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Blow-up results for the weakly coupled system of semilinear wave equations with weak dampings

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Abstract. This work is devoted to the study of initial boundary value problem for k-component system of semilinear wave equations with several fundamental boundary conditions (namely, the Dirichlet, Neumann, and Robin boundary conditions). Blow-up results and lifespan estimates of solutions to the problem with two different types of weak damping terms and power nonlinearities in the sub-critical and critical cases on exterior domain are obtained. The test function technique is performed in the proofs. It is worth observing that our results in Theorem 1.1 in this article contain the results in [6] as a special case when $\theta = 0$. To the best of our knowledge, the results in Theorems 1.1-1.2 are new.

§1 Introduction

Our main purpose in this work is to investigate the following initial boundary value problem for the k-component system

$$\begin{cases} \partial_t^2 u_1(x,t) - \Delta u_1(x,t) + c(x,t)\partial_t u_1(x,t) = |u_k(x,t)|^{p_1}, \quad (x,t) \in \Omega^c \times (0,T), \\ \partial_t^2 u_2(x,t) - \Delta u_2(x,t) + c(x,t)\partial_t u_2(x,t) = |u_1(x,t)|^{p_2}, \quad (x,t) \in \Omega^c \times (0,T), \\ \quad \vdots \\ \partial_t^2 u_k(x,t) - \Delta u_k(x,t) + c(x,t)\partial_t u_k(x,t) = |u_{k-1}(x,t)|^{p_k}, \ (x,t) \in \Omega^c \times (0,T), \\ \alpha \dfrac{\partial u_l}{\partial n^+}(x,t) + \beta u_l(x,t) = 0, \quad (x,t) \in \partial\Omega^c \times (0,T), \ l = 1,2,\dots,k, \\ u_l(x,0) = \varepsilon u_{0,l}(x), \ \partial_t u_l(x,0) = \varepsilon u_{1,l}(x), \quad x \in \Omega^c, \ l = 1,2,\dots,k, \end{cases} \tag{1.1}$$

where

$$c(x,t) = a(x) = (1+|x|^2)^{-\frac{\theta}{2}} \, (\theta \le 1),$$

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or

$$c(x,t)=\frac{a(x)}{1+t}=(1+|x|^2)^{-\frac{\theta}{2}}(1+t)^{-1}\,(\theta<0),$$

respectively. The indexes of nonlinear terms satisfy $1<p_l<\infty$ with $l=1,\,2,\,\cdots,\,k\,(k\geq 2)$. $\Delta=\Delta_{\alpha,\,\beta}$ stands for the Laplace operator dependent on the boundary condition on exterior domain. Here, α, and β are real constants which satisfy $(\alpha,\,\beta)\neq(0,\,0)$. The boundary conditions are called the Dirichlet boundary condition ($\alpha=0$), the Neumann boundary condition ($\beta=0$) and the Robin boundary condition otherwise. Let $\Omega=B_1(0)=\{x\in\mathbb{R}^n\big|\,|x|\leq 1\}$ and $\Omega^c=\mathbb{R}^n\backslash B_1(0)$. We suppose $B_R(0)=\{x\in\mathbb{R}^n\big|\,|x|\leq R\}$, where $R>2$. Moreover, functions $u_{0,\,l}$ and $u_{1,\,l}$ with $l=1,\,2,\,\cdots,\,k\,(k\geq 2)$ represent shape of the initial values, which satisfy $\operatorname{supp}(u_{0,l},\,u_{1,l})\subset\Omega^c\cap B_R(0)$. The positive constant ε describes the size of initial values.

We sketch some historical background regarding the Cauchy problem of nonlinear heat equation

$$\begin{cases}u_t-\Delta u=|u|^p, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=\varepsilon f(x), & x\in\mathbb{R}^n,\end{cases}\tag{1.2}$$

and the Cauchy problem of nonlinear wave equation

$$\begin{cases}u_{tt}-\Delta u=|u|^p, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=\varepsilon f(x),\ u_t(x,0)=\varepsilon g(x), & x\in\mathbb{R}^n.\end{cases}\tag{1.3}$$

Problem (1.2) possesses the Fujita exponent $p_F(n)=1+\frac{2}{n}$. There exists a unique global (in time) weak solution if $p>p_F(n)$ and local (in time) weak solution blows up if $1<p\leq 1+\frac{2}{n}$. Problem (1.3) admits the Strauss exponent $p_S(n)=\frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$ for $n\geq 2$ and $p_S(n)=\infty$ for $n=1$. More precisely, if $1<p\leq p_S(n)$, the solution of the Cauchy problem with small initial values blows up in a finite time. If $p>p_S(n)$, then the solution exists globally (in time). One can refer to the references [28, 40, 42, 43] for more details.

The Cauchy problem of linear damped wave equation

$$\begin{cases}u_{tt}-\Delta u+c(x,t)u_t=0, & x\in\mathbb{R}^n,\ t>0,\\ (u,\,u_t)(x,0)=\varepsilon(u_0,\,u_1)(x), & x\in\mathbb{R}^n,\end{cases}\tag{1.4}$$

causes great concern, where the coefficient in damping term satisfies

$$c(x,t)=a_0a(x)b(t)=a_0(1+|x|)^{-\alpha}(1+t)^{-\beta},$$

while a_0 is a positive constant. It is the space dependent damping when $\alpha\in\mathbb{R}$, $\beta=0$. Behavior of solution can be classified in the following cases. If $\alpha\in(-\infty,1)$, the solution behaves like that of heat equation. The damping is scaling invariant weak damping in the case of $\alpha=1$. If $\alpha\in(1,\infty)$, the damping is scattering. This indicates that the solution behaves like that of classical wave equation. On the other hand, it is the time-dependent damping when $\alpha=0$, $\beta\in\mathbb{R}$. The solution does not decay to zero in general when $\beta\in(-\infty,-1)$. When $\beta\in[-1,1)$, the solution behaves like that of heat equation. The damping is scale invariant when $\beta=1$. In the case of $\beta\in(1,\infty)$, the solution behaves like that of wave equation. Asymptotic behavior of solution depends on the constant a_0.

Many scholars are committed to the Cauchy problem of semilinear wave equation with space or time-dependent damping

$$\begin{cases} u_{tt} - \Delta u + c(x,t)u_t = f(u,\, u_t), & x \in \mathbb{R}^n,\, t > 0, \\ (u,\, u_t)(x,0) = \varepsilon(u_0,\, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

where $c(x,t) = a_0(1+|x|^2)^{-\frac{\alpha}{2}}(1+t)^{-\beta}$ $(a_0 > 0)$, $f(u,\, u_t) = |u|^p,\, |u_t|^p,\, |u_t|^p + |u|^q$ (see [3, 10–12, 17–21, 23, 24, 26, 27, 29, 36, 39, 41] and the references therein). Todorova et al. [38] investigate the existence of global (in time) solution to classical damped wave equation in the energy space when $p > 1 + \frac{2}{n}$ and blow-up phenomenon of solution when $1 < p < 1 + \frac{2}{n}$. Fino et al. [8] show blow-up result of solution to problem (1.5) with $\alpha = 0$, $\beta = 0$ in the critical case, where the initial values have compact supports. Without the compactness of initial values, Lai et al. [25] present blow-up result for the initial boundary value problem of semilinear damped wave equation on exterior domain in high dimensions ($n \geq 3$) critical case. Blow-up dynamic and lifespan estimate of solution to semilinear wave equation with space dependent damping $D(x)u_t$ $(0 \leq D(x) \leq \frac{\mu}{(1+|x|)^{\alpha}},\, \alpha > 1)$ and potential term are derived in [22]. The main tool employed in the proof is the test function method. Ming et al. [30] verify blow-up result of solution to the variable-coefficient wave equation with scattering damping term $\frac{\mu}{(1+t)^{\beta}}u_t$ $(\beta > 1)$ and divergence form nonlinearities in the sub-critical and critical cases. Upper bound lifespan estimates of solutions to the problem are deduced by taking advantage of the rescaled test function technique and iteration method. Ikeda et al. [16] establish upper bound and lower bound lifespan estimates of solutions to semilinear wave equation with general time dependent effective damping term, where the test function approach is performed. Making use of the test function method $(\varphi(x,t) = D^{\delta}_{t|T}(\varphi_1^{\ell}(x)\varphi_2(t)))$, Dannawi et al. [4] acquire blow-up result of solution to problem (1.5) with $a_0 > 0$, $\alpha,\, \beta \geq 0$, $\alpha + \beta < 1$, $\alpha\beta = 0$ and nonlinear memory term under positive initial value conditions. Non-existence of global solution to problem (1.5) with $\alpha \in [0,1)$, $\beta \in (-1,1)$, $\alpha\beta = 0$ and $f(u,\, u_t) = |u|^p$ in the critical case is obtained by utilizing the test function technique $(\psi_R^k(x,t) = \eta(tR^{-\frac{2-\alpha}{1+\beta}})\phi(\frac{x}{R}))$ (see [13]).

