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Surface family with a common involute of a spacelike curve in Minkowski 3-space

SHEN Ting JIANG Ping* WANG Jun
KONG Da-li SUN Bei-bei

Abstract. In Minkowski 3-space, we construct a surfaces family interpolating the involute of a spacelike curve as a common line of curvature, geodesic or asymptotic curve. Moreover, the conditions are examined for the ruled surface to be developable. And we support our conclusion with some examples.

§1 Introduction

Curve and surface geometric modeling design are widely used in industrial production. In differential geometry, which are three characteristic curves that affect the properties of surfaces, which are the line of curvature, asymptotic curve and geodesic [19]. The geodesic curve is an inherent geometric feature and the shortest path between two points in a small area of surface, which is used in many geometric operations. The asymptotic curve appears in astrophysics and astronomy to resolve problems like finding escape orbits for a group of stars in a stellar system [2]. The line of curvature is an indispensable tool to display changes in the main direction on the surface. It can be used to guide surface analysis, and geometric design [1,20,30].

In recent years, interpolating a given curve as a characteristic curve has obtained many results. In Euclidean 3-space, Wang and Tang [34] discussed the conditions satisfied by the scale functions for constructing the surface pencil through a geodesic curve, and Kasap et al. discussed different scale functions in [21]. Li and Liu et al. constructed a cluster of surfaces that interpolated a common line of curvature or asymptotic curve [25,26]. Bayram and Kasap constructed the family of surfaces with an asymptotic curve [11]. Atalay and Kasap interpolated the smarandache geodesic curve to generate the surface family with different frames [4,5]. Güler [15] interpolated an asymptotic curve to design the offset surface family. Atalay [2,3] obtained surface pencil pair through the Mannheim pair as asymptotic and geodesic curves, respectively. In Galilean space, Ref. [22,23] designed a surface family passing through a common

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*Corresponding author.

geodesic and asymptotic curve, respectively. Jiang et al. investigated the case of interpolating pairs of Bertrand curves [17].

Since Einstein proposed the theory of relativity, the Minkowski model of spacetime has received a lot of attention. Non-zero vectors in Minkowski space fall into one of three categories. The spacelike curve and timelike curve, respectively, describe the trajectories of observers traveling faster and slower than the velocity of light, while a lightlike curve represents the trajectories of observers traveling at the velocity of light. In Minkowski 3-space (R_1^3), Kasap et al. [6, 7, 18] constructed surface family through a common null asymptotic curve, geodesic and null geodesic, respectively. Ergün et al. [12] discussed the condition that a surface pencil interpolates a spacelike (timelike) line of curvature.

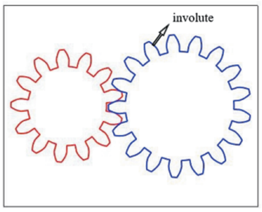


Figure 1. Gear with involute teeth.

For special curves, like natural lift and involute, due to the complexity of space, there are few studies on these special curves in Minkowski 3-space. Evren and Bayram et al. [13] studied surface family interpolating a natural lift of a given curve as a common asymptotic curve or geodesic. Huygens studied involute when considering the use of clocks without slings on ships at sea [14], when a straight line on a plane rolls along a fixed circle, the locus of any point on the straight line is the involute of the circle. Since the involute tooth profile is smooth meshing, has good transmission continuity, and is relatively simple to manufacture, it is widely used in the machinery industry. The gear with involute teeth is shown in Fig. 1. Bilici and Bayram have only studied involute in Euclidean space, and they derived the conditions for the scale function satisfied when the involute acts as a common asymptotic curve or line of curvature [8, 14]. At present, as far as we know, there is no research on interpolating involute as a characteristic curve in Minkowski 3-space.

A surface produced by a family of continuously changing lines is called a ruled surface. A developable surface is a special ruled surface, and it can be used in geometric design and manufacturing systems. Moreover, developable surfaces are also used in products of flexible materials such as leather, paper, sheet metal. Liu and Li et al. constructed developable surfaces that interpolated a common asymptotic curve and line of curvature, respectively [26, 27]. Wang et al. have studied developable surface pencil pairs in [32].

In this paper, the preliminaries in Minkowski 3-space are introduced in Sec. 2. In Sec. 3, we discuss the surface family with the involute of the spacelike curve as a common line of curvature, asymptotic curve and geodesic, respectively. In Sec. 4, we give the expression of a ruled surface with $\beta(v)$ as the asymptotic curve, and discuss the conditions for the ruled surface to be developable. In the end, some examples are given in Sec. 5.

§2 Preliminaries

Definition 1. *Let* $\boldsymbol{a} = (x_1, y_1, z_1)$ *and* $\boldsymbol{c} = (x_2, y_2, z_2)$ *be vectors in* R_1^3*, the inner product [24], the vector product [28] of* $\boldsymbol{a}$ *and* $\boldsymbol{c}$ *are defined by*

$$(\boldsymbol{a}, \boldsymbol{c}) = -x_1x_2 + y_1y_2 + z_1z_2,$$

and

$$\boldsymbol{a} \times \boldsymbol{c} = (y_1z_2 - z_1y_2, x_1z_2 - z_1x_2, x_2y_1 - x_1y_2). \tag{1}$$

Definition 2. *[24] Suppose that* $\boldsymbol{a} = (x_1, y_1, z_1) \in R_1^3$ *is a vector, then* $\boldsymbol{a}$ *is said to be*

(1) lightlike if $(\boldsymbol{a}, \boldsymbol{a}) = 0$ *and* $\boldsymbol{a} \neq 0$*;*
(2) spacelike if $(\boldsymbol{a}, \boldsymbol{a}) > 0$ *or* $\boldsymbol{a} = 0$*;*
(3) timelike if $(\boldsymbol{a}, \boldsymbol{a}) < 0$*.*

