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The blow-up of solutions for porous medium equations with viscoelastic term under positive initial energy

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Abstract. This paper deals with homogeneous Dirichlet boundary value problem to a class of porous medium equations with viscoelastic term

$$\frac{\partial u}{\partial t} - \Delta u^m + \int_0^t g(t-s)\Delta u^m(x,s)\mathrm{d}s = u^p, \qquad x \in \Omega, \quad t \geq 0,$$

where $p > m$ $(m > 0)$. We prove that the weak solutions of the above problem blow up in finite time when the initial energy is positive and the function g satisfies suitable conditions. Our result generalizes that of S.A. Messaoudi in [1].

§1 Introduction

We consider the following equations with viscoelastic term

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m + \displaystyle\int_0^t g(t-s)\Delta u^m(x,s)\mathrm{d}s = u^p, & x \in \Omega, \quad t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a bounded domain, $\partial\Omega$ is Lipschitz contiuous, $u_0 \geq 0$, $m > 0$ and $g:\ \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function. Problem (1) arises from a variety of mathematical model in engineering and physical sciences.

A purely elastic material has a capacity to store mechanical energy with no dissipation of the energy [2]. This is the case for viscoelastic material. Thus, it is widely used in physics and other fields, such as the diffusion of electrorheological fluids [3], heat conduction with memory materials and viscous flow in viscoelastic materials [4], and the types of materials with dissipation and storage of mechanical energy [5]. In the past few decades, the study of equations with viscoelastic terms has drawn a considerable attention. When $m = 1$, there are

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many results have been obtained. We refer the interested readers to [6]-[11]. In [1], Messaoudi considered the following initial boundary value problem

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = \mid u\mid^{q-2} u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2}$$

He proved that the solution of (2) blows up in finite time when the initial energy is negative and q satisfies the following conditions

$$2 < q \le \frac{2(n-1)}{n-2},\ n > 2; \quad q > 2\ (n = 1, 2).$$

The absence of the viscoelastic term ($g = 0$) for Problem (1)

$$\begin{cases} u_t - \Delta u^m = u^p & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3}$$

has been studied by various authors and several results have been established [12]-[16]. But the viscoelastic term makes the Problem (1) more complicated. Specially in the existence of viscoelastic term, we need to estimate the terms

$$\int_0^t g(t-s)\|\nabla u^m(t) - \nabla u^m(s)\|_2^2 \mathrm{d}s, \quad \int_0^t g'(t-s)\|\nabla u^m(t) - \nabla u^m(s)\|_2^2 \mathrm{d}s$$

and

$$\int_0^t g(t-s)\int_\Omega \nabla u^m(x,t)\cdot(\nabla u^m(x,s) - \nabla u^m(x,t))\mathrm{d}x\mathrm{d}s.$$

Motivated by the above works, we investigate a more general porous media equation than Problem (2). By using the differential inequality technique, we establish the blow-up of weak solutions for Problem (1) with positive initial energy, and obtain an upper bound of blow-up time, which enrich the results of weak solutions blow up under negative initial energy in [1].

§2 Blow up in finite time

First, we need to introduce the following definitions.

Definition 2.1 A function $u^m(x,t) \in L^\infty(\Omega\times(0,T)) \bigcap L^2(0,T;H_0^1(\Omega))$ is called solution of Problem (1) if and only if the following equality holds

$$-\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x - \int\int_{Q_T}(u\varphi_t - \nabla u^m\cdot\nabla\varphi)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega(\int_0^t g(t-s)\nabla u^m(x,s)\nabla\varphi\mathrm{d}s)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_\Omega u^p\varphi\mathrm{d}x\mathrm{d}t,$$

for all $\varphi \in \Phi = \{\varphi \mid \varphi \in H^1(\Omega\times(0,T)),\ \varphi(x,T) = 0,\ \varphi(x,t)\mid_{\partial\Omega} = 0\},\ u_0 \in L^\infty(\Omega)\bigcap H_0^1(\Omega)$.

