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# Periodic solutions and fixed-time synchronization of discontinuous time-varying delayed Cohen-Grossberg neural networks

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**Abstract**. In this paper, a class of discontinuous Cohen-Grossberg neural networks with time-varying delays is considered. Firstly, under the extended Filippov differential inclusions framework, the problem of periodic solutions of the considered neural networks with more relaxed conditions imposed on the amplification functions is analyzed by using set-valued mapping and Kakutani's fixed point theorem, which has rarely been used to study such problem. Secondly, the fixed-time synchronization of the error system of the considered neural networks is also investigated by designing a novel control strategy, which can improve not only the previous ones with sign function greatly, but also can reduce the chattering phenomenon. Finally, two numerical examples are presented to further illustrate the validity of the obtained results.

## §1 Introduction

Cohen-Grossberg neural networks (CGNNs) were first proposed in 1983 [8], and then became one of the most commonly used and representative neurodynamic models. Such as Hopfield neural networks [38], cellular neural networks [7], and Lotka-Volterra system [29] are all special cases. It has captured increasing attention due to its wide applications in classification [10], engineering [17], parallel computing [3], associative memory [18], especially in solving some optimization problems [6], and so on. Clearly, these applications heavily rely on the existence of equilibrium points, as well as the qualitative features of stability. Therefore, qualitative characterizations of these dynamical behaviors are critical.

Actually, discontinuities can cause bad effects on qualitative analysis since traditional theories, particularly the existence theory of solutions, are no longer valid for discontinuous systems.

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Filippov, a Soviet mathematician, pioneered the differential inclusion theory for studying discontinuous systems in 1988 [13]. The solution of the discontinuous differential equation can be turned into a solution of its associated differential inclusion by creating an appropriate set-valued mapping. It is generally known that Forti and Nistri were the first to deal with the global stability of neural networks modeled by a differential equation with a discontinuous right-hand side [14], which was based on the theory of Filippov differential inclusions. Neural networks with discontinuous neuron activations, as Forti and Nistri pointed out, are significant and appear often in applications. Exploring the dynamical dynamics of discontinuous neural networks is so crucial in practice. In reality, periodic solutions play a significant role in terms of practical importance, such as associative memory [35], pattern recognition [12], machine learning [16], many biological and cognitive activities (for example, heartbeat, locomotion, respiration, mastication, and memorization) require repetition. Despite the importance of the periodic solutions of neural networks, there are few works focusing on the periodic solutions of the CGNNs, see [26, 39] and there are many almost-periodic solutions, see [2, 15, 32]. However, the definitions between periodic solution and almost-periodic solution are different. Besides, the CGNNs studied in [26, 39] had continuous activation functions. In recent years, periodic solutions of discontinuous CGNNs have been rarely considered, see [9, 25, 34]. However, the conditions imposed on the amplification functions in [9, 25, 34] were all bounded. It is limited. So, providing some relaxed conditions imposed on the amplification functions and further considering the periodic solutions of discontinuous CGNNs are vital. It is the first target of this paper.

In previous studies, the proof of the existence of periodic solution of CGNNs model was passed by contraction mapping fixed point theorem [37], Mawhin's continuation theorem of the set-valued theorem [27], and so on. However, Kakutani's fixed point theorem [21] has never been used to prove the periodic solution of the CGNNs model, so this paper aims to fill the gap. It is the second target of this paper.

On the other hand, the synchronization of real systems has been extensively investigated due to its wide applications in communication security and neuroscience [33]. Asymptotic synchronization and exponential synchronization are incorporated in the category of the infinite time synchronization [5, 36]. In many practical applications, the synchronization is required to be realized in some finite-time and fixed-time instead of asymptotically, which leads to the study of finite-time and fixed-time synchronization of CGNNs via some designed controllers, see, for example [4, 22] and the references therein. Moreover, the finite-time and fixed-time synchronization can be reduced to the finite-time and fixed-time stabilization of the corresponding error systems. However, the previously designed controllers all contained the sign function, which can bring chattering phenomena [11]. Therefore, new control law should be designed, which can help achieve synchronization and reduce the chattering phenomena. It is the third target of this paper.

Motivated by the above discussions, this paper investigates the periodic solutions of a class of discontinuous time-varying delayed CGNNs via Kakutani's fixed point theorem under more relaxed conditions imposed on the amplification functions. Moreover, fixed-time synchronization is also considered by designing a new control law without containing the sign function.

The major contributions of the paper can be stated as following aspects.

- Some relaxed conditions imposed on the amplification functions are given, which can improve the bounded conditions in the previous works, such as [2], [15], [26] and [36].
- Kakutani's fixed point theorem is first used to prove the periodic solution of the formulated discontinuous time-varying delayed CGNNs.
- A novel control law is designed, which can help achieve synchronization and reduce the chattering phenomena.
- A more accurate estimation of the settling-time is given in compared to the previous ones when studying the fixed-time synchronization of discontinuous time-varying delayed CGNNs.

## §2 Preliminaries and model formulation

*Notation.* Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. The superscript $T$ represents the transpose operator. For $x \in \mathbb{R}^n$, $\|x\|$ represents any vector norm of $x$. Given a set $\mathbb{E} \subset \mathbb{R}^n$ , by meas($\mathbb{E}$) we mean the Lebesgue measure of set $\mathbb{E}$ in $\mathbb{R}^n$ and $\overline{co}[\mathbb{E}]$ represents the closure of the convex hull of $\mathbb{E}$. If $z \in \mathbb{R}^n$ and $\delta > 0, \mathfrak{B}(z, \delta) = \{z^* \in \mathbb{R}^n : \|z^* - z\| \leq \delta\}$ denotes the ball of $\delta$ about $z$. Let $L^1([0, \xi), \mathbb{R}^n)$, $\xi \leq +\infty$ denote the Banach space of the Lebesgue integrable functions $g : [0, \xi) :\to \mathbb{R}^n$ equipped with the norm $\int_0^\xi \|g(t)\| \mathrm{d}t$ .

For any continuous $\omega$-periodic function $h(t)$ defined on $\mathbb{R}$, we denote

$$\bar{h} = \frac{1}{\omega} \int_0^\omega h(t) \mathrm{d}t, \ h^M = \sup_{t \in [0, \omega]} |h(t)|, \ h^L = \inf_{t \in [0, \omega]} |h(t)|.$$

### 2.1 Preliminaries

Let $\mathbb{R}^n (n \geq 1)$ denote an $n$-dimensional real Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Given $X \subseteq \mathbb{R}^n$, let us introduce following notations

$$\mathcal{P}_0(X) = \{A \subset X : A \neq \emptyset\}, \mathcal{P}(X) = \mathcal{P}_0(X) \bigcup \{\emptyset\},$$
$$\mathcal{P}_c(X) = \{A \subset X : \text{nonempty closed and convex}\},$$
$$\mathcal{P}_{kc}(X) = \{A \subset X : \text{nonempty compact and convex}\}.$$

For convenience, we sometimes denote $2^X = \mathcal{P}_0(X)$. Suppose $X \subseteq \mathbb{R}^n$, map $x \mapsto F(x)$ is called a set-valued map from $X \hookrightarrow p(\mathbb{R}^n)$, if to every point $x$ of the set $X$, there corresponds a nonempty set $F(x) \subset \mathbb{R}^n$.

Consider the following non-autonomous functional differential equation

$$\dot{z}_i(t) = f(t, z_t), \tag{2.1}$$

where $x_t(\cdot)$ denotes the history of the state from time $t-\tau$, up to the present time $t$; $\dot{z}_i(t)$ denotes the time derivative of $z$ and $f:\mathbb{R}\times C\to\mathbb{R}^n$ is measurable and essentially locally bounded. In this case, $f(t,z_t)$ denotes a vector field which is allowed to be discontinuous.

Let us construct the Filippov set-valued map $F:\mathbb{R}\times C\to 2^{\mathbb{R}^n}$ given as follows:

$$F(t,x_t)=\bigcap_{\delta>0}\ \bigcap_{means(\mathcal{N})=0}\overline{co}[f(t,\mathfrak{B}(z_t,\delta)\backslash\mathcal{N})].$$

Here means($\mathcal{N}$) stands for the Lebesgue measure of set $\mathcal{N}$, intersection is taken over all sets $\mathcal{N}$ of Lebesgue measure zero and over all $\delta>0$, where $\mathfrak{B}(z_t,\delta):=\{z_t^{'}\in C_\tau;\|z_t^{'}-z_t\|_C<\delta|\}$.

**Definition 2.1.** ([13]) A vector-valued function $z(t)$ defined on a non-degenerate interval $I\subseteq\mathbb{R}$ is said to be a Filippov solution for functional differential equation (2.1), if it is absolutely continuous on any compact subinterval $[t_1,t_2]$ of $\mathbb{I}$, and $z(t)$ satisfies the following functional differential inclusion

$$\dot{z}_i(t)\in F(t,z_t),\quad for\ a.e.\ t\in\mathbb{I}. \tag{2.2}$$

***Lemma* 2.1.** ([1, Kakutani's fixed point theorem]) If $\Omega$ is a compact convex subset of a Banach space $Z$ and the set-valued map $\varphi:\Omega\to\mathcal{P}_{kc}(\Omega)$ is an upper semicontinuous convex compact map, then $\varphi$ has a fixed point in $\Omega$, that is to say, there exists $z\in\Omega$ such that $z\in\varphi(z)$.

For convenience, we denote $\mathbb{D}=[0,\omega]$.