Recently, the investigation of weakly coupled system of semilinear wave equations

$$\begin{cases} u_{tt} - \Delta u + g(u_t) = f_1(v,\, v_t), & x \in \mathbb{R}^n,\, t > 0, \\ v_{tt} - \Delta v + g(v_t) = f_2(u,\, u_t), & x \in \mathbb{R}^n,\, t > 0, \\ (u,\, u_t,\, v,\, v_t)(x,0) = \varepsilon(u_0,\, u_1,\, v_0,\, v_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

attracts more and more attention. Chen et al. [1] explore blow-up and lifespan estimates of solutions to problem (1.6) with $g(u_t) = u_t$, $g(v_t) = v_t$ and $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$ in the critical case. Upper bound lifespan estimates of solutions to the problem are derived by applying the test function method associated with nonlinear differential inequalities. Lower bound lifespan estimates of solutions in low space dimension are documented by introducing polynomial logarithmic type time weighted Sobolev spaces. Taking advantage of the test function technique and iteration method, Palmieri et al. [35] show blow-up dynamics and lifespan estimates of solutions to problem (1.6) with scattering dampings $g(u_t) = b_1(t)u_t$, $g(v_t) = b_2(t)v_t$ and $f_1(v,\, v_t) = |v|^p$, $f_2(u,\, u_t) = |u|^q$ in the sub-critical and critical cases. Ming et al. [31] illustrate formation of singularities of solutions to problem (1.6) with scattering damping terms $g(u_t) = \frac{\mu}{(1+t)^{\beta}}u_t$, $g(v_t) = \frac{\mu}{(1+t)^{\beta}}v_t$ $(\beta > 1)$ and combined nonlinearities $f_1(v,\, v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f_2(u,\, u_t) = |u_t|^{p_2} + |u|^{q_2}$. Upper bound lifespan estimates of solutions are documented by making use of the test function method and iteration argument. We refer readers to the references [2, 5, 7, 9, 15, 32, 33] for more details.

Enlightened by the works in [1, 6, 14, 34, 37], our main aim of this article is to show blow-up results and lifespan estimates of solutions to problem (1.1) on exterior domain in the cases of several fundamental boundary conditions, which contain two types of weak damping terms

$$c(x,t)\partial_t u_i = (1+|x|^2)^{-\frac{\theta}{2}}\partial_t u_i \, (\theta \le 1,\ 1 \le i \le k,\ i \in \mathbb{N}^*), \tag{1.7}$$

$$c(x,t)\partial_t u_i = (1+|x|^2)^{-\frac{\theta}{2}}(1+t)^{-1}\partial_t u_i \, (\theta < 0,\ 1 \le i \le k,\ i \in \mathbb{N}^*), \tag{1.8}$$

and power nonlinearities, respectively. It is worth observing that sharp lifespan estimates of solutions to the weakly coupled system of semilinear classical damped wave equations in the critical case are documented by employing the test function method ($\psi = [\eta(\frac{t^2+|x|^4}{R^4})]^{\mu+2}$) (see [1]). Making use of the test function technique ($\psi = \phi(\frac{t^2+|x|^4}{R^4})$), Takeda et al. [37] investigate blow-up phenomenons of solutions to the k-component system of semilinear damped wave equations in the whole space. Upper bound lifespan estimates of solutions to the initial boundary value problems with three types of boundary conditions (the Dirichlet, Neumann, and Robin boundary conditions) for the k-component system of semilinear classical damped wave equations with power nonlinearities are considered in [6]. The proof is based on the test function method ($\psi = \Psi(x)[\varphi(\frac{t^2+(|x|-1)^4}{R^4})]^{\lambda+2}$). Ikeda et al. [14] derive lifespan estimate of solution to the initial boundary value problem of semilinear wave equation with weak damping term $a(x)u_t$ ($|a(x)| \le a_0\langle x\rangle^{-\alpha}$, $\alpha \in [0,1]$) and power nonlinearity $|u|^p$, where the test function method is applied ($\psi_R(x,t) = [\eta(\frac{t+|x|^{2-\alpha}}{R})]^{2p'}$). Nishihara et al. [34] discuss blow-up of solution to the Cauchy problem of semilinear wave equation with damping term $c(x,t)u_t$ ($c(x,t) = a_0(1+|x|^2)^{-\frac{\alpha}{2}}(1+t)^{-\beta}$). Upper bound lifespan estimate of solution to the problem in the case of $\alpha < 0$, $\beta = 1$ is obtained by taking advantage of the test function technique ($\psi_R(x,t) = (1+t)[\eta(\frac{t^2+|x|^{2-\alpha}}{R^2})]^{2p'}$). Moreover, blow-up dynamic and lifespan estimate of solution to the problem in the case of $\alpha < 0$, $\beta = 0$ are deduced by utilizing the test function approach ($\psi_R(x,t) = [\eta(\frac{t+|x|^{2-\alpha}}{R})]^{2p'}$). From our observation, there is no related results for upper bound lifespan estimates of solutions to problem (1.1) with time or space dependent damping terms and three types of boundary conditions. To fill this gap, we extend the problem studied in [14] to problem (1.1) with damping in (1.7) by exploiting the test function technique ($\psi(x,t) = \Psi(x)[\eta(\frac{t+(|x|-1)^{2-\theta}}{R})]^{\lambda+2}$), which is different from the test functions in [6, 14, 34] (see Theorem 1.1). The problem investigated in [34] is extended to problem (1.1) with damping in (1.8). The proof is based on the test function technique ($\psi(x,t) = \Psi(x)[\eta(\frac{t^2+(|x|-1)^{2-\theta}}{R^2})]^{\lambda+2}$), which is different from the test functions in [6, 34] (see Theorem 1.2). It is worth noting that our results in Theorem 1.1 in this article contain the results in [6] as a special case when $\theta = 0$. To the best of our knowledge, the results in Theorems 1.1-1.2 are new.

Throughout the paper, we define the matrix

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & p_1 \\ p_2 & 0 & \dots & 0 & 0 \\ 0 & p_3 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & p_k & 0 \end{pmatrix}$$

and the column vector $\gamma = (\gamma_1, \gamma_2, ..., \gamma_k)^t = (P - I_k)^{-1} \underbrace{(1, 1, \ldots, 1)^t}_{k\,times}$, where I_k stands for the identity matrix. $(\alpha_1, \alpha_2, \ldots, \alpha_k)^t$ is the transposition vector of vector $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. We assume $\det(P - I_k) = (-1)^{k+1}(\prod_{l=1}^{k} p_l - 1) \neq 0$ with $p_l > 1\,(l = 1, 2, \ldots, k)$, which implies that the inverse matrix $(P - I_k)^{-1}$ exists. Thus, we set $\gamma_{\max} = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. A uniform constant $C = C(f(x))$ implies that the constant C depends on the function $f(x)$, which is different from line to line. $A \lesssim B$ denotes that there exists a positive constant C such that $A \leq CB$.

Related lemma, definition of weak solutions and main results in this paper are illustrated as follows.