For the convenience, we define $\delta(*)$ as

$$\delta(*) = \begin{cases} -1, & \text{if } * \text{ is timelike vector,} \\ 1, & \text{if } * \text{ is spacelike vector.} \end{cases} \tag{2}$$

Suppose that $\boldsymbol{\alpha}(v)$ is a spacelike curve in R_1^3, let $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ be the Frenet frame of $\boldsymbol{\alpha}(v)$, where $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$ are tangent, principal normal and binormal vectors of $\boldsymbol{\alpha}(v)$, respectively. Then, $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ satisfy the following formulas [31, 33]

$$\frac{\mathrm{d}}{\mathrm{d}v}\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \delta(\boldsymbol{B})\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}, \tag{3}$$

$$\begin{aligned} &(\boldsymbol{T}, \boldsymbol{T}) = 1, (\boldsymbol{N}, \boldsymbol{N}) = \delta(\boldsymbol{B}), (\boldsymbol{B}, \boldsymbol{B}) = -\delta(\boldsymbol{B}), \\ &(\boldsymbol{T}, \boldsymbol{N}) = (\boldsymbol{N}, \boldsymbol{B}) = (\boldsymbol{T}, \boldsymbol{B}) = 0, \end{aligned} \tag{4}$$

where τ and κ are the torsion and curvature of $\boldsymbol{\alpha}(v)$, respectively. $\delta(\boldsymbol{B})$ is defined as Eq. (2).

Definition 3. *[29] Darboux vector* $\boldsymbol{W}$ *of* $\boldsymbol{\alpha}(v)$ *is defined as* $\boldsymbol{W}=\tau\boldsymbol{T}+\kappa\boldsymbol{B}$*, where* $\theta = \langle\boldsymbol{W}, \boldsymbol{B}\rangle$ *(see Fig. 2) is the angle between* $\boldsymbol{W}$ *and* $\boldsymbol{B}$*,* $\kappa = |\boldsymbol{W}|cos\theta$*,* $\tau = |\boldsymbol{W}|sin\theta$*.*

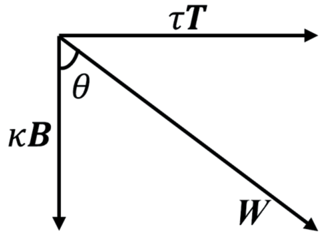


Figure 2. Darboux vector $\boldsymbol{W}$ of $\boldsymbol{\alpha}(v)$.

Remark 2.1. If a curve is a pseudo null curve, then it is a spacelike curve whose binormal vector is lightlike in R_1^3 [33]. For the pseudo null curve, by Ref. [31], we know κ only has two values: if $\boldsymbol{\alpha}(v)$ is a straight line, then $\kappa = 0$, otherwise, $\kappa = 1$.

Definition 4. *[16] An orthogonal track of the tangents to a curve is called the involute of that curve. The involute $\boldsymbol{\beta}(v)$ of a curve $\boldsymbol{\alpha}(v)$ is defined by*

$$\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + (m - v)\boldsymbol{T}(v), S_1 \le v \le S_2, \tag{5}$$

where m is an arbitrary constant, S_1, S_2 are real numbers.

Lemma 2.1. [31] In R_1^3, if $\boldsymbol{\alpha}(v)$ is a pseudo null curve with $\kappa = 1$ and $\tau \neq 0$, then there is no involute of the curve.

Since there is no involute when $\boldsymbol{\alpha}(v)$ is a nonlinear pseudo null curve, we only discuss the surface family with the involute of the spacelike curve with spacelike and timelike $\boldsymbol{B}$ in the next sections.

Let $\boldsymbol{\beta}(v)$ be the involute of $\boldsymbol{\alpha}(v)$, $\{\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*\}$ be the Frenet frame of $\boldsymbol{\beta}(v)$. By Ref. [9] [10], we can obtain the following relation between the Frenet frames of $\boldsymbol{\alpha}(v)$ and $\boldsymbol{\beta}(v)$. Let $\boldsymbol{X} = (\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$, $\boldsymbol{Y} = (\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*)$, then

$$\boldsymbol{Y}^T = \boldsymbol{A_i}\boldsymbol{X}^T, \tag{6}$$

where $\boldsymbol{X}^T$ represents the transpose of the matrix $\boldsymbol{X}$, and

$$\boldsymbol{A}_i = \begin{cases} \boldsymbol{A}_1, & \text{if } \boldsymbol{B} \text{ is timelike and } \boldsymbol{W} \text{ is spacelike,} \\ \boldsymbol{A}_2, & \text{if } \boldsymbol{B} \text{ and } \boldsymbol{W} \text{ are timelike,} \\ \boldsymbol{A}_3, & \text{if } \boldsymbol{B} \text{ is spacelike,} \end{cases}$$

$$\boldsymbol{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & -\cosh\theta \\ -\cosh\theta & 0 & \sinh\theta \end{pmatrix}, \boldsymbol{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & \cosh\theta \end{pmatrix}, \boldsymbol{A_3} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Remark 3.1. By Ref. [10], if $\boldsymbol{\alpha}$ is a spacelike curve with spacelike $\boldsymbol{B}$, then the $\boldsymbol{W}$ of $\boldsymbol{\alpha}$ is spacelike, and θ is a Lorentzian spacelike angle between $\boldsymbol{B}$ and $\boldsymbol{W}$.