Let $\mathbb{F}(T,Z)=(\mathbb{F}_1(T,Z),\mathbb{F}_2(T,Z),...,\mathbb{F}_n(T,Z))^\top$ be a set-valued function and $L^1(\mathbb{D},\mathbb{R}^n)$ denotes the Banach space of all functions $\varsigma=(\varsigma_1,\varsigma_2,...,\varsigma_n)^\top\ :\ \mathbb{D}\to\mathbb{R}^n$ which are Lebesgue integrable.

Define the following set-valued operator

$$\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2,...,\mathcal{F}_n)^\top\ :\ X\to L^1(\mathbb{D},\mathbb{R}^n), \tag{2.3}$$

where

$$\mathcal{F}_i(z)=\{\varsigma_i\in L^1(\mathbb{D},\mathbb{R}^n):\varsigma_i(t)\in\mathbb{F}_i(t,z(t))\ for\ a.e.\ t\in\mathbb{D}\},\quad i\in\mathbb{N}=\{1,2,...,n\}.$$

***Lemma* 2.2.** ([19, 28]) Consider $\mathbb{D}=[0,\omega]$ which is a compact real interval. Let $\mathbb{F}$ be a Carathéodory set-valued map with $\mathcal{F}(z)\neq\emptyset$ for each fixed $z\in Z$ and let $\mathcal{L}:\ L^1(\mathbb{D},\mathbb{R}^n)\to C(\mathbb{D})$ be a continuous linear mapping. Then the operator $\mathcal{L}\circ\mathcal{F}:\ C(\mathbb{D})\to 2^{C(\mathbb{D})}$ is a closed graph operator in $C(\mathbb{D})\times C(\mathbb{D})$.

**Definition 2.2.** ([31]) The origin of error system is said to be globally fixed-time stable if it is globally uniformly finite-time stable and the settling time $T$ is globally bounded, *i.e.*, $\exists\,T_{\max}\in\mathbb{R}_+$ such that $T(e_0)\le t_{\max},\ \forall e_0\in\mathbb{R}^n$.

***Lemma* 2.3.** ([30]) If there exists a continuous radically unbounded function $V:\mathbb{R}^n\to\mathbb{R}_+\bigcup\{0\}$ such that

(1) $V(z)=0\Rightarrow z=0$;

(2) Any solution $z(t)$ of (2.1) satisfies $\dot{\tilde{V}}(z(t))\le-\check{a}V^\pi(z(t))-\check{b}V^\varpi(z(t))$, for some $\check{a},\check{b}>0,\ 0<\varpi<1<\pi$, where $\dot{\tilde{V}}$ is the set-valued Lie derivative of $V$. Then, $V(z(t))=0,\ t\ge T(z_0)$, with the settling time bounded by

$$T(z_0)\le T^1_{\max}=\frac{1}{\check{a}(\pi-1)}+\frac{1}{\check{b}(1-\varpi)}.$$

***Lemma* 2.4.** ([24]) Suppose $V(\cdot):\mathbb{R}^n\to\mathbb{R}_+\bigcup\{0\}$ is a continuous radially unbounded function, and the following condition hold $\dot{V}(z(t))\le \rho V(z(t))-\check{a}V^{\pi}(z(t))-\check{b}V^{\varpi}(z(t))-\hbar$, where $\check{a},\check{b},\hbar>0$ and $0\le\varpi<1<\pi$. Then the zero solution of system (2.1) is stable in a fixed-time and the settling time is

$$T^2_{\max}=\frac{2^{\pi-1}(\check{a}^{\frac{1}{\pi}}+\hbar^{\frac{1}{\pi}})^{1-\pi}}{\check{a}^{\frac{1}{\pi}}(\pi-1)}+\frac{(\check{b}^{\frac{1}{\varpi}}+\hbar^{\frac{1}{\varpi}})^{1-\varpi}-\hbar^{\frac{1-\varpi}{\varpi}}}{\check{b}^{\frac{1}{\varpi}}(1-\varpi)}.$$

***Lemma* 2.5.** ([22]) Let $\epsilon_1,\epsilon_2,...,\epsilon_p\ge 0$, $0<m\le 1$. Then the following inequality holds

$$\sum_{i=1}^{p}\epsilon_i^m\ge\Big(\sum_{i=1}^{p}\epsilon_i\Big)^m.$$

## 2.2 Model formulation

In this paper, we consider the following CGNNs with time-varying delays

$$\dot{x_i}(t)=-d_i(x_i(t))\Big(a_i(x_i(t))-\sum_{j=1}^{N}b_{ij}f_j(x_j(t))-\sum_{j=1}^{N}c_{ij}g_j(x_j(t-\tau_j(t)))-I_i\Big),\tag{2.4}$$

with initial conditions $x_{i0}(\theta)=\phi_i(\theta)$, $\theta\in[-\tau,0]$, where $i\in\mathbb{N}=\{1,2,...,N\}$, $N\ge 2$ is the number of neurons in the network, $x_i(t)$ represents the state variable of the $i$ th neuron, $d_i(x_i(t))$ denotes the amplification function, $a_i(x_i(t))$is an appropriately behaved function, matrices $B=(b_{ij})_{n\times n}\in\mathbb{R}^{n\times n}$ and $C=(c_{ij})_{n\times n}\in\mathbb{R}^{n\times n}$ are the connection weight matrix and delayed connection weight matrix, respectively, $f_j$ and $g_j$ denote the activation functions. $\tau_j(t)$ corresponds to the time varying delays resulting from the finite speed of the axonal signal transmission and satisfies $\tau=\max\limits_{1\le j\le n}\sup\limits_{t\in\mathbb{R}}|\tau_j(t)|$, $I_i$ is the external input to the $i$th neuron.

To derive the main results, the following conditions are introduced.

(H1) For $i\in\mathbb{N}$, $d_i(x)\ne 0$.

(H2) For every $i\in\mathbb{N}$, $f_i$(or $g_i$): $\mathbb{R}\to\mathbb{R}$ is continuous except on a countable set of isolated points $\{\rho_k^i\}$, where there exist finite right and left limits, $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$ ($g_i^+(\rho_k^i)$ and $g_i^-(\rho_k^i)$), respectively. Moreover, $f_i$ (or $g_i$) has at most a finite number of discontinuities on any compact interval of $\mathbb{R}$.

(H3) There exist nonnegative constants $\alpha_i,\alpha_i^*$ and $\beta_i,\beta_i^*$ such that

$$\sup_{\gamma_i\in\overline{co}[f_i(x_i)]}|\,\gamma_i|\le\alpha_i|x_i|+\beta_i,\ \forall x_i\in\mathbb{R},$$

$$\sup_{\gamma_i^*\in\overline{co}[g_i(x_i)]}|\,\gamma_i^*|\le\alpha_i^*|x_i|+\beta_i^*,\ \forall x_i\in\mathbb{R},$$

where, for $\theta\in\mathbb{R}$,

$$\overline{co}[f_i(\theta)]=[\min\{f_i^-(\theta),f_i^+(\theta)\},\max\{f_i^-(\theta),f_i^+(\theta)\}],$$

$$\overline{co}[g_i(\theta)]=[\min\{g_i^-(\theta),g_i^+(\theta)\},\max\{g_i^-(\theta),g_i^+(\theta)\}].$$

**Remark 2.1.** The study of periodic solutions of discontinuous CGNNs is insufficient, see [9, 34]. However, the conditions imposed on the amplification functions in [9, 34] were all bounded. It is limited. Condition (H1) is more relaxed. Thus, some previous works can be improved.

Choose the transformation function $h_i(x)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}(h_i(x)) = \frac{1}{d_i(x)},\ h_i(0)=0. \tag{2.5}$$

According to (H1), $\frac{1}{d_i(x)}$ exists, which is positive and continuous for all $x \in \mathbb{R}$, thus, $h_i(x)$ is a strictly increasing function with respect to $x$ . Letting $z_i(t) = h_i(x_i(t))$, it can be obtained directly that $\dot{z}_i(t) = \dot{h}_i(x_i(t))\dot{x}_i(t) = \frac{1}{d_i(x_i(t))}\dot{x}_i(t)$, $x_i(t) = h_i^{-1}(z_i(t))$. Substituting the above variable transformations into systems (2.4) can get

$$\dot{z}_i(t) = -a_i(h_i^{-1}(z_i(t))) + \sum_{j=1}^{N} b_{ij} f_j(h_j^{-1}(z_j(t))) + \sum_{j=1}^{N} c_{ij} g_j(h_j^{-1}(z_i(t-\tau_j(t)))) + I_i. \tag{2.6}$$

For later discussion, we always assume that $\tau_j(t)$, $j \in \mathbb{N}$, are continuously $\omega$-periodic functions in $\mathbb{R}$.

For each $i \in \mathbb{N}$, let $a_i(x_i(t)) = v_i(t)h_i(x_i(t))$, where $v_i(t)$ is continuously $\omega$-periodic function in $\mathbb{R}$. Substituting the above variable transformations into (2.6) gives

$$\dot{z}_i(t) = -v_i(t)z_i(t) + \sum_{j=1}^{N} b_{ij} f_j(h_j^{-1}(z_j(t))) + \sum_{j=1}^{N} c_{ij} g_j(h_j^{-1}(z_i(t-\tau_j(t)))) + I_i. \tag{2.7}$$

By using [1], there exists a measurable function $\gamma_j(t) \in \overline{co}[f_j(h_j^{-1}(z_j(t)))]$, $\gamma_j^*(t) \in \overline{co}[g_j(h_j^{-1}(z_j(t)))]$, such that

$$\dot{z}_i(t) = -v_i(t)z_i(t) + \sum_{j=1}^{N} b_{ij}\gamma_j(t) + \sum_{j=1}^{N} c_{ij}\gamma_j^*(t-\tau_j(t)) + I_i. \tag{2.8}$$

## §3 Periodic solutions analysis

In this section, Kakutani's fixed point theorem of set-valued maps is used to study the existence of periodic solutions of (2.4).