Lemma 1.1. *(Existence of the local solution) Let $n \geq 1$. Suppose $1 < p_l < \infty$ with $l = 1, 2, \cdots, k\,(k \geq 2)$ if $n = 1, 2$. Assume $1 < p_l \leq \frac{n}{n-2}$ with $l = 1, 2, \cdots, k\,(k \geq 2)$ if $n \geq 3$. Suppose that the initial values satisfy $u_{0,l}(x) \in H^1(\Omega^c)$, $u_{1,l}(x) \in L^2(\Omega^c)$, supp $(u_{0,l}, u_{1,l}) \subset \Omega^c \cap B_R(0)$ for $R > 2$. Then, there exist positive constant T and uniquely determined local (in time) mild solutions to problem* (1.1)

$$\begin{aligned}&(u_1, u_2, \ldots, u_k)\\ &\in C([0,T]; H^1(\Omega^c)) \cap C^1([0,T]; L^2(\Omega^c))\\ &\quad \times C([0,T]; H^1(\Omega^c)) \cap C^1([0,T]; L^2(\Omega^c))\\ &\quad\quad \vdots\\ &\quad \times C([0,T]; H_0^1(\Omega^c)) \cap C^1([0,T]; L^2(\Omega^c))\end{aligned}$$

which satisfy supp $(u_l, \partial_t u_l) \subset \{(x,t) \in \Omega^c \times [0,T) \big| \, |x| \leq t + R\}$. Moreover, if the lifespan estimates of solutions $T(\varepsilon) < \infty$, it holds that

$$\lim_{t \to T(\varepsilon)^-} \| (u_l,\, \partial_t u_l) \|_{H^1(\Omega^c) \times L^2(\Omega^c)} = \infty$$

for $l = 1, 2, \cdots, k\,(k \geq 2)$.

Proof of Lemma 1.1. The proof of Lemma 1.1 is similar to the Proposition 1.1 in the reference [34]. We omit its detailed proof.

Definition 1.1. *Assume that $(u_{0,l}, u_{1,l}) \in H^1(\Omega^c) \times L^2(\Omega^c)$. If*

$$\begin{aligned}&(u_1, u_2, \ldots, u_k)\\ &\in \big(C([0,T), H_0^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c)) \cap L^{p_2}_{loc}([0,T) \times \Omega^c)\big)\\ &\quad \times \big(C([0,T), H_0^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c)) \cap L^{p_3}_{loc}([0,T) \times \Omega^c)\big)\\ &\quad\quad \vdots\\ &\quad \times \big(C([0,T), H_0^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c)) \cap L^{p_1}_{loc}([0,T) \times \Omega^c)\big)\end{aligned}$$

satisfy

$$\int_0^T \int_{\Omega^c} |u_k(x,s)|^{p_1} \phi(x,s) dx ds + \int_{\Omega^c} u_{1,l}(x) \phi(x,0) dx$$

$$
\begin{aligned}
&= \int_0^T \int_{\Omega^c} \big(\nabla u_1(x,s)\nabla\phi(x,s) - \partial_t u_1(x,s)\partial_t\phi(x,s) \\
&\quad + c(x,t)\partial_t u_1(x,s)\phi(x,s)\big)dxds \qquad (1.9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \int_{\Omega^c} |u_l(x,s)|^{p_{l+1}}\phi(x,s)dxds + \int_{\Omega^c} u_{1,l+1}(x)\phi(x,0)dx \\
&= \int_0^T \int_{\Omega^c} \big(\nabla u_{l+1}(x,s)\nabla\phi(x,s) - \partial_t u_{l+1}(x,s)\partial_t\phi(x,s) \\
&\quad + c(x,t)\partial_t u_{l+1}(x,s)\phi(x,s)\big)dxds \qquad (1.10)
\end{aligned}
$$

with $l = 1, 2, \ldots, k-1\,(k \geq 2)$, *where* $\phi(x,s) \in C^2([0,T)\times\Omega^c)$ *with* $supp\,\phi \subset [0,T)\times\Omega^c$ *such that* $(\alpha\frac{\phi}{\partial n^+} + \beta\phi)(t)|_{\partial\Omega^c} = 0$, *then* $(u_1, u_2, \ldots, u_k)$ *are called weak solutions to problem* (1.1).

Theorem 1.1. *Let* $n \geq 1$, $p_l > 1$ *with* $l = 1, 2, \ldots, k\,(k \geq 2)$ *and*

$$
\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} \geq \frac{n-\theta}{2}. \qquad (1.11)
$$

Suppose that $u_{0,l}\Psi(x) \in L^1(\Omega^c)$, $u_{1,l}\Psi(x) \in L^1(\Omega^c)$ *and*

$$
\int_{\Omega^c} (a(x)u_{0,l}(x) + u_{1,l}(x))\Psi(x)dx > 0, \qquad (1.12)
$$

where $\Psi(x)$ *is defined in* (2.2). *Then, there exists a positive constant*

$$
\varepsilon_0 = \varepsilon_0(n, p_1, \ldots, p_k, \alpha, \beta, R, \theta, u_{0,l}, u_{1,l})
$$

such that for all $\varepsilon \in (0, \varepsilon_0]$, *the lifespan estimates of solutions* $T(\varepsilon)$ *to problem* (1.1) *with* $c(x,t) = a(x)\,(a(x) = (1+|x|^2)^{-\frac{\theta}{2}},\ \theta \leq 1)$ *satisfy*

$$
T(\varepsilon) \leq \begin{cases}
C\varepsilon^{-\left(\frac{2}{2-\theta}\max\{\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_k\}-1\right)^{-1}}, & \max\{\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_k\} > \frac{2-\theta}{2}, \\
& \beta \neq 0,\ n = 1, \\
C(\varepsilon^{-1}\log(\varepsilon^{-1}))^{\frac{2-\theta}{2\Gamma(2,p_1,p_2,\ldots,p_k)}}, & \Gamma(2, p_1, p_2, \ldots, p_k) > 0,\ \beta \neq 0, \\
\exp\big(\exp(C\varepsilon^{-(p_1-1)})\big), & \Gamma(2, p_1, p_2, \ldots, p_k) = 0,\ \beta \neq 0, \\
& p_i = p_j,\ i, j \in \{1, 2, \ldots, k\}, \\
C\varepsilon^{-(\frac{2}{2-\theta}\Gamma(n,p_1,p_2,\ldots,p_k))^{-1}}, & \Gamma(n, p_1, p_2, \ldots, p_k) > 0,\ \beta \neq 0,\ n \geq 3, \\
\exp(C\varepsilon^{-(p_1p_2\cdots p_k-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta \neq 0,\ n \geq 3, \\
& p_i \neq p_j,\ i, j \in \{1, 2, \ldots, k\},\ i \neq j, \\
\exp(C\varepsilon^{-(p_1-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta \neq 0,\ n \geq 3, \\
& p_i = p_j,\ i, j \in \{1, 2, \ldots, k\}, \\
C\varepsilon^{-(\frac{2}{2-\theta}\Gamma(n,p_1,p_2,\ldots,p_k))^{-1}}, & \Gamma(n, p_1, p_2, \ldots, p_k) > 0,\ \beta = 0,\ n \geq 1, \\
\exp(C\varepsilon^{-(p_1p_2\cdots p_k-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta = 0,\ n \geq 1, \\
& p_i \neq p_j,\ i, j \in \{1, 2, \ldots, k\},\ i \neq j, \\
\exp(C\varepsilon^{-(p_1-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta = 0,\ n \geq 1, \\
& p_i = p_j,\ i, j \in \{1, 2, \ldots, k\},
\end{cases} \qquad (1.13)
$$

where $\Gamma(n, p_1, p_2, \ldots, p_k) = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - \frac{n-\theta}{2} \geq 0$, *and* C *is a positive constant inde-*

pendent of ε.

