G \cong DS(n-3,1)$, where F stands for either J or SJ index.

In [11], Das also investigated the J index and SJ index for unicyclic connected chain graphs, identifying the structures of these graphs that possess the maximum and minimum values for the two indices.

Theorem 1.2. *([11]) For a unicyclic connected chain graph G with n vertices and n edges,*

$$F\left(G\left(1,1,\lfloor\frac{n}{2}\rfloor;1,1,\lfloor\frac{n}{2}\rfloor\right)\right) \leq F(G) \leq F(G(1,1;2,n-4)),$$

where the equality on the left holds if and only if $G \cong G\left(1,1,\lfloor\frac{n}{2}\rfloor;1,1,\lfloor\frac{n}{2}\rfloor\right)$, and the equality on the right holds if and only if $G \cong G(1,1;2,n-4)$, where F stands for either J or SJ index.

Building upon this result, we examine the Balaban index and sum-Balaban index for bicyclic connected chain graphs, characterizing the structures that achieve extremal values for the two indices. Additionally, we provide an ordering for all bicyclic connected chain graphs with $n \geq 7$ vertices based on the size of their Balaban index and sum-Balaban index. The paper is structured into three sections: the present section introduces basic concepts and reviews relevant research results, the second section conducts preliminary analysis on the structure of bicyclic connected chain graphs. We present the main results in the third section.

§2 The structure of bicyclic connected chain graphs

Let $G = G(s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_k)$ be a bicyclic connected chain graph with n vertices and $m = n+1$ edges. In this section, we will show that

$$G \in \{K_{2,3},\ G(1,2;2,n-5),\ G(1,1;3,n-5),\ G(1,2,p;1,1,n-5-p)\},$$

where $1 \leq p \leq n-6$.

Firstly we prove that $1 \leq k \leq 3$. From the definition of chain graphs, the number of edges of G is

$$m = \frac{1}{2}\left[\sum_{j=1}^{k} t_j \sum_{i=1}^{k+1-j} s_i + \sum_{i=1}^{k} s_i \sum_{j=1}^{k+1-i} t_j\right].$$

Since G is also a bicyclic graph, then $m = n+1$. If $k \geq 4$, given $s_i \geq 1, t_i \geq 1$ ($1 \leq i \leq k$), we have

$$m = \frac{1}{2}\left[\sum_{j=1}^{k} t_j \sum_{i=1}^{k+1-j} s_i + \sum_{i=1}^{k} s_i \sum_{j=1}^{k+1-i} t_j\right]$$

$$\geq \frac{1}{2}\left[t_1\sum_{i=1}^{k}s_i+t_2\sum_{i=1}^{k-1}s_i+t_3\sum_{i=1}^{k-2}s_i+t_ks_1+s_1\sum_{j=1}^{k}t_j+s_2\sum_{j=1}^{k-1}t_j+s_3\sum_{j=1}^{k-2}t_j+s_kt_1\right]$$

$$=\frac{1}{2}\left[\left(t_1\sum_{i=1}^{k}s_i+t_2\sum_{i=1}^{k-1}s_i+t_1s_k\right)+\left(s_1\sum_{j=1}^{k}t_j+s_2\sum_{j=1}^{k-1}t_j+s_1t_k\right)+\left(t_3\sum_{i=1}^{k-2}s_i+s_3\sum_{j=1}^{k-2}t_j\right)\right]$$

$$\geq \frac{1}{2}\left[2\sum_{i=1}^{k}(s_i+t_i)+4\right]=n+2.$$

This contradiction implies that $1\leq k\leq 3$.

If $k=1$, the bicyclic connected chain graph is a complete bipartite graph K_{s_1,t_1}. Since the number of edges exceeds the number of vertices by 1, we have $s_1>1, t_1>1, s_1t_1=s_1+t_1+1$, hence $(s_1-1)(t_1-1)=2$, which means $s_1=2, t_1=3$ or $s_1=3, t_1=2$. That is, when $k=1$, $n=5$ and the corresponding bicyclic connected chain graph $G\cong G(3,2)\cong K_{3,2}$. It is straightforward to calculate that

$$J(G(3,2))=\frac{n+1}{3}\times\frac{6}{\sqrt{6\times 5}}=\frac{2\sqrt{30}}{5},$$
$$SJ(G(3,2))=\frac{n+1}{3}\times\frac{6}{\sqrt{6+5}}=\frac{12\sqrt{11}}{11}.$$

If $k=2$, we consider the structure of the bicyclic connected chain graph $G(s_1,s_2;t_1,t_2)$ by repeatedly using the fact that the number of edges in G is $m=s_1t_1+s_1t_2+s_2t_1=n+1=(s_1+t_1+s_2+t_2)+1$. If $s_1\geq 2$ and $t_1\geq 2$, then

$$\begin{aligned}m&=s_1t_1+s_1t_2+s_2t_1\geq 4+(s_1-1)t_2+t_2+s_2(t_1-1)+s_2\\&\geq 4+(s_1-1)+t_2+(t_1-1)+s_2=s_1+s_2+t_1+t_2+2=n+2,\end{aligned}$$

which contradicts $m=n+1$. Therefore $s_1\leq 1$ or $t_1\leq 1$ holds. However, if $s_1=t_1=1$, then $m=s_1t_1+s_1t_2+s_2t_1=1+t_2+s_2=n-1$, also a contradiction. Thus, we have $s_1\geq 2$ or $t_1\geq 2$. If $s_1=1$ and $t_1\geq 4$, then $m=s_1t_1+s_1t_2+s_2t_1=t_1+t_2+s_2t_1\geq t_1+t_2+4s_2\geq t_1+t_2+1+s_2+2s_2\geq n+2$, a contradiction. By symmetry, if $s_1\geq 4, t_1=1$, we also have $m\geq n+2$, which is a contradiction. So we must have $s_1=1,\ 2\leq t_1\leq 3$ or $2\leq s_1\leq 3,\ t_1=1$. If $s_1=1, t_1=2$, then from $m=s_1t_1+s_2t_1+s_1t_2=2+2s_2+t_2=n+1$, we can obtain that $s_2=2,\ t_2=n-5$. Therefore, we know that $G\cong G(1,2;2,n-5)$ and $n\geq 6$. Similarly, if $s_1=1$ and $t_1=3$, then $s_2=1,\ t_2=n-5$. It follows that $G\cong G(1,1;3,n-5)$ and $n\geq 6$. By symmetry, it is obvious that $G(1,2;2,n-5)\cong G(2,n-5;1,2)$ and $G(1,1;3,n-5)\cong G(3,n-5;1,1)$. So if $k=2$, then $n\geq 6$ and the corresponding bicyclic connected chain graph $G\cong G(1,2;2,n-5)$ or $G\cong G(1,1;3,n-5)$.

If $k=3$, we consider the structure of the bicyclic connected chain graph $G(s_1,s_2,s_3;t_1,t_2,t_3)$ by repeatedly using the fact that $m=s_1t_1+s_1t_2+s_1t_3+s_2t_1+s_2t_2+s_3t_1=n+1$. If $s_1\geq 2$ or $t_1\geq 2$, without loss of generality, assume that $s_1\geq 2$. Then

$$\begin{aligned}m&=s_1t_1+s_1t_2+s_1t_3+s_2t_1+s_2t_2+s_3t_1\\&\geq (s_1-1)t_1+t_1+(s_1-1)t_2+t_2+(s_1-1)t_3+t_3+s_2+s_2+s_3\\&\geq (s_1-1)+t_1+1+t_2+1+t_3+s_2+1+s_3=n+2,\end{aligned}$$

which contradicts $m=n+1$. Therefore $s_1=t_1=1$ holds. Since $s_1=t_1=1$, then from

$m = s_1t_1 + s_1t_2 + s_1t_3 + s_2t_1 + s_2t_2 + s_3t_1 = 1 + t_2 + t_3 + s_2 + s_2t_2 + s_3 = n+1$, we can deduce that $s_2t_2 = 2$, hence $s_2 = 1,\ t_2 = 2$ or $s_2 = 2,\ t_2 = 1$. By symmetry, we can conclude that if $k = 3$, then $n \geq 7$ and the corresponding bicyclic connected chain graph $G \cong G(1, 2, p; 1, 1, q)$, where $p + q = n - 5,\ 1 \leq p,\ q \leq n - 6$.

In the following section, we will compare the J index and SJ index for the bicyclic connected chain graphs $G(1, 1; 3, n-5)$, $G(1, 2; 2, n-5)$, and $G(1, 2, p; 1, 1, q)$ for different values of p and q, and rank all bicyclic connected chain graphs according to the size of their J index and SJ index.

§3 Main results

Before presenting the main results, we first provide the following two lemmas. These results will be frequently used in the proofs of the subsequent theorems.

