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Dynamics analysis of octonion-valued stochastic shunting inhibitory cellular neural networks with varying delays

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Abstract. In this paper, we use a direct method to study the almost periodic dynamics of an octonion-valued stochastic shunting inhibitory cellular neural network with variable delays. By using the fixed point method and inequality technique, the existence, uniqueness and stability of almost periodic solutions in the sense of distribution of the neural network under consideration are obtained. Our results are brand new.

§1 Introduction

Octonion algebra is a nonassociative generalization of quaternion algebra and is not covered by Clifford algebra [1, 2]. Quaternion-valued neural networks [3–13] and Clifford-valued neural networks [14–25] have gradually become a hotspot in the field of neural network research because of their importance in theory and practical application as neural networks with multi-dimensional values. Octonion-valued neural networks were first proposed by C.A. Popa [26]. Due to their important potential application value, at present, the qualitative research of mathematical models of octonion-valued neural networks has begun to attract the attention of scholars [27–31]. However, because the multiplication of octonion algebra does not satisfy the commutative law and associative law, it brings great difficulties to the dynamics research of octonion-valued neural networks. Because of this difficulty, the current results on the dynamics of octonion-valued neural networks are obtained by decomposing them into real-valued neural networks, and then studying them as real-valued neural networks. Such results are not suitable for direct application to octonion-valued neural networks. Therefore, it is of great theoretical and practical value to study the dynamics of octonion-valued neural networks by direct methods.

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On the one hand, the dynamics of real-valued shunting inhibitory cellular networks has been the focus of many researchers due to their important applications in psychophysics, adaptive pattern recognition, and image processing [32–39]. However, since their mathematical models include the multiplication of three terms, namely, connection weight function, activation function and state variable, and the multiplication of octonion algebra does not meet the associative law, there is no report on the dynamics of octonion-valued shunting inhibitory cellular neural networks. Therefore, it is an interesting and challenging work to study the dynamics of octonion-valued shunting inhibitory cellular neural networks.

On the other hand, in the real world, a neural network system is always disturbed by many random factors, so considering the neural network with random disturbances is more consistent with the real situation. Indeed, stochastic neural networks including fractional-order stochastic neural networks have been widely studied [40–48]. However, the research on the dynamics of octonion-valued stochastic neural networks has not been reported yet.

In addition, almost periodic oscillation is one of the important qualitative properties of neural networks, and time delays often occur in real systems.

Based on the above observations, the main purpose of this paper is to study the existence and global exponential stability of almost periodic solutions in the distribution sense for a class of octonion-valued stochastic shunting inhibitory cellular neural networks with time-varying delays. To the best of the author's knowledge, this is the first article to study the almost periodic solutions of octonion-valued stochastic neural networks.

The rest of the paper is organized as follows. In the second section, we introduce some preliminary knowledge and give a description of the model. In Section 3, we study the existence and global exponential stability of almost periodic solutions in the distribution sense of the considered neural networks. In Section 4, we give an example to illustrate the validity of our results.

§2 Model description and preliminaries

The algebra of octonions is defined as

$$\mathbb{O} := \left\{ x = \sum_{p=0}^{7} [x]_p e_p \,\middle|\, [x]_0, [x]_1, \cdots [x]_7 \in \mathbb{R} \right\},$$

where $e_p, p = 0, 1, 2, \ldots, 7$ represent the octonion units, which satisfy the following multiplication table [1].

The addition of octonions is defined by $x + y = \sum\limits_{p=0}^{7} ([x]_p + [y]_p)e_p$, and the scalar multiplication is given by $\alpha x = \sum\limits_{p=0}^{7} (\alpha[x]_p)e_p$.

Equipped with the above operations, $\mathbb{O}$ is a real algebra. Based on the table, one can find that $\mathbb{O}$ is neither commutative nor associative. Moreover, $e_0 = 1$ and for $x = \sum\limits_{p=0}^{7} [x]_p e_p \in \mathbb{O}$, we denote $x^c = \sum\limits_{p \neq 0} [x]_p e_p$ and $x^r = x - x^c$.

Table 1. Multiplication table of octonions.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

For $y = \sum\limits_{p=0}^{7}[y]_p e_p \in \mathbb{O}$, we define $\|y\|_{\mathbb{O}} = \sqrt{\sum\limits_{p=0}^{7}[y]_p^2}$. Let $\Xi = \{ij, i = 1, 2, \dots, m, j = , 2, \dots, n\}$ and for $z = (z_{11}, z_{12}, \cdots z_{mn})^T \in \mathbb{O}^{m\times n}$, we define $\|z\|_{\mathbb{O}^{m\times n}} = \max\limits_{ij\in\Xi}\{\|z_{ij}\|_{\mathbb{O}}\}$, then both $(\mathbb{O}, \|\cdot\|_{\mathbb{O}})$ and $(\mathbb{O}^{m\times n}, \|\cdot\|_{\mathbb{O}^{m\times n}})$ are Banach spaces.

Throughout this paper, we stipulate that for any $x, y, z \in \mathbb{O}$, $xyz = (xy)z$.

The model we focus on in this work is the following Octonion-valued stochastic shunting inhibitory cellular neural network with varying delays

$$\begin{aligned} dx_{ij}(t) =& \Big(- a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t - \nu_{ij}(t)))x_{ij}(t) + I_{ij}(t)\Big)dt \\ &+ \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t)\sigma_{ij}(x_{ij}(t))dw_{ij}(t), \end{aligned} \tag{2.1}$$

where $ij \in \Xi$, C_{ij} denotes the cell at the (i, j) position of the lattice, the q-neighborhood $N_r(i, j)$ of C_{ij} is

$$N_r(i, j) = \{C_{kl} : \max(|k - i|, |l - j|) \le q, 1 \le k \le m, 1 \le l \le n\},$$

and $N_r(i, j)$ is similarly defined; $x_{ij}(t) \in \mathbb{O}$ represents the activity of the cell C_{ij} at time t; $I_{ij}(t) \in \mathbb{O}$ stands for the external input to C_{ij} at time t; $a_{ij}(t) \in \mathbb{O}$ is the passive decay rate of the cell activity at time t; $C_{ij}^{kl} \in \mathbb{O}$ and $B_{ij}^{kl} \in \mathbb{O}$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij} at time t; the activity function f is a continuous function representing the output or firing rate of the cell C_{kl}; $w_{ij}(t)$ is the $m\times n$-dimentional Brownian motion defined on a complete probability space; σ_{ij} is the diffusion coefficient; and $\nu_{ij}(t) \ge 0$ represent the transmission delay.

Denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$ a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\ge0}$ meeting the usual conditions. Denote by $CB_{\mathcal{F}_0}([-\gamma, 0], \mathbb{O}^{m\times n})$ the family of all bounded, $\mathcal{F}_0$-measurable, $C([-\gamma, 0], \mathbb{O}^{m\times n})$-valued random variables.

System (2.1) is supplemented with the initial values

$$x_{ij}(s) = \varphi_{ij}(s), s \in [-\gamma, 0],\ ij \in \Xi,$$

where $\varphi_{ij} \in CB_{\mathcal{F}_0}([-\gamma, 0], \mathbb{O}^{m\times n})$, $\gamma = \max\limits_{ij\in\Xi}\{\sup\limits_{t\in\mathbb{R}} \nu_{ij}(t)\}$.

Definition 2.1. *[49] A continuous function $f : \mathbb{R} \to \mathbb{O}^{m\times n}$ is said to be almost periodic, if for every $\varepsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that*

$$\|f(t+\tau) - f(t)\|_{\mathbb{O}^{m\times n}} < \varepsilon, t \in \mathbb{R}.$$

Denote by $AP(\mathbb{R}, \mathbb{O}^{m\times n})$ the set of all such functions.

Let $\mathcal{B}(\mathbb{O}^{m\times n})$ be the σ-algebra of Borel sets of $\mathbb{O}^{m\times n}$ and $\mathcal{P}(\mathbb{O}^{m\times n})$ be the set of all probability measures defined on $\mathcal{B}(\mathbb{O}^{m\times n})$. We denote by $C_B(\mathbb{O}^{m\times n})$ the set of all Lipschitz continuous functions $f : \mathbb{O}^{m\times n} \to \mathbb{R}$ with $\|f\|_\infty = \sup\limits_{x\in\mathbb{O}^{m\times n}} |f(x)| < \infty$.

For $f \in C_B(\mathbb{O}^{m\times n})$, $\mu, \nu \in \mathcal{P}(\mathbb{O}^{m\times n})$, we define

$$d_{BL}(\mu,\nu) := \sup_{\|f\|_{BL}\leq 1} \int_{\mathbb{E}} f d(\mu - \nu),$$

where $\|f\|_L = \sup\limits_{a\neq b} \frac{|f(a)-f(b)|}{\|a,b\|_{\mathbb{O}^{m\times n}}}$, $\|f\|_{BL} = \max\{\|f\|_\infty, \|f\|_L\}$.

For a random variable $X : (\Omega, \mathcal{F}, P) \to \mathbb{O}^{m\times n}$, we will use $\mu(X) := P \circ X^{-1}$ and $E(X)$ to represent its distribution and its expectation, respectively.

Let $\mathcal{L}^2(\Omega, \mathbb{O}^{m\times n})$ be the space of all $\mathbb{O}^{m\times n}$-value random variables such that $E\|X\|^2_{\mathbb{O}^{m\times n}} = \int_\Omega \|X\|^2_{\mathbb{O}^{m\times n}} dP < \infty$.

Definition 2.2. *[50] For a stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^{m\times n})$, if for any $t_0 \in \mathbb{R}$,*

$$\lim_{t\to t_0} E\|X(t) - X(t_0)\|^2_{\mathbb{O}^{m\times n}} = 0,$$

we call it $\mathcal{L}^2$-continuous. If $\sup\limits_{t\in\mathbb{R}} E\|X(t)\|^2_{\mathbb{O}^{m\times n}} < \infty$, we call it $\mathcal{L}^2$-bounded.

Definition 2.3. *[50] A stochastic process $X : \mathbb{R} \to \mathbb{O}^{m\times n}$ is said to be almost periodic in the sense of distribution, if for every $\varepsilon > 0$, there exists a positive number l such that every interval of length l contains a number τ such that*

$$d_{BL}(P \circ [X(t+\tau)]^{-1}, P \circ [X(t)]^{-1}) < \varepsilon,\ t \in \mathbb{R}.$$

In the rest of this paper, we will use the following notations:

$$\bar{a}^r_{ij} = \inf_{t\in\mathbb{R}} |a^r_{ij}(t)|, \quad \bar{a}^c_{ij} = \sup_{t\in\mathbb{R}} \|a^c_{ij}(t)\|_{\mathbb{O}}, \quad \tilde{a}^r_{ij} = \sup_{t\in\mathbb{R}} |a^r_{ij}(t)|, \bar{C}^{kl}_{ij} = \sup_{t\in\mathbb{R}} \|C^{kl}_{ij}(t)\|_{\mathbb{O}},$$
$$\bar{B}^{kl}_{ij} = \sup_{t\in\mathbb{R}} \|B^{kl}_{ij}(t)\|_{\mathbb{O}}, \quad \nu^+_{ij} = \sup_{t\in\mathbb{R}} \nu_{ij}(t), \quad I^+_{ij} = \sup_{t\in\mathbb{R}} \|I_{ij}(t)\|_{\mathbb{O}}, \quad \bar{a}^r = \min_{(i,j)}\{\bar{a}^r_{ij}\}.$$

The assumptions used in this paper are as follows:

(A_1) For $ij \in \Xi$, $a^r_{ij} \in AP(\mathbb{R}, \mathbb{R}^+)$ with $\bar{a}^r_{ij} > 0$, $a^c_{ij}, C^{kl}_{ij}, B^{kl}_{ij}, I_{ij} \in AP(\mathbb{R}, \mathbb{O})$, $\nu_{ij} \in AP(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$ and there exist positive constants $\dot{\nu}^+_{ij}$ such that for $t \in \mathbb{R}$, $\dot{\nu}_{ij}(t) \leq \dot{\nu}^+_{ij} < 1$;

(A_2) For $ij \in \Xi$, $f, \sigma_{ij} \in C(\mathbb{O}, \mathbb{O})$, there exist constants $M_f, M^\sigma_{ij} > 0$ such that for all $x, y \in \mathbb{O}$,

$$\|f(x) - f(y)\|_{\mathbb{O}} \leq M_f \|x - y\|_{\mathbb{O}}, \quad \|\sigma_{ij}(x) - \sigma_{ij}(y)\|_{\mathbb{O}} \leq M^\sigma_{ij} \|x - y\|_{\mathbb{O}},$$

$L_f = \sup\limits_{t\in\mathbb{R}} \|f(x(t))\|_{\mathbb{O}}$ and $f(0) = \sigma_{ij}(0) = 0$.