Definition 2.2 Let $u(x,t)$ be a weak solution of Problem (1), we call the function $u(x,t)$ blows up in finite time if the maximal existence time T is finite and

$$\lim_{t\to T^-}\|u(.,t)\|_{L^m(\Omega)} = +\infty.$$

The energy functional corresponding to Problem (1) is

$$E(t)=\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u^m\|_2^2-\frac{m}{m+p}\|u\|_{m+p}^{m+p}+\frac{1}{2}\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s. \quad (4)$$

For the relaxation function g and the number p, we assume that

$$g(s)\geq 0,\quad g'(s)\leq 0,\quad 1-\int_0^\infty g(s)ds=l>0, \quad (5)$$

$$m<p\leq\frac{m(n+2)}{n-2},\quad n\geq 3. \quad (6)$$

We also set

$$\alpha_1=B^{-\frac{p+m}{p-m}},\qquad E_1=(\frac{1}{2}-\frac{m}{m+p})\alpha_1^2, \quad (7)$$

where $B=\frac{C^*}{l^{\frac{1}{2}}}$ for C^* the best constant of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{1+\frac{p}{m}}(\Omega)$.

Multiplying the Equation (1) by $(u^m)_t$ and integrating over Ω, we get

$$\begin{aligned}E'(t)&=\frac{1}{2}\int_0^t g'(t-\tau)\|\nabla^m(t)-\nabla u^m(\tau)\|_2^2\mathrm{d}\tau-\frac{1}{2}g(t)\|\nabla u^m\|_2^2-m\int_\Omega u^m\cdot(u_t)^2\mathrm{d}x\\&=\frac{1}{2}\int_0^t g'(t-\tau)\|\nabla^m(t)-\nabla u^m(\tau)\|_2^2\mathrm{d}\tau-\frac{1}{2}g(t)\|\nabla u^m\|_2^2\\&\quad-\frac{4m}{(m+1)^2}\int_\Omega\left(u^{\frac{m+1}{2}}\right)_t^2dx\\&\leq 0.\end{aligned} \quad (8)$$

Lemma 2.1 Let u be a solution of (1) with initial data satisfying

$$E(0)<E_1,\qquad \|\nabla u_0^m\|_2>\alpha_1. \quad (9)$$

Then there exists a constant $\beta>\alpha_1$ such that

$$\left[(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2+\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right]^{\frac{1}{2}}\geq\beta, \quad (10)$$

$$\|u\|_{m+p}\geq B^{\frac{1}{m}}\beta^{\frac{1}{m}},\qquad \forall t\in[0,T). \quad (11)$$

Proof. By the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{1+\frac{p}{m}}(\Omega)$, we have

$$\|u\|_{m+p}^m=\|u^m\|_{1+\frac{p}{m},\Omega}\leq C^*\|\nabla u^m\|_{2,\Omega},\qquad m<p\leq\frac{m(n+2)}{n-2},\quad (n\geq 3). \quad (12)$$

Combining the above inequality and (4), we can get

$$\begin{aligned}E(t)&=\frac{1}{2}(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2-\frac{m}{m+p}\|u\|_{m+p}^{m+p}\\&\quad+\frac{1}{2}\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\\&\geq\frac{1}{2}(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2+\frac{1}{2}\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\\&\quad-\frac{m}{m+p}C^{*\frac{m+p}{m}}\|\nabla u^m\|_2^{\frac{m+p}{m}}.\end{aligned} \quad (13)$$