§3 Surface family with involute as common characteristic curves

In this section, we discuss the surface family interpolating the involute $\boldsymbol{\beta}(v)$ of a given spacelike curve $\boldsymbol{\alpha}(v)$ as three kinds of characteristic curves (line of curvature, asymptotic curve and geodesic) in Minkowski 3-space.

Suppose that surface family $\boldsymbol{P}(v, l)$ interpolating $\boldsymbol{\beta}(v)$ is given by [14]

$$\begin{aligned} \boldsymbol{P}(v,l) = \boldsymbol{\beta}(v) + x(v,l)\boldsymbol{T}^*(v) + y(v,l)\boldsymbol{N}^*(v) + z(v,l)\boldsymbol{B}^*(v), \\ S_1 \le v \le S_2, S_3 \le l \le S_4, \end{aligned} \tag{7}$$

where $x(v,l)$, $y(v,l)$ and $z(v,l)$ are called marching-scale functions and S_3 and S_4 are real numbers.

Since $\boldsymbol{\beta}(v)$ is on the $\boldsymbol{P}(v,l)$, there exists $l_0 \in [S_3, S_4]$, so that $\boldsymbol{P}(v, l_0) = \boldsymbol{\beta}(v)$, that is

$$x(v, l_0) = y(v, l_0) = z(v, l_0) = 0, S_1 \le v \le S_2, S_3 \le l_0 \le S_4. \tag{8}$$

In order to obtain the surface family, we give the following lemmas [19] which introduce the condition for the given curve to be characteristic curves.

Lemma 3.1. A curve on a surface is a line of curvature if and only if the normal lines of the surface along this curve form a developable surface.

Lemma 3.2. A curve is an asymptote on the surface $\boldsymbol{P}(v,l)$ if and only if along the curve the normal vector of $\boldsymbol{P}(v,l)$ and the principal normal vector of the curve are orthogonal to each other.

Lemma 3.3. A curve is a geodesic curve on the surface $\boldsymbol{P}(v,l)$ if and only if the normal vector of $\boldsymbol{P}(v,l)$ and the principal normal vector of the curve are parallel to each other.

According to the lemmas, we first discuss the normal vector $\boldsymbol{n}(v,l)$ of $\boldsymbol{P}(v,l)$, which is given by

$$\boldsymbol{n}(v,l)=\boldsymbol{P}_v(v,l)\times\boldsymbol{P}_l(v,l), \tag{9}$$

where $\boldsymbol{P}_v(v,l)$ represents the partial derivative of $\boldsymbol{P}(v,l)$.

From Eqs.(6)-(9), along the curve the normal vector $\boldsymbol{n}(v,l_0)$ can be expressed as the following equations.

(1) If $\boldsymbol{B}$ is timelike,

$$\begin{aligned}\boldsymbol{n}(v,l_0)=&\delta(\boldsymbol{W})(m-v)\kappa\,[(y_l(v,l_0)\sinh 2\theta-\delta(\boldsymbol{W})z_l(v,l_0)\cosh 2\theta)\boldsymbol{N}^*\\&+(y_l(v,l_0)\cosh 2\theta-\delta(\boldsymbol{W})z_l(v,l_0)\sinh 2\theta)\boldsymbol{B}^*]\ ;\end{aligned} \tag{10}$$

(2) If $\boldsymbol{B}$ is spacelike, then

$$\boldsymbol{n}(v,l_0)=-(m-v)\kappa\,[(z_l(v,l_0)\boldsymbol{N}^*-y_l(v,l_0)\boldsymbol{B}^*]\ , \tag{11}$$

where $y_l(v,l)$ and $z_l(v,l)$ represent the partial derivatives of $y(v,l)$ and $z(v,l)$ with respect to l, respectively.

3.1 Surface family with common involute as line of curvature

In this part, we discuss the conditions for the involute $\boldsymbol{\beta}(v)$ of $\boldsymbol{\alpha}(v)$ being a line of curvature on $\boldsymbol{P}(v,l)$.

Let the normal surface of $\boldsymbol{\beta}(v)$ be $\boldsymbol{\psi}(v,l)=\boldsymbol{\beta}(v)+l\boldsymbol{n}_1(v)$, where $\boldsymbol{n}_1(v)$ is a vector orthogonal to the curve $\boldsymbol{\beta}(v)$, $\vartheta=\langle\boldsymbol{N}^*,\boldsymbol{n}_1(v)\rangle$, let $\boldsymbol{n}_1=\boldsymbol{n}_1(v)$, and by Ref. [12], we know

$$\boldsymbol{n}_1=\begin{cases}\cosh\vartheta\boldsymbol{N}^*+\sinh\vartheta\boldsymbol{B}^*, & \text{if } \boldsymbol{\beta}(v) \text{ is a spacelike curve,}\\ \cos\vartheta\boldsymbol{N}^*+\sin\vartheta\boldsymbol{B}^*, & \text{if } \boldsymbol{\beta}(v) \text{ is a timelike curve.}\end{cases} \tag{12}$$

According to *Lemma 3.1.*, if $\boldsymbol{\psi}(v,l)$ is developable and $\boldsymbol{n}(v,l_0)\parallel\boldsymbol{n}_1$, then $\boldsymbol{\beta}(v)$ is a line of curvature on $\boldsymbol{P}(v,l)$.

Firstly, we discuss the conditions that marching-scale function satisfies when $\boldsymbol{n}(v,l_0)\parallel\boldsymbol{n}_1$.