§3 Main results

Let $B = UBC(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^{m\times n}))$ be the space of all $\mathcal{L}^2$-bounded and uniformly $\mathcal{L}^2$-continuous functions from $\mathbb{R}$ to $\mathcal{L}^2(\Omega, \mathbb{O}^{m\times n})$. Then, $(B, \|\cdot\|_B)$ is a Banach space, where $\|x\|_B = \big(\sup_{t\in\mathbb{R}} E\|x(t)\|^2_{\mathbb{O}^{m\times n}}\big)^{\frac{1}{2}}$ for $x \in B$.

Set $x^* = (x^*_{11}, x^*_{12}, \cdots x^*_{mn})$, where $x^*_{ij}(t) = \int_{-\infty}^t e^{-\int_s^t a^r_{ij}(u)du} I_{ij}(s)ds, t \in \mathbb{R}$ and take a constant k such that $\|x^*\|_B \le k$.

For $ij \in \Xi$, we denote

$$F_{ij}(t,x) = -a^c_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_q(i,j)} C^{kl}_{ij}(t) f(x_{kl}(t-\nu_{ij}(t)))x_{ij}(t) + I_{ij}(t),$$

$$G_{ij}(t,x) = \sum_{C_{kl}\in N_r(i,j)} B^{kl}_{ij}(t)\sigma_{ij}(x_{ij}(t)). \tag{3.2}$$

It is easy to check that if $x = (x_{11}, x_{12}, \cdots x_{mn})^T$ is a solution of the following system:

$$x_{ij}(t) = \int_{-\infty}^t e^{-\int_s^t a^r_{ij}(u)du} F_{ij}(s,x)ds + \int_{-\infty}^t e^{-\int_s^t a^r_{ij}(u)du} G_{ij}(s,x)dw_{ij}(s),\ ij \in \Xi, \tag{3.3}$$

then x is a solution of system (2.1).

Let $B_k = \{x \in B | \|x - x^*\|_B < k\}$, then

$$\|x\|_B \le \|x - x^*\|_B + \|x^*\|_B \le 2k.$$

Theorem 3.1. *Assume that* (A_1)*-*(A_4) *are fulfilled, suppose further that the following conditions are satisfied:*

(A_3)

$$\Pi := \max_{ij\in\Xi}\Bigg\{\frac{3(\bar{a}^c_{ij})^2}{(\bar{a}^r_{ij})^2} + \frac{3}{(\bar{a}^r_{ij})^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2 + \frac{6}{(\bar{a}^r_{ij})^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2 + \frac{3}{2\bar{a}^r_{ij}}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}^{kl}_{ij})^2\sum_{C_{kl}\in N_r(i,j)}(M^\sigma_{ij})^2\Bigg\} < \frac{1}{4}.$$

(A_4)

$$\Theta := \max_{ij\in\Xi}\Bigg\{\frac{12}{\bar{a}^r_{ij}}(\bar{a}^c_{ij})^2 + \frac{12}{\bar{a}^r_{ij}}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2 + 12\sum_{C_{kl}\in N_r(i,j)}(\bar{B}^{kl}_{ij})^2\sum_{C_{kl}\in N_r(i,j)}(M^\sigma_{ij})^2 + \frac{24}{\bar{a}^r_{ij}}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2\frac{e^{\bar{a}^r_{ij}\nu^+_{ij}}}{1-\dot{\nu}^+_{ij}}\Bigg\} < \bar{a}^r.$$

Then there exists a unique almost periodic solution in the sense of distribution to system (2.1), *which lies in* $B_k = \{x \in B | \|x - x^*\|_B < k\}$.

Proof. Consider the mapping $\Psi : B_k \to B$, $\Psi x = (\Psi_{11}x, \Psi_{12}x, \cdots, \Psi_{mn}x)^T$, where $x \in B_k$ and

$$(\Psi_{ij}x)(t) = \int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} F_{ij}(s,x)ds + \int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} G_{ij}(s,x)dw_{ij}(s),\ t \in \mathbb{R}, ij \in \Xi.$$

Step 1, we need to verify that Ψ is a self-mapping. Indeed, for $x \in B_k$, we have

$$\begin{aligned}
&\|\Psi x - x^*\|_B^2 \\
&\le 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\left\{E\left\|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} a_{ij}^c(s)x_{ij}(s)ds\right\|_{\mathbb{O}}^2\right\} \\
&\quad + 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\left\{E\left\|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)f(x_{kl}(s-\nu_{ij}(s)))x_{ij}(s)ds\right\|_{\mathbb{O}}^2\right\} \\
&\quad + 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\left\{E\left\|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))dw_{ij}(s)\right\|_{\mathbb{O}}^2\right\} \\
&\le 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\left\{E\left(\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} \bar{a}_{ij}^c\|x\|_B ds\right)^2\right\} \\
&\quad + 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\left\{E\left(\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} \sum_{C_{kl}\in N_q(i,j)} \bar{C}_{ij}^{kl} L_f\|x\|_B ds\right)^2\right\} \\
&\quad + 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\left\{E\left(\int_{-\infty}^{t} e^{-2\int_s^t a_{ij}^r(u)du} \left\|\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))\right\|_{\mathbb{O}}^2 ds\right)\right\} \\
&\le \max_{ij\in\Xi}\left\{\frac{3(\bar{a}_{ij}^c)^2}{(\bar{a}_{ij}^r)^2} + \frac{3}{(\bar{a}_{ij}^r)^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2\right. \\
&\quad \left. + \frac{3}{2\bar{a}_{ij}^r}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2\right\}\|x\|_B^2 \\
&< \Pi(2k)^2 = k^2.
\end{aligned}$$