Lemma 3.1. *([8]) For any positive real number $a > 1$,*

$$1 - \frac{1}{2a} < \sqrt{\frac{a}{a+1}} < 1 - \frac{1}{2a} + \frac{3}{8a^2}.$$

Lemma 3.2. *Let $f(x) = \dfrac{1}{\sqrt{x-2}} + \dfrac{1}{\sqrt{x+2}} - \dfrac{2}{\sqrt{x}}$. Then $f(x) > 0$ and $f(x)$ is monotonically decreasing when $x > 2$.*

Proof. Suppose $g(x) = x^{-3/2}$, then $g'(x) = -\frac{3}{2}x^{-5/2}, g''(x) = \frac{15}{4}x^{-7/2}$. So we can see that $g''(x) > 0$ for $x > 0$, which indicates that $g(x)$ is a convex function when $x > 0$. Therefore, when $x > 2$, we have $g(x-2) + g(x+2) > 2g(x)$. And since

$$f'(x) = -\frac{1}{2}\left[(x-2)^{-3/2} + (x+2)^{-3/2} - 2x^{-3/2}\right] = -\frac{1}{2}\left[g(x-2) + g(x+2) - 2g(x)\right] < 0$$

for $x > 2$, we can immediately obtain that $f(x)$ is monotonically decreasing when $x > 2$. Similarly, let $h(x) = x^{-1/2}$, then $h'(x) = -\frac{1}{2}x^{-3/2}, h''(x) = \frac{3}{4}x^{-5/2}$. So we can see that $h''(x) > 0$ for $x > 0$, which indicates that $h(x)$ is a convex function when $x > 0$. Therefore, when $x > 2$, we have $h(x-2) + h(x+2) > 2h(x)$, which implies that $f(x) > 0$ when $x > 2$. ■

Firstly, we compare the size of J index and SJ index of $G(1, 1; 3, n-5)$, and $G(1, 2; 2, n-5)$.

Theorem 3.3. *$J(G(1, 1; 3, n-5)) > J(G(1, 2; 2, n-5))$ and $SJ(G(1, 1; 3, n-5)) > SJ(G(1, 2; 2, n-5))$.*

Proof. Let $p = n - 5$. We prove the result regarding the J index. According to the definition of J index, we can express the J index of $G(1, 1; 3, n-5)$ and of $G(1, 2; 2, n-5)$ as follows, respectively

$$J(G(1, 1; 3, p)) = \frac{n+1}{3}\left(\frac{p}{\sqrt{(2p+8)(p+5)}} + \frac{3}{\sqrt{(p+5)(2p+6)}} + \frac{3}{\sqrt{(2p+6)(3p+5)}}\right),$$

$$J(G(1, 2; 2, p)) = \frac{n+1}{3}\left(\frac{p}{\sqrt{(2p+9)(p+6)}} + \frac{2}{\sqrt{(p+6)(2p+5)}} + \frac{4}{\sqrt{(2p+5)(3p+6)}}\right).$$

Obviously $(2p+8)(p+5) < (2p+9)(p+6)$, $(p+5)(2p+6) < (p+6)(2p+5)$, hence

$$\frac{p}{\sqrt{(2p+8)(p+5)}} > \frac{p}{\sqrt{(2p+9)(p+6)}}, \frac{2}{\sqrt{(p+5)(2p+6)}} > \frac{2}{\sqrt{(p+6)(2p+5)}}.$$

Denote $A = (p+5)(2p+6)$, $B = (2p+6)(3p+5)$ and $C = (2p+5)(3p+6)$. Then $C - A = 4p^2 + 9p > 3p$, $B - C = p$, $A < B$ and so

$$\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{C}} = \frac{4p^2+9p}{\sqrt{AC}(\sqrt{A}+\sqrt{C})} > \frac{3p}{\sqrt{BC}(\sqrt{B}+\sqrt{C})} = \frac{3}{\sqrt{C}} - \frac{3}{\sqrt{B}}.$$

Therefore,

$$\frac{1}{\sqrt{(p+5)(2p+6)}} + \frac{3}{\sqrt{(2p+6)(3p+5)}} > \frac{4}{\sqrt{(2p+5)(3p+6)}}.$$

Based on the above proof, we know $J(G(1,1;3,n-5)) > J(G(1,2;2,n-5))$.

Similarly, according to the definition of SJ index, we can express the SJ index of $G(1,1;3,n-5)$ and $G(1,2;2,n-5)$ as follows, respectively

$$SJ(G(1,1;3,p)) = \frac{n+1}{3}\left(\frac{p}{\sqrt{3p+13}} + \frac{3}{\sqrt{3p+11}} + \frac{3}{\sqrt{5p+11}}\right),$$

$$SJ(G(1,2;2,p)) = \frac{n+1}{3}\left(\frac{p}{\sqrt{3p+15}} + \frac{2}{\sqrt{3p+11}} + \frac{4}{\sqrt{5p+11}}\right).$$

Thus we have

$$\begin{aligned}&SJ(G(1,1;3,p)) - SJ(G(1,2;2,p))\\ =\ &\frac{n+1}{3}\left(\frac{p}{\sqrt{3p+13}} - \frac{p}{\sqrt{3p+15}} + \frac{1}{\sqrt{3p+11}} - \frac{1}{\sqrt{5p+11}}\right) > 0,\end{aligned}$$

which implies $SJ(G(1,1;3,n-5)) > SJ(G(1,2;2,n-5))$. ■

In the following five theorems, we compare the size of J index and SJ index of $G(1,2,p;1,1,q)$ for different values of p and q.

Theorem 3.4. *Let $G(1,2,p;1,1,q)$ be a connected chain graph of order n ($n = p+q+5, 1 \le p, q \le n-6$). If $p \ge q$, then*

$$F(G(1,2,p;1,1,q)) < F(G(1,2,p+1;1,1,q-1)),$$

where $F = J$, or SJ.

Proof. We first prove the result for the J index. According to the definition of J index, the expressions for J index of $G(1,2,p;1,1,q)$ and $G(1,2,p+1;1,1,q-1)$ are as follows:

$$\begin{aligned}J(G(1,2,p;1,1,q)) =&\frac{n+1}{3}\left[\frac{2}{\sqrt{(3p+2q+5)(2p+3q+6)}} + \frac{2}{\sqrt{(2p+3q+6)(p+2q+5)}}\right.\\ &+ \frac{1}{\sqrt{(p+2q+5)(2p+q+6)}} + \frac{1}{\sqrt{(3p+2q+5)(2p+q+6)}}\\ &\left.+ \frac{p}{\sqrt{(p+2q+5)(2p+3q+8)}} + \frac{q}{\sqrt{(2p+q+6)(3p+2q+9)}}\right], \quad (1)\end{aligned}$$

$$\begin{aligned}&J(G(1,2,p+1;1,1,q-1))\\ =&\frac{n+1}{3}\left[\frac{2}{\sqrt{(3p+2q+6)(2p+3q+5)}} + \frac{2}{\sqrt{(2p+3q+5)(p+2q+4)}}\right.\end{aligned}$$

$$+\frac{1}{\sqrt{(p+2q+4)(2p+q+7)}}+\frac{1}{\sqrt{(3p+2q+6)(2p+q+7)}}$$
$$+\left.\frac{p+1}{\sqrt{(p+2q+4)(2p+3q+7)}}+\frac{q-1}{\sqrt{(2p+q+7)(3p+2q+10)}}\right].$$

So we have

$$\begin{aligned}&\frac{3}{n+1}\cdot(J(G(1,2,p+1;1,1,q-1))-J(G(1,2,p;1,1,q)))\\&=X_1+X_2+X_3+X_4+X_5+X_6,\end{aligned}$$

where

$$\begin{aligned}X_1&=\frac{2}{\sqrt{(3p+2q+6)(2p+3q+5)}}-\frac{2}{\sqrt{(3p+2q+5)(2p+3q+6)}},\\X_2&=\frac{2}{\sqrt{(2p+3q+5)(p+2q+4)}}-\frac{2}{\sqrt{(2p+3q+6)(p+2q+5)}},\\X_3&=\frac{1}{\sqrt{(p+2q+4)(2p+q+7)}}-\frac{1}{\sqrt{(p+2q+5)(2p+q+6)}},\\X_4&=\frac{1}{\sqrt{(3p+2q+6)(2p+q+7)}}-\frac{1}{\sqrt{(3p+2q+5)(2p+q+6)}},\\X_5&=\frac{p+1}{\sqrt{(p+2q+4)(2p+3q+7)}}-\frac{p}{\sqrt{(p+2q+5)(2p+3q+8)}},\\X_6&=\frac{q-1}{\sqrt{(2p+q+7)(3p+2q+10)}}-\frac{q}{\sqrt{(2p+q+6)(3p+2q+9)}}.\end{aligned}$$

Claim 1. $X_1\geq 0$ and $X_3>0$.