First of all, let $Z = \{z(t) = (z_1(t), z_2(t), ..., z_n(t))^\top \in C(\mathbb{R}, \mathbb{R}^n) :\ z(t+\omega) = z(t)\}$. For a positive constant vector $\eta = (\eta_1, \eta_2, ..., \eta_n)^\top \in \mathbb{R}^n$ with $\eta_i > 0$, let us define the norm

$$\|z\|_Z = \max_{1\le i\le n} |z_i|_\eta,\ |z_i|_\eta = \sup_{t\in[0,\omega]} \Big\{\frac{1}{\eta_i}|z_i(t)|\Big\}.$$

Then $Z$ is a Banach space endowed with the above norm $\|\cdot\|_Z$.

If $z(t) = (z_1(t), z_2(t), ..., z_n(t))^\top \in Z$ is a Filippov solution of (2.7), then it follows from (2.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[z_i(t)\exp\Big\{\int_0^t v_i(s)\mathrm{d}s\Big\}\Big] \in \exp\Big\{\int_0^t v_i(s)\mathrm{d}s\Big\}\mathbb{F}_i(t,z), \tag{2.9}$$

where

$$\mathbb{F}_i(t,z) = \sum_{j=1}^{N} b_{ij}\overline{co}[f_j(h_j^{-1}(z_j(t)))] + \sum_{j=1}^{N} c_{ij}\overline{co}[g_j(h_j^{-1}(z_j(t-\tau_j(t))))] + I_i.$$

By integrating both sides of differential inclusion (3.1) over the interval $[t, t+\omega]$, we can get the following nonlinear integral inclusions

$$z_i(t) \in \int_t^{t+\omega} G_i(t,s)\mathbb{F}_i(s,z(s))\mathrm{d}s, \tag{2.10}$$

where $G_i(t,s)$ denotes the Green function and it is given by

$$G_i(t,s)=\frac{1}{1-\exp\{-\omega\overline{v}_i\}}\exp\Big\{-\int_s^{t+\omega}v_i(\sigma)\mathrm{d}\sigma\Big\}, \tag{2.11}$$

for $t\le s\le t+\omega$, and

$$G_i(t,s)\le\frac{1}{1-\exp\{-\omega\overline{v}_i\}}\triangleq\bar{G}_i. \tag{2.12}$$

Obviously, the existence of $\omega$-periodic solution of (2.7) is equivalent to that of (3.2).

Before proving the main results, the following condition is also needed

(H4) There exist positive constants $\eta_1,\eta_2,...,\eta_n$ such that

$$\mathcal{C}=\max_{1\le i\le n}\sup_{t\in[0,\omega]}\Big\{\frac{1}{\eta_i}\Big[\sum_{j=1}^N\eta_jd_j^M(\alpha_jb_{ij}^M+\alpha_j^*c_{ij}^M)\Big]\int_t^{t+\omega}G_i(t,s)\mathrm{d}s\Big\}<1. \tag{2.13}$$

And, there exists a sufficient large $\mathcal{B}\ge1$ such that

$$\mathcal{C}\le1-\frac{\mathcal{A}}{\mathcal{B}}, \tag{2.14}$$

where

$$\mathcal{A}=\max_{1\le i\le n}\Big\{\frac{\omega\bar{G}_i}{\eta_i}\Big[\sum_{j=1}^n(b_{ij}^M\beta_j+c_{ij}^M\beta_j^*)+I_i^M\Big]\Big\}. \tag{2.15}$$

Define a compact convex subset $\Omega\subset Z$ with

$$\Omega=\{z(t)=(z_1(t),z_2(t),...,z_n(t))^\top\in Z:\ \|z\|_Z\le\mathcal{B}\}, \tag{2.16}$$

and further define a set-valued map $\Phi:\ Z\to P_{kc}(Z)$,

$$\Phi(z)(t)=(\Phi_1(z)(t),\Phi_2(z)(t),...,\Phi_n(z)(t))^\top, \tag{2.17}$$

where

$$\Phi_i(z)(t)=\int_t^{t+\omega}G_i(t,s)\mathbb{F}_i(s,z(s))\mathrm{d}s,\ i\in\mathbb{N}. \tag{2.18}$$

***Theorem* 3.1.** $\Phi(z)\in P_{kc}(\Omega)$ for each fixed $z\in\Omega$.

*Proof.* Let $z=(z_1,z_2,...,z_n)^\top\in\Omega$ and $u=(u_1,u_2,...,u_n)^\top\in\Phi(z)$. Then there exists measurable functions $\gamma_i(t)\in\overline{co}[f_i(h_i^{-1}(z_i(t)))]$, $\gamma_i^*(t-\tau_i(t))\in\overline{co}[g_i(h_i^{-1}(z_i(t-\tau_i(t))))]$ such that

$$u_i(t)=\int_t^{t+\omega}G_i(t,s)\xi_i(s)\mathrm{d}s\in\int_t^{t+\omega}G_i(t,s)\mathbb{F}_i(s,z(s))\mathrm{d}s=\Phi_i(z)(t), \tag{2.19}$$

where

$$\xi_i(s)=\sum_{j=1}^Nb_{ij}\gamma_j(s)+\sum_{j=1}^Nc_{ij}\gamma_j^*(s-\tau_j(s))+I_i. \tag{2.20}$$

For any $z\in\Omega$, based on (H1)-(H4) and in view of (3.6)-(3.8), (3.11) and (3.12), we have

$$\begin{aligned}\Big|\frac{1}{\eta_i}u_i(t)\Big|=&\Big|\frac{1}{\eta_i}\int_t^{t+\omega}G_i(t,s)\Big(\sum_{j=1}^Nb_{ij}\gamma_j(s)+\sum_{j=1}^Nc_{ij}\gamma_j^*(s-\tau_j(s))+I_i\Big)\mathrm{d}s\Big|\\ \le&\frac{1}{\eta_i}\int_t^{t+\omega}G_i(t,s)\Big[\sum_{j=1}^N|b_{ij}|(\alpha_j|h_j^{-1}(z_j(s))|+\beta_j)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\eta_i}\int_t^{t+\omega}G_i(t,s)\Big[\sum_{j=1}^N(b_{ij}^M\beta_j+c_{ij}^M\beta_j^*)+I_i^M\Big]\mathrm{d}s\\
\leq&\frac{\mathcal{B}}{\eta_i}\sum_{j=1}^N\eta_j d_j^M(\alpha_j b_{ij}^M+\alpha_j^*c_{ij}^M)\int_t^{t+\omega}G_i(t,s)\mathrm{d}s\\
&+\frac{\omega\bar{G}_i}{\eta_i}\Big[\sum_{j=1}^N(b_{ij}^M\beta_j+c_{ij}^M\beta_j^*)+I_i^M\Big]\\
\leq&\mathcal{BC}+\mathcal{A}\leq\mathcal{B}(1-\frac{\mathcal{A}}{\mathcal{B}})+\mathcal{A}=\mathcal{B},
\end{aligned}
$$

which implies that $\|u(t)\|_Z=\max_{1\leq i\leq n}\sup_{t\in[0,\omega]}\left\{\frac{1}{\eta_i}|u_i(t)|\right\}\leq\mathcal{B}$, $\forall z\in\Omega$. Hence, it is clear that $\Phi:\Omega\to P_{kc}(\Omega)$. □

***Theorem* 3.2.** If the conditions (H1)-(H4) are satisfied, then $\Phi(z)$ defined in (3.9) is convex for each $z\in\Omega$.

*Proof.* Let $z=(z_1,z_2,...,z_n)^\top\in\Omega$ and $u=(u_1,u_2,...,u_n)^\top\in\Phi(z)$, $u^c=(u_1^c,u_2^c,...,u_n^c)^\top\in\Phi(z)$. Then there exists measurable functions $\gamma_i(t)\in\overline{co}[f_i(h_i^{-1}(z_i(t)))]$, $\gamma_i^*(t-\tau_i(t))\in\overline{co}[g_i(h_i^{-1}(z_i(t-\tau_i(t))))]$, $\gamma_i^c(t)\in\overline{co}[f_i(h_i^{-1}(z_i(t)))]$, $\gamma_i^{c*}(t-\tau_i(t))\in\overline{co}[g_i(h_i^{-1}(z_i(t-\tau_i(t))))]$ such that

$$
u_i^c(t)=\int_t^{t+\omega}G_i(t,s)\xi_i^c(s)\mathrm{d}s\in\int_t^{t+\omega}G_i(t,s)\mathbb{F}_i(s,z(s))\mathrm{d}s=\Phi_i(z)(t),
$$

where

$$
\xi_i^c(s)=\sum_{j=1}^N b_{ij}\gamma_j^c(s)+\sum_{j=1}^N c_{ij}\gamma_j^{c*}(s-\tau_j(s))+I_i.
$$