Moreover, let $x \in B_k$ and $t_1, t_2 \in \mathbb{R}$ satisfying $t_1 > t_2$, we infer that

$$\begin{aligned}
&E\|(\Psi x)(t_1) - (\Psi x)(t_2)\|_{\mathbb{O}^{m\times n}}^2 \\
&= \max_{ij\in\Xi}\left\{E\left\|\int_{-\infty}^{t_2}\left(e^{-\int_s^{t_1} a_{ij}^r(u)du} - e^{-\int_s^{t_2} a_{ij}^r(u)du}\right)\right.\right. \\
&\times\left(-a_{ij}^c(s)x_{ij}(s) - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)f(x_{kl}(s-\nu_{ij}(s)))x_{ij}(s) + I_{ij}(s)\right)ds \\
&+ \int_{t_2}^{t_1} e^{-\int_s^{t_1} a_{ij}^r(u)du}\left(-a_{ij}^c(s)x_{ij}(s) - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)f(x_{kl}(s-\nu_{ij}(s)))x_{ij}(s) + I_{ij}(s)\right)ds \\
&+ \int_{-\infty}^{t_2}\left(e^{-\int_s^{t_1} a_{ij}^r(u)du} - e^{-\int_s^{t_2} a_{ij}^r(u)du}\right)\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))dw_{ij}(s) \\
&\left.\left. + \int_{t_2}^{t_1} e^{-\int_s^{t_1} a_{ij}^r(u)du}\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))dw_{ij}(s)\right\|_{\mathbb{O}}^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{ij\in\Xi}\bigg\{E\bigg[\int_{-\infty}^{t_2}\left|e^{-\int_s^{t_1}a_{ij}^r(u)du}-e^{-\int_s^{t_2}a_{ij}^r(u)du}\right|\bigg(\bar{a}_{ij}^c\|x\|_B+\sum_{C_{kl}\in N_q(i,j)}\bar{C}_{ij}^{kl}L_f\|x\|_B+I_{ij}^+\bigg)ds\\
&\quad+\int_{t_2}^{t_1}e^{-\int_s^{t_1}a_{ij}^r(u)du}\bigg(\bar{a}_{ij}^c\|x\|_B+\sum_{C_{kl}\in N_q(i,j)}\bar{C}_{ij}^{kl}L_f\|x\|_B+I_{ij}^+\bigg)ds\\
&\quad+\bigg\|\int_{-\infty}^{t_2}\left|e^{-\int_s^{t_1}a_{ij}^r(u)du}-e^{-\int_s^{t_2}a_{ij}^r(u)du}\right|\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))dw_{ij}(s)\\
&\quad+\int_{t_2}^{t_1}e^{-\int_s^{t_1}a_{ij}^r(u)du}\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))dw_{ij}(s)\bigg\|_{\mathbb{O}}\bigg]^2\bigg\}\\
&\leq \max_{ij\in\Xi}8\bigg[(\bar{a}_{ij}^c2k)^2+\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2(2k)^2+(I_{ij}^+)^2\bigg]\\
&\quad\times\bigg[\bigg(\int_{-\infty}^{t_2}e^{-\bar{a}_{ij}^r(t_2-s)}\left|\int_s^{t_2}a_{ij}^r(u)du-\int_s^{t_1}a_{ij}^r(u)du\right|ds\bigg)^2+\bigg(\int_{t_2}^{t_1}e^{-\int_s^{t_1}a_{ij}^r(u)du}ds\bigg)^2\bigg]\\
&\quad+\max_{ij\in\Xi}8\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2(2k)^2\bigg[\int_{-\infty}^{t_2}e^{-2\bar{a}_{ij}^r(t_2-s)}\\
&\quad\times\left|\int_s^{t_2}a_{ij}^r(u)du-\int_s^{t_1}a_{ij}^r(u)du\right|^2ds+\int_{t_2}^{t_1}e^{-2\int_s^{t_1}a_{ij}^r(u)du}ds\bigg]\\
&\leq \max_{ij\in\Xi}\bigg\{8\bigg[(\bar{a}_{ij}^c2k)^2+\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2(2k)^2+(I_{ij}^+)^2\bigg]\\
&\quad\times\bigg[\bigg(\frac{\tilde{a}_{ij}^r}{\bar{a}_{ij}^r}\bigg)^2+1\bigg]+\frac{4(\tilde{a}_{ij}^r)^2}{\bar{a}_{ij}^r}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2(2k)^2\bigg\}|t_1-t_2|^2\\
&\quad+\max_{ij\in\Xi}\bigg\{8\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2(2k)^2\bigg\}|t_1-t_2|\\
&\leq K_1|t_1-t_2|^2+K_2|t_1-t_2|,
\end{aligned}
$$

which implies that Tx is uniformly $\mathcal{L}^2$-continuous. Consequently, Ψ is well defined.

Step 2, we will prove that Ψ is a contraction mapping. For any $x, y \in B_k$, we deduce that