(1) If $\boldsymbol{B}$ is timelike, by Eqs.(10) and (12), and since $\kappa\neq 0$, $m-v\neq 0$, $\boldsymbol{n}(v,l_0)\parallel\boldsymbol{n}_1$ if and only if there exists a nonzero function $\eta(v)$, such that

$$\begin{cases}(y_l(v,l_0)\sinh 2\theta-\delta(\boldsymbol{W})z_l(v,l_0)\cosh 2\theta)=\eta(v)\cosh\vartheta,\\(y_l(v,l_0)\cosh 2\theta-\delta(\boldsymbol{W})z_l(v,l_0)\sinh 2\theta)=\eta(v)\sinh\vartheta,\end{cases}$$

where $\delta(\boldsymbol{W})$ is defined as Eq. (2).

(2) If $\boldsymbol{B}(v)$ is spacelike, similarly, by Eqs. (11) and (12), we obtain

$$y_l(v,l_0)=\eta(v)\cos\vartheta,\ z_l(v,l_0)=\eta(v)\sin\vartheta.$$

On the other hand, the surface $\boldsymbol{\psi}(v,l)$ is developable if [19]

$$det\ [\boldsymbol{\beta}',\boldsymbol{n}_1,\boldsymbol{n}_1']=0, \tag{13}$$

where det is the determinant of the matrix.

(1) If $\boldsymbol{B}$ is timelike, then $\boldsymbol{\beta}(v)$ is a spacelike curve [9],

(i) while $\boldsymbol{W}$ is spacelike, taking the derivative of Eq. (12), we have

$$\boldsymbol{n}_1'=\sinh\vartheta\,\vartheta'\,\boldsymbol{N}^*+\cosh\vartheta\,\boldsymbol{N}^{*\prime}+\cosh\vartheta\,\vartheta'\,\boldsymbol{B}^*+\sinh\vartheta\,\boldsymbol{B}^{*\prime}. \tag{14}$$

Substitute Eqs.(3) and (6) to Eq. (14), we obtain

$$\boldsymbol{n}_1'=(\vartheta'-\theta')(\sinh\vartheta\,\boldsymbol{N}^*+\cosh\vartheta\,\boldsymbol{B}^*)-(\kappa\sinh(\vartheta-\theta)-\tau\cosh(\vartheta-\theta))\boldsymbol{T}^*.$$

Then, from $\boldsymbol{\beta}'=\boldsymbol{T}^*$ and by the properties of the determinant, Eq. (13) can be rewritten as

$$\begin{aligned}0&=det\ [\boldsymbol{\beta}',\boldsymbol{n}_1,\boldsymbol{n}_1']\\&=det\ [\boldsymbol{T}^*,\sinh\vartheta\,\boldsymbol{B}^*,(\vartheta'-\theta')\sinh\vartheta\,\boldsymbol{N}^*]+det\ [\boldsymbol{T}^*,\cosh\vartheta\,\boldsymbol{N}^*,(\vartheta'-\theta')\cosh\vartheta\,\boldsymbol{B}^*]\ .\end{aligned}$$

Since $\boldsymbol{T}^*\nparallel\boldsymbol{N}^*$ and $\boldsymbol{T}^*\nparallel\boldsymbol{B}^*$, the equation holds when $\vartheta'-\theta'=0$, that is,

$$\vartheta(v)-\theta(v)=constant.$$

(ii) While $\boldsymbol{W}$ is timelike, substitute Eqs.(3) and (6) into Eq. (14),

$$\boldsymbol{n}_1'=(\vartheta'+\theta')(\sinh\vartheta\,\boldsymbol{N}^*+\cosh\vartheta\,\boldsymbol{B}^*)+(\tau\sinh(\theta+\vartheta)-\kappa\cosh(\theta+\vartheta))\boldsymbol{T}^*.$$

Similar to the proof of (i), we obtain $\vartheta(v)+\theta(v)=constant$.

(2) If $\boldsymbol{B}$ is spacelike, then $\boldsymbol{\beta}(v)$ is a timelike curve [10], taking the derivative of Eq. (12), and pluging in Eqs.(3) and (6), we have

$$\boldsymbol{n}_1'=(\theta'-\vartheta')(\sin\vartheta\,\boldsymbol{N}^*+\cos\vartheta\,\boldsymbol{B}^*)-(\cos(-\theta+\vartheta)\kappa-\sin(\vartheta-\theta)\tau)\boldsymbol{T}^*.$$

Similarly, we get $\vartheta(v)-\theta(v)=constant$.

Then the above conclusions can be expressed as follows.

Theorem 3.1. Let $\boldsymbol{\beta}(v)$ be involute of the given spacelike curve $\boldsymbol{\alpha}(v)$. Then $\boldsymbol{\beta}(v)$ is a line of curvature on the surface $\boldsymbol{P}(v,l)$ (Eq. (7)) if the marching-scale functions satisfy $x(v,l_0)=y(v,l_0)=z(v,l_0)=0$, and

(1) if $\boldsymbol{B}(v)$ is timelike, then

$$\begin{cases}\vartheta(v)-\delta(\boldsymbol{W})\,\theta(v)=constant,\\(y_l(v,l_0)\sinh2\theta-\delta(\boldsymbol{W})\,z_l(v,l_0)\cosh2\theta)=\eta(v)\cosh\vartheta,\\(y_l(v,l_0)\cosh2\theta-\delta(\boldsymbol{W})\,z_l(v,l_0)\sinh2\theta)=\eta(v)\sinh\vartheta,\end{cases}$$

(2) if $\boldsymbol{B}(v)$ is spacelike, then

$$\begin{cases}\vartheta(v)-\theta(v)=constant,\\y_l(v,l_0)=\eta(v)\cos\vartheta,\ z_l(v,l_0)=\eta(v)\sin\vartheta,\end{cases}$$

where $S_1\le v\le S_2, S_3\le l,l_0\le S_4$ (l_0 fixed), $\theta=\theta(v)=\langle\boldsymbol{W},\boldsymbol{B}\rangle$, $\vartheta=\vartheta(v)=\langle\boldsymbol{N}^*,\boldsymbol{n}_1\rangle$, $\delta(\boldsymbol{W})$ is defined as Eq. (2), and $\eta(v)$ is a nonzero function.