Let $C^* = Bl^{\frac{1}{2}}$ then

$$
\begin{aligned}
E(t) &\geq \frac{1}{2}(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \frac{1}{2}\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s \\
&\quad - \frac{m}{m+p}B^{\frac{m+p}{m}} l^{\frac{m+p}{2m}} \|\nabla u^m\|_2^{\frac{m+p}{m}} \\
&\geq \frac{1}{2}\left[(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right] \\
&\quad - \frac{mB^{\frac{m+p}{m}}}{m+p}\left[(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\right]^{\frac{m+p}{2m}} \\
&= \frac{\alpha^2}{2} - \frac{m}{m+p}B^{\frac{m+p}{m}}\alpha^{\frac{m+p}{m}} \doteq h(\alpha),
\end{aligned} \tag{14}
$$

where

$$
\alpha = \left[(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right]^{\frac{1}{2}}.
$$

It is easy to verify that h is increasing for $0<\alpha<\alpha_1$, decreasing for $\alpha>\alpha_1$, $h(\alpha)\to-\infty$ as $\alpha\to+\infty$ and

$$
h(\alpha_1) = (\frac{1}{2}-\frac{m}{m+p})B^{-\frac{2(p+m)}{p-m}} = E_1, \tag{15}
$$

Therefore, since $E(0)<E_1$, there exists $\beta>\alpha_1$ such that $h(\beta)=E(0)$. By using (14), we have

$$
h(\|\nabla u_0^m\|_2) \leq E(0) = h(\beta), \tag{16}
$$

which implies that $\|\nabla u_0^m\| \geq \beta$.

We suppose by contradiction that

$$
\left[(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right]^{\frac{1}{2}} < \beta \tag{17}
$$

for some $t_0>0$ and by the continuity of

$$
(1-\int_0^t g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s,
$$

we can choose t_0 such that

$$
\left[(1-\int_0^{t_0} g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^{t_0} g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right]^{\frac{1}{2}} > \alpha_1.
$$

So we can get by (14)

$$
\begin{aligned}
E(t_0) &\geq h\left(\left[(1-\int_0^{t_0} g(s)\mathrm{d}s)\|\nabla u^m\|_2^2 + \int_0^{t_0} g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right]^{\frac{1}{2}}\right) \\
&> h(\beta) = E(0).
\end{aligned}
$$

This is impossible since $E(t)\leq E(0)$ for all $t\in[0,T)$. So (10) is established.

To prove (11), we obtain from (4) and (9)

$$
\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u^m\|_2^2 + \frac{1}{2}\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s \leq E(0) + \frac{m}{m+p}\|u\|_{m+p}^{m+p},
$$

consequently

$$\begin{aligned}\frac{m}{m+p}\|u\|_{m+p}^{m+p} &\geq \frac{1}{2}\left[(1-\int_0^t g(s))\|\nabla u^m\|_2^2+\int_0^t g(t-s)\|\nabla u^m(t)-\nabla u^m(s)\|_2^2\mathrm{d}s\right]-E(0)\\ &\geq \frac{1}{2}\beta^2-E(0)=\frac{m}{m+p}B^{\frac{m+p}{m}}\beta^{\frac{m+p}{m}}.\end{aligned}$$

Therefore (11) yields the desired result. The proof is completed. □

Theorem 2.1 Assume that (5), (6) and (7) hold. Give $u_0\in H_0^1(\Omega)\bigcap L^\infty(\Omega)$ satisfying

$$\|\nabla u_0\|_2>\alpha_1,\qquad E(0)<E_1. \tag{18}$$

If

$$\int_0^\infty g(s)\mathrm{d}s<\frac{1-c_0}{1-\frac{3}{4}c_0},\qquad c_0=\frac{2m}{m+p}+\frac{p-m}{p+m}(\frac{\alpha}{\beta})^{\frac{m+p}{m}}<1, \tag{19}$$

then any solution of (1) blows up in finite time.