Theorem 1.2. *Let* $n \geq 1$, $p_l > 1$ *with* $l = 1, 2, \ldots, k\,(k \geq 2)$ *and*

$$\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} \geq \frac{n-\theta}{2}. \tag{1.14}$$

Assume that $u_{0,l}\Psi(x) \in L^1(\Omega^c)$, $u_{1,l}\Psi(x) \in L^1(\Omega^c)$ *and*

$$\int_{\Omega^c} ((a(x)-1)u_{0,l}(x) + u_{1,l}(x))\Psi(x)dx > 0, \tag{1.15}$$

where $\Psi(x)$ *is defined in* (2.2). *Then, there exists a positive constant*

$$\varepsilon_0 = \varepsilon_0(n, p_1, \ldots, p_k, \alpha, \beta, R, \theta, u_{0,l}, u_{1,l})$$

such that for all $\varepsilon \in (0, \varepsilon_0]$, *the lifespan estimates of solutions* $T(\varepsilon)$ *to problem* (1.1) *with* $c(x,t) = \frac{a(x)}{1+t}$ $(a(x) = (1+|x|^2)^{-\frac{\theta}{2}}, \theta \leq 0)$ *satisfy*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\left(\frac{4}{2-\theta}\max\{\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_k\}-2\right)^{-1}}, & \max\{\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_k\} > \dfrac{2-\theta}{2}, \\ & \beta \neq 0,\ n = 1, \\ C(\varepsilon^{-1}\log(\varepsilon^{-1}))^{\frac{2-\theta}{4\Gamma(2,p_1,p_2,\ldots,p_k)}}, & \Gamma(2, p_1, p_2, \ldots, p_k) > 0,\ \beta \neq 0, \\ \exp\left(\exp(C\varepsilon^{-(p_1-1)})\right), & \Gamma(2, p_1, p_2, \ldots, p_k) = 0,\ \beta \neq 0, \\ & p_i = p_j,\ i, j \in \{1, 2, \ldots, k\}, \\ C\varepsilon^{-(\frac{4}{2-\theta}\Gamma(n,p_1,p_2,\ldots,p_k))^{-1}}, & \Gamma(n, p_1, p_2, \ldots, p_k) > 0,\ \beta \neq 0,\ n \geq 3, \\ \exp(C\varepsilon^{-(p_1p_2\ldots p_k-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta \neq 0,\ n \geq 3, \\ & p_i \neq p_j,\ i, j \in \{1, 2, \ldots, k\},\ i \neq j, \\ \exp(C\varepsilon^{-(p_1-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta \neq 0,\ n \geq 3, \\ & p_i = p_j,\ i, j \in \{1, 2, \ldots, k\}, \\ C\varepsilon^{-(\frac{4}{2-\theta}\Gamma(n,p_1,p_2,\ldots,p_k))^{-1}}, & \Gamma(n, p_1, p_2, \ldots, p_k) > 0,\ \beta = 0,\ n \geq 1, \\ \exp(C\varepsilon^{-(p_1p_2\ldots p_k-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta = 0,\ n \geq 1, \\ & p_i \neq p_j,\ i, j \in \{1, 2, \ldots, k\},\ i \neq j, \\ \exp(C\varepsilon^{-(p_1-1)}), & \Gamma(n, p_1, p_2, \ldots, p_k) = 0,\ \beta = 0,\ n \geq 1, \\ & p_i = p_j,\ i, j \in \{1, 2, \ldots, k\}, \end{cases} \tag{1.16}$$

where $\Gamma(n, p_1, p_2, \ldots, p_k) = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - \frac{n-\theta}{2} \geq 0$, *and* C *is a positive constant independent of* ε.

Remark 1.1. It is worth noticing that Nishihara et al. [34] consider blow-up phenomenon of solution to the Cauchy problem of semilinear wave equation with damping term $c(x,t)u_t$ $(c(x,t) = a_0(1+|x|^2)^{-\frac{\alpha}{2}}(1+t)^{-\beta})$ in the cases of $\alpha < 0$, $\beta = 1$ and $\alpha < 0$, $\beta = 0$. Upper bound lifespan estimates of solutions to the problem are derived by making use of the test function technique. But there is no related results for blow-up and lifespan estimates of solutions to problem (1.1) with time or space dependent damping terms and three types of boundary conditions. We establish upper bound lifespan estimates of solutions to problem (1.1) with two types of damping terms in (1.7) and (1.8) by applying the test function method, respectively (see Theorems

1.1-1.2). Moreover, we observe that our results in Theorems 1.1 are exactly coincide with the results in [6] when $\theta = 0$.

§2 Related lemmas and Proof of Theorem 1.1

Let $\eta(t) \in C^\infty([0,\infty))$ satisfy

$$\eta(t) = \begin{cases} 1, & t \le \frac{1}{2}, \\ decreasing, & \frac{1}{2} < t < 1, \\ 0, & t \ge 1, \end{cases} \qquad \eta^*(t) = \begin{cases} 0, & t \le \frac{1}{2}, \\ \eta(t), & t > \frac{1}{2}. \end{cases}$$

We set two test functions $\phi_R(x,t)$ and $\phi_R^*(x,t)$ which are used to derive the lifespan estimates of solutions to problem (1.1). Namely,

$$\phi_R(x,t) = (\eta(\frac{t + (|x|-1)^{2-\theta}}{R}))^{\lambda+2}, \ \ \phi_R^*(x,t) = (\eta^*(\frac{t + (|x|-1)^{2-\theta}}{R}))^{\lambda+2}, \tag{2.1}$$

where $\theta \le 1$, $\lambda \ge \max\limits_{1\le l\le k} \frac{2}{p_l - 1}$.

We present a lemma which will be applied in the proof.

Lemma 2.1. *[6] Let $R_1 > 0$, $C_0 > 0$, $\sigma \ge 0$ and $\mu \in \mathbb{R}$. Assume that $0 \le \varphi(x,t) \in L^1_{loc}([0,T), L^1(\Omega^c))$ for $T > R_1$, which satisfies*

$$\begin{aligned} & C\varepsilon + \int_0^T \int_{\Omega^c} \varphi(x,s)\phi_R(x,s)dxds \\ & \le C_0 R^{-\frac{\sigma}{p'}} (\log R)^{\frac{\mu}{p'}} (\int_0^T \int_{\Omega^c} \varphi(x,s)\phi_R^*(x,s)dxds)^{\frac{1}{p}} \end{aligned}$$

for all $R \in [R_1, T)$. Here, $\phi_R(x,s)$ and $\phi_R^*(x,s)$ are defined in (2.1). It holds that

$$T(\varepsilon) \le \begin{cases} C\varepsilon^{-\frac{1}{\sigma}} (\log(\varepsilon^{-1}))^{\frac{\mu}{\sigma}}, & \sigma > 0, \ \mu \in \mathbb{R}, \\ \exp(C\varepsilon^{-\frac{p-1}{1-\mu(p-1)}}), & \sigma = 0, \ \mu < \frac{1}{p-1}, \\ \exp\exp(C\varepsilon^{-(p-1)}), & \sigma = 0, \ \mu = \frac{1}{p-1}, \end{cases}$$

where C is a positive constant independent of ε.

Proof of Theorem 1.1. We define the function

$$\Psi(x) = \begin{cases} |x| - 1 + \frac{\alpha}{\beta}, & \beta \ne 0, \ n = 1, \\ \log|x| + \frac{\alpha}{\beta}, & \beta \ne 0, \ n = 2, \\ 1 - |x|^{2-n} + \frac{\alpha}{\beta}(n-2), & \beta \ne 0, \ n \ge 3, \\ 1, & \beta = 0, \ n \ge 1. \end{cases} \tag{2.2}$$

Direct computation gives rise to

$$\nabla\Psi(x)=\begin{cases}1, \quad \beta\neq 0,\ n=1,\\ \dfrac{x}{|x|^2}, \quad \beta\neq 0,\ n=2,\\ (n-2)\dfrac{x}{|x|^n}, \quad \beta\neq 0,\ n\geq 3,\\ 0, \quad \beta=0,\ n\geq 1.\end{cases} \tag{2.3}$$

It is deduced from (2.1) that

$$|\partial_t\phi_R(x,t)|\lesssim R^{-1}(\phi_R^*(x,t))^{\frac{\lambda+1}{\lambda+2}}, \tag{2.4}$$

$$|\partial_t^2\phi_R(x,t)|\lesssim R^{-2}(\phi_R^*(x,t))^{\frac{\lambda}{\lambda+2}}, \tag{2.5}$$

$$|\nabla\phi_R(x,t)|\lesssim R^{-1}(|x|-1)^{1-\theta}\frac{x}{|x|}(\phi_R^*(x,t))^{\frac{\lambda+1}{\lambda+2}}, \tag{2.6}$$

$$|\Delta\phi_R(x,t)|\lesssim R^{-1}(|x|-1)^{-\theta}(\phi_R^*(x,t))^{\frac{\lambda}{\lambda+2}}. \tag{2.7}$$