Let $0\leq\lambda\leq1$. Note that $\lambda\gamma_j(t)+(1-\lambda)\gamma_j^c(t)\in\overline{co}[f_j(h_j^{-1}(z_j(t)))]$ and $\lambda\gamma_j^*(t-\tau_j(t))+(1-\lambda)\gamma_j^{c*}(t-\tau_j(t))\in\overline{co}[g_j(h_j^{-1}(z_j(t-\tau_j(t))))]$, then it follows that

$$
\begin{aligned}
&\lambda\xi_i(t)+(1-\lambda)\xi_i^c(t)\\
=&\sum_{j=1}^N b_{ij}[\lambda\gamma_j(t)+(1-\lambda)\gamma_j^c(t)]+\sum_{j=1}^N c_{ij}[\lambda\gamma_j^*(t-\tau_j(t))+(1-\lambda)\gamma_j^{c*}(t-\tau_j(t))]\\
&+I_i\in\sum_{j=1}^N b_{ij}\overline{co}[f_i(h_i^{-1}(z_i(t)))]+\sum_{j=1}^N c_{ij}\overline{co}[g_j(h_j^{-1}(z_j(t-\tau_j(t))))]+I_i=\mathbb{F}_i(t,z).
\end{aligned}
$$

So, for all $t\in[0,\omega]$, we obtain

$$
\begin{aligned}
\lambda u_i(t)+(1-\lambda)u_i^c(t)=&\int_t^{t+\omega}G_i(t,s)[\lambda\xi_i(s)+(1-\lambda)\xi_i^c(s)]\mathrm{d}s\\
\in&\int_t^{t+\omega}G_i(t,s)\mathbb{F}_i(s,z(s))\mathrm{d}s=\Phi_i(z)(t).
\end{aligned}
$$

Thus, $\lambda u(t)+(1-\lambda)u^c(t)\in\Phi(z)$. Therefore, $\Phi(z)$ is convex, $\forall z\in\Omega$. □

***Theorem* 3.3.** If the conditions (H1)-(H4) are satisfied, then $\Phi:\Omega\to P_{kc}(\Omega)$ is a compact map.

*Proof.* Accoding to Ascoli-Arzela Theorem, we need to prove that $\Phi(\Omega)$ is uniformly bounded and equi-continuous. The following two steps can help finish the proof.

*Step 1.* We first show that $\Phi(\Omega)$ is a uniformly bounded set. Let $z=(z_1,z_2,...,z_n)^\top\in\Omega$ and $u=(u_1,u_2,...,u_n)^\top\in\Phi(z)$ be arbitrary. Then there exists measurable functions $\gamma_i(t)\in\overline{co}[f_i(h_i^{-1}(z_i(t)))]$, $\gamma_i^*(t-\tau_i(t))\in\overline{co}[g_i(h_i^{-1}(z_i(t-\tau_i(t))))]$, such that

$$|\gamma_i|\le\sup_{\gamma_i\in\overline{co}[f_i(h_i^{-1}(z_i(t)))]}|\gamma_i|\le\alpha_i|h_i^{-1}(z_i(t))|+\beta_i\le\alpha_i|d_i(t_0)||z_j(t)|+\beta_i\le\alpha_i\eta_i d_i^M\mathcal{B}+\beta_i,$$

$$\forall x_i\in\mathbb{R},$$

and

$$\begin{aligned}|\gamma_i^*|\le&\sup_{\gamma_i^*\in\overline{co}[g_i(h_i^{-1}(z_i(t-\tau_i(t))))]}|\gamma_i^*|\le\alpha_i^*|h_i^{-1}(z_i(t-\tau_i(t)))|+\beta_i^*\\ &\le\alpha_i^*|d_i(t_0)||z_i(t-\tau_i(t))|+\beta_i^*\le\alpha_i^*\eta_i d_i^M\mathcal{B}+\beta_i^*,\forall x_i\in\mathbb{R}.\end{aligned}$$

Then, for any $z\in\Omega$, we can get from (3.4), (3.11) and (3.12) that

$$\begin{aligned}\Big|\frac{1}{\eta_i}u_i(t)\Big|=&\Big|\frac{1}{\eta_i}\int_t^{t+\omega}G_i(t,s)\Big[\sum_{j=1}^N b_{ij}\gamma_j(s)+\sum_{j=1}^N c_{ij}\gamma_j^*(s-\tau_j(s))+I_i\Big]\mathrm{d}s\Big|\\ \le&\mathcal{BC}+\mathcal{A}\le\mathcal{B}(1-\frac{\mathcal{A}}{\mathcal{B}})+\mathcal{A}=\mathcal{B},\end{aligned}$$

which yields $\|u(t)\|_Z=\max_{1\le i\le n}\sup_{t\in[0,\omega]}\left\{\frac{1}{\eta_i}|u_i(t)|\right\}\le\mathcal{B},\ \forall z\in\Omega$. So, $\Phi(\Omega)$ is a uniformly bounded set for all $z\in\Omega$.

*Step 2.* We will conclude that $\Phi(\Omega)$ is an equi-continuous set. Let $t\in[0,\omega]$, for any $u\in\Phi(\Omega)$ and each $i\in\mathbb{N}$, it follows from (3.4) that

$$\begin{aligned}|u_i(t)-u_i(t^*)|=&\Big|\int_t^{t+\omega}G_i(t,s)\xi_i(s)\mathrm{d}s-\int_{t^*}^{t^*+\omega}G_i(t^*,s)\xi_i(s)\mathrm{d}s\Big|\\ \le&\Big|\int_t^{t+\omega}G_i(t,s)\xi_i(s)\mathrm{d}s-\int_t^{t+\omega}G_i(t^*,s)\xi_i(s)\mathrm{d}s\Big|\\ &+\Big|\int_t^{t+\omega}G_i(t^*,s)\xi_i(s)\mathrm{d}s-\int_{t^*}^{t^*+\omega}G_i(t^*,s)\xi_i(s)\mathrm{d}s\Big|\\ \le&\max_{t\le s\le t+\omega}\{|G_i(t,s)-G_i(t^*,s)|\}\int_0^\omega|\xi_i(s)|\mathrm{d}s\\ &+\bar{G}_i\Big|\int_t^{t^*}|\xi_i(s)|\mathrm{d}s\Big|+\bar{G}_i\Big|\int_{t+\omega}^{t^*+\omega}|\xi_i(s)|\mathrm{d}s\Big|,\ \forall z\in\Omega.\end{aligned}\tag{2.21}$$

For any $z\in\Omega$ and $i\in\mathbb{N}$, by using (H2), (H3) and (3.12), we get

$$\begin{aligned}|\xi_i(t)|\le&\sum_{j=1}^N|b_{ij}||\gamma_j(t)|+\sum_{j=1}^N|c_{ij}||\gamma_j^*(t-\tau_j(t))|+|I_i|\\ \le&\sum_{j=1}^n[\eta_j d_j^M\mathcal{B}(b_{ij}^M\alpha_j+c_{ij}^M\alpha_j^*)+(b_{ij}^M\beta_j+c_{ij}^M\beta_j^*)]+I_i^M\triangleq\mathcal{Y}_i.\end{aligned}\tag{2.22}$$

It follows from (3.13) and (3.14) that

$$|u_i(t)-u_i(t^*)|\le\max_{t\le s\le t+\omega}\{|G_i(t,s)-G_i(t^*,s)|\}\omega\mathcal{Y}_i+2\bar{G}_i\mathcal{Y}_i|t-t^*|,\quad\forall z\in\Omega.$$

Let $t \to t^*$, the right-hand side of the above inequality tends to zero. So, $\|u_i(t) - u_i(t^*)\| \to 0$ as $t \to t^*$, where $\|\cdot\|$ denotes any vector norm. Thus, $\Phi(\Omega)$ is equi-continuous. Therefore, $\Phi : \Omega \to P_{kc}(\Omega)$ is a compact map by the above two steps. □

***Theorem* 3.4.** If the conditions (H1)-(H4) are satisfied, then $\Phi : \Omega \to P_{kc}(\Omega)$ is upper semi-continuous.

*Proof.* According to [20], we only need to prove that $\Phi$ is a closed graph operator. Define a continuous linear operator $\mathcal{L} : L^1(\mathbb{D}, \mathbb{R}^n) \to C(\mathbb{D})$ given by

$$\mathcal{L}\xi(t) = \begin{pmatrix} \int_t^{t+\omega} G_1(t,s)\xi_1(s)\mathrm{d}s \\ \int_t^{t+\omega} G_2(t,s)\xi_2(s)\mathrm{d}s \\ \cdots \\ \int_t^{t+\omega} G_n(t,s)\xi_n(s)\mathrm{d}s \end{pmatrix}, t \in \mathbb{D}.$$

By *Lemma* 2.2, it follows that $\Phi = \mathcal{L} \circ \mathcal{F}$ is a closed graph operator. So, $\Phi$ is upper semi-continuous. □

***Theorem* 3.5.** Suppose that the conditions (H1)-(H4) are satisfied, then (2.4) has at least one $\omega$-periodic solution.

*Proof.* By using *Theorems* 3.1-3.4, it gives that all the conditions in *Lemma* 2.1 hold. So, (2.4) has at least one $\omega$-periodic solution. □

***Corollary* 3.1.** Suppose that (H1)-(H3) are satisfied and assume further that there exist positive constants $\eta_1, \eta_2, ..., \eta_n$ such that

$$\max_{1 \le i \le n} \Big\{ \frac{1}{\eta_i v_i^L} \sum_{j=1}^{N} \eta_j d_j^M (\alpha_j b_{ij}^M + \alpha_j^* c_{ij}^M) \Big\} < 1. \tag{2.23}$$

Then (2.7) has at least one $\omega$-periodic solution.