Since $p\geq q\geq 1$, we have

$$(3p+2q+6)(2p+3q+5)\leq(3p+2q+5)(2p+3q+6),$$
$$(p+2q+4)(2p+q+7)<(p+2q+5)(2p+q+6).$$

Therefore,

$$X_1=\frac{2}{\sqrt{(3p+2q+6)(2p+3q+5)}}-\frac{2}{\sqrt{(3p+2q+5)(2p+3q+6)}}\geq 0,$$

and

$$X_3=\frac{1}{\sqrt{(p+2q+4)(2p+q+7)}}-\frac{1}{\sqrt{(p+2q+5)(2p+q+6)}}>0.$$

Claim 2. $X_2+X_4>0$.

Denote $A_1=2p+3q+5$, $B_1=p+2q+4$, $C_1=3p+2q+5$ and $D_1=2p+q+6$. Then $A_1\leq C_1$ and $B_1<D_1$ since $p\geq q\geq 1$. Therefore, we have

$$\begin{aligned}X_2=&\frac{2}{\sqrt{A_1B_1}}-\frac{2}{\sqrt{(A_1+1)(B_1+1)}}\\=&\frac{6p+10q+20}{\sqrt{A_1B_1(A_1+1)(B_1+1)}\left[\sqrt{A_1B_1}+\sqrt{(A_1+1)(B_1+1)}\right]}\\&>\frac{5p+3q+12}{\sqrt{C_1D_1(C_1+1)(D_1+1)}\left[\sqrt{C_1D_1}+\sqrt{(C_1+1)(D_1+1)}\right]}\end{aligned}$$

$$=\frac{1}{\sqrt{C_1D_1}}-\frac{1}{\sqrt{(C_1+1)(D_1+1)}}=-X_4.$$

It follows that $X_2+X_4>0$.

Claim 3. $X_5+X_6>0$.

Denote $a=p+2q+4$, $b=2p+3q+7$, $c=2p+q+6$ and $d=3p+2q+9$. It is obvious that $a<c$ and $b<d$ since $p\geq q$. From Lemma 3.1, we can deduce that

$$\begin{aligned}X_5=&\frac{p+1}{\sqrt{ab}}-\frac{p}{\sqrt{(a+1)(b+1)}}=\frac{1}{\sqrt{ab}}\left[(p+1)-p\sqrt{\frac{a}{a+1}}\cdot\sqrt{\frac{b}{b+1}}\right]\\ >&\frac{1}{\sqrt{ab}}\left[(p+1)-p\left(1-\frac{1}{2a}+\frac{3}{8a^2}\right)\left(1-\frac{1}{2b}+\frac{3}{8b^2}\right)\right]\\ =&\frac{1}{\sqrt{ab}}+\frac{p}{\sqrt{ab}}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{3}{8a^2}-\frac{3}{8b^2}-\frac{1}{4ab}+\frac{3}{16ab^2}+\frac{3}{16a^2b}-\frac{9}{64a^2b^2}\right)\\ >&\frac{1}{\sqrt{ab}}+\frac{p}{\sqrt{ab}}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{3}{8a^2}-\frac{3}{8b^2}-\frac{1}{4ab}\right)\\ =&\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{ab}}\left(\frac{p-1}{2a}+\frac{p-1}{2b}\right)+\frac{1}{\sqrt{ab}}\left(\frac{1}{2a}-\frac{3p}{8a^2}-\frac{3p}{8b^2}+\frac{1}{2b}-\frac{p}{4ab}\right),\end{aligned}$$

and

$$\begin{aligned}X_6=&\frac{q-1}{\sqrt{(c+1)(d+1)}}-\frac{q}{\sqrt{cd}}=\frac{1}{\sqrt{cd}}\left[(q-1)\sqrt{\frac{c}{c+1}}\cdot\sqrt{\frac{d}{d+1}}-q\right]\\ >&\frac{1}{\sqrt{cd}}\left[(q-1)\left(1-\frac{1}{2c}\right)\left(1-\frac{1}{2d}\right)-q\right]\\ =&\frac{-1}{\sqrt{cd}}\left[1+(q-1)\left(\frac{1}{2c}+\frac{1}{2d}-\frac{1}{4cd}\right)\right]\\ >&\frac{-1}{\sqrt{ab}}\left[1+(q-1)\left(\frac{1}{2c}+\frac{1}{2c}-\frac{1}{4cd}\right)\right]\\ >&\frac{-1}{\sqrt{ab}}-\frac{1}{\sqrt{ab}}\left(\frac{q-1}{2c}+\frac{q-1}{2d}\right).\end{aligned}$$

Based on the definition of a and b, we can see that $4ab^2-3pa^2-3pb^2=3b^2(a-p)+a(b^2-3pa)>0$, which implies $4ab^2>3p(a^2+b^2)$. Therefore, we have $\frac{1}{2a}>\frac{3p}{8a^2}+\frac{3p}{8b^2}$. Since $a>p$, it follows that $\frac{1}{2b}>\frac{p}{4ab}$. By recalling that $p\geq q$, $a<c$ and $b<d$, we have

$$X_5+X_6>\frac{1}{\sqrt{ab}}\left(\frac{p-1}{2a}-\frac{q-1}{2c}+\frac{p-1}{2b}-\frac{q-1}{2d}+\frac{1}{2a}-\frac{3p}{8a^2}-\frac{3p}{8b^2}+\frac{1}{2b}-\frac{p}{4ab}\right)>0.$$

Incorporating the results of Claims 1, 2, and 3, we can conclude that

$$J(G(1,2,p;1,1,q))<J(G(1,2,p+1;1,1,q-1))$$

when $p\geq q\geq 1$.

According to the definition of SJ index, the expressions for SJ index of $G(1,2,p;1,1,q)$ and $G(1,2,p+1;1,1,q-1)$ are as follows:

$$\begin{aligned}SJ(G(1,2,p;1,1,q)) \quad = \quad & \frac{n+1}{3}\left[\frac{2}{\sqrt{5p+5q+11}}+\frac{2}{\sqrt{3p+5q+11}}+\frac{1}{\sqrt{3p+3q+11}}\right.\\ &\left.+\frac{1}{\sqrt{5p+3q+11}}+\frac{p}{\sqrt{3p+5q+13}}+\frac{q}{\sqrt{5p+3q+15}}\right], \qquad (2)\end{aligned}$$

$$SJ(G(1,2,p+1;1,1,q-1)) = \frac{n+1}{3}\left[\frac{2}{\sqrt{5p+5q+11}} + \frac{2}{\sqrt{3p+5q+9}} + \frac{1}{\sqrt{3p+3q+11}} + \frac{1}{\sqrt{5p+3q+13}} + \frac{p+1}{\sqrt{3p+5q+11}} + \frac{q-1}{\sqrt{5p+3q+17}}\right].$$

Therefore, we know that $\frac{3}{n+1}(SJ(G(1,2,p+1;1,1,q-1)) - SJ(G(1,2,p;1,1,q))) = Y_1 + Y_2$, where

$$Y_1 = \frac{2}{\sqrt{3p+5q+9}} - \frac{2}{\sqrt{3p+5q+11}} + \frac{1}{\sqrt{5p+3q+13}} - \frac{1}{\sqrt{5p+3q+11}},$$
$$Y_2 = \frac{p+1}{\sqrt{3p+5q+11}} - \frac{p}{\sqrt{3p+5q+13}} + \frac{q-1}{\sqrt{5p+3q+17}} - \frac{q}{\sqrt{5p+3q+15}}.$$