For $l=1,2,\dots,k-1$, we denote two functions

$$I_R[u_l]=\int_0^T\int_{\Omega^c}|u_l(x,s)|^{p_{l+1}}\psi(x,s)dxds,$$

$$I_R[u_k]=\int_0^T\int_{\Omega^c}|u_k(x,s)|^{p_1}\psi(x,s)dxds.$$

Replacing ϕ in (1.9)-(1.10) by $\psi=\Psi(x)\phi_R(x,t)$, we derive

$$\begin{aligned}&I_R[u_k]+\varepsilon\int_{\Omega^c}(a(x)u_{0,1}(x)+u_{1,1}(x))\Psi(x)\phi_R(x,0)dx\\ &=\int_0^T\int_{\Omega^c}u_1(x,s)\big(\partial_t^2(\Psi(x)\phi_R(x,s))\\ &\quad-\Delta(\Psi(x)\phi_R(x,s))-a(x)\partial_t(\Psi(x)\phi_R(x,s))\big)dxds\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&I_R[u_l]+\varepsilon\int_{\Omega^c}(a(x)u_{0,l+1}(x)+u_{1,l+1}(x))\Psi(x)\phi_R(x,0)dx\\ &=\int_0^T\int_{\Omega^c}u_{l+1}(x,s)\big(\partial_t^2(\Psi(x)\phi_R(x,s))\\ &\quad-\Delta(\Psi(x)\phi_R(x,s))-a(x)\partial_t(\Psi(x)\phi_R(x,s))\big)dxds\end{aligned} \tag{2.9}$$

with $l=1,2,\dots,k-1$.

Making use of (2.2)-(2.7) yields

$$\begin{aligned}&|\partial_t^2(\Psi\phi_R)-\Delta(\Psi\phi_R)-a(x)\partial_t(\Psi\phi_R)|\\ &=|\Psi\partial_t^2\phi_R-2\nabla\Psi\nabla\phi_R-\Psi\Delta\phi_R-a(x)\Psi\partial_t\phi_R|\\ &\lesssim(\frac{1}{R^2}+\frac{(|x|-1)^{-\theta}}{R})\Psi(\phi_R^*)^{\frac{\lambda}{\lambda+2}},\end{aligned} \tag{2.10}$$

where we have employed the fact

$$\begin{cases}1-\dfrac{1}{|x|}\leq\log|x|, \quad n=2,\\ (n-2)\dfrac{x}{|x|^n}\geq 2|x|, \quad n\geq 3.\end{cases} \tag{2.11}$$

It is worth noticing that

$$\lim_{R\to\infty}\int_{\Omega^c}(a(x)u_{0,l}(x)+u_{1,l}(x))\Psi(x)\phi_R(x,0)dx$$
$$=\int_{\Omega^c}(a(x)u_{0,l}(x)+u_{1,l}(x))\Psi(x)dx$$

for all $l=1,2,\ldots,k\,(k\geq 2)$. Thus, there exists a sufficiently large constant R_0 such that

$$\int_{\Omega^c}(a(x)u_{0,l}(x)+u_{1,l}(x))\Psi(x)\phi_R(x,0)dx\geq C_{1,l}>0 \tag{2.12}$$

for $R>R_0$. Here, $C_{1,l}\,(l=1,2,\ldots,k\,(k\geq 2))$ are positive constants, where we have performed the conditions $u_{0,l}(x)\Psi(x)\in L^1(\Omega^c)$, $u_{1,l}(x)\Psi(x)\in L^1(\Omega^c)$ and (1.12).

From (2.8), (2.10)-(2.12) and the condition $\lambda\geq\max\limits_{1\leq l\leq k}\frac{2}{p_l-1}$, we obtain

$$\begin{aligned}I_R[u_k]+C_{1,1}\varepsilon &\lesssim R^{-\frac{2}{2-\theta}}\Big(\int_0^T\int_{\Omega^c}\Psi dxds\Big)^{\frac{1}{p_2'}}\\ &\quad\times\Big(\int_0^T\int_{\Omega^c}|u_1|^{p_2}\Psi(\phi_R^*)^{\frac{p_2\lambda}{\lambda+2}}dxds\Big)^{\frac{1}{p_2}}\\ &\lesssim I_{p_2}(R)\Big(\int_0^T\int_{\Omega^c}|u_1|^{p_2}\Psi(\phi_R^*)^{\frac{p_2\lambda}{\lambda+2}}dxds\Big)^{\frac{1}{p_2}}\\ &\lesssim I_{p_2}(R)\Big(\int_0^T\int_{\Omega^c}|u_1|^{p_2}\Psi(\phi_R^*)dxds\Big)^{\frac{1}{p_2}},\end{aligned} \tag{2.13}$$

where $I_{p_2}(R)$ is defined by

$$I_{p_2}(R)=\begin{cases}R^{1-\frac{4-\theta}{(2-\theta)p_2}}, & \beta\neq 0,\ n=1,\\ R^{1-\frac{4-\theta}{(2-\theta)p_2}}(\log R^{\frac{1}{2-\theta}})^{\frac{p_2-1}{p_2}}, & \beta\neq 0,\ n=2,\\ R^{\frac{n-\theta}{2-\theta}-\frac{n+2-\theta}{(2-\theta)p_2}}, & \beta\neq 0,\ n\geq 3,\\ R^{\frac{n-\theta}{2-\theta}-\frac{n+2-\theta}{(2-\theta)p_2}}, & \beta=0,\ n\geq 1.\end{cases}$$

In a similar way, we acquire

$$I_R[u_1]+C_{1,2}\varepsilon\lesssim I_{p_3}(R)\Big(\int_0^T\int_{\Omega^c}|u_2|^{p_3}\Psi\phi_R^*dxds\Big)^{\frac{1}{p_3}}, \tag{2.14}$$

$$I_R[u_2]+C_{1,3}\varepsilon\lesssim I_{p_4}(R)\Big(\int_0^T\int_{\Omega^c}|u_3|^{p_4}\Psi\phi_R^*dxds\Big)^{\frac{1}{p_4}}, \tag{2.15}$$

$$\vdots$$

$$I_R[u_{k-1}]+C_{1,k}\varepsilon\lesssim I_{p_1}(R)\Big(\int_0^T\int_{\Omega^c}|u_k|^{p_1}\Psi\phi_R^*dxds\Big)^{\frac{1}{p_1}}. \tag{2.16}$$

For the sake of brevity, we suppose $\gamma_{\max}=\max\{\gamma_1,\gamma_2,\ldots,\gamma_k\}$. Direct calculation shows

$$\gamma_k=\frac{1+p_k+p_{k-1}p_k+\cdots+p_2p_3\cdots p_k}{\prod\limits_{l=1}^{k}p_l-1}.$$

Utilizing (1.11) leads to $\Gamma(n,p_1,p_2,\ldots,p_k)=\gamma_{\max}-\frac{n-\theta}{2}\geq 0$.

Thus, we need to discuss the following seven cases in the next Table 1.

Table 1. Combination of $\Gamma(n,p_1,p_2,\dots,p_k)$, β and n.