*Proof.* From the inequality (3.4) and (3.5), we can obtain

$$\begin{aligned} &\frac{1}{\eta_i}\Big[\sum_{j=1}^{N} \eta_j d_j^M (\alpha_j b_{ij}^M + \alpha_j^* c_{ij}^M)\Big] \int_t^{t+\omega} G_i(t,s)\mathrm{d}s \\ =&\frac{1}{\eta_i(1-\exp\{-\omega\overline{v}_i\})}\Big[\sum_{j=1}^{N} \eta_j d_j^M (\alpha_j b_{ij}^M + \alpha_j^* c_{ij}^M)\Big] \int_t^{t+\omega} \frac{v_i(s)\exp\{-\int_s^{t+\omega} v_i(\sigma)\mathrm{d}\sigma\}}{v_i(s)}\mathrm{d}s \\ \le&\frac{1}{\eta_i v_i^L(1-\exp\{-\omega\overline{v}_i\})}\Big[\sum_{j=1}^{N} \eta_j d_j^M (\alpha_j b_{ij}^M + \alpha_j^* c_{ij}^M)\Big] \int_t^{t+\omega} v_i(s)\exp\Big\{-\int_s^{t+\omega} v_i(\sigma)\mathrm{d}\sigma\Big\}\mathrm{d}s \\ =&\frac{1}{\eta_i v_i^L}\sum_{j=1}^{N} \eta_j d_j^M (\alpha_j b_{ij}^M + \alpha_j^* c_{ij}^M) < 1. \end{aligned}$$

This implies that the condition (H4) holds. According to *Theorem* 3.5, the CGNNs system (2.7) has at least one $\omega$-periodic solution. The proof is complete. □

**Remark 3.1.** In the past several years, almost-periodic solutions of the CGNNs have been widely considered, but few works focusing on the periodic solutions of the CGNNs have been studied (see [26, 39]). As we know, the definitions between periodic solution and almost-periodic solution are different. Besides, the CGNNs studied in [26, 39] had continuous activation functions. Periodic solutions of discontinuous CGNNs have been rarely considered, see [9, 34]. However, the conditions imposed on the amplification functions in [9, 25, 34] were all bounded. It is limited. This paper has provided a new approach to further study the discontinuous CGNNs with more relaxed conditions imposed on the amplification functions by using Kakutani's fixed point theorem. Actually, Kakutani's fixed point theorem has never been used to prove the periodic solution of CGNNs.

Consequently, some previous works on the periodic solutions of the CGNNs in [9, 25, 26, 34, 39], can be improved.

## §4 Fixed-time synchronization via no-chattering control

Let system (2.4) be the drive system and construct a controlled response system described by

$$\dot{y}_i(t) = -d_i(y_i(t))\Big(a_i(y_i(t)) - \sum_{j=1}^{N} b_{ij} f_j(y_j(t)) - \sum_{j=1}^{N} c_{ij} g_j(y_j(t-\tau_j(t))) - I_i\Big) + u_i(t), \tag{4.1}$$

with initial conditions $y_{i0}(\theta) = \varphi_i(\theta),\ \theta \in [-\tau, 0]$.

Substituting the above variable transformations same as (2.6) into the response system (4.1), we can get

$$\begin{aligned}\dot{w}_i(t) = &- a_i(h_i^{-1}(z_i(t))) + \sum_{j=1}^{N} b_{ij} f_j(h_j^{-1}(w_i(t))) \\ &+ \sum_{j=1}^{N} c_{ij} g_j(h_j^{-1}(w_i(t-\tau_i(t)))) + I_i + \frac{u_i(t)}{d_i(h_i^{-1}(w_i(t)))}.\end{aligned} \tag{4.2}$$

Denote $e_i(t) = w_i(t) - z_i(t)$, the error dynamics are obtained as

$$\begin{aligned}\dot{e}_i(t) = &- [a_i(h_i^{-1}(w_i(t))) - a_i(h_i^{-1}(z_i(t)))] + \sum_{j=1}^{N} [b_{ij}(f_j(h_j^{-1}(w_j(t))) - f_j(h_j^{-1}(z_j(t))))] \\ &+ \sum_{j=1}^{N} [c_{ij}(g_j(h_j^{-1}(w_j(t-\tau_j(t)))) - g_j(h_j^{-1}(z_j(t-\tau_j(t)))))] + \frac{u_i(t)}{d_i(h_i^{-1}(w_i(t)))},\end{aligned} \tag{4.3}$$

with initial conditions $e_{i0}(\theta) = h_i(\phi_i(\theta)) - h_i(\varphi_i(\theta))$, $\theta \in [-\tau, 0]$. Let $e_{i0} = (e_{10}(\theta), e_{20}(\theta), ..., e_{N0}(\theta))^\top$, $\theta \in [-\tau, 0]$.

By using differential inclusion theory, we have

$$\begin{aligned}\dot{e}_i(t) \in &- [a_i(h_i^{-1}(w_i(t))) - a_i(h_i^{-1}(z_i(t)))] + \sum_{j=1}^{N} [b_{ij}(\overline{co}[f_j(h_j^{-1}(w_j(t)))] - \overline{co}[f_j(h_j^{-1}(z_j(t)))])] \\ &+ \sum_{j=1}^{N} [c_{ij}(\overline{co}[g_j(h_j^{-1}(w_j(t-\tau_j(t))))] - \overline{co}[g_j(h_j^{-1}(z_j(t-\tau_j(t))))])] + \frac{u_i(t)}{d_i(h^{-1}(w_i(t)))}.\end{aligned} \tag{4.4}$$

If $e_i(t)$ is the Filippov solutions of system (4.3), then there exists a measurable functions $\gamma_{1j}(t) \in \overline{co}[f_j(h_j^{-1}(z_j(t)))]$, $\gamma_{2j}(t) \in \overline{co}[f_j(h_j^{-1}(w_j(t)))]$, $\gamma_{1j}^*(t-\tau_j(t)) \in \overline{co}[g_j(h_j^{-1}(z_j(t-\tau_j(t))))]$ and $\gamma_{2j}^*(t-\tau_j(t)) \in \overline{co}[g_j(h_j^{-1}(w_j(t-\tau_j(t))))]$, such that

$$\begin{aligned}\dot{e}_i(t) =& -[a_i(h_i^{-1}(w_i(t))) - a_i(h_i^{-1}(z_i(t)))] + \sum_{j=1}^N b_{ij}[\gamma_{2j}(t) - \gamma_{1j}(t)] \\ &+ \sum_{j=1}^N c_{ij}[\gamma_{2j}^*(t-\tau_j(t)) - \gamma_{1j}^*(t-\tau_j(t))] + \frac{u_i(t)}{d_i(h^{-1}(w_i(t)))}.\end{aligned} \tag{4.5}$$

In order to achieve the fixed-time synchronization, the following discontinuous control law is designed

$$u_i(t) = -\lambda_i - \frac{\varepsilon_i(t)}{|\varepsilon_i(t)|}(\xi_i|\varepsilon_i(t)| + \eta_i|\varepsilon_i(t-\tau_i(t))| + \kappa_i|\varepsilon_i(t)|^\pi + \gamma_i|\varepsilon_i(t)|^\varpi), \tag{4.6}$$

where $\varepsilon_i(t) = y_i(t) - x_i(t), 0 < \varpi < 1 < \pi, \lambda_i, \xi_i, \eta_i, \kappa_i$, and $\gamma_i$ are the parameters to be designed later.

(H5) The derivative of the amplification function $a_i(x)$ has a positive lower bound, i.e., there exists a positive constant $a_i$ such that $\dot{a}_i(x) \geq a_i > 0,\ x \in \mathbb{R}$.

***Theorem* 4.1.** Suppose that (H1), (H3), and (H5) hold, then the fixed-time synchronization of (2.4) and (4.1) under (4.6) is achieved if the design parameters are appropriately selected as following $\lambda_i > 0,\ \kappa_i > 0,\ \gamma_i > 0$,

$$\xi_i \geq \bar{d}_i\Big(-a_i + \Big(\sum_{j=1}^N b_{ji}^M \alpha_i\Big)\Big),\quad \eta_i \geq \bar{d}_i\Big(\sum_{j=1}^N c_{ji}^M \alpha_i^*\Big),\quad i \in \mathbb{N}.$$

Additionally, the settling time is given as

$$T_{\max}^1 = \frac{1}{(\pi-1)\Big(\min_i\Big\{\frac{\underline{d}_i^\pi}{\bar{d}_i}\kappa_i N^{1-\pi}\Big\}\Big)} + \frac{1}{(1-\varpi)\Big(\min_i\Big\{\frac{\underline{d}_i^\varpi}{\bar{d}_i}\gamma_i\Big\}\Big)}.$$