$$
\begin{aligned}
&\|\Psi x-\Psi y\|_B^2\\
&\leq 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u)du}a_{ij}^c(s)(x_{ij}(s)-y_{ij}(s))ds\bigg\|_{\mathbb{O}}^2\bigg\}\\
&\quad+3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u)du}\\
&\quad\times\sum_{C_{kl}\in N_q(i,j)}C_{ij}^{kl}(s)(f(x_{kl}(s-\nu_{ij}(s)))x_{ij}(s)-f(y_{kl}(s-\nu_{ij}(s)))y_{ij}(s))ds\bigg\|_{\mathbb{O}}^2\bigg\}\\
&\quad+3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u)du}\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)(\sigma_{ij}(x_{ij}(s))-\sigma_{ij}(y_{ij}(s)))dw_{ij}(s)\bigg\|_{\mathbb{O}}^2\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\le 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg(\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du}\bar{a}_{ij}^c\|x-y\|_B ds\bigg)^2\bigg\} \\
&\quad + 6\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg\|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du} \\
&\quad \times \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)(f(x_{kl}(s-\nu_{ij}(s)))-f(y_{kl}(s-\nu_{ij}(s))))x_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\bigg\} \\
&\quad + 6\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg\|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u)du}\sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)f(y_{kl}(s-\nu_{ij}(s)))(x_{ij}(s)-y_{ij}(s))ds\bigg\|_{\mathbb{O}}^2\bigg\} \\
&\quad + 3\sup_{t\in\mathbb{R}}\max_{ij\in\Xi}\bigg\{E\bigg(\int_{-\infty}^{t} e^{-2\int_s^t a_{ij}^r(u)du}\bigg\|\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)(\sigma_{ij}(x_{ij}(s))-\sigma_{ij}(y_{ij}(s)))\bigg\|_{\mathbb{O}}^2 ds\bigg)\bigg\} \\
&\le \max_{ij\in\Xi}\bigg\{\frac{3(\bar{a}_{ij}^c)^2}{(\bar{a}_{ij}^r)^2}+\frac{6}{(\bar{a}_{ij}^r)^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2+\frac{6}{(\bar{a}_{ij}^r)^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2 \\
&\quad \times \sum_{C_{kl}\in N_q(i,j)}(L_f)^2+\frac{3}{2\bar{a}_{ij}^r}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^{\sigma})^2\bigg\}\|x-y\|_B^2 \\
&< \Pi\|x-y\|_B^2 < \frac{1}{4}\|x-y\|_B^2,
\end{aligned}
$$

which implies that Ψ is a contraction mapping. Hence, system (2.1) has a unique solution x in B_k.

Step 3, we will show that this x is almost periodic in the sense of distribution.

Due to $x\in UC_b(\mathbb{R},\mathcal{L}^2(\Omega,\mathbb{O}^{m\times n}))$, for any $\varepsilon>0$, there exists $\delta\in(0,\varepsilon)$ such that $E\|x(t_1)-x(t_2)\|_{\mathbb{O}^{m\times n}}^2<\varepsilon$ for $|t_1-t_2|<\delta$. In virtue of (A_1), for the δ above, we can infer that there exists $\tau(\delta)$ such that for $t\in\mathbb{R}$,

$$
\begin{aligned}
&|\nu_{ij}(t+\tau)-\nu_{ij}(t)|<\delta,\quad |a_{ij}^r(t+\tau)-a_{ij}^r(t)|<\delta,\quad \|a_{ij}^c(t+\tau)-a_{ij}^c(t)\|_{\mathbb{O}}^2<\delta, \\
&\|C_{ij}^{kl}(t+\tau)-C_{ij}^{kl}(t)\|_{\mathbb{O}}^2<\delta,\quad \|B_{ij}^{kl}(t+\tau)-B_{ij}^{kl}(t)\|_{\mathbb{O}}^2<\delta,\quad \|I_{ij}(t+\tau)-I_{ij}(t)\|_{\mathbb{O}}^2<\delta,
\end{aligned}
$$

and as a consequent, $E\|x(t-\nu_{ij}(t+\tau))-x(t-\nu_{ij}(t))\|_{\mathbb{O}^{m\times n}}^2<\varepsilon$.

Invoking (3.3), we can gain

$$
\begin{aligned}
x_{ij}(t+\tau)=&\int_{-\infty}^{t} e^{-\int_{s+\tau}^{t+\tau} a_{ij}^r(u)du}\bigg[-a_{ij}^c(s+\tau)x_{ij}(s+\tau) \\
&-\sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s+\tau)f(x_{kl}(s+\tau-\nu_{ij}(s+\tau)))x_{ij}(s+\tau)+I_{ij}(s+\tau)\bigg]ds \\
&+\int_{-\infty}^{t} e^{-\int_{s+\tau}^{t+\tau} a_{ij}^r(u)du}\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s+\tau)\sigma_{ij}(x_{ij}(s+\tau))d(w_{ij}(s+\tau)-w_{ij}(\tau)),
\end{aligned}
$$

where $w_{ij}(s+\tau)-w_{ij}(\tau)$ is a Brownian motion with the same law as $w_{ij}(s)$. Hence,

$$
\begin{aligned}
&E\|x(t+\tau)-x(t)\|_{\mathbb{O}^{m\times n}}^2 \\
&\le 12\max_{ij\in\Xi} E\bigg\|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}^r(u+\tau)du}a_{ij}^c(s+\tau)(x_{ij}(s+\tau)-x_{ij}(s))ds\bigg\|_{\mathbb{O}}^2
\end{aligned}
$$

$$
\begin{aligned}
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}(a_{ij}^c(s+\tau)-a_{ij}^c(s))x_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}\left(e^{-\int_s^t a_{ij}^r(u+\tau)du}-e^{-\int_s^t a_{ij}^r(u)du}\right)a_{ij}^c(s)x_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}\\
&\times\sum_{C_{kl}\in N_q(i,j)}C_{ij}^{kl}(s+\tau)f(x_{kl}(s+\tau-\nu_{ij}(s+\tau)))(x_{ij}(s+\tau)-x_{ij}(s))ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}\\
&\times\sum_{C_{kl}\in N_q(i,j)}(C_{ij}^{kl}(s+\tau)-C_{ij}^{kl}(s))f(x_{kl}(s+\tau-\nu_{ij}(s+\tau)))x_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}\\
&\times\sum_{C_{kl}\in N_q(i,j)}C_{ij}^{kl}(s)(f(x_{kl}(s+\tau-\nu_{ij}(s+\tau)))-f(x_{kl}(s-\nu_{ij}(s))))x_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}\left(e^{-\int_s^t a_{ij}^r(u+\tau)du}-e^{-\int_s^t a_{ij}^r(u)du}\right)\\
&\times\sum_{C_{kl}\in N_q(i,j)}C_{ij}^{kl}(s)f(x_{kl}(s-\nu_{ij}(s)))x_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s+\tau)\\
&\times(\sigma_{ij}(x_{ij}(s+\tau))-\sigma_{ij}(x_{ij}(s)))dw_{ij}(s)\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}\sum_{C_{kl}\in N_r(i,j)}(B_{ij}^{kl}(s+\tau)-B_{ij}^{kl}(s))\sigma_{ij}(x_{ij}(s))dw_{ij}(s)\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}\left(e^{-\int_s^t a_{ij}^r(u+\tau)du}-e^{-\int_s^t a_{ij}^r(u)du}\right)\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)\sigma_{ij}(x_{ij}(s))dw_{ij}(s)\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}e^{-\int_s^t a_{ij}^r(u+\tau)du}(I_{ij}(s+\tau)-I_{ij}(s))ds\bigg\|_{\mathbb{O}}^2\\
&+12\max_{ij\in\Xi}E\bigg\|\int_{-\infty}^{t}\left(e^{-\int_s^t a_{ij}^r(u+\tau)du}-e^{-\int_s^t a_{ij}^r(u)du}\right)I_{ij}(s)ds\bigg\|_{\mathbb{O}}^2\\
:=&\sum_{q=1}^{12}S_q(t).
\end{aligned}
$$