3.2 Surface family with common involute as an asymptotic curve

In this part, we discuss the conditions for the involute $\boldsymbol{\beta}(v)$ of $\boldsymbol{\alpha}(v)$ being an asymptotic curve on $\boldsymbol{P}(v,l)$.

According to *Lemma 3.2.*, if $\boldsymbol{\beta}(v)$ is an asymptotic curve on $\boldsymbol{P}(v,l)$, then we know $\langle \boldsymbol{n}(v,l_0), \boldsymbol{N}^*\rangle = 0$. Thus, while $\boldsymbol{B}$ is timelike, by Eq. (10), $\kappa \neq 0$ and $m - v \neq 0$, we obtain

$$y_l(v,l_0)\sinh 2\theta - \delta(\boldsymbol{W})z_l(v,l_0)\cosh 2\theta = 0,\ y_l(v,l_0)\cosh 2\theta - \delta(\boldsymbol{W})z_l(v,l_0)\sinh 2\theta \neq 0,$$

and if $\boldsymbol{B}(v)$ is spacelike, we have $z_l(v,l_0) = 0$, $y_l(v,l_0) \neq 0$.

From the above conclusions and by Eq. (8), we conclude as follows.

Theorem 3.2. Let $\boldsymbol{\beta}(v)$ be involute of the given spacelike curve $\boldsymbol{\alpha}(v)$. Then $\boldsymbol{\beta}(v)$ is an asymptotic curve on $\boldsymbol{P}(v,l)$ (Eq. (7)) if the marching-scale functions satisfy $x(v,l_0) = y(v,l_0) = z(v,l_0) = 0$, and

(1) if $\boldsymbol{B}$ is timelike, then

$$\begin{cases} y_l(v,l_0)\sinh 2\theta - \delta(\boldsymbol{W})\,z_l(v,l_0)\cosh 2\theta = 0, \\ y_l(v,l_0)\cosh 2\theta - \delta(\boldsymbol{W})\,z_l(v,l_0)\sinh 2\theta \neq 0, \end{cases}$$

(2) if $\boldsymbol{B}$ is spacelike, then

$$y_l(v,l_0) \neq 0,\ z_l(v,l_0) = 0,$$

where $S_1 \leq v \leq S_2, S_3 \leq l, l_0 \leq S_4$ (l_0 fixed), $\theta = \theta(v) = \langle \boldsymbol{W}, \boldsymbol{B}\rangle$, and $\delta(\boldsymbol{W})$ is defined as Eq. (2).

3.3 Surface family with common involute as a geodesic

In this part, we discuss the conditions for the involute $\boldsymbol{\beta}(v)$ of $\boldsymbol{\alpha}(v)$ being a geodesic on $\boldsymbol{P}(v,l)$.

According to *Lemma 3.3.*, if $\boldsymbol{\beta}(v)$ is a geodesic curve on $\boldsymbol{P}(v,l)$, we know $\boldsymbol{n}(v,l_0) \parallel \boldsymbol{N}^*$. Thus, if $\boldsymbol{B}$ is a timelike vector and $\boldsymbol{W}$ is spacelike, by Eq. (10), $\kappa \neq 0$, and $m - v \neq 0$, we have

$$y_l(v,l_0)\sinh 2\theta - \delta(\boldsymbol{W})z_l(v,l_0)\cosh 2\theta \neq 0,\ y_l(v,l_0)\cosh 2\theta - \delta(\boldsymbol{W})z_l(v,l_0)\sinh 2\theta = 0,$$

and if $\boldsymbol{B}$ is spacelike, by Eq. (11), then $z_l(v,l_0) \neq 0$, $y_l(v,l_0) = 0$.

From the above conclusions and by Eq. (8), we conclude as follows.

Theorem 3.3. Let $\boldsymbol{\beta}(v)$ be involute of the given spacelike curve $\boldsymbol{\alpha}(v)$. Then $\boldsymbol{\beta}(v)$ is a geodesic curve on $\boldsymbol{P}(v,l)$ (Eq. (7)) if $x(v,l_0) = y(v,l_0) = z(v,l_0) = 0$, and

(1) if $\boldsymbol{B}(v)$ is timelike, then

$$\begin{cases} y_l(v,l_0)\sinh 2\theta - \delta(\boldsymbol{W})\,z_l(v,l_0)\cosh 2\theta \neq 0, \\ y_l(v,l_0)\cosh 2\theta - \delta(\boldsymbol{W})\,z_l(v,l_0)\sinh 2\theta = 0, \end{cases}$$

(2) if $\boldsymbol{B}(v)$ is spacelike, then

$$y_l(v,l_0) = 0,\ z_l(v,l_0) \neq 0,$$

where $S_1 \leq v \leq S_2$, $S_3 \leq l, l_0 \leq S_4$ (l_0 fixed), and $\theta = \theta(v) = \langle \boldsymbol{W}, \boldsymbol{B}\rangle$, and $\delta(\boldsymbol{B})$ is defined as Eq. (2).

Particularly, if marching-scale functions can be expressed as

$$x(v,l) = \rho(v)H(l), y(v,l) = \varrho(v)R(l), z(v,l) = \sigma(v)Q(l), \tag{15}$$

where $\rho(v)$, $\sigma(v)$, $\varrho(v)$, $H(l)$, $Q(l)$, $R(l)$ are C^1 functions and $\rho(v)$, $\sigma(v)$, $\varrho(v)$ are not uniformly zero, $S_1 \le v \le S_2$, $S_3 \le l, l_0 \le S_4$.