*Proof.* If $e_i = 0,\ i \in \mathbb{N}$, the zero solution of (4.3) achieves the fixed-time synchronization for all $t > 0$ . If $e_i \neq 0,\ i \in \mathbb{N}$, consider the following Lyapunov function candidate

$$V(e(t)) = \sum_{i=1}^N |e_i(t)| = \sum_{i=1}^N sgn(e_i(t))e_i(t).$$

The Lie derivative of $V(e(t))$ along the error dynamics (4.5) can be calculated

$$\begin{aligned}\dot{\tilde{V}} =& \sum_{i=1}^N sgn(e_i(t))\dot{e}_i(t) \\ =& \sum_{i=1}^N -sgn(e_i(t))[a_i(h_i^{-1}(w_i(t))) - a_i(h_i^{-1}(z_i(t)))] \\ &+ \sum_{i=1}^N sgn(e_i(t))\Big(\sum_{j=1}^N b_{ij}[\gamma_{2j}(t) - \gamma_{1j}(t)]\Big) \\ &+ \sum_{i=1}^N sgn(e_i(t))\Big(\sum_{j=1}^N c_{ij}[\gamma_{2j}^*(t-\tau_j(t)) - \gamma_{1j}^*(t-\tau_j(t))]\Big)\end{aligned}$$

$$+\sum_{i=1}^{N} sgn(e_i(t))\frac{u_i(t)}{d_i(h_i^{-1}(w_i(t)))}.$$

Since the behaved function $a_i(x_i(t))$ and the transformation function $w_i(x_i(t))$ are both strictly monotonically increasing, differentiable and $a_i(0) = 0$, $h_i(0) = 0$, $a_i(h_i^{-1}(z_i(t)))$ is also strictly monotonically increasing and differentiable with respect to $t$, thus, there exists a constant $\theta \in \mathbb{R}$ such that

$$\sum_{i=1}^{N} -sgn(e_i(t))[a_i(h_i^{-1}(w_i(t))) - a_i(h_i^{-1}(z_i(t)))]$$
$$= \sum_{i=1}^{N} -sgn(e_i(t))\dot{a}_i(h_i^{-1}(\theta))(y_i(t) - x_i(t)) \leq \sum_{i=1}^{N} -a_i|\varepsilon_i(t)|,$$

where $\varepsilon_i(t) = y_i(t) - x_i(t)$, and noting the fact that $h_i^{-1}(z_i(t))$ is strictly monotone increasing and $h_i^{-1}(0) = 0$, and one gets $sgn(\varepsilon_i(t)) = sgn(e_i(t))$.

According to (H3), we get

$$\sum_{i=1}^{N}\sum_{j=1}^{N} sgn(e_i(t))b_{ij}[\gamma_{2j}(t) - \gamma_{1j}(t)] \leq \sum_{i=1}^{N}\sum_{j=1}^{N} sgn(e_i(t))b_{ij}(t)[\alpha_j|h_j^{-1}(w_j(t))$$
$$- h_j^{-1}(z_j(t))|] \leq \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} b_{ji}^{M}\alpha_i\Big)|\varepsilon_i(t)|,$$

and

$$\sum_{i=1}^{N}\sum_{j=1}^{N} sgn(e_i(t))c_{ij}[\gamma_{2j}^{*}(t-\tau_j(t)) - \gamma_{1j}^{*}(t-\tau_j(t))]$$
$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N} sgn(e_i(t))c_{ij}(t)[\alpha_j^{*}|h_j^{-1}(w_j(t-\tau_j(t))) - h_j^{-1}(z_j(t-\tau_j(t)))|]$$
$$\leq \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} c_{ji}^{M}\alpha_i^{*}\Big)|\varepsilon_i(t-\tau_i(t))|.$$

Furthermore, it follows that

$$\sum_{i=1}^{N} sgn(e_i(t))\frac{u_i(t)}{d_i(h_i^{-1}(w_i(t)))}\sum_{i=1}^{N} sgn(e_i(t))\frac{1}{d_i(h_i^{-1}(w_i(t)))}\Big[-\lambda_i - \frac{\varepsilon_i(t)}{|\varepsilon_i(t)|}\cdot(\xi_i|\varepsilon_i(t)|$$
$$+ \eta_i|\varepsilon_i(t-\tau_i(t))| + \kappa_i|\varepsilon_i(t)|^{\pi} + \gamma_i|\varepsilon_i(t)|^{\varpi})\Big]$$
$$\leq -\sum_{i=1}^{N}\frac{\lambda_i}{\bar{d}_i} - \sum_{i=1}^{N}\frac{1}{\bar{d}_i}\xi_i|\varepsilon_i(t)| - \sum_{i=1}^{N}\frac{1}{\bar{d}_i}\eta_i|\varepsilon_i(t-\tau_{ij}(t))| - \sum_{i=1}^{N}\frac{\underline{d}_i^{\pi}}{\bar{d}_i}\kappa_i|e_i(t)|^{\pi} - \sum_{i=1}^{N}\frac{\underline{d}_i^{\varpi}}{\bar{d}_i}\gamma_i|e_i(t)|^{\varpi}.$$

Substituting the above inequalities into $\dot{V}$, it follows that

$$\dot{V} \leq \sum_{i=1}^{N}|\varepsilon_i(t)|\Big[-a_i + \Big(\sum_{j=1}^{N} b_{ji}^{M}\alpha_i\Big) - \frac{1}{\bar{d}_i}\xi_i\Big] + \sum_{i=1}^{N}|\varepsilon_i(t-\tau_i(t))|\Big[\Big(\sum_{j=1}^{N} c_{ji}^{M}\alpha_i^{*}\Big) - \frac{1}{\bar{d}_i}\eta_i\Big]$$

$$-\sum_{i=1}^{N}\frac{d_i^{\pi}}{\bar{d}_i}\kappa_i|e_i(t)|^{\pi}-\sum_{i=1}^{N}\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i|e_i(t)|^{\varpi}-\sum_{i=1}^{N}\frac{\lambda_i}{\bar{d}_i}\leq-\sum_{i=1}^{N}\frac{d_i^{\pi}}{\bar{d}_i}\kappa_i|e_i(t)|^{\pi}-\sum_{i=1}^{N}\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i|e_i(t)|^{\varpi}$$
$$\leq-\Big(\min_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}\Big)\Big(\sum_{i=1}^{N}|e_i(t)|\Big)^{\pi}-\Big(\min_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}\Big)\Big(\sum_{i=1}^{N}|e_i(t)|\Big)^{\varpi},\tag{4.7}$$

which together with *Lemma* 2.3 the drive-response system achieve the robust fixed-time synchronization, and the settling time is

$$T_{\max}^1=\frac{1}{(\pi-1)\Big(\min_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}\Big)}+\frac{1}{(1-\varpi)\Big(\min_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}\Big)}.$$

□

***Theorem* 4.2.** Suppose that (H1), (H3), and (H5) hold, then the fixed-time synchronization of (2.4) and (4.1) under (4.6) is achieved if the design parameters are appropriately selected as following $\lambda_i>0,\ \kappa_i>0,\ \gamma_i>0$,

$$\xi_i<\bar{d}_i\Big(-a_i+\Big(\sum_{j=1}^{N}b_{ji}^{M}\alpha_i\Big)\Big),\ \rho=\Big[-a_i+\Big(\sum_{j=1}^{N}b_{ji}^{M}\alpha_i\Big)-\frac{1}{\bar{d}_i}\xi_i\Big]<0,\quad i\in\mathbb{N}.$$

And the settling time is given as

$$T_{\max}^2=\frac{2^{\pi-1}\Big(\min\limits_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}^{\frac{1}{\pi}}+\hbar^{\frac{1}{\pi}}\Big)^{1-\pi}}{\min\limits_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}^{\frac{1}{\pi}}(\pi-1)}+\frac{\Big(\min\limits_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}^{\frac{1}{\varpi}}+\hbar^{\frac{1}{\varpi}}\Big)^{1-\varpi}-\hbar^{\frac{1-\varpi}{\varpi}}}{\min\limits_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}^{\frac{1}{\varpi}}(1-\varpi)},$$

where $\hbar=\sum_{i=1}^{N}\frac{\lambda_i}{\bar{d}_i}>0$.

*Proof.* According to the proof of *Theorem* 4.1, we can get

$$\begin{aligned}\dot{V}\leq&\sum_{i=1}^{N}|\varepsilon_i(t)|\Big[-a_i+\Big(\sum_{j=1}^{N}b_{ji}^{M}\alpha_i\Big)-\frac{1}{\bar{d}_i}\xi_i\Big]+\sum_{i=1}^{N}|\varepsilon_i(t-\tau_i(t))|\Big[\Big(\sum_{j=1}^{N}c_{ji}^{M}\alpha_i^*\Big)-\frac{1}{\bar{d}_i}\eta_i\Big]\\&-\sum_{i=1}^{N}\frac{d_i^{\pi}}{\bar{d}_i}\kappa_i|e_i(t)|^{\pi}-\sum_{i=1}^{N}\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i|e_i(t)|^{\varpi}-\sum_{i=1}^{N}\frac{\lambda_i}{\bar{d}_i}.\\=&\sum_{i=1}^{N}\rho|\varepsilon_i(t)|-\sum_{i=1}^{N}\frac{d_i^{\pi}}{\bar{d}_i}\kappa_i|e_i(t)|^{\pi}-\sum_{i=1}^{N}\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i|e_i(t)|^{\varpi}-\hbar\\\leq&\rho V(t)-\Big(\min_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}\Big)\Big(\sum_{i=1}^{N}|e_i(t)|\Big)^{\pi}-\Big(\min_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}\Big)\Big(\sum_{i=1}^{N}|e_i(t)|\Big)^{\varpi}-\hbar.\end{aligned}$$

Based on the *Lemma* 2.4, we see that drive-response system can achieve fixed-time synchronization, and the settling time is

$$T_{\max}^2=\frac{2^{\pi-1}\Big(\min\limits_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}^{\frac{1}{\pi}}+\hbar^{\frac{1}{\pi}}\Big)^{1-\pi}}{\min\limits_i\Big\{\frac{d_i^{\pi}}{\bar{d}_i}\kappa_iN^{1-\pi}\Big\}^{\frac{1}{\pi}}(\pi-1)}+\frac{\Big(\min\limits_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}^{\frac{1}{\varpi}}+\hbar^{\frac{1}{\varpi}}\Big)^{1-\varpi}-\hbar^{\frac{1-\varpi}{\varpi}}}{\min\limits_i\Big\{\frac{d_i^{\varpi}}{\bar{d}_i}\gamma_i\Big\}^{\frac{1}{\varpi}}(1-\varpi)}.$$