Noticing the fact that

$$\int_{-\infty}^{t}|e^{-\int_s^t a_{ij}^r(u+\tau)du}-e^{-\int_s^t a_{ij}^r(u)du}|ds\le\int_{-\infty}^{t}e^{-\bar{a}_{ij}^r(t-s)}\int_s^t|a_{ij}^r(u+\tau)-a_{ij}^r(u)|duds$$

$$\le\varepsilon\int_{-\infty}^{t}e^{-\bar{a}_{ij}^r(t-s)}(t-s)ds=\frac{\varepsilon}{\left(\bar{a}_{ij}^r\right)^2}\int_0^{+\infty}e^{-\omega}\omega d\omega=\frac{\Gamma(2)\varepsilon}{\left(\bar{a}_{ij}^r\right)^2}=\frac{\varepsilon}{\left(\bar{a}_{ij}^r\right)^2},$$

$$\int_{-\infty}^{t}|e^{-\int_s^t a_{ij}^r(u+\tau)du}-e^{-\int_s^t a_{ij}^r(u)du}|^2ds\le\int_{-\infty}^{t}e^{-2\bar{a}_{ij}^r(t-s)}\left(\int_s^t|a_{ij}^r(u+\tau)-a_{ij}^r(u)|du\right)^2ds$$

$$\le\varepsilon^2\int_{-\infty}^{t}e^{-2\bar{a}_{ij}^r(t-s)}(t-s)^2ds=\frac{\varepsilon^2}{(2\bar{a}_{ij}^r)^3}\int_0^{+\infty}e^{-\omega}\omega^2d\omega=\frac{\Gamma(3)\varepsilon^2}{(2\bar{a}_{ij}^r)^3}=\frac{\varepsilon^2}{4(\bar{a}_{ij}^r)^3}$$

and invoking Cauchy-Schwarz's inequality and Itô's isometry, we infer

$$\begin{aligned}
&E\|x(t+\tau)-x(t)\|_{\mathbb{O}^{m\times n}}^2\\
&\le\max_{ij\in\Xi}\left\{\frac{12(\bar{a}_{ij}^c)^2}{\bar{a}_{ij}^r}\int_{-\infty}^{t}e^{-\bar{a}_{ij}^r(t-s)}E\|x(s+\tau)-x(s)\|_{\mathbb{O}}^2ds\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{12\varepsilon}{(\bar{a}_{ij}^r)^2}(2k)^2\right\}+\max_{(i,j)}\left\{\frac{12\varepsilon^2}{(\bar{a}_{ij}^r)^4}(\bar{a}_{ij}^c)^2(2k)^2\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{12}{\bar{a}_{ij}^r}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2\int_{-\infty}^{t}e^{-\bar{a}_{ij}^r(t-s)}E\|x(s+\tau)-x(s)\|_{\mathbb{O}}^2ds\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{12nm\varepsilon}{(\bar{a}_{ij}^r)^2}\sum_{C_{kl}\in N_q(i,j)}(L_f)^2(2k)^2\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{24}{\bar{a}_{ij}^r}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2\frac{e^{\bar{a}_{ij}^r\nu_{ij}^+}}{1-\dot{\nu}_{ij}^+}\right.\\
&\quad\left.\times\int_{-\infty}^{t}e^{-\bar{a}_{ij}^r(t-s)}E\|x(s+\tau)-x(s)\|_{\mathbb{O}}^2ds\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{24\varepsilon}{(\bar{a}_{ij}^r)^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{12\varepsilon^2}{(\bar{a}_{ij}^r)^4}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2(2k)^2\right\}\\
&\quad+\max_{ij\in\Xi}\left\{12\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2\int_{-\infty}^{t}e^{-2\bar{a}_{ij}^r(t-s)}E\|x(s+\tau)-x(s)\|_{\mathbb{O}}^2ds\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{6nm\varepsilon}{\bar{a}_{ij}^r}\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2(2k)^2\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{3\varepsilon^2}{(\bar{a}_{ij}^r)^3}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^\sigma)^2(2k)^2\right\}\\
&\quad+\max_{ij\in\Xi}\left\{\frac{12\varepsilon}{(\bar{a}_{ij}^r)^2}\right\}+\max_{(i,j)}\left\{\frac{12\varepsilon^2}{(\bar{a}_{ij}^r)^4}(I_{ij}^+)^2\right\}.
\end{aligned}$$

Hence, we can conclude that

$$E\|x(t+\tau)-x(t)\|^2_{\mathbb{O}^{m\times n}} \le \Lambda\varepsilon + \Theta\int_{-\infty}^{t} e^{-\bar{a}^r(t-s)}E\|x(s+\tau)-x(s)\|^2_{\mathbb{O}^{m\times n}}ds, \tag{3.4}$$

where Θ is defined in (A_4) and

$$\begin{aligned}\Lambda = \max_{ij\in\Xi}\Big\{&\frac{12}{(\bar{a}^r_{ij})^2}(2k)^2+\frac{12}{(\bar{a}^r_{ij})^4}(\bar{a}^c_{ij})^2(2k)^2\varepsilon+\frac{12nm}{(\bar{a}^r_{ij})^2}\sum_{C_{kl}\in N_q(i,j)}(L_f)^2(2k)^2\\
&+\frac{24}{(\bar{a}^r_{ij})^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2\\
&+\frac{12}{(\bar{a}^r_{ij})^4}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2(2k)^2\varepsilon+\frac{6nm}{\bar{a}^r_{ij}}\sum_{C_{kl}\in N_r(i,j)}(M^\sigma_{ij})^2(2k)^2\\
&+\frac{3}{(\bar{a}^r_{ij})^3}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}^{kl}_{ij})^2\sum_{C_{kl}\in N_r(i,j)}(M^\sigma_{ij})^2(2k)^2\varepsilon+\frac{12}{(\bar{a}^r_{ij})^2}+\frac{12}{(\bar{a}^r_{ij})^4}(I^+_{ij})^2\varepsilon\Big\}.\end{aligned}$$