Denote $a_1 = 3p+5q+9$, $b_1 = 5p+3q+11$, $c_1 = 3p+5q+11$ and $d_1 = 5p+3q+15$. Since $p \geq q \geq 1$, it is obvious that $b_1 > a_1 \geq 17$, $c_1 > 3p$ and $d_1 > c_1 \geq 19$. We can deduce from Lemma 3.1 that

$$\begin{aligned} Y_1 =& \frac{2}{\sqrt{a_1}} - \frac{2}{\sqrt{a_1+2}} + \frac{1}{\sqrt{b_1+2}} - \frac{1}{\sqrt{b_1}} = \frac{2}{\sqrt{a_1}}\left[1 - \sqrt{\frac{\frac{a_1}{2}}{\frac{a_1}{2}+1}}\right] - \frac{1}{\sqrt{b_1}}\left[1 - \sqrt{\frac{\frac{b_1}{2}}{\frac{b_1}{2}+1}}\right] \\ >& \frac{2}{\sqrt{a_1}}\left(\frac{1}{a_1} - \frac{3}{2a_1^2}\right) - \frac{1}{\sqrt{b_1}} \times \frac{1}{b_1} = \frac{1}{a_1\sqrt{a_1}} - \frac{1}{b_1\sqrt{b_1}} + \frac{1}{a_1\sqrt{a_1}} - \frac{6}{2a_1^2\sqrt{a_1}} > 0, \\ Y_2 =& \frac{p+1}{\sqrt{c_1}} - \frac{p}{\sqrt{c_1+2}} + \frac{q-1}{\sqrt{d_1+2}} - \frac{q}{\sqrt{d_1}} \\ =& \frac{1}{\sqrt{c_1}}\left(p+1-p\sqrt{\frac{\frac{c_1}{2}}{\frac{c_1}{2}+1}}\right) - \frac{1}{\sqrt{d_1}}\left[q-(q-1)\sqrt{\frac{\frac{d_1}{2}}{\frac{d_1}{2}+1}}\right] \\ >& \frac{1}{\sqrt{c_1}}\left(1 + \frac{p}{c_1} - \frac{3p}{2c_1^2}\right) - \frac{1}{\sqrt{d_1}} \times \left[q - (q-1)(1 - \frac{1}{d_1})\right] \\ =& \frac{1}{\sqrt{c_1}}\left(1 + \frac{p}{c_1} - \frac{3p}{2c_1^2}\right) - \frac{1}{\sqrt{d_1}} \times \left(1 + \frac{q-1}{d_1}\right) \\ =& \left(\frac{1}{\sqrt{c_1}} - \frac{1}{\sqrt{d_1}}\right) + \left(\frac{p-1}{c_1\sqrt{c_1}} - \frac{q-1}{d_1\sqrt{d_1}}\right) + \left(\frac{1}{c_1\sqrt{c_1}} - \frac{3p}{2c_1^2\sqrt{c_1}}\right) > 0. \end{aligned}$$

Incorporating the above two inequalities, we can immediately obtain that if $p \geq q \geq 1$, then

$$SJ(G(1,2,p+1;1,1,q-1)) > SJ(G(1,2,p+1;1,1,q-1)).$$

■

Theorem 3.5. *Let $G(1,2,p;1,1,q)$ be a connected chain graph of order n ($n = p+q+5, 1 \leq p, q \leq n-6$). If $1 \leq p \leq q$, then*

$$F(G(1,2,p;1,1,q)) < F(G(1,2,p-1;1,1,q+1)),$$

where $F = J$, or SJ.

Proof. Following (1) in the proof of Theorem 3.4, we can similarly obtain that

$$\begin{aligned} & \frac{3}{n+1}(J(G(1,2,p-1;1,1,q+1)) - J(G(1,2,p;1,1,q))) \\ =& X_7 + X_8 + X_9 + X_{10} + X_{11} + X_{12} + X_{13}, \end{aligned}$$

where

$$X_7 = \frac{2}{\sqrt{(3p+2q+4)(2p+3q+7)}} - \frac{2}{\sqrt{(3p+2q+5)(2p+3q+6)}},$$

$$
\begin{aligned}
X_8 &= \frac{2}{\sqrt{(2p+3q+7)(p+2q+6)}} - \frac{2}{\sqrt{(2p+3q+6)(p+2q+5)}},\\
X_9 &= \frac{1}{\sqrt{(p+2q+6)(2p+q+5)}} - \frac{1}{\sqrt{(p+2q+5)(2p+q+6)}},\\
X_{10} &= \frac{1}{\sqrt{(3p+2q+4)(2p+q+5)}} - \frac{1}{\sqrt{(3p+2q+5)(2p+q+6)}},\\
X_{11} &= \frac{p}{\sqrt{(p+2q+6)(2p+3q+9)}} - \frac{p}{\sqrt{(p+2q+5)(2p+3q+8)}},\\
X_{12} &= \frac{q}{\sqrt{(2p+q+5)(3p+2q+8)}} - \frac{q}{\sqrt{(2p+q+6)(3p+2q+9)}},\\
X_{13} &= \frac{1}{\sqrt{(2p+q+5)(3p+2q+8)}} - \frac{1}{\sqrt{(p+2q+6)(2p+3q+9)}}.
\end{aligned}
$$

Claim I. $X_7 > 0$ and $X_9 \geq 0$.

Since

$$
(3p+2q+4)(2p+3q+7) < (3p+2q+5)(2p+3q+6),
$$
$$
(p+2q+6)(2p+q+5) \leq (p+2q+5)(2p+q+6),
$$

it follows that $X_7 > 0$ and $X_9 \geq 0$.

Claim II. If $p = q$, then $X_{11} + X_{12} = 0$; If $p < q$, then $X_{11} + X_{12} > 0$.

It is obvious that $X_{11} + X_{12} = 0$ if $p = q$. If $q \geq p+1$, denote $a_2 = p+2q+5$, $b_2 = 2p+3q+8$, $c_2 = 2p+q+5$ and $d_2 = 3p+2q+8$. Then $4c_2 > 3q$, $d_2 - c_2 = p+q+3$ and $(12p+3q+32)c_2 > 2p(p+q+3)$. Therefore, we have

$$
\begin{aligned}
&\frac{1}{2c_2} + \frac{1}{2d_2} - \frac{3q}{8c_2^2} - \frac{3q}{8d_2^2} - \frac{q}{4c_2d_2}\\
=&\frac{1}{8c_2^2d_2^2}\left[(4c_2-3q)d^2 + (4d_2-3q)c^2 - 2qc_2d_2)\right]\\
=&\frac{1}{8c_2^2d_2^2}\left[(8p+q+20)d_2^2 + (12p+5q+32)c_2^2 - 2qc_2(c_2+p+q+3)\right]\\
=&\frac{1}{8c_2^2d_2^2}\left[(8p+q+20)d_2^2 + (12p+3q+32)c_2^2 - 2q(p+q+3)c_2\right] > 0.
\end{aligned}
$$

Since $1 \leq p \leq q-1$, we know that $9 \leq c_2 < a_2$ and $15 \leq d_2 < b_2$. We can deduce from Lemma 3.1 that

$$
\begin{aligned}
X_{11} =& \frac{p}{\sqrt{(a_2+1)(b_2+1)}} - \frac{p}{\sqrt{a_2b_2}}\\
=& \frac{p}{\sqrt{a_2b_2}}\left(\sqrt{\frac{a_2}{a_2+1}}\cdot\sqrt{\frac{b_2}{b_2+1}} - 1\right) > \frac{p}{\sqrt{a_2b_2}}\left[\left(1-\frac{1}{2a_2}\right)\left(1-\frac{1}{2b_2}\right) - 1\right]\\
=& \frac{p}{\sqrt{a_2b_2}}\left(\frac{1}{4a_2b_2} - \frac{1}{2a_2} - \frac{1}{2b_2}\right) > -\frac{p}{\sqrt{a_2b_2}}\left(\frac{1}{2a_2} + \frac{1}{2b_2}\right),\\
X_{12} =& \frac{q}{\sqrt{c_2d_2}} - \frac{q}{\sqrt{(c_2+1)(d_2+1)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q}{\sqrt{c_2 d_2}}\left(1-\sqrt{\frac{c_2}{c_2+1}}\cdot\sqrt{\frac{d_2}{d_2+1}}\right)\\
&> \frac{q}{\sqrt{c_2 d_2}}\left[1-\left(1-\frac{1}{2c_2}+\frac{3}{8c_2^2}\right)\left(1-\frac{1}{2d_2}+\frac{3}{8d_2^2}\right)\right]\\
&> \frac{q}{\sqrt{c_2 d_2}}\left(\frac{1}{2c_2}+\frac{1}{2d_2}-\frac{3}{8c_2^2}-\frac{3}{8d_2^2}-\frac{1}{4c_2d_2}\right)\\
&\geq \frac{p+1}{\sqrt{c_2 d_2}}\left(\frac{1}{2c_2}+\frac{1}{2d_2}\right)-\frac{q}{\sqrt{c_2 d_2}}\left(\frac{3}{8c_2^2}+\frac{3}{8d_2^2}+\frac{1}{4c_2d_2}\right)\\
&> \frac{p}{\sqrt{a_2 b_2}}\left(\frac{1}{2a_2}+\frac{1}{2b_2}\right)+\frac{1}{\sqrt{c_2 d_2}}\left(\frac{1}{2c_2}+\frac{1}{2d_2}-\frac{3q}{8c_2^2}-\frac{3q}{8d_2^2}-\frac{q}{4c_2d_2}\right)\\
&> \frac{p}{\sqrt{a_2 b_2}}\left(\frac{1}{2a_2}+\frac{1}{2b_2}\right).
\end{aligned}
$$

These two inequalities lead to the conclusion that $X_{11} + X_{12} > 0$.

Claim III. $X_8 + X_{10} + X_{13} > 0$.