□

***Theorem* 4.3.** If $\breve{a} < \hbar$, then $T^2_{\max} < T^1_{\max}$.

*Proof.* Since $\pi > 1$, we have

$$\frac{2^{\pi-1}(\breve{a}^{\frac{1}{\pi}} + \hbar^{\frac{1}{\pi}})^{1-\pi}}{\breve{a}^{\frac{1}{\pi}}(\pi-1)} = \frac{2^{\pi-1}\Big(\frac{1}{\breve{a}^{\frac{1}{\pi}}+\hbar^{\frac{1}{\pi}}}\Big)^{\pi-1}}{\breve{a}^{\frac{1}{\pi}}(\pi-1)} = \frac{\Big(\frac{2}{\breve{a}^{\frac{1}{\pi}}+\hbar^{\frac{1}{\pi}}}\Big)^{\pi-1}}{\breve{a}^{\frac{1}{\pi}}(\pi-1)},$$

which together with $\breve{a} < \hbar$ gives

$$\frac{2^{\pi-1}(\breve{a}^{\frac{1}{\pi}} + \hbar^{\frac{1}{\pi}})^{1-\pi}}{\breve{a}^{\frac{1}{\pi}}(\pi-1)} \le \frac{\Big(\frac{2}{2\breve{a}^{\frac{1}{\pi}}}\Big)^{\pi-1}}{\breve{a}^{\frac{1}{\pi}}(\pi-1)} \le \frac{1}{\breve{a}(\pi-1)}. \tag{4.8}$$

From $0 < \varpi < 1$, it follows that $0 < 1-\varpi < 1$. According to *Lemma* 2.5 we obtain

$$\frac{\Big(\breve{b}^{\frac{1}{\varpi}} + \hbar^{\frac{1}{\varpi}}\Big)^{1-\varpi} - \hbar^{\frac{1-\varpi}{\varpi}}}{\breve{b}^{\frac{1}{\varpi}}(1-\varpi)} \le \frac{\breve{b}^{\frac{1-\varpi}{\varpi}} + \hbar^{\frac{1-\varpi}{\varpi}} - \hbar^{\frac{1-\varpi}{\varpi}}}{\breve{b}^{\frac{1}{\varpi}}(1-\varpi)} = \frac{1}{\breve{b}(1-\varpi)}. \tag{4.9}$$

Thus, (4.8) and (4.9) can obtain the conclusion. □

**Remark 4.1.** In *Lemma* 2.3 and *Lemma* 2.4, the settling times have been estimated. But from *Theorem* 4.3, it follows that $T^2_{\max} < T^1_{\max}$ when $\breve{a} < \hbar$. Thus, the condition in *Lemma* 2.4 is more relaxed, and *Lemma* 2.4 takes more advantages than *Lemma* 2.3.

**Remark 4.2.** The designed control input (4.6) can help effectively realize the fixed-time synchronization of the derive-response neural network system. One may notice that the control input are composed of several terms in which both current and past state information are utilized. It can be seen from the proof of *Theorem* 4.1 that each term in (4.6) contributes to the realization of fixed-time synchronization. But different from the previous control input, the bad chattering phenomenon is weakened. Consequently, some previous results containing sign functions in the control laws, such as [4, 23, 24, 36], can be improved greatly.

## §5 Examples and simulations

In this section, we provide two concrete examples and simulations to demonstrate the validity of the theoretical results.

**Example 5.1.** Consider the CGNNs as the drive system

$$\dot{x}_i(t) = -d_i(x_i(t))\Big(a_i(x_i(t)) - \sum_{j=1}^{N} b_{ij}f_j(x_j(t)) - \sum_{j=1}^{N} c_{ij}g_j(x_j(t-\tau_j(t))) - I_i\Big), \quad i = 1,2, \tag{4.10}$$

and the response system described as

$$\dot{y}_i(t) = -d_i(y_i(t))\Big(a_i(y_i(t)) - \sum_{j=1}^{N} b_{ij}f_j(y_j(t)) - \sum_{j=1}^{N} c_{ij}g_j(y_j(t-\tau_j(t))) - I_i\Big), \quad i = 1,2, \tag{4.11}$$

where $d_1(\theta) = 2 + 0.1\sin(\theta)$, $d_2(\theta) = 3 - 0.1\cos(\theta)$, $\tau_1(t) = \sin t$, $\tau_2(t) = \cos t$, $I_1 = I_2 = 1$ and

$$a_1(\theta) = 4 \cdot \frac{2}{\sqrt{4-0.01}} \arctan\Big(\frac{2\tan\frac{\theta}{2} + 0.1}{\sqrt{4-0.01}}\Big),$$

$$a_2(\theta) = 4 \cdot \frac{2}{\sqrt{9-0.01}} \arctan\Big(\sqrt{\frac{3+0.1}{3-0.1}} \tan\frac{\theta}{2}\Big),$$

$$f_i(\theta_i) = \begin{cases} \tanh(\theta_i) - \frac{1}{5} & \theta_i \geq 0, \\ \tanh(\theta_i) + \frac{1}{5} & \theta_i < 0, \end{cases} \quad i = 1,2, \; g_i(\theta_i) = \begin{cases} \tanh(\theta_i) - \frac{1}{10} & \theta_i \geq 0, \\ \tanh(\theta_i) + \frac{1}{10} & \theta_i < 0, \end{cases} \quad i = 1,2,$$

$$B = (b_{ij})_{2\times 2} = \begin{pmatrix} 0.3 + 0.1\sin t & 0.2 + 0.1\sin t \\ 0.3 + 0.1\cos t & 0.2 + 0.1\cos t \end{pmatrix},$$

$$C = (c_{ij})_{2\times 2} = \begin{pmatrix} 0.1 + 0.2\sin t & 0.1 + 0.2\cos t \\ 0.1 + 0.2\cos t & 0.1 + 0.2\sin t \end{pmatrix}.$$

Obviously, for each $i = 1,2$, the discontinuous neuron activation satisfies condition (H2) and is non-monotonic function. Moreover, 0 is a discontinuous point of the neuron activation function $f_i(\cdot)$ , $g_i(\cdot)$ and $\overline{co}[f_i(0)] = [-\frac{1}{5}, \frac{1}{5}]$, $\overline{co}[g_i(0)] = [-\frac{1}{10}, \frac{1}{10}]$.

On the one hand, it is easy to check that the linear growth condition (H3) holds by letting $\alpha_1 = \alpha_2 = \alpha_1^* = \alpha_2^* = 1$ and $\beta_1 = \beta_2 = \frac{1}{5}$, $\beta_1^* = \beta_2^* = \frac{1}{10}$. Take $\eta_1 = \eta_2 = 1$. By simple computation, we obtain

$$\max_{1\leq i\leq n} \Big\{ \frac{1}{\eta_i v_i^L} \sum_{j=1}^{N} \eta_j d_j^M (\alpha_j b_{ij}^M + \alpha_j^* c_{ij}^M) \Big\} = 0.91 < 1.$$

This shows that the inequality (3.15) is satisfied. According to *Corollary* 3.1, we can conclude that the system (5.1) has at least one $2\pi$-periodic solution. Consider the initial condition of system (5.1): $\phi = (-2,2)^\top$ for $t \in [-1,0]$. The numerical simulations are shown in Fig. 1(a) which also confirms the existence of a periodic solution for system (5.1).

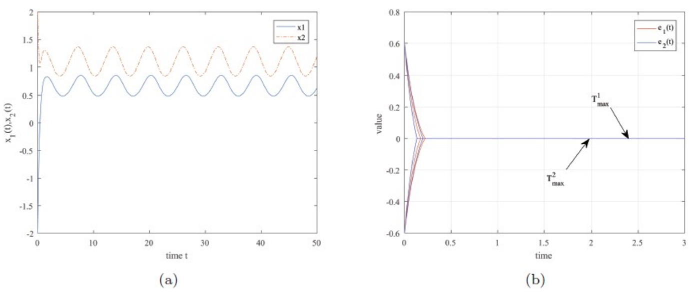


Figure 1. (a) Time-domain behaviors of $x_1(t), x_2(t)$ for Example 1 of system (5.1); (b) Synchronization error between (5.1) and (5.2) under control (5.3).

Consider the fixed-time synchronization of discontinuous drive-response system we can get $\bar{d}_1 = 2.1$, $\bar{d}_2 = 3.1$, $\underline{d}_1 = 1.9$, $\underline{d}_2 = 2.9$, $a_1 = \frac{1}{2.1}$, $a_2 = \frac{1}{3.1}$. Thus, the control inputs of the response system are formulated as

$$\begin{aligned} u_1(t) =& -0.5sgn(\varepsilon_1(t)) - \frac{\varepsilon_1(t)}{|\varepsilon_1(t)|}(0.68|\varepsilon_1(t)| + 1.26|\varepsilon_1(t-\tau_1(t))| + 2|\varepsilon_1(t)|^2 + 2|\varepsilon_1(t)|^{0.5}), \\ u_2(t) =& -0.7sgn(\varepsilon_2(t)) - \frac{\varepsilon_2(t)}{|\varepsilon_2(t)|}(0.86|\varepsilon_2(t)| + 1.86|\varepsilon_2(t-\tau_2(t))| + 2|\varepsilon_2(t)|^2 + 2|\varepsilon_2(t)|^{0.5}). \end{aligned} \tag{4.12}$$

Up to now, we can see that all the conditions in *Theorem* 4.1 are satisfied. Moreover, when the adaptive control (5.3) is appended to the response system, the response system converges to the drive system with in a finite time, and the states of the error system remain zero thereafter. This fact can be illustrated by Figure. 1(b). Furthermore, according to *Theorems* 4.1-4.2, we obtain

$$T^1_{\max} \approx 2.4021s,\ T^2_{\max} \approx 1.9730s.$$

Clearly, one can see that $T^2_{\max} < T^1_{\max}$. Up to now, we can see that all the conditions in *Theorems* 4.1-4.2 are satisfied.