Define

$$W_\tau = \sup_{t\in\mathbb{T}}E\|x(t+\tau)-x(t)\|^2_{\mathbb{O}^{m\times n}}.$$

Then, from (3.4), we deduce that

$$E\|x(t+\tau)-x(t)\|^2_{\mathbb{O}^{m\times n}} \le W_\tau \le \Lambda\varepsilon\frac{\bar{a}^r}{\bar{a}^r-\Theta}. \tag{3.5}$$

Noting that

$$\begin{aligned}&d_{BL}(P\circ[x(t+\tau)]^{-1},P\circ[x(t))]^{-1})\\
=&\sup_{\|f\|_{BL}\le 1}\Big|\int_\Omega[f(x(t+\tau))-f(x(t)))]dP\Big|\\
\le&\int_\Omega\|x(t+\tau)-x(t)\|_{\mathbb{O}^{m\times n}}dP\\
\le&(E\|x(t+\tau)-x(t)\|^2_{\mathbb{O}^{m\times n}})^{\frac{1}{2}},\end{aligned}$$

which combined with (3.5) yields that

$$d_{BL}(P\circ[x(t+\tau)]^{-1},P\circ[x(t))]^{-1}) \le \sqrt{\Lambda\varepsilon\frac{\bar{a}^r}{\bar{a}^r-\Theta}},$$

that is, $x(t)$ is almost periodic solution in the sense of distribution. This ends the proof. □

Remark 3.1. *Since the set of all almost periodic random processes in the sense of distribution does not constitute a complete normed linear space, but space $(B,\|\cdot\|_B)$ is a Banach space, we first prove that system* (2.1) *has a unique solution in B, and then use the definition to prove that this solution is an almost periodic solution in the sense of distribution.*

Similar to the proof of Theorem 3.2 in [51], one can readily show the following global exponential stability result about the almost periodic solution of (2.1).

Theorem 3.2. *Let* (A_1)-(A_4) *and the following condition*

(A_5)

$$C := \max_{ij\in\Xi}\Big\{\frac{5}{(\bar{a}^r_{ij})^2}(\bar{a}^c_{ij})^2+\frac{5}{(\bar{a}^r_{ij})^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}^{kl}_{ij})^2\sum_{C_{kl}\in N_q(i,j)}(L_f)^2$$

$$+\frac{5}{(\bar{a}_{ij}^r)^2}\sum_{C_{kl}\in N_q(i,j)}(\bar{C}_{ij}^{kl})^2\sum_{C_{kl}\in N_q(i,j)}(M_f)^2(2k)^2$$
$$+\frac{5}{2\bar{a}_{ij}^r}\sum_{C_{kl}\in N_r(i,j)}(\bar{B}_{ij}^{kl})^2\sum_{C_{kl}\in N_r(i,j)}(M_{ij}^{\sigma})^2\Bigg\}<1,$$

be satisfied. Let x be the almost periodic solution in the sense of distribution to system (2.1) *with initial value φ and y an arbitrary solution of system* (2.1) *with initial value ψ. Then there exist constants $\lambda>0$ and $M>1$ such that*

$$E\|x(t)-y(t)\|_{\mathbb{O}^{m\times n}}^2\le ME\|\varphi-\psi\|_0^2e^{-\lambda t},\ t>0,$$

where $E\|\varphi-\psi\|_0^2=\sup\limits_{s\in[-\gamma,0]}E\|\varphi(s)-\psi(s)\|_{\mathbb{O}^{m\times n}}^2$.