If $p = q$, recalling $f(x)$ defined in Lemma 3.2, we can immediately obtain that $f(5p+6) > f(5p+7) > 0$. Therefore,

$$
\begin{aligned}
& X_8 + X_{10} + X_{13}\\
&= \frac{2}{\sqrt{(5p+7)(3p+6)}}-\frac{2}{\sqrt{(5p+6)(3p+5)}}+\frac{1}{\sqrt{(5p+4)(3p+5)}}\\
&\quad -\frac{1}{\sqrt{(5p+5)(3p+6)}}+\frac{1}{\sqrt{(3p+5)(5p+8)}}-\frac{1}{\sqrt{(3p+6)(5p+9)}}\\
&= \frac{1}{\sqrt{3p+5}}\left(\frac{1}{\sqrt{5p+4}}+\frac{1}{\sqrt{5p+8}}-\frac{2}{\sqrt{5p+6}}\right)\\
&\quad -\frac{1}{\sqrt{3p+6}}\left(\frac{1}{\sqrt{5p+5}}+\frac{1}{\sqrt{5p+9}}-\frac{2}{\sqrt{5p+7}}\right)\\
&= \frac{1}{\sqrt{3p+5}}f(5p+6)-\frac{1}{\sqrt{3p+6}}f(5p+7)\\
&> \left(\frac{1}{\sqrt{3p+5}}-\frac{1}{\sqrt{3p+6}}\right)f(5p+7) > 0.
\end{aligned}
$$

If $q \geq p+1$, denote $A_2 = 2p+3q+6$, $B_2 = p+2q+5$, $C_2 = 3p+2q+4$ and $D_2 = 2p+q+5$. Then it is obvious that $A_2 > C_2$ and $B_2 > D_2$, so we have

$$
\begin{aligned}
X_8 &= \frac{2}{\sqrt{(A_2+1)(B_2+1)}}-\frac{2}{\sqrt{A_2B_2}}\\
&= \frac{-6p-10q-24}{\sqrt{A_2B_2(A_2+1)(B_2+1)}\left[\sqrt{A_2B_2}+\sqrt{(A_2+1)(B_2+1)}\right]},
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
X_{10} &= \frac{1}{\sqrt{C_2D_2}}-\frac{1}{\sqrt{(C_2+1)(D_2+1)}}\\
&= \frac{5p+3q+10}{\sqrt{C_2D_2(C_2+1)(D_2+1)}\left[\sqrt{C_2D_2}+\sqrt{(C_2+1)(D_2+1)}\right]}\\
&> \frac{5p+3q+10}{\sqrt{A_2B_2(A_2+1)(B_2+1)}\left[\sqrt{A_2B_2}+\sqrt{(A_2+1)(B_2+1)}\right]}.
\end{aligned}
\tag{4}
$$

Let $r = \left(\frac{12}{7}\right)^{2/3} \approx 1.43237 > 1.4$, and recall that $1 \le p < q$. We can deduce that

$$r(A_2+1) - (C_2+4) > -0.2p + 2.2q + 1.8 > 0, \tag{5}$$

and

$$\frac{(B_2+1)(A_2+3)}{A_2B_2} = \left(1+\frac{3}{A_2}\right)\left(1+\frac{1}{B_2}\right) \le \frac{17}{14}\cdot\frac{11}{10} < r. \tag{6}$$

It follows from $q > p$, $D_2 < B_2+1$, and the inequalities in (5) and (6) that

$$\begin{aligned} X_{13} &= \frac{1}{\sqrt{D_2(C_2+4)}} - \frac{1}{\sqrt{(B_2+1)(A_2+3)}} \\ &= \frac{2(q-p+1)(2q+2p+7)}{\sqrt{D_2(C_2+4)(B_2+1)(A_2+3)}\left(\sqrt{D_2(C_2+4)}+\sqrt{(B_2+1)(A_2+3)}\right)} \\ &> \frac{2r^{-3/2}(q-p+1)(2q+2p+7)}{\sqrt{A_2B_2(A_2+1)(B_2+1)}\left[\sqrt{A_2B_2}+\sqrt{(A_2+1)(B_2+1)}\right]}. \end{aligned} \tag{7}$$

Based on the results in (3), (4) and (7), in order to establish that $X_8 + X_{10} + X_{13} > 0$, we only need to show that

$$2(q-p+1)(2q+2p+7) - r^{3/2}(p+7q+14) > 0.$$

Denote $g_1(p,q) = 2(q-p+1)(2q+2p+7) - r^{3/2}(p+7q+14)$. Then $g_1(p,p+1) = (16p+36) - \frac{12}{7}(8p+21) > 0$. Since $\frac{\partial g_1}{\partial q} = 8q+6 > 0$, then for any positive integer $1 \le p < q$, we have $g_1(p,q) \ge g_1(p,p+1) > 0$. Thus $X_8 + X_{10} + X_{13} > 0$ follows immediately.

Incorporating the results of Claims I, II, III, we can conclude that if $1 \le p \le q$, then

$$J(G(1,2,p;1,1,q)) < J(G(1,2,p-1;1,1,q+1)).$$

Next we prove the similar result for SJ index. Following (2) in the proof of Theorem 3.4, we know that

$$\begin{aligned} &\frac{3}{n+1}\cdot\left(SJ(G(1,2,p-1;1,1,q+1)) - SJ(G(1,2,p;1,1,q))\right) \\ &= \frac{2}{\sqrt{3p+5q+13}} - \frac{2}{\sqrt{3p+5q+11}} + \frac{1}{\sqrt{5p+3q+9}} - \frac{1}{\sqrt{5p+3q+11}} \\ &\quad + \frac{p-1}{\sqrt{3p+5q+15}} - \frac{p}{\sqrt{3p+5q+13}} + \frac{q+1}{\sqrt{5p+3q+13}} - \frac{q}{\sqrt{5p+3q+15}} \\ &= Y_3 + Y_4 + Y_5, \end{aligned}$$

where

$$\begin{aligned} Y_3 &= -\left(\frac{p}{\sqrt{3p+5q+13}} - \frac{p}{\sqrt{3p+5q+15}}\right) + \frac{q}{\sqrt{5p+3q+13}} - \frac{q}{\sqrt{5p+3q+15}}, \\ Y_4 &= \frac{2}{\sqrt{3p+5q+13}} - \frac{2}{\sqrt{3p+5q+11}} + \frac{1}{\sqrt{5p+3q+9}} - \frac{1}{\sqrt{5p+3q+11}}, \\ Y_5 &= \frac{1}{\sqrt{5p+3q+13}} - \frac{1}{\sqrt{3p+5q+15}}. \end{aligned}$$

We begin the proof by showing that $Y_3 \ge 0$. It is obvious that $Y_3 = 0$ if $p = q$. If $q \ge p+1$, let $h(x) = \frac{1}{\sqrt{x}}$. Then $h'(x) = -\frac{1}{2x\sqrt{x}} < 0$, $h''(x) = \frac{3}{4x^2\sqrt{x}} > 0$ when $x > 0$. Therefore, from the inequality $3p+5q+13 \geq 5p+3q+15$ and the properties of convex functions, it follows that

$$0 < h(3p+5q+13) - h(3p+5q+15) < h(5p+3q+13) - h(5p+3q+15).$$

Given that $1 \leq p < q$, we can conclude that $Y_3 > 0$.

We also prove $Y_4 + Y_5 > 0$. If $p = q$,

$$\begin{aligned} Y_4 + Y_5 &= \frac{3}{\sqrt{8p+13}} - \frac{3}{\sqrt{8p+11}} + \frac{1}{\sqrt{8p+9}} - \frac{1}{\sqrt{8p+15}} \\ &= -\frac{6}{\sqrt{8p+11}\sqrt{8p+13}(\sqrt{8p+11}+\sqrt{8p+13})} \\ &\quad + \frac{6}{\sqrt{8p+9}\sqrt{8p+15}(\sqrt{8p+9}+\sqrt{8p+15})}. \end{aligned}$$

Obviously $(8p+11)(8p+13) > (8p+9)(8p+15)$, thus

$$\sqrt{8p+11}\sqrt{8p+13}(\sqrt{8p+11}+\sqrt{8p+13}) > \sqrt{8p+9}\sqrt{8p+15}(\sqrt{8p+9}+\sqrt{8p+15}),$$

and it follows that $Y_4 + Y_5 > 0$. If $1 \leq p < q$, denote $A_3 = 3p+5q+11$ and $B_3 = 5p+3q+9$. Then it is obvious that $20 \leq B_3 < A_3 - 2$. Therefore,

$$\begin{aligned} Y_4 &= \frac{-4}{\sqrt{A_3(A_3+2)}\left(\sqrt{A_3+2}+\sqrt{A_3}\right)} + \frac{2}{\sqrt{B_3(B_3+2)}\left(\sqrt{B_3+2}+\sqrt{B_3}\right)} \\ &> \frac{-2}{\sqrt{A_3(A_3+2)}\left(\sqrt{A_3+2}+\sqrt{A_3}\right)}. \end{aligned} \tag{8}$$

Recalling that $r = \left(\frac{12}{7}\right)^{2/3} \approx 1.43237$, it is easy to check that $A_3 + 4 < r(A_3+2)$ and $B_3 + 4 < rA_3$. So we have

$$\begin{aligned} Y_5 &= \frac{2(q-p+1)}{\sqrt{(A_3+4)(B_3+4)}\left(\sqrt{A_3+4}+\sqrt{B_3+4}\right)} \\ &> \frac{2r^{-3/2}(q-p+1)}{\sqrt{A_3(A_3+2)}\left(\sqrt{A_3+2}+\sqrt{A_3}\right)} > \frac{2}{\sqrt{A_3(A_3+2)}\left(\sqrt{A_3+2}+\sqrt{A_3}\right)}. \end{aligned} \tag{9}$$

According to the inequalities in (8) and (9), we can deduce that $Y_4 + Y_5 > 0$.