Proposition 3.1. Let $\boldsymbol{\beta}(v)$ be involute of the given spacelike curve $\boldsymbol{\alpha}(v)$. Then $\boldsymbol{\beta}(v)$ is a line of curvature, an asymptotic curve or a geodesic on $\boldsymbol{P}(v,l)$ (Eq. (7)) if $y_l(v,l_0) = \varrho(v)R^{'}(l_0)$, $z_l(v,l_0) = \sigma(v)Q^{'}(l_0)$ in *Theorem 1*, *Theorem 2* or *Theorem 3*, respectively.

§4 Ruled surface family with common involute as an asymptotic curve

In this section, we give an expression for the ruled surface $\boldsymbol{P}(v,l)$ with $\boldsymbol{\beta}(v)$ as an asymptotic curve; moreover, the conditions are examined for the ruled surface $\boldsymbol{P}(v,l)$ to be developable.

The surface $\boldsymbol{P}(v,l)$ as a ruled surface can be expressed as a parameter form

$$\boldsymbol{P}(v,l) = \boldsymbol{\beta}(v) + (l-l_0)\boldsymbol{M}, \tag{16}$$

where $\boldsymbol{M}$ is the direction of the rulings.

Comparing Eqs. (7) and (16), we can easily obtain

$$\begin{aligned} (l-l_0)\boldsymbol{M} = x(v,l)\boldsymbol{T}^* + y(v,l)\boldsymbol{N}^* + z(v,l)\boldsymbol{B}^*,\\ S_1 \le v \le S_2, S_3 \le l \le S_4, \end{aligned} \tag{17}$$

and we can get $x(v,l)$, $y(v,l)$ and $z(v,l)$ as follows:

$$\begin{cases} x(v,l) = (l-l_0)det(\boldsymbol{M},\boldsymbol{B}^*,\boldsymbol{N}^*),\\ y(v,l) = (l-l_0)det(\boldsymbol{M},\boldsymbol{T}^*,\boldsymbol{B}^*),\\ z(v,l) = (l-l_0)det(\boldsymbol{M},\boldsymbol{N}^*,\boldsymbol{T}^*). \end{cases} \tag{18}$$

If $\boldsymbol{\beta}(v)$ is asymptotic on $\boldsymbol{P}(v,l)$, the marching-scale functions satisfies *Theorem 2*, then

$$det(\boldsymbol{M},\boldsymbol{T}^*,\boldsymbol{B}^*) = 0, det(\boldsymbol{M},\boldsymbol{N}^*,\boldsymbol{T}^*) \neq 0.$$

Thus, at any point on the curve $\boldsymbol{\beta}(v)$, $\boldsymbol{M}$ must be in the plane formed by $\boldsymbol{T}^*$ and $\boldsymbol{B}^*$. Moreover, $\boldsymbol{M}$ and $\boldsymbol{T}^*$ must not be parallel. Thus, we assume that

$$\boldsymbol{M} = a(v)\boldsymbol{T}^* + d(v)\boldsymbol{B}^*, \tag{19}$$

where $a(v)$ and $d(v)$ are real functions and $d(v) \neq 0$ for all $v \in [S_1, S_2]$.

Substituting Eq. (19) in Eq. (18), we have

$$x(v,l) = (l-l_0)a(v),\ y(v,l) = 0,\ z(v,l) = (l-l_0)d(v),$$

where $z(v,l) \neq 0$ for all $v \in [S_1, S_2]$.

Thus, we conclude as follows.

Theorem 4.1. Let $\boldsymbol{\beta}(v)$ be involute of the given spacelike curve $\boldsymbol{\alpha}(v)$. Then the ruled surface with $\boldsymbol{\beta}(v)$ as common asymptotic directrix is given by

$$\boldsymbol{P}(v,l) = \boldsymbol{\beta}(v) + (l-l_0)(a(v)\boldsymbol{T}^* + d(v)\boldsymbol{B}^*), \tag{20}$$

where $a(v)$ and $d(v)$ are real functions, $d(v) \neq 0$ for all $v \in [S_1, S_2]$.

Theorem 4.2. Let $\boldsymbol{\beta}(v)$ be the involute of the given spacelike curve $\boldsymbol{\alpha}(v)$, then the ruled surface $\boldsymbol{P}(v,l)$ (Eq. (20)) possessing $\boldsymbol{\beta}(v)$ as a geodesic curve is a developable surface if $\theta = constant$

and $a(v)=0$ or $\frac{\tau}{\kappa}=q$, where $a(v)$ is a real function, and

$$q=\begin{cases}-\cot\theta, & \text{if } \boldsymbol{B} \text{ is spacelike,}\\ \tanh\theta, & \text{if } \boldsymbol{B} \text{ and } \boldsymbol{W} \text{ is timelike.}\\ \coth\theta, & \text{if } \boldsymbol{B} \text{ is timelike and } \boldsymbol{W} \text{ is spacelike.}\end{cases}$$

Proof. The ruled surface is decided by the direction vector $\boldsymbol{M}$, if the ruled surface is developable, then

$$det[\boldsymbol{\beta}^{'}(v),\boldsymbol{M},\boldsymbol{M}^{'}]=0 \tag{21}$$

Next, we only discuss the condition where $\boldsymbol{B}$ as a timelike binormal vector and $\boldsymbol{W}$ is timelike binormal.