§4 Example

Example 4.1. *In system* (2.1), *for $n=2$ and $i,j=1,2$, take*

$$\begin{aligned}
x_{ij}=&[x_{ij}]_0e_0+[x_{ij}]_1e_1+[x_{ij}]_2e_2+[x_{ij}]_3e_3+[x_{ij}]_4e_4+[x_{ij}]_5e_5+[x_{ij}]_6e_6+[x_{ij}]_7e_7,\\
f(x_{ij})=&0.01e_0\sin[x_{ij}]_0+0.03e_1\sin[x_{ij}]_1+0.02e_2\sin[x_{ij}]_2+0.05e_3\sin[x_{ij}]_3\\
&+0.02e_4\sin[x_{ij}]_4+0.03e_5\sin[x_{ij}]_5+0.01e_6\sin[x_{ij}]_6+0.01e_7\sin[x_{ij}]_7,\\
a_{11}(t)=&(1.2+0.2\sin t)e_0+0.015e_1\sin\sqrt{3}t+0.01e_2\cos\sqrt{2}t+0.01e_3\sin\sqrt{5}t\\
&+0.02e_4\cos\sqrt{7}t+0.03e_5\sin\sqrt{7}t+0.01e_6\cos\sqrt{5}t+0.02e_7\sin\sqrt{3}t,\\
a_{12}(t)=&(1.3+0.3\sin t)e_0+0.02e_1\sin\sqrt{2}t+0.015e_2\cos\sqrt{3}t+0.01e_3\sin\sqrt{7}t\\
&+0.03e_4\cos\sqrt{2}t+0.02e_5\cos t+0.04e_6\sin\sqrt{3}t+0.03e_7\sin\sqrt{5}t,\\
a_{21}(t)=&(1.1+0.1\sin t)e_0+0.015e_1\sin\sqrt{3}t+0.02e_2\sin\sqrt{2}t+0.015e_3\sin\sqrt{5}t\\
&+0.02e_4\sin\sqrt{7}t+0.01e_5\cos\sqrt{2}t+0.03e_6\cos\sqrt{3}t+0.01e_7\cos t,\\
a_{22}(t)=&(1.4+0.4\sin t)e_0+0.015e_1\sin\sqrt{2}t+0.01e_2\cos\sqrt{5}t+0.01e_3\sin\sqrt{3}t\\
&+0.03e_4\cos\sqrt{3}t+0.02e_5\sin\sqrt{7}t+0.015e_6\cos t+0.03e_7\cos\sqrt{2}t,\\
C_{11}(t)=&0.01e_0\sin t+0.02e_2\cos\sqrt{2}t, C_{12}(t)=0.02e_1\sin\sqrt{3}t+0.01e_3\cos t,\\
C_{21}(t)=&0.03e_4\cos t+0.01e_6\sin\sqrt{2}t, C_{22}(t)=0.01e_5\cos\sqrt{3}t+0.03e_7\sin t,\\
B_{11}(t)=&0.02e_0\sin\sqrt{2}t+0.01e_2\cos\sqrt{5}t, B_{12}(t)=0.03e_1\cos\sqrt{2}t+0.01e_3\sin t,\\
B_{21}(t)=&0.01e_4\sin t+0.02e_6\sin\sqrt{3}t, B_{22}(t)=0.02e_5\sin t+0.01e_7\cos\sqrt{3}t,\\
\sigma_{ij}(x_{ij})=&0.01e_0\sin[x_{ij}]_0+0.02e_1\sin[x_{ij}]_1+0.01e_2\sin[x_{ij}]_2+0.03e_3[x_{ij}]_3\\
&+0.02e_4\sin[x_{ij}]_4+0.01e_5\sin[x_{ij}]_5+0.03e_6\sin[x_{ij}]_6+0.02e_7\sin[x_{ij}]_7,\\
I_{11}(t)=&0.01e_0\sin\sqrt{3}t+0.03e_1\sin\sqrt{2}t+0.02e_2\cos t+0.01e_3\sin\sqrt{6}t\\
&+0.02e_4\cos\sqrt{2}t+0.01e_5\sin\sqrt{7}t+0.03e_6\sin t+0.01e_7\cos\sqrt{5}t,\\
I_{12}(t)=&0.01e_0\sin\sqrt{3}t+0.03e_1\sin t+0.01e_2\cos\sqrt{2}t+0.01e_{12}\sin\sqrt{3}t\\
&+0.03e_4\sin\sqrt{7}t+0.02e_5\cos\sqrt{3}t+0.01e_6\sin\sqrt{2}t+0.03e_7\cos t,\\
I_{21}(t)=&0.03e_0\cos t+0.01e_1\sin\sqrt{3}t+0.01e_2\sin\sqrt{2}t+0.05e_3\cos\sqrt{2}t
\end{aligned}$$

$$
\begin{aligned}
&+ 0.02e_4 \sin \sqrt{5}t + 0.03e_5 \sin t + 0.02e_6 \cos \sqrt{3}t + 0.01e_7 \cos \sqrt{5}t, \\
I_{22}(t) =&0.02e_0 \sin \sqrt{2}t + 0.01e_1 \cos \sqrt{3}t + 0.03e_2 \sin \sqrt{3}t + 0.02e_3 \sin \sqrt{5}t \\
&+ 0.01e_4 \sin t + 0.03e_5 \sin \sqrt{7}t + 0.02e_6 \cos \sqrt{2}t + 0.01e_7 \cos t, \\
\nu_{ij}(t) =&0.03 \sin t.
\end{aligned}
$$

Take $k = 0.0548$, then, we have

$$\Pi \approx 0.012182 < \frac{1}{4}, \Theta \approx 0.04873 < 1, C \approx 0.020304 < 1.$$

Therefore, system(2.1) has a unique almost periodic solution in the sense of distribution that is global exponentially stable (see Figures [1-3]).

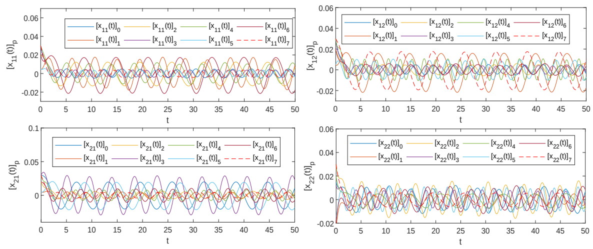

Figure 1. States $[x_{11}]_p, [x_{12}]_p, [x_{21}]_p, [x_{22}]_p$ of (2.1) with different initial values.

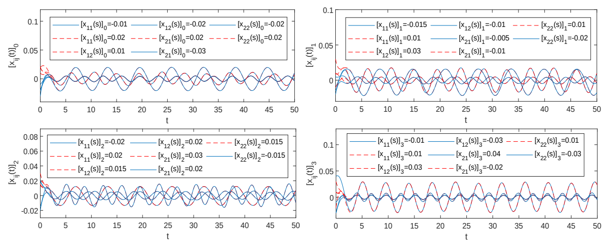

Figure 2. Stability of states $[x_{ij}]_0, [x_{ij}]_1, [x_{ij}]_2, [x_{ij}]_3$ of (2.1) with different initial values.

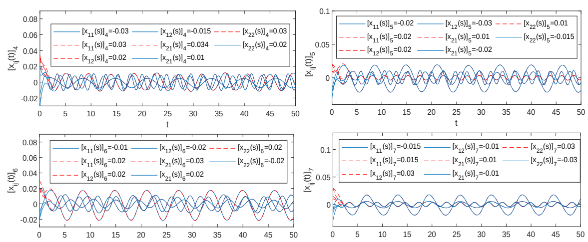

Figure 3. Stability of states $[x_{ij}]_4, [x_{ij}]_5, [x_{ij}]_6, [x_{ij}]_7$ of (2.1) with different initial values.

Remark 4.1. *The conclusion of Example 1 cannot be deduced from [12, 34], nor from any known results.*

§5 Conclusion

In this paper, we have established the existence and stability of almost periodic solutions in the sense of distribution for a class of octonion-valued stochastic shunting inhibitory neural networks by using the direct method. This is the first time to consider the dynamics of octonion-valued stochastic neural networks, so the results of this paper are completely new. The method presented in this paper can be used to study the existence of almost automorphic solutions in the sense of distribution of octonion-valued stochastic neural networks, and the existence and stability of almost periodic solutions in the sense of distribution of fractional octonion-valued stochastic neural networks.

Declarations

Conflict of interest The authors declare no conflict of interest.

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