From the above, it can be concluded that if $1 \leq p \leq q$, then

$$SJ(G(1,2,p;1,1,q)) < SJ(G(1,2,p-1;1,1,q+1)).$$

■

Theorem 3.6. *Let $G(1,2,p;1,1,q)$ be a connected chain graph of order n ($n = p+q+5, 1 \leq p, q \leq n-6$). If $1 \leq p < q$, then*

$$F(G(1,2,p;1,1,q)) < F(G(1,2,q;1,1,p))$$

where $F = J$, or SJ.

Proof. Following (1) in the proof of Theorem 3.4, we can similarly obtain that

$$\begin{aligned} &\frac{3}{n+1}(J(G(1,2,q;1,1,p)) - J(G(1,2,p;1,1,q))) \\ =\ & X_{14} + X_{15} + X_{16} + X_{17} + X_{18} + X_{19}, \end{aligned}$$

where

$$X_{14} = \frac{2}{\sqrt{(2p+3q+5)(3p+2q+6)}} - \frac{2}{\sqrt{(3p+2q+5)(2p+3q+6)}},$$

$$X_{15} = \frac{2}{\sqrt{(3p+2q+6)(2p+q+5)}} - \frac{2}{\sqrt{(2p+3q+6)(p+2q+5)}},$$
$$X_{16} = \frac{1}{\sqrt{(2p+q+5)(p+2q+6)}} - \frac{1}{\sqrt{(p+2q+5)(2p+q+6)}},$$
$$X_{17} = \frac{1}{\sqrt{(2p+3q+5)(p+2q+6)}} - \frac{1}{\sqrt{(3p+2q+5)(2p+q+6)}},$$
$$X_{18} = \frac{p}{\sqrt{(p+2q+6)(2p+3q+9)}} - \frac{p}{\sqrt{(p+2q+5)(2p+3q+8)}},$$
$$X_{19} = \frac{q}{\sqrt{(2p+q+5)(3p+2q+8)}} - \frac{q}{\sqrt{(2p+q+6)(3p+2q+9)}}.$$

Firstly we show that $X_{14}+X_{16}>0$. Denote $A_4=(2p+3q+5)(3p+2q+6)$, $B_4=(3p+2q+5)(2p+3q+6)$, $C_4=(2p+q+5)(p+2q+6)$ and $D_4=(p+2q+5)(2p+q+6)$. It is easy to check that $B_4-A_4=p-q$ and $D_4-C_4=q-p$. Then we have

$$X_{14} = \frac{2}{\sqrt{A_4}} - \frac{2}{\sqrt{B_4}} = \frac{-2(q-p)}{\sqrt{A_4B_4}\left(\sqrt{A_4}+\sqrt{B_4}\right)},$$
$$X_{16} = \frac{1}{\sqrt{C_4}} - \frac{1}{\sqrt{D_4}} = \frac{q-p}{\sqrt{C_4D_4}\left(\sqrt{C_4}+\sqrt{D_4}\right)}.$$

Let $r_1=1.6>2^{2/3}$. Noticing that

$$A_4-r_1C_4=\left(2.8p^2+2.8q^2+5pq-18\right)+(2.4q-0.2p)>0,$$
$$B_4-r_1D_4=\left(2.8p^2+2.6q^2+5pq-18\right)+(2.4p+0.2q^2-0.2q)>0,$$

we have $\sqrt{A_4B_4}(\sqrt{A_4}+\sqrt{B_4})>2\sqrt{C_4D_4}(\sqrt{C_4}+\sqrt{D_4})$, and thus $X_{14}+X_{16}>0$.

Next we clarify that $X_{15}+X_{17}>0$. Denote $A_5=(3p+2q+6)(2p+q+5)$, $B_5=(2p+3q+6)(p+2q+5)$, $C_5=(2p+3q+5)(p+2q+6)$ and $D_5=(3p+2q+5)(2p+q+6)$. It is easy to check that $B_5-A_5=(q-p)(4p+4q+11)$ and $D_5-C_5=-(q-p)(4p+4q+11)$. Then we have

$$X_{15} = \frac{2}{\sqrt{A_5}} - \frac{2}{\sqrt{B_5}} = \frac{2(q-p)(4p+4q+11)}{\sqrt{A_5B_5}\left(\sqrt{A_5}+\sqrt{B_5}\right)}$$
$$X_{17} = \frac{1}{\sqrt{C_5}} - \frac{1}{\sqrt{D_5}} = \frac{-(q-p)(4p+4q+11)}{\sqrt{C_5D_5}\left(\sqrt{C_5}+\sqrt{D_5}\right)}$$

Noticing that $C_5-B_5=D_5-A_5=p+q>0$, so we have $2\sqrt{C_5D_5}(\sqrt{C_5}+\sqrt{D_5})>\sqrt{A_5B_5}(\sqrt{A_5}+\sqrt{B_5})$, which implies that $X_{15}+X_{17}>0$.

Finally we prove that $X_{18}+X_{19}>0$. Denote $A_6=(p+2q+6)(2p+3q+9)$, $B_6=(p+2q+5)(2p+3q+8)$, $C_6=(2p+q+5)(3p+2q+8)$ and $D_6=(2p+q+6)(3p+2q+9)$. It is easy to check that $A_6-C_6=4q^2-4p^2+18q-10p+5>0$, $B_6-D_6=4q^2-4p^2+10q-18p-5\geq 14(q-p)-5>0$, $B_6-A_6=-3p-5q-5$ and $D_6-C_6=5p+3q+5$. Then we have

$$X_{18} = \frac{p}{\sqrt{A_6}} - \frac{p}{\sqrt{B_6}} = \frac{-p(3p+5q+5)}{\sqrt{A_6B_6}\left(\sqrt{A_6}+\sqrt{B_6}\right)},$$
$$X_{19} = \frac{q}{\sqrt{C_6}} - \frac{q}{\sqrt{D_6}} = \frac{q(5p+3q+5)}{\sqrt{C_6D_6}\left(\sqrt{C_6}+\sqrt{D_6}\right)}.$$

Noticing that $q>p$, $q(5p+3q+5)>p(3p+5q+5)$, $C_6<A_6$ and $D_6<B_6$, so we have

$X_{18} + X_{19} > 0$.

From the above, we can conclude that

$$J(G(1,2,p;1,1,q)) < J(G(1,2,q;1,1,p))$$

when $1 \le p < q$.

Now we prove the similar result for the SJ index. According to the definition of SJ index and similarly as that in the proof of Theorem 3.4, we know that

$$\begin{aligned}
&\frac{3}{n+1} \cdot (SJ(G(1,2,q;1,1,p)) - SJ(G(1,2,p;1,1,q))) \\
&= \frac{p}{\sqrt{3p+5q+15}} - \frac{p}{\sqrt{3p+5q+13}} + \frac{q}{\sqrt{5p+3q+13}} - \frac{q}{\sqrt{5p+3q+15}} \\
&\quad + \frac{1}{\sqrt{5p+3q+11}} - \frac{1}{\sqrt{3p+5q+11}} \\
&= -\frac{2p}{\sqrt{(3p+5q+15)(3p+5q+13)}\left(\sqrt{3p+5q+15}+\sqrt{3p+5q+13}\right)} \\
&\quad + \frac{2q}{\sqrt{(5p+3q+13)(5p+3q+15)}\left(\sqrt{5p+3q+13}+\sqrt{5p+3q+15}\right)} \\
&\quad + \frac{1}{\sqrt{5p+3q+11}} - \frac{1}{\sqrt{3p+5q+11}}
\end{aligned}$$

Given that $p < q$, it is easy to see that $3p+5q+15 > 5p+3q+15$, $3p+5q+13 > 5p+3q+13$, $3p+5q+11 > 5p+3q+11$. Thus we can deduce that if $1 \le p < q$, then

$$SJ(G(1,2,p;1,1,q)) < SJ(G(1,2,q;1,1,p)).$$

■

Theorem 3.7. *If $n \equiv 0 (\mod 2)$, denote $p = \frac{n-6}{2}$, then for any $1 \le k \le p-1$, we have*

$$F\left(G\left(1,2,p+k;1,1,p-k+1\right)\right) < F\left(G\left(1,2,p-k;1,1,p+k-1\right)\right),$$

where $F = J$, SJ.