Proof. We define

$$G(t)=\frac{1}{m+1}\int_\Omega u^{m+1}\mathrm{d}x,$$

and differentiate $G(t)$ to get

$$\begin{aligned}G'(t) &= \int_\Omega u^m u_t\mathrm{d}x\\ &= \int_\Omega u^m(\Delta u^m-\int_0^t g(t-s)\Delta u^m\mathrm{d}s+u^p)\mathrm{d}x\\ &= -\int_\Omega |\nabla u^m|^2\,\mathrm{d}x+\int_\Omega u^{m+p}\mathrm{d}x+\int_\Omega\int_0^t g(t-s)\nabla u^m(x,t)\nabla u^m(x,s)\mathrm{d}s\mathrm{d}x\\ &= -\int_\Omega |\nabla u^m|^2\,\mathrm{d}x+\int_\Omega\int_0^t g(t-s)\,|\nabla u^m(x,t)|^2\,\mathrm{d}s\mathrm{d}x\\ &\quad -\int_\Omega\int_0^t g(t-s)\nabla u^m(x,t)(\nabla u^m(x,t)-\nabla u^m(x,s))\mathrm{d}s\mathrm{d}x+\|u\|_{m+p}^{m+p}.\end{aligned}$$

By the Hölder inequality and Young's inequality, we can arrive at

$$\begin{aligned}G'(t) &\geq -(1-\frac{3}{4}\int_0^t g(t-s)\mathrm{d}s)\|\nabla u^m\|_2^2\\ &\quad -\int_0^t g(t-s)\|\nabla u^m(x,t)-\nabla u^m(x,s)\|_2^2\mathrm{d}s+\|u\|_{m+p}^{m+p}.\end{aligned} \tag{20}$$

Let $H(t)=E_1-E(t)$, $t\geq 0$. We substitute for $\|\nabla u^m\|_2^2$ from (4), so (20) becomes

$$\begin{aligned}G'(t) &\geq \|u\|_{m+p}^{m+p}+\frac{2(1-\frac{3}{4}\int_0^t g(s)\mathrm{d}s)}{1-\int_0^t g(s)\mathrm{d}s}H(t)-\frac{2(1-\frac{3}{4}\int_0^t g(s)\mathrm{d}s)}{1-\int_0^t g(s)\mathrm{d}s}E_1\\ &\quad +\left(\frac{1-\frac{3}{4}\int_0^t g(s)\mathrm{d}s}{1-\int_0^t g(s)\mathrm{d}s}-1\right)\int_0^t g(t-\tau)\|\nabla u^m(x,t)-\nabla u^m(x,\tau)\|_2^2\mathrm{d}\tau\\ &\quad -\frac{2m}{m+p}\frac{1-\frac{3}{4}\int_0^t g(s)\mathrm{d}s}{1-\int_0^t g(s)\mathrm{d}s}\|u\|_{m+p}^{m+p}\geq\gamma\int_\Omega u^{m+p}\mathrm{d}x,\end{aligned} \tag{21}$$

where

$$\gamma=1-\left[\frac{2m}{m+p}+\frac{p-m}{p+m}(\frac{\alpha}{\beta})^{\frac{m+p}{m}}\right]\frac{1-\frac{3}{4}\int_0^\infty g(s)\mathrm{d}s}{1-\int_0^\infty g(s)\mathrm{d}s}>0. \tag{22}$$

By the embedding theorem $L^{m+p}(\Omega) \hookrightarrow L^{m+1}(\Omega)$, we obtain that

$$\|u\|_{m+1} \leq C_1 \|u\|_{m+p}. \tag{23}$$

Inserting (23) into (21), we can derive that

$$G^{'}(t) \geq C_2 G(t)^{\frac{m+p}{m+1}}, \tag{24}$$

where $C_2 = \gamma C_1^{-(m+p)}(m+1)^{\frac{m+p}{m+1}}$, C_1 is a best embedding constant. By a direct integration of (24), we can get

$$G^{\frac{p-1}{m+1}}(t) \geq \frac{1}{G^{\frac{1-p}{1+m}}(0) - \frac{p-1}{m+1} C_2 t}.$$

The above inequality implies that $G(t)$ blows up at finite time $T^* \leq \frac{(m+1)G^{\frac{1-p}{1+m}}(0)}{(p-1)C_2}$. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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