	$\Gamma(n,p_1,p_2,\dots,p_k)$	β	n (dimensions)
Case 1	$\Gamma(n,p_1,p_2,\dots,p_k)>0$	$\beta\neq 0$	$n\geq 3$
Case 2	$\Gamma(n,p_1,p_2,\dots,p_k)=0$ and $p_i\neq p_j$ $(i,j\in\{1,2,\dots,k\}$ and $i\neq j)$	$\beta\neq 0$	$n\geq 3$
Case 3	$\Gamma(n,p_1,p_2,\dots,p_k)=0$ and $p_i=p_j$ $(i,j\in\{1,2,\dots,k\})$	$\beta\neq 0$	$n\geq 3$
Case 4	$\Gamma(2,p_1,p_2,\dots,p_k)>0$	$\beta\neq 0$	$n=2$
Case 5	$\Gamma(2,p_1,p_2,\dots,p_k)=0$ and $p_i=p_j$ $(i,j\in\{1,2,\dots,k\})$	$\beta\neq 0$	$n=2$
Case 6	$\max\{\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_k\}>\frac{2-\theta}{2}$	$\beta\neq 0$	$n=1$
Case 7	$\Gamma(1,p_1,p_2,\dots,p_k)\geq 0$	$\beta=0$	$n\geq 1$

In Case 1, taking advantage of (2.13)-(2.16), we deduce

$$\begin{aligned}
&I_R[u_{k-1}]+C_{1,k}\varepsilon\\
&\lesssim I_{p_1}(I_R[u_k])^{\frac{1}{p_1}}\\
&\lesssim I_{p_1}(I_{p_2}(I_R[u_1])^{\frac{1}{p_2}})^{\frac{1}{p_1}}\\
&=I_{p_1}(I_{p_2})^{\frac{1}{p_1}}(I_R[u_1])^{\frac{1}{p_1p_2}}\\
&\lesssim R^{\frac{n-\theta}{2-\theta}-\frac{2}{(2-\theta)p_1}-\frac{n+2-\theta}{(2-\theta)p_1p_2}}(I_R[u_1])^{\frac{1}{p_1p_2}}\\
&\quad\vdots\\
&\lesssim R^{\frac{n-\theta}{2-\theta}-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_{k-1}})-\frac{n+2-\theta}{(2-\theta)p_1p_2\cdots p_k}}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\
&=R^{-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+\frac{n-\theta}{2-\theta}(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}},
\end{aligned}\tag{2.17}$$

which results in

$$\begin{aligned}
C_{1,k}\varepsilon&\lesssim R^{-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+\frac{n-\theta}{2-\theta}(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\
&\quad-I_R[u_{k-1}]\\
&\lesssim R^{-\frac{2}{2-\theta}\gamma_k+\frac{n-\theta}{2-\theta}}
\end{aligned}\tag{2.18}$$

for $R\geq R_0$. Sending $R\to T(\varepsilon)$ in (2.18) yields

$$T(\varepsilon)\leq C\varepsilon^{-(\frac{2}{2-\theta}\Gamma(n,p_1,p_2,\dots,p_k))^{-1}}.$$

As a consequence, we conclude the fourth lifespan estimate in (1.13).

In Case 2, similar to the derivation in (2.17), we arrive at

$$\begin{aligned}
&I_R[u_{k-1}]+C_{1,k}\varepsilon\\
&\lesssim R^{-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+\frac{n-\theta}{2-\theta}(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\
&=(\int_0^T\int_{\Omega^c}|u_{k-1}(x,s)|^{p_k}\Psi(x)\phi_R^*(x,s)dxds)^{\frac{1}{p_1p_2\cdots p_k}}.
\end{aligned}\tag{2.19}$$

We define the auxiliary functions

$$h_{p_k}=h_{p_k}(r)=\int_0^T\int_{\Omega^c}|u_{k-1}(x,s)|^{p_k}\Psi(x)\phi_r^*(x,s)dxds,\tag{2.20}$$

$$H_{p_k} = H_{p_k}(R) = \int_0^R h_{p_k}(r) r^{-1} dr. \tag{2.21}$$

It follows from (2.20)-(2.21) that

$$H_{p_k} \le \frac{\log 2}{4} I_R[u_{k-1}], \qquad h_{p_k}(R) = R H'_{p_k}(R). \tag{2.22}$$

Taking into account (2.19) and (2.22), we achieve

$$\frac{4}{\log 2} H_{p_k}(R) + C_{1,k}\varepsilon \lesssim (R H'_{p_k}(R))^{\frac{1}{p_1 p_2 \cdots p_k}}.$$

Therefore, we derive the fifth lifespan estimate in (1.13).

In Case 3, we obtain $p = p_1 = p_2 = \cdots = p_k = 1 + \frac{2}{n-\theta}$. It is worth observing that

$$\begin{aligned}
&\partial_t^2(u_1 + u_2 + \cdots + u_k) - \Delta(u_1 + u_2 + \cdots + u_k) + a(x)\partial_t(u_1 + u_2 + \cdots + u_k) \\
&= |u_1|^p + |u_2|^p + \cdots + |u_k|^p \\
&\ge C|u_1 + u_2 + \cdots + u_k|^p.
\end{aligned} \tag{2.23}$$

We discuss problem (1.1) with $c(x,t) = (1+|x|^2)^{-\frac{\theta}{2}}$ $(\theta \le 1)$ as the single equation in the critical case $p = 1 + \frac{2}{n-\theta}$. It holds that

$$T(\varepsilon) \le \exp\{C\varepsilon^{-(p_1-1)}\}.$$

In case 4, similar to the derivation in the Case 1, we acquire

$$\begin{aligned}
&I_R[u_{k-1}] + C_{1,k}\varepsilon \\
&\lesssim I_{p_1}(I_R[u_k])^{\frac{1}{p_1}} \\
&\lesssim I_{p_1}(I_{p_2})^{\frac{1}{p_1}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\lesssim R^{1-\frac{2}{(2-\theta)p_1}-\frac{4-\theta}{(2-\theta)p_1 p_2}}(\log R^{\frac{1}{2-\theta}})^{1-\frac{1}{p_1 p_2}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\quad \cdots \\
&\lesssim R^{-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1 p_2}+\cdots+\frac{1}{p_1 p_2 \cdots p_k})+(1-\frac{1}{p_1 p_2 \cdots p_k})} \\
&\quad \times(\log R^{\frac{1}{2-\theta}})^{1-\frac{1}{p_1 p_2 \cdots p_k}}(I_R[u_{k-1}])^{\frac{1}{p_1 p_2 \cdots p_k}}.
\end{aligned} \tag{2.24}$$

Applying (2.24), Lemma 2.1 and taking $\sigma = \frac{2}{2-\theta}\Gamma(2, p_1, p_2, \ldots, p_k)$, $\mu = 1$, $p = p_1 p_2 \cdots p_k$ give rise to the second lifespan estimate in (1.13).

In Case 5, we observe $p = p_1 = p_2 = \cdots = p_k = 1 + \frac{2}{2-\theta}$. Utilizing Lemma 2.1 with $\sigma = 0$, $\mu = 1$, $p = 1 + \frac{2}{2-\theta}$, we conclude the third lifespan estimate in (1.13).

In Case 6, similar to the derivation in the Case 2, we deduce

$$\begin{aligned}
&I_R[u_{k-1}] + C_{1,k}\varepsilon \\
&\lesssim I_{p_1}(I_R[u_k])^{\frac{1}{p_1}} \\
&\lesssim I_{p_1}(I_{p_2})^{\frac{1}{p_1}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\lesssim R^{1-\frac{2}{(2-\theta)p_1}-\frac{4-\theta}{(2-\theta)p_1 p_2}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\quad \cdots \\
&\lesssim R^{-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1 p_2}+\cdots+\frac{1}{p_1 p_2 \cdots p_k})+(1-\frac{1}{p_1 p_2 \cdots p_k})} \\
&\quad \times(I_R[u_{k-1}])^{\frac{1}{p_1 p_2 \cdots p_k}},
\end{aligned} \tag{2.25}$$

which yields

$$\begin{aligned}C_{1,k}\varepsilon &\lesssim R^{-\frac{2}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\&\quad - I_R[u_{k-1}]\\&\lesssim R^{-\frac{2}{2-\theta}\gamma_k+1}\end{aligned} \tag{2.26}$$

for $R \geq R_0$. Sending $R \to T(\varepsilon)$ in (2.26) leads to the first lifespan estimate in (1.13).

In Case 7, similar to the derivation in Cases 1-3, we arrive at the seventh-ninth lifespan estimates in (1.13). The proof of Theorem 1.1 is finished. ■

§3 Proof of Theorem 1.2

We set

$$\psi_R(x,t)=(\eta(\frac{t^2+(|x|-1)^{2-\theta}}{R^2}))^{\lambda+2}, \tag{3.1}$$

$$\psi_R^*(x,t)=(\eta^*(\frac{t^2+(|x|-1)^{2-\theta}}{R^2}))^{\lambda+2}, \tag{3.2}$$

where $\theta<0$, $\lambda \geq \max\limits_{1\leq l\leq k}\frac{2}{p_l-1}$.