Proof. Following (1) in the proof of Theorem 3.4, we can similarly obtain that

$$\begin{aligned}
&\frac{3}{n+1} \cdot (J(G(1,2,p-k;1,1,p+k+1)) - J(G(1,2,p+k;1,1,p-k+1))) \\
&= \frac{2}{\sqrt{(5p-k+7)(5p+k+9)}} - \frac{2}{\sqrt{(5p+k+7)(5p-k+9)}} \\
&\quad + \frac{2}{\sqrt{(5p+k+9)(3p+k+7)}} - \frac{2}{\sqrt{(5p-k+9)(3p-k+7)}} \\
&\quad + \frac{1}{\sqrt{(5p-k+7)(3p-k+7)}} - \frac{1}{\sqrt{(5p+k+7)(3p+k+7)}} \\
&\quad + \frac{1}{\sqrt{(3p-k+7)(5p-k+11)}} - \frac{1}{\sqrt{(3p+k+7)(5p+k+11)}}.
\end{aligned}$$

It is easy to calculate that $(5p+k+7)(5p-k+9) - (5p-k+7)(5p+k+9) = 4k > 0$, so we have

$$\frac{2}{\sqrt{(5p-k+7)(5p+k+9)}} - \frac{2}{\sqrt{(5p+k+7)(5p-k+9)}} > 0.$$

Recalling that $f(x)=\frac{1}{\sqrt{x-2}}+\frac{1}{\sqrt{x+2}}-\frac{2}{\sqrt{x}}$ defined in Lemma 3.2, we can deduce that

$$\begin{aligned}&\frac{3}{n+1}\cdot(J(G(1,2,p-k;1,1,p+k+1))-J(G(1,2,p+k;1,1,p-k+1)))\\ >&-\frac{1}{\sqrt{3p+k+7}}f(5p+k+9)+\frac{1}{\sqrt{3p-k+7}}f(5p-k+9)\\ >&f(5p+k+9)\left(\frac{1}{\sqrt{3p-k+7}}-\frac{1}{\sqrt{3p+k+7}}\right)>0.\end{aligned}$$

This confirms that $J\left(G\left(1,2,p+k;1,1,p-k+1\right)\right)<J\left(G\left(1,2,p-k;1,1,p+k-1\right)\right)$.

Now we come to the similar result for SJ index. Similarly as that in the proof of Theorem 3.4 and the definition of SJ index, we can obtain that

$$\begin{aligned}&\frac{3}{n+1}\cdot(SJ(G(1,2,p-k;1,1,p+k+1))-SJ(G(1,2,p+k;1,1,p-k+1)))\\ =&\frac{1}{\sqrt{8p-2k+14}}+\frac{1}{\sqrt{8p-2k+18}}-\frac{2}{\sqrt{8p-2k+16}}\\ &-\left(\frac{1}{\sqrt{8p+2k+14}}+\frac{1}{\sqrt{8p+2k+18}}-\frac{2}{\sqrt{8p+2k+16}}\right)\\ =&f(8p-2k+16)-f(8p+2k+16)>0.\end{aligned}$$

This confirms that $SJ(G(1,2,p+k;1,1,p-k+1))<SJ(G(1,2,p-k;1,1,p+k+1))$. ■

Theorem 3.8. *If* $n\equiv 1(\mod 2)$, *denote* $p=\dfrac{n-5}{2}$, *then for any* $2\le k\le p$, *we have*

$$F\left(G\left(1,2,p+k-1;1,1,p-k+1\right)\right)<F\left(G\left(1,2,p-k;1,1,p+k\right)\right),$$

where $F=J$, *or* SJ.

Proof. Following (1) in the proof of Theorem 3.4, we can similarly obtain that

$$\begin{aligned}&\frac{3}{n+1}\cdot(J(G(1,2,p-k;1,1,p+k))-J(G(1,2,p+k-1;1,1,p-k+1)))\\ =&\frac{2}{\sqrt{(5p-k+5)(5p+k+6)}}-\frac{2}{\sqrt{(5p+k+4)(5p-k+7)}}\\ &+\frac{2}{\sqrt{(5p+k+6)(3p+k+5)}}-\frac{2}{\sqrt{(5p-k+7)(3p-k+6)}}\\ &+\frac{1}{\sqrt{(3p+k+5)(3p-k+6)}}-\frac{1}{\sqrt{(3p-k+6)(3p+k+5)}}\\ &+\frac{1}{\sqrt{(5p-k+5)(3p-k+6)}}-\frac{1}{\sqrt{(5p+k+4)(3p+k+5)}}\\ &+\frac{p-k}{\sqrt{(3p+k+5)(5p+k+8)}}-\frac{p+k-1}{\sqrt{(3p-k+6)(5p-k+9)}}\\ &+\frac{p+k}{\sqrt{(3p-k+6)(5p-k+9)}}-\frac{p-k+1}{\sqrt{(3p+k+5)(5p+k+8)}}.\end{aligned}$$

It is easy to calculate that $(5p+k+4)(5p-k+7)-(5p-k+5)(5p+k+6)=4k-2>0$, so we have

$$\frac{2}{\sqrt{(5p-k+5)(5p+k+6)}}-\frac{2}{\sqrt{(5p+k+4)(5p-k+7)}}>0.$$

Recalling that $f(x)=\frac{1}{\sqrt{x-2}}+\frac{1}{\sqrt{x+2}}-\frac{2}{\sqrt{x}}$ defined in Lemma 3.2, we can deduce that

$$\begin{aligned}
&\frac{3}{n+1}\cdot\left(J(G(1,2,p-k;1,1,p+k))-J(G(1,2,p+k-1;1,1,p-k+1))\right)\\
&>\frac{1}{\sqrt{3p-k+6}}\left(\frac{1}{\sqrt{5p-k+5}}+\frac{1}{\sqrt{5p-k+9}}-\frac{2}{\sqrt{5p-k+7}}\right)\\
&-\frac{1}{\sqrt{3p+k+5}}\left(\frac{1}{\sqrt{5p+k+4}}+\frac{1}{\sqrt{5p+k+8}}-\frac{2}{\sqrt{5p+k+6}}\right)\\
&=\frac{1}{\sqrt{3p-k+6}}f(5p-k+7)-\frac{1}{\sqrt{3p+k+5}}f(5p+k+6)\\
&>\left(\frac{1}{\sqrt{3p-k+6}}-\frac{1}{\sqrt{3p+k+5}}\right)f(5p+k+6)>0.
\end{aligned}$$

This confirms that $J\left(G\left(1,2,p+k-1;1,1,p-k+1\right)\right)<J\left(G\left(1,2,p-k;1,1,p+k\right)\right).$

Now we come to the similar result for the SJ index. Similarly as that in the proof of Theorem 3.4 and the definition of SJ index, we can obtain that

$$\begin{aligned}
&\frac{3}{n+1}\cdot\left(SJ(G(1,2,p-k;1,1,p+k))-SJ(G(1,2,p+k-1;1,1,p-k+1))\right)\\
&=\frac{1}{\sqrt{8p-2k+11}}+\frac{1}{\sqrt{8p-2k+15}}-\frac{2}{\sqrt{8p-2k+13}}\\
&-\left(\frac{1}{\sqrt{8p+2k+9}}+\frac{1}{\sqrt{8p+2k+13}}-\frac{2}{\sqrt{8p+2k+11}}\right)\\
&=f(8p-2k+13)-f(8p+2k+11)>0.
\end{aligned}$$

This confirms that $SJ(G(1,2,p+k-1;1,1,p-k+1))<SJ(G(1,2,p-k;1,1,p+k))$. ∎

Conclusions. If $k=3$, Theorem 3.4 and Theorem 3.5 indicate that the larger the absolute difference of $|p-q|$, the larger the J index and SJ index of the bicyclic connected chain graph $G(1,2,p;1,1,q)$ are. Combining this with Theorem 3.6, we can deduce that

$$F\left(G\left(1,2,\left\lfloor\frac{n-5}{2}\right\rfloor;1,1,\left\lceil\frac{n-5}{2}\right\rceil\right)\right)\le F(G(1,2,p;1,1,,q))\le F\left(G\left(1,2,n-6;1,1,1\right)\right),$$

where $F=J$, or SJ. So before no further research, the equality on the left holds if and only if $p=\left\lfloor\frac{n-5}{2}\right\rfloor$, and $q=\left\lceil\frac{n-5}{2}\right\rceil$ and the equality on the right holds if and only if $p=n-6$, and $q=1$.