Since $\boldsymbol{M}=a\boldsymbol{T}^{*}+d\boldsymbol{B}^{*}$, from Eqs. (3) and (6), we obtain

$$\begin{aligned}\boldsymbol{M}'=&(a^{'}-\kappa d\cosh\theta+\tau d\sinh\theta)\boldsymbol{T}^{*}\\&+(-a\kappa\sinh\theta+a\tau\cosh\theta-d\theta^{'})\boldsymbol{N}^{*}\\&+(-a\kappa\cosh\theta+a\tau\sinh\theta+d^{'})\boldsymbol{B}^{*}.\end{aligned}$$

Substituting into Eq. (21), we get

$$det[\boldsymbol{\beta}^{'}(v),\boldsymbol{M},\boldsymbol{M}^{'}]=[\boldsymbol{T}^{*},d\boldsymbol{B}^{*},(-d\theta^{'}-a\kappa\sinh\theta+a\tau\cosh\theta)\boldsymbol{N}^{*}]=0.$$

Since $\boldsymbol{T}^{*}$ is not parallel to $\boldsymbol{N}^{*}$ and $\boldsymbol{B}^{*}$, the equation holds if $-d\theta^{'}-a\kappa\sinh\theta+a\tau\cosh\theta=0$. Specially, by $d\neq 0$, if $\theta=constant$ and $\frac{\tau}{\kappa}=\tanh\theta$ or $\theta=constant$ and $a(v)=0$, then $\boldsymbol{P}(v,t)$ is a developable surface.

Similarly, when $\boldsymbol{B}$ is spacelike, we prove that if $\theta=constant$ and $\frac{\tau}{\kappa}=-\cot\theta$ or $a(v)=0$, $\boldsymbol{P}(v,l)$ is a developable surface, and when $\boldsymbol{B}$ is timelike and $\boldsymbol{W}$ is spacelike, if $\theta=constant$ and $\frac{\tau}{\kappa}=\coth\theta$ or $a(v)=0$, $\boldsymbol{P}(v,l)$ is a developable surface.

§5 Examples

In this section, we give examples to construct the surface family with involute $\boldsymbol{\beta}(v)$ of a spacelike curve $\boldsymbol{\alpha}(v)$ as three kinds of characteristic curves.

Example 5.1. Given a spacelike curve $\boldsymbol{\alpha}(v)=(0,\cos v,\sin v)$, the $\boldsymbol{B}$ of $\boldsymbol{\alpha}(v)$ is a timelike vector, and

$$\boldsymbol{T}=(0,-\sin v,\cos v),\ \boldsymbol{N}=(0,-\cos v,-\sin v),\ \boldsymbol{B}=(1,0,0),$$

and $\kappa=1$, $\theta=0$, $\tau=0$. Let $m=0$, we get the involute of $\boldsymbol{\alpha}(v)$ is $\boldsymbol{\beta}(v)=(0,\cos v+v\sin v,\sin v-v\cos v)$ and

$$\boldsymbol{T}^{*}=(0,-\cos v,-\sin v),\ \boldsymbol{N}^{*}=(0,\sin v,-\cos v),\ \boldsymbol{B}^{*}=(1,0,0).$$

If we choose $x(v,l)=\frac{v}{3}l^{2}$, $y(v,l)=\frac{1}{2}l^{3}$, $z(v,l)=\frac{v}{2}\sin 2l$ and $\vartheta=0$, $\eta(v)=\frac{1}{3}v$, $l_{0}=0$, satisfying *Theorem 3.1.* and construct the surface family interpolating $\boldsymbol{\beta}(v)$ as a line of curvature, then we have

$$P_{1}(v,l)=(\frac{v}{6}\sin 2l,\ (1-\frac{v}{3}l^{2})\cos v+\sin v(\frac{1}{2}l^{3}+v),\ (1-\frac{v}{3}l^{2})\sin v-\cos v(\frac{1}{2}l^{3}+v)),$$

where $-2\leq v\leq 2, -1.5\leq l\leq 0$, (see Fig. 3(a)).

Choosing $x(v,l)=\frac{v}{3}l^{2}$, $y(v,l)=\frac{1}{2}\sin vl$, $z(v,l)=\frac{1}{2}l^{3}$ satisfying *Theorem 3.2.* and constructing the surface family interpolating $\boldsymbol{\beta}(v)$ an asymptotic curve, we can get

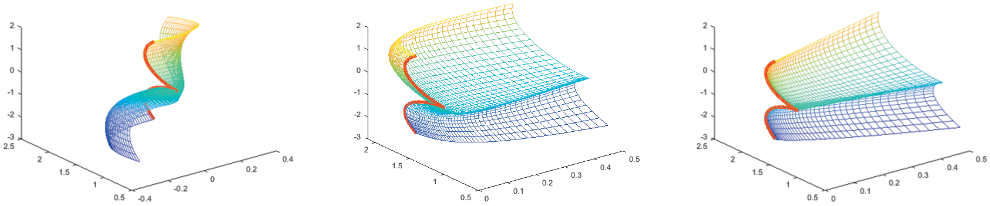
$$P_2(v,l)=(\frac{1}{2}l^3,\ (1-\frac{v}{3}l^2)\cos v+\sin v(\frac{1}{2}\sin 2l+v),\ (1-\frac{v}{3}l^2)\sin v-\cos v(\frac{1}{2}\sin 2l+v)),$$

where $-2\leq v\leq 2, 0\leq l\leq 2$, (see Fig. 3(b)).