It follows from (3.1) and (3.2) that

$$|\partial_t\psi_R(x,t)|\lesssim R^{-2}(1+t)(\psi_R^*(x,t))^{\frac{\lambda+1}{\lambda+2}}, \tag{3.3}$$

$$|\partial_t^2\psi_R(x,t)|\lesssim R^{-2}(\psi_R^*(x,t))^{\frac{\lambda}{\lambda+2}}, \tag{3.4}$$

$$|\nabla\psi_R(x,t)|\lesssim R^{-2}(|x|-1)^{1-\theta}\frac{x}{|x|}(\psi_R^*(x,t))^{\frac{\lambda+1}{\lambda+2}}, \tag{3.5}$$

$$|\Delta\psi_R(x,t)|\lesssim R^{-2}(|x|-1)^{-\theta}(\psi_R^*(x,t))^{\frac{\lambda}{\lambda+2}}. \tag{3.6}$$

Replacing ϕ in (1.9)-(1.10) by $\psi(x,t)=(1+t)\Psi(x)\psi_R(x,t)$, we acquire

$$\begin{aligned}\int_{\Omega^c}|u_k|^{p_1}\psi dx &= \frac{d}{dt}\int_{\Omega^c}\big(\partial_t u_1\psi-u_1\partial_t\psi+\frac{a(x)}{1+t}u_1\psi\big)dx\\&\quad+\int_{\Omega^c}u_1\big(\partial_t^2\psi-\Delta\psi-\partial_t(\frac{a(x)}{1+t}\psi)\big)dx,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_{\Omega^c}|u_l|^{p_{l+1}}\psi dx &= \frac{d}{dt}\int_{\Omega^c}\big(\partial_t u_{l+1}\psi-u_{l+1}\partial_t\psi+\frac{a(x)}{1+t}u_{l+1}\psi\big)dx\\&\quad+\int_{\Omega^c}u_{l+1}\big(\partial_t^2\psi-\Delta\psi-\partial_t(\frac{a(x)}{1+t}\psi)\big)dx.\end{aligned} \tag{3.8}$$

Utilizing (2.2)-(2.3), (2.11), (3.3)-(3.6) and (3.7) gives rise to

$$\begin{aligned}&|\partial_t^2\psi-\Delta\psi-\partial_t(\frac{a(x)}{1+t}\psi)|\\&=|2\Psi(x)\partial_t\psi_R(x,t)+(1+t)\Psi(x)\partial_t^2\psi_R(x,t)\\&\quad-(1+t)(\Psi(x)\Delta\psi_R(x,t)+2\nabla\Psi(x)\nabla\psi_R(x,t))\\&\quad-a(x)\Psi(x)\partial_t\psi_R(x,t)|\\&\lesssim(1+t)R^{-\frac{4}{2-\theta}}\Psi(x)(\psi_R(x,t))^{\frac{\lambda}{\lambda+2}}.\end{aligned} \tag{3.9}$$

It is worth to notice that

$$\begin{aligned}&\lim_{R\to\infty}\int_{\Omega^c}((a(x)-1)u_{0,l}(x)+u_{1,l}(x))\Psi(x)\psi_R(x,0)dx\\&=\int_{\Omega^c}((a(x)-1)u_{0,l}(x)+u_{1,l}(x))\Psi(x)dx\end{aligned}$$

for all $l=1,2,\dots,k\,(k\geq 2)$.

Therefore, there exists a sufficiently large constant R_1 such that

$$\int_{\Omega^c}((a(x)-1)u_{0,l}(x)+u_{1,l}(x))\Psi(x)\psi_R(x,0)dx\geq C_{2,l}>0 \tag{3.10}$$

for $R>R_1$.

Taking advantage of (3.7), (3.9)-(3.10) and the condition $\lambda\geq\max\limits_{1\leq l\leq k}\frac{2}{p_l-1}$, we deduce

$$\begin{aligned}I_R[u_k]+C_{2,1}\varepsilon&\lesssim R^{-\frac{4}{2-\theta}}(\int_0^T\int_{\Omega^c}(1+s)\Psi dxds)^{\frac{1}{p_2'}}\\&\quad\times(\int_0^T\int_{\Omega^c}|u_1|^{p_2}(1+s)\Psi(\psi_R^*)^{\frac{p_2\lambda}{\lambda+2}}dxds)^{\frac{1}{p_2}}\\&\lesssim J_{p_2}(R)(\int_0^T\int_{\Omega^c}|u_1|^{p_2}(1+s)\Psi(\psi_R^*)^{\frac{p_2\lambda}{\lambda+2}}dxds)^{\frac{1}{p_2}}\\&\lesssim J_{p_2}(R)(\int_0^T\int_{\Omega^c}|u_1|^{p_2}(1+s)\Psi(\psi_R^*)dxds)^{\frac{1}{p_2}},\end{aligned}\tag{3.11}$$

where $J_{p_2}(R)$ is denoted by

$$J_{p_2}(R)=\begin{cases}R^{2-\frac{8-2\theta}{(2-\theta)p_2}}, & \beta\neq 0,\ n=1,\\ R^{2-\frac{8-2\theta}{(2-\theta)p_2}}(\log R^{\frac{2}{2-\theta}})^{\frac{1}{p_2'}}, & \beta\neq 0,\ n=2,\\ R^{\frac{2(n-\theta)}{2-\theta}-\frac{2n+4-2\theta}{(2-\theta)p_2}}, & \beta\neq 0,\ n\geq 3,\\ R^{\frac{2(n-\theta)}{2-\theta}-\frac{2n+4-2\theta}{(2-\theta)p_2}}, & \beta=0,\ n\geq 1.\end{cases}$$

Analogously, we achieve

$$I_R[u_1]+C_{2,2}\varepsilon\lesssim J_{p_3}(R)(\int_0^T\int_{\Omega^c}|u_2|^{p_3}(1+s)\Psi\psi_R^*dxds)^{\frac{1}{p_3}},\tag{3.12}$$

$$I_R[u_2]+C_{2,3}\varepsilon\lesssim J_{p_4}(R)(\int_0^T\int_{\Omega^c}|u_3|^{p_4}(1+s)\Psi\psi_R^*dxds)^{\frac{1}{p_4}},\tag{3.13}$$

$$\vdots$$

$$I_R[u_{k-1}]+C_{2,k}\varepsilon\lesssim J_{p_1}(R)(\int_0^T\int_{\Omega^c}|u_k|^{p_1}(1+s)\Psi\psi_R^*dxds)^{\frac{1}{p_1}}.\tag{3.14}$$

For the sake of brevity, we assume $\gamma_{\max}=\max\{\gamma_1,\gamma_2,\dots,\gamma_k\}$. Direct computation gives rise to

$$\gamma_k=\frac{1+p_k+p_{k-1}p_k+\cdots+p_2p_3\cdots p_k}{\prod\limits_{l=1}^{k}p_l-1}.$$

Utilizing (1.14) shows $\Gamma(n,p_1,p_2,\dots,p_k)=\gamma_{\max}-\frac{n-\theta}{2}\geq 0$.

Therefore, we need to consider the following seven cases in the next Table 2.

Table 2. Combination of $\Gamma(n,p_1,p_2,\ldots,p_k)$, β and n.