If $k=1$, there is only a unique bicyclic chain graph $G(2;3)\cong K_{2,3}$ of order $n=5$, and no further research is necessary. If $n\ge 7$ and $k=2$ or $k=3$, it is not difficult to see that $G(1,2,n-5;1,1,0)\cong G(1,1;3,n-5)$ and $G(1,2,0;1,1,n-5)\cong G(1,2;2,n-5)$. Note that we require $p\ge q\ge 1$ in Theorem 3.4, therefore Theorem 3.4 implies that

$$F(G(1,1;3,n-5))=F(G(1,2,n-5;1,1,0))>F(G(1,2,n-6;1,1,1)),$$

where F stands for either J or SJ index. Similarly, since we require $1\le p\le q$ in Theorem 3.5, it implies that

$$F(G(1,2;2,n-5))=F(G(1,2,0;1,1,n-5))>F(G(1,2,1;1,1,n-6)),$$

where F stands for either J or SJ index. Together with Theorem 3.3 and Theorem 3.6, we can conclude that among all bicyclic connected chain graphs of order $n\ge 7$, the graph with the minimum J index and SJ index is

$$G\left(1,2,\left\lfloor\frac{n-5}{2}\right\rfloor;1,1,\left\lceil\frac{n-5}{2}\right\rceil\right).$$

The graph with the maximum J index and SJ index is

$$G\left(1,1;3,n-5\right).$$

In fact, we can order all bicyclic connected chain graphs of order $n \geq 7$ by J index and SJ index, the ordering for the J index can be derived from Theorem 3.3 $\sim$ 3.8, and for the SJ index, the ordering is identical.

If $n \equiv 0(\mod 2)$, let $p = \dfrac{n-6}{2}$, then

$$\begin{aligned}
& J\left(G\left(1,2,p;1,1,p+1\right)\right) < J(G(1,2,p+1;1,1,p)) \\
< \;& J\left(G\left(1,2,p-1;1,1,p+2\right)\right) < J(G(1,2,p+2;1,1,p-1)) \\
< \;& \ldots \\
< \;& J\left(G\left(1,2,1;1,1,2p\right)\right) < J\left(G\left(1,2,2p;1,1,1\right)\right) \\
< \;& J(G(1,2,0;1,1,2p+1)) = J(G(1,2;2,n-5)) \\
< \;& J(G(1,2,2p+1;1,1,0)) = J(G(1,1;3,n-5)),
\end{aligned}$$

If $n \equiv 1(\mod 2)$, let $p = \dfrac{n-5}{2}$, then

$$\begin{aligned}
& J\left(G\left(1,2,p;1,1,p\right)\right) < J(G(1,2,p-1;1,1,p+1)) \\
< \;& J\left(G\left(1,2,p+1;1,1,p-1\right)\right) < J(G(1,2,p-2;1,1,p+2)) \\
< \;& \ldots \\
< \;& J(G(1,2,2p-1;1,1,1)) < J(G(1,2,0;1,1,2p)) = J(G(1,2;2,n-5)) \\
< \;& J(G(1,2,2p;1,1,0)) = J\left(G\left(1,1;3,n-5\right)\right).
\end{aligned}$$

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] A Alazemi, M Anđelić, S K Simić. *Eigenvalue location for chain graphs*, Linear Algebra and its Applications, 2016, 505: 194-210.

[2] A Alazemi, M Anđelić, K C Das, et al. *Chain graph sequences and Laplacian spectra of chain graphs*, Linear and Multilinear Algebra, 2023, 71(4): 569-585.

[3] A Alazemi, M Anđelić, T Koledin, et al. *Eigenvalues-free intervals of distance matrices of threshold graphs and chain graphs*, Linear and Multilinear Algebra, 2021, 69(16): 2959-2975.

[4] A Alazemi, M Anđelić, A Salim. *On main eigenvalues of chain graphs*, Computational and Applied Mathematics, 2021, 40: 268.

[5] M Anđelić, E Andrade, D M Cardoso, et al. *Some new considerations about double nested graphs*, Linear Algebra and its Applications, 2015, 483: 323-341.

[6] M Aouchiche, P Hansen. *Distance spectra of graphs: a survey*, Linear Algebra and its Applications, 2014, 458: 301-386.

[7] F K Bell, D Cvetković, P Rowlinson, et al. *Graphs for which the least eigenvalue is minimal, I*, Linear Algebra and its Applications, 2008, 429(1): 234-241.

[8] F K Bell, D Cvetković, P Rowlinson, et al. *Graphs for which the least eigenvalue is minimal, II*, Linear Algebra and its Applications, 2008, 429(8-9): 2168-2179.

[9] J A Bondy, U S R Murty. *Graph theory*, New York: Springer, 2008.

[10] A E Brouwer, W H Haemers. *Spectra of Graphs*, Heidelberg: Spinger, 2012.

[11] K C Das. *On the Balaban Index of Chain Graphs*, Bulletin of the Malaysian Mathematical Sciences Society, 2021, 44: 2123-2138.

[12] K C Das, K Xu. *On Relation Between Kirchhoff Index, Laplacian-energy-like invariant and Laplacian energy of graphs*, Bulletin of the Malaysian Mathematical Sciences Society, 2016, 39: 59-75.

[13] K C Das, A Alazemi, M Anđelić. *On energy and Laplacian energy of chain graphs*, Discrete Applied Mathematics, 2020, 284: 391-400.

[14] H Deng. *On the Balaban Index of Trees*, MATCH Communications in Mathematical and in Computer Chemistry, 2011, 66(1): 253-260.

[15] H Deng. *On the sum-Balaban Index*, MATCH Communications in Mathematical and in Computer Chemistry, 2011, 66(1): 273-284.

[16] B Deng, X Zhu. *Minimal Balaban index on two classes of graphs*, Ars Combinatoria, 2019, 147: 203-213.

[17] J Devillers, A T Balaban. *Topological Indices and Related Descriptors in QSAR and QSPR*, Singapore: Gordon and Breach Science, 1999.

[18] C Godsil, G Royel. *Algebraic Graph Theory*, Berlin: Springer, 2001.

[19] O Ivancius, T Ivanciuc, A T Balaban. *Design of topological indices. Part 10. Parameters based on electronegativity and vovalent radius for the computation of molecular graph descriptors for heteroatom-containing molecules*, Journal of Chemical Information and Compututer Sciences, 1998, 38: 395-495.

[20] H Lei, H Yang. *Bounds for the Sum-Balaban index and (revised) Szeged index of regular graphs*, Applied Mathematics and Computation, 2015, 268: 1259-1266.

[21] F Li, Q Ye, J Rada. *The augmented Zagreb indices of fluoranthene-type benzenoid systems*, Bulletin of the Malaysian Mathematical Sciences Society, 2019, 42: 1119-1141.

[22] J Liu, C Wang, S Wang, et al. *Zagreb Indices and Multiplicative Zagreb Indices of Eulerian Graphs*, Bulletin of the Malaysian Mathematical Sciences Society, 2019, 42: 67-78.

[23] L Lu, Q Huang, X Huang. *The graphs with exactly two distance eigenvalues different from* -1 *and* -3, Journal of Algebraic Combinatorics, 2017, 45: 629-647.

[24] Y Mei, C Gao, M Liu. *The bounds of the energy and Laplacian energy of chain graphs*, AIMS Mathematics, 2021, 6(5): 4847-4859.

[25] M Mogharrab, M J Nadjafi-Arani, G H Fath-Tabar, et al. *Some bounds on Balaban index of a graph*, Utilitas Mathematica, 2011, 84: 325-331.

[26] L Sun. *Bounds on the Balaban index of trees*, MATCH Communications in Mathematical and in Computer Chemistry, 2010, 63(3): 813-818.

[27] R Xing, B Zhou, A Gravac. *On sum-Balaban index*, Ars Combinatoria, 2012, 104: 211-223.

[28] Z Xiong, Y Hou. *On the eigenvalues and Seidel eigenvalues of chain graphs*, Discrete Applied Mathematics, 2024, 351: 44-53.

[29] B Zhou, N Trinajsti. *Bounds on the Balaban index*, Croatica Chemica Acta, 2008, 81: 319-323.

[1]School of Mathematics, Renmin University of China, Beijing 100872, China.
Email: xzlv@ruc.edu.cn
[2]Beijing No.11 High School, Beijing 100062, China.
Email: mmy971126@163.com
[3]School of Information, Renmin University of China, Beijing 100872, China.
Email: megaowier@sjtu.edu.cn