If we choose $x(v,l)=\frac{v}{3}l^2$, $y(v,l)=\frac{1}{3}l^2$, $z(v,l)=\frac{1}{2}\sin 2l$ satisfying *Theorem 3.3.* and construct the surface family interpolating $\boldsymbol{\beta}(v)$ as a geodesic curve, then we have

$$P_3(v,l)=(\frac{1}{2}\sin 2l,\ (1-\frac{v}{3}l^2)\cos v+\sin v(\frac{1}{2}l^3+v),\ (1-\frac{v}{3}l^2)\sin v-\cos v(v+\frac{1}{2}l^3)),$$

where $-1.5\leq v\leq 3, 0\leq l\leq 1$. (see Fig. 3(c)).



(a) The surface interpolating $\boldsymbol{\beta}(v)$ as a curvature curve.

(b) The surface interpolating $\boldsymbol{\beta}(v)$ as an asymptotic curve.

(c) The surface interpolating $\boldsymbol{\beta}(v)$ as a geodesic curve.

Figure 3. The surface family with involute of $\boldsymbol{\alpha}(v)$ with timelike $\boldsymbol{B}(v)$ as a common Characteristic curve. (The red curve represents $\boldsymbol{\beta}(v)$.)

Example 5.2 Let $\boldsymbol{\alpha}(v)=(\cos v,\sin v,0)$ be a spacelike curve, and its binormal vector $\boldsymbol{B}$ is timelike vector, we get

$$\boldsymbol{T}=(-\sin v,\cos v,0),\ \boldsymbol{N}=(-\cos v,-\sin v,0),\ \boldsymbol{B}=(0,0,1),$$

then $\kappa=1$, $\tau=0$ and $\theta=0$. Let $m=0$, the involute of the curve $\boldsymbol{\alpha}(v)$ is $\boldsymbol{\beta}(v)=(\cos v+v\sin v,\sin v-v\cos v,0)$ and

$$\boldsymbol{T}^*=(-\cos v,-\sin v,0),\ \boldsymbol{N}^*=(\sin v,-\cos v,0),\ \boldsymbol{B}^*=(0,0,-1),$$

We choose $x(v,l)=vl$, $y(v,l)=\frac{v}{4}\sin l$, $z(v,l)=\frac{v}{3}l^2$ and $\vartheta=0$, $\eta(v)=\frac{v}{4}$, $l_0=0$, satisfying *Theorem 3.1.* and construct the surface family interpolating $\boldsymbol{\beta}(v)$ as a line of curvature, then we have

$$P_1(v,l)=((1-vl)\cos v+(v+\frac{v}{4}\sin l)\sin v,\ (1-vl)\sin v-\cos v(\frac{v}{4}\sin l+v),\ -\frac{v}{3}l^2),$$

where $-2\leq v\leq 2, 0\leq l\leq 0.8$, (see Fig. 4(a)).

Choosing $x(v,l)=vl$, $y(v,l)=\frac{1}{2}\sin l$, $z(v,l)=\frac{1}{3}l^3$ satisfying *Theorem 3.2.* and constructing the surface family interpolating $\boldsymbol{\beta}(v)$ as an asymptotic curve, then we have

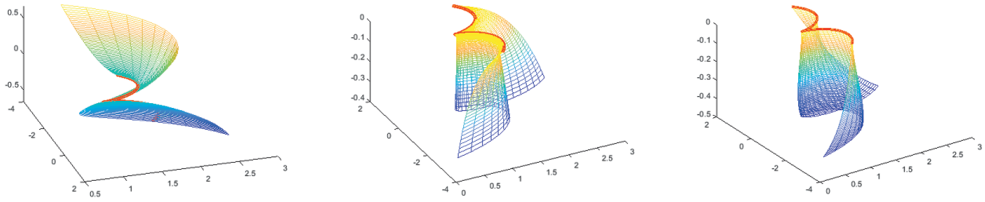
$$P_2(v,l)=((1-vl)\cos v+\sin v(\frac{1}{2}\sin l+v),\ (1-vl)\sin v-\cos v(\frac{1}{2}\sin l+v),\ -\frac{1}{3}l^3),$$

where $-2\leq v\leq 2, 0\leq l\leq 0.8$, (see and Fig. 4(b)).

If we choose $x(v,l)=vl$, $y(v,l)=\frac{1}{3}l^3$, $z(v,l)=\frac{1}{2}\sin l$ satisfying *Theorem 3.3.* and constructing the surface family interpolating $\boldsymbol{\beta}(v)$ as a geodesic curve, then we have

$$P_3(v,l)=((1-vl)\cos v+\sin v(\frac{1}{3}l^3+v),\ (1-vl)\sin v-\cos v(\frac{1}{3}l^3+v),\ -\frac{1}{2}\sin l),$$

where $-2 \leq v \leq 2$, $0 \leq l \leq 0.8$, (see Fig. 4(c)).



(a) The surface interpolating $\boldsymbol{\beta}(v)$ as a curvature curve. (b) The surface interpolating $\boldsymbol{\beta}(v)$ as an asymptotic curve. (c) The surface interpolating $\boldsymbol{\beta}(v)$ as a geodesic curve.

Figure 4. The surface family with involute of $\boldsymbol{\alpha}(v)$ with spacelike $\boldsymbol{B}(v)$ as a common characteristic curve. (The red curve represents $\boldsymbol{\beta}(v)$.)

§6 Conclusion

In R_1^3, we discuss the surface family interpolating the involute of a spacelike curve as three kinds of characteristic curves and discuss the conditions that ruled surface family with a common involute as an asymptotic curve. However, the fact that the curve is timelike curve needs to be further explored. In addition, the same study may be done for implicitly defined surfaces, and it is possible to consider other spaces.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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School of Mathematics, Hefei University of technology, Hefei 230601, China.
Email: hfutus@hfut.edu.cn