	$\Gamma(n,p_1,p_2,\ldots,p_k)$	β	n (dimensions)
Case 1	$\Gamma(n,p_1,p_2,\ldots,p_k)>0$	$\beta\neq 0$	$n\geq 3$
Case 2	$\Gamma(n,p_1,p_2,\ldots,p_k)=0$ and $p_i\neq p_j$ $(i,j\in\{1,2,\ldots,k\}$ and $i\neq j)$	$\beta\neq 0$	$n\geq 3$
Case 3	$\Gamma(n,p_1,p_2,\ldots,p_k)=0$ and $p_i=p_j$ $(i,j\in\{1,2,\ldots,k\})$	$\beta\neq 0$	$n\geq 3$
Case 4	$\Gamma(2,p_1,p_2,\ldots,p_k)>0$	$\beta\neq 0$	$n=2$
Case 5	$\Gamma(2,p_1,p_2,\ldots,p_k)=0$ and $p_i=p_j$ $(i,j\in\{1,2,\ldots,k\})$	$\beta\neq 0$	$n=2$
Case 6	$\max\{\gamma_1,\,\gamma_2,\,\cdots,\,\gamma_k\}>\frac{2-\theta}{4}$	$\beta\neq 0$	$n=1$
Case 7	$\Gamma(1,p_1,p_2,\ldots,p_k)\geq 0$	$\beta=0$	$n\geq 1$

In Case 1, according to (3.11)-(3.14), we have

$$\begin{aligned}
&I_R[u_{k-1}]+C_{2,k}\varepsilon\\
&\lesssim J_{p_1}(I_R[u_k])^{\frac{1}{p_1}}\\
&\lesssim J_{p_1}(J_{p_2}(I_R[u_1])^{\frac{1}{p_2}})^{\frac{1}{p_1}}\\
&=J_{p_1}(J_{p_2})^{\frac{1}{p_1}}(I_R[u_1])^{\frac{1}{p_1p_2}}\\
&\lesssim R^{\frac{2(n-\theta)}{2-\theta}-\frac{4}{(2-\theta)p_1}-\frac{2n+4-2\theta}{(2-\theta)p_1p_2}}(I_R[u_1])^{\frac{1}{p_1p_2}}\\
&\quad\vdots\\
&\lesssim R^{\frac{2(n-\theta)}{2-\theta}-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_{k-1}})-\frac{2n+4-2\theta}{(2-\theta)p_1p_2\cdots p_k}}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\
&=R^{-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+\frac{2(n-\theta)}{2-\theta}(1-\frac{1}{p_1p_2\cdots p_k})}\\
&\quad\times(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}},
\end{aligned}\tag{3.15}$$

which yields

$$\begin{aligned}
C_{2,k}\varepsilon&\lesssim R^{-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+\frac{2(n-\theta)}{2-\theta}(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\
&\quad-I_R[u_{k-1}]\\
&\lesssim R^{-\frac{4}{2-\theta}\gamma_k+\frac{2(n-\theta)}{2-\theta}}
\end{aligned}\tag{3.16}$$

for $R\geq R_0$. Letting $R\to T(\varepsilon)$ in (3.16), we obtain

$$T(\varepsilon)\leq C\varepsilon^{-(\frac{4}{2-\theta}\Gamma(n,p_1,p_2,\ldots,p_k))^{-1}}.$$

In Case 2, similar to the derivation in (3.15), we arrive at

$$\begin{aligned}
&I_R[u_{k-1}]+C_{2,k}\varepsilon\\
&\lesssim R^{-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+\frac{2(n-\theta)}{2-\theta}(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}}\\
&=(\int_0^T\int_{\Omega^c}|u_{k-1}(x,s)|^{p_k}\Psi(x)(1+s)\psi_R^*(x,s)dxds)^{\frac{1}{p_1p_2\cdots p_k}}.
\end{aligned}\tag{3.17}$$

We denote the auxiliary functions

$$h_{p_k}=h_{p_k}(r)=\int_0^T\int_{\Omega^c}|u_{k-1}(x,s)|^{p_k}\Psi(x)(1+s)\psi_r^*(x,s)dxds,\tag{3.18}$$

$$H_{p_k} = H_{p_k}(R) = \int_0^R h_{p_k}(r) r^{-1} dr. \tag{3.19}$$

Exploiting (3.18) and (3.19), we derive

$$H_{p_k} \le \frac{\log 2}{4} I_R[u_{k-1}], \qquad h_{p_k}(R) = R H'_{p_k}(R). \tag{3.20}$$

Making use of (3.17)-(3.20) leads to

$$\frac{4}{\log 2} H_{p_k}(R) + C_{2,k}\varepsilon \lesssim (R H'_{p_k}(R))^{\frac{1}{p_1 p_2 \cdots p_k}}.$$

As a consequence, we conclude the fifth lifespan estimate in (1.16).

In Case 3, we have $p = p_1 = p_2 = \cdots = p_k = 1 + \frac{2}{n-\theta}$. It is worth noticing that

$$\begin{aligned}
&\partial_t^2(u_1 + u_2 + \cdots + u_k) - \Delta(u_1 + u_2 + \cdots + u_k) + \frac{a(x)}{1+t}\partial_t(u_1 + u_2 + \cdots + u_k) \\
&= |u_1|^p + |u_2|^p + \cdots + |u_k|^p \\
&\ge C|u_1 + u_2 + \cdots + u_k|^p.
\end{aligned} \tag{3.21}$$

We consider problem (1.1) with $c(x,t) = (1+|x|^2)^{-\frac{\theta}{2}}(1+t)^{-1}$ $(\theta < 0)$ as the single equation in the critical case $p = 1 + \frac{2}{n-\theta}$. It holds that

$$T(\varepsilon) \le \exp\{C\varepsilon^{-(p_1-1)}\}.$$

In Case 4, similar to the derivation in (3.15), we observe

$$\begin{aligned}
&I_R[u_{k-1}] + C_{2,k}\varepsilon \\
&\lesssim J_{p_1}(I_R[u_k])^{\frac{1}{p_1}} \\
&\lesssim J_{p_1}(J_{p_2})^{\frac{1}{p_1}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\lesssim R^{2-\frac{4}{(2-\theta)p_1}-\frac{8-2\theta}{(2-\theta)p_1 p_2}}(\log R^{\frac{2}{2-\theta}})^{1-\frac{1}{p_1 p_2}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\quad \cdots \\
&\lesssim R^{-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1 p_2}+\cdots+\frac{1}{p_1 p_2 \cdots p_k})+2(1-\frac{1}{p_1 p_2 \cdots p_k})} \\
&\quad \times(\log R^{\frac{2}{2-\theta}})^{1-\frac{1}{p_1 p_2 \cdots p_k}}(I_R[u_{k-1}])^{\frac{1}{p_1 p_2 \cdots p_k}}.
\end{aligned} \tag{3.22}$$

An application of (3.22), Lemma 2.1 and taking $\sigma = \frac{4}{2-\theta}\Gamma(2, p_1, p_2, \ldots, p_k)$, $\mu = 1$, $p = p_1 p_2 \cdots p_k$ show the second lifespan estimate in (1.16).

In Case 5, we acquire $p = p_1 = p_2 = \cdots = p_k = 1 + \frac{2}{2-\theta}$. Making use of Lemma 2.1 with $\sigma = 0$, $\mu = 1$, $p = 1 + \frac{2}{2-\theta}$, we deduce the third lifespan estimate in (1.16).

In Case 6, similar to the derivation in (3.15), we get

$$\begin{aligned}
&I_R[u_{k-1}] + C_{2,k}\varepsilon \\
&\lesssim J_{p_1}(I_R[u_k])^{\frac{1}{p_1}} \\
&\lesssim J_{p_1}(J_{p_2})^{\frac{1}{p_1}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\lesssim R^{2-\frac{4}{(2-\theta)p_1}-\frac{8-2\theta}{(2-\theta)p_1 p_2}}(I_R[u_1])^{\frac{1}{p_1 p_2}} \\
&\quad \cdots \\
&\lesssim R^{-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1 p_2}+\cdots+\frac{1}{p_1 p_2 \cdots p_k})+2(1-\frac{1}{p_1 p_2 \cdots p_k})} \\
&\quad \times(I_R[u_{k-1}])^{\frac{1}{p_1 p_2 \cdots p_k}},
\end{aligned} \tag{3.23}$$

which results in

$$
\begin{aligned}
C_{2,k}\varepsilon &\lesssim R^{-\frac{4}{2-\theta}(\frac{1}{p_1}+\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k})+2(1-\frac{1}{p_1p_2\cdots p_k})}(I_R[u_{k-1}])^{\frac{1}{p_1p_2\cdots p_k}} \\
&\quad - I_R[u_{k-1}] \\
&\lesssim R^{-\frac{4}{2-\theta}\gamma_k+2} \qquad (3.24)
\end{aligned}
$$

for $R \geq R_0$. Sending $R \to T(\varepsilon)$ in (3.24) leads to the first lifespan estimate in (1.16).

In Case 7, similar to the derivation in cases 1-3, we derive the seventh-ninth lifespan estimates in (1.16). This completes the proof of Theorem 1.2. ■

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Declarations

Conflict of interest The authors declare no conflict of interest.

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