**Example 5.2.** Consider the CGNNs with time-varying delays network system with the drive system given by

$$\dot{x_i}(t) = -\,d_i(x_i(t))\Big(a_i(x_i(t)) - \sum_{j=1}^{N} b_{ij} f_j(x_j(t)) - \sum_{j=1}^{N} c_{ij} g_j(x_j(t-\tau_j(t))) - I_i\Big),\ i = 1, 2, 3, \tag{4.13}$$

and the response system described as

$$\dot{y_i}(t) = -\,d_i(y_i(t))\Big(a_i(y_i(t)) - \sum_{j=1}^{N} b_{ij} f_j(y_j(t)) - \sum_{j=1}^{N} c_{ij} g_j(y_j(t-\tau_j(t))) - I_i\Big),\ i = 1, 2, 3, \tag{4.14}$$

where $d_1(\theta) = 3 + 0.1\sin(\theta)$, $d_2(\theta) = d_3(\theta) = 4 + 0.1\cos(\theta)$, $\tau_1 = \sin t,\ \tau_2 = \cos t,\ \tau_3 = \sin 0.5t$, $I_1 = I_2 = I_3 = 1$ and

$$a_1(\theta) = 10 \cdot \frac{2}{\sqrt{9-0.01}} \arctan\Big(\frac{3\tan\frac{\theta}{2} + 0.1}{\sqrt{9-0.01}}\Big),$$

$$a_2(\theta) = a_3(\theta) = 10 \cdot \frac{2}{\sqrt{16-0.01}} \arctan\Big(\sqrt{\frac{4-0.1}{4+0.1}}\tan\frac{\theta}{2}\Big),$$

$$f_i(\theta_i) = \begin{cases} \tanh(\theta_i) + \frac{1}{5}, & \theta_i \geq 0, \\ \tanh(\theta_i) - \frac{1}{5}, & \theta_i < 0, \end{cases} \quad g_i(\theta_i) = \begin{cases} \tanh(\theta_i) + \frac{1}{10}, & \theta_i \geq 0, \\ \tanh(\theta_i) - \frac{1}{10}, & \theta_i < 0, \end{cases} \quad i = 1, 2, 3,$$

$$(b_{ij})_{3\times 3} = \begin{pmatrix} 0.2 + 0.1\sin t & 0.1 + 0.1\sin t & 0.1 + 0.2\cos t \\ 0.2 + 0.1\cos t & 0.1 + 0.2\sin t & 0.1 + 0.3\sin t \\ 0.3 + 0.1\sin t & 0.1 + 0.3\cos t & 0.1 + 0.1\cos t \end{pmatrix},$$

$$(c_{ij})_{3\times 3} = \begin{pmatrix} 0.1 + 0.2\cos t & 0.1 + 0.1\cos t & 0.2 + 0.1\sin t \\ 0.2 + 0.1\sin t & 0.2 + 0.1\cos t & 0.1 + 0.3\cos t \\ 0.3 + 0.1\cos t & 0.1 + 0.3\sin t & 0.1 + 0.1\sin t \end{pmatrix}.$$

Obviously, for each $i = 1, 2, 3$, the discontinuous neuron activation satisfies condition (H2) and is non-monotonic function. Moreover, 0 is a discontinuous point of the neuron activation function $f_i(\cdot)$ , $g_i(\cdot)$ and $\overline{co}[f_i(0)] = [-\frac{1}{5}, \frac{1}{5}]$, $\overline{co}[g_i(0)] = [-\frac{1}{10}, \frac{1}{10}]$.

On the one hand, it is easy to check that the linear growth condition (H3) holds by letting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha^*_1 = \alpha^*_2 = \alpha^*_3 = 1$ and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{5}$, $\beta^*_1 = \beta^*_2 = \beta^*_3 = \frac{1}{10}$. Take $\eta_1 = \eta_2 = \eta_3 = 1$. By simple computation, we obtain

$$\max_{1\leq i\leq n}\Big\{\frac{1}{\eta_i v^L_i}\sum_{j=1}^{N}\eta_j d^M_j(\alpha_j b^M_{ij} + \alpha^*_j c^M_{ij})\Big\} = 0.842 < 1.$$

This shows that the inequality (3.15) is satisfied. According to *Corollary* 3.1, we can conclude that the system (5.3) has at least one $2\pi$-periodic solution. Consider the initial condition of system (5.3): $\phi = (-2, 2, 2)^\top$ for $t \in [-1, 0]$. The numerical simulations are shown in Fig. 2(a) which also confirms the existence of a periodic solution for system (5.3).

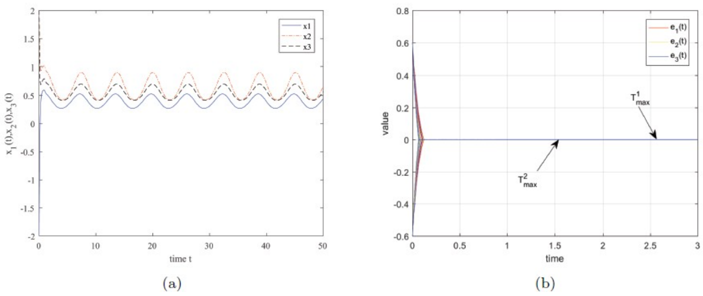


Figure 2. (a) Time-domain behaviors of $x_1(t)$, $x_2(t)$, $x_3(t)$ for Example 2 of system (5.4); (b) Synchronization error between (5.4) and (5.5) under control (5.6).

Consider the fixed-time synchronization of discontinuous drive-response system we can gets $\bar{d}_1 = 3.1$, $\bar{d}_2 = \bar{d}_3 = 4.1$, $\underline{d}_1 = 2.9$, $\underline{d}_2 = \underline{d}_3 = 3.9$, $a_1 = \frac{1}{3.1}$, $a_2 = a_2 = \frac{1}{4.1}$ same as Example 1 the control inputs of the response system are formulated as

$$\begin{aligned}
u_1(t) &= -0.75sgn(\varepsilon_1(t)) - \frac{\varepsilon_1(t)}{|\varepsilon_1(t)|}(2.1|\varepsilon_1(t)| + 3.1|\varepsilon_1(t-\tau_1(t))| + 2|\varepsilon_1(t)|^2 + 2|\varepsilon_1(t)|^{0.5}), \\
u_2(t) &= -1.23sgn(\varepsilon_2(t)) - \frac{\varepsilon_2(t)}{|\varepsilon_2(t)|}(2.69|\varepsilon_2(t)| + 3.69|\varepsilon_2(t-\tau_2(t))| + 2|\varepsilon_2(t)|^2 + 2|\varepsilon_2(t)|^{0.5}), \\
u_3(t) &= -1.23sgn(\varepsilon_3(t)) - \frac{\varepsilon_3(t)}{|\varepsilon_3(t)|}(2.69|\varepsilon_3(t)| + 3.69|\varepsilon_3(t-\tau_3(t))| + 2|\varepsilon_3(t)|^2 + 2|\varepsilon_3(t)|^{0.5}).
\end{aligned} \tag{4.15}$$

Up to now, we can see that all the conditions in *Theorem* 4.1 are satisfied. Moreover, when the adaptive control (5.6) is appended to the response system, the response system converges to the drive system within a finite time, and the states of the error system remain zero thereafter. This fact can be illustrated by Fig. 2(b). Furthermore, according to *Theorems* 4.1-4.2, we obtain

$$T^1_{\max} \approx 2.6071s, \ T^2_{\max} \approx 1.6001s.$$

Clearly, one can see that $T^2_{\max} < T^1_{\max}$. Up to now, we can see that all the conditions in *Theorems* 4.1-4.2 are satisfied.

**Remark 5.1.** Based on the above two numerical examples, it follows that the error system (4.1) can achieve fixed-time synchronization by *Theorems* 4.1-4.2. Based on *Theorems* 4.1-4.2, two settling times $T^1_{\max}$ and $T^2_{\max}$ of fixed-time synchronization are estimated. But $T^2_{\max} < T^1_{\max}$ by direct computation. The fact can also be shown by Fig.1(b) and Fig.2(b). This implies that the accuracy of the settling time given by *Lemma* 2.4 is higher.

## §6 Conclusion

This paper has addressed the periodic and fixed-time synchronization problem for the discontinuous time-varying delayed CGNNs. Firstly, by using a relaxed condition imposed on the amplification functions, set-valued mapping, and Kakutani's fixed point theorem, the existence of periodic solutions has been proved, and Kakutani's fixed point theorem is firstly used to prove the periodic solution of discontinuous time-varying delayed CGNNs. Based on the existence of periodic solutions, fixed-time synchronization has been further studied via a novel control law, which not only can help achieve synchronization but also reduce the chattering phenomena. Moreover, a more accurate estimation of the settling time is given in comparison to the previous ones. Finally, the effectiveness of the main analytical results has been validated by numerical simulations.

The methods and tools provided by this paper can be applied to another network model, impulsive CGNNs, and stochastic CGNNs. These are the further topics.

### Declarations

**Conflict of interest** The authors declare no conflict of interest.

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