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A selective survey on mathematical programming in macroeconomics

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Abstract. This paper surveys the literature for the optimization problems in both discrete and continuous time models in macroeconomics, and provides an overview over some related computational methods to solve the models linearly and nonlinearly, and to compute the transition dynamics and the impulse response functions. Also, the introduction of the financial sectors, the continuous time analysis, and the advanced mathematical tools into the general equilibrium framework expands greatly the scope of the interdisciplinary research to mathematics, statistics and econometrics, and creates further space for exploration and collaboration. Finally, some future research issues related to this topic are highlighted.

§1 Introduction

Since the linear programming was introduced from mathematics to economics to help economists decide the allocation of limited resource, both mathematical programming methods and macroeconomics theories have made much more progress. In the frontier of these progress, the mathematical theories, the computational methods, and the macroeconomic models promote each other and make the boundaries among the three different subjects melt. This survey provides a selective overview on how the relation between mathematical programming and macroeconomics evolve from the past and about what the latest trend in this interdisciplinary subject is nowadays.

Our review on this relation starts from the dynamic stochastic general equilibrium (DSGE) models in discrete time dated back in 1970's. The well known Bellman equations were applied to solve the optimization problems of the agents within different microeconomic sectors in the macroeconomy. In the mean time, the finance theory in continuous time was thriving due to its popularity in asset pricing and the elegant combination of mathematics and portfolio theories.

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It was until the burst of the financial crisis of 2008 that financial sectors were added to the macroeconomic models to help people better understand how the real sector of the economy and the business cycles are influenced by the financial market's turbulence. One critical tool to construct such a connection is through the continuous time framework. Different from the optimization methods in discrete time, the stochastic calculus, the stochastic control, the numerical methods to solve the stochastic differential equations (SDEs), the ordinary differential equations (ODEs), and the partial differential equations (PDEs) are often needed to solve the macroeconomic optimization problems in continuous time. Therefore, the second half of this survey provide an overview of the latest progress of the macroeconomic modeling in continuous time, especially the heterogeneous agent models (HAM), as well as the advanced computational methods to solve the models' key equations, the transition dynamics, and the impulse response functions. Thanks to the fast-growing computation power, the solutions that we can get from the nonlinear models are global solutions.

The scope of the questions that macroeconomics can explore has been greatly expanded due to the advances in the mathematical modeling and computational tools, from the macroeconomics models without micro-foundations, to the models that depict the households', firms', and governments' decision-making; from the static analysis to the dynamic analysis; from the deterministic equilibrium to the equilibrium with uncertainty; from the first-order approximation solutions locally, to the nonlinear solutions globally. Throughout this survey, we can find the efforts made by generations of scholars that clarify and promote our knowledge on these matters continuously. Even though the current coverings in this interdisciplinary subject can be thought as scattered, we look forward to more future studies coming to fill the gap in the unknown space.

The rest of this paper is organized as follows. Section 2 gives a review on mathematical programming for macroeconomics models with micro-foundations. Section 3 outlines some challenges for the discrete-time nonlinear dynamic stochastic general equilibrium models with detailed derivation of the equations relegated to the Appendix. Section 4 describes the dynamic programming methods for continuous-time macro-finance models. Finally, Section 5 concludes the paper with some remarks for future research.

§2 Mathematical Programming for Macroeconomics Models with Micro-Foundations

The micro-foundation was first introduced in the macroeconomics models by the school of the new classical macroeconomics. The two generations of the new classical macroeconomists are the school of the rational expectation, such as Thomas J. Sargent, Robert E. Lucas Jr., and the school of the real business cycle, such as Edward C. Prescott and Neil Wallace, respectively. There are three key assumptions in the new classical macroeconomics: the agents' maximization, the rational expectation, and the market clearing. The decision problems of the microeconomic sectors of the economy are combined in a general equilibrium model, thus serving as the micro-foundation of the macroeconomics model. Specifically, the determination of the equilibrium output and prices are from the solutions of the optimization problems of the households, the producers, the retailers, and the government. To solve these optimization problems in discrete time, the dynamic programming, such as the Bellman equation, is necessary.

The real business cycle school pointed that it is the real factors causing the economic fluctuations, rather than the monetary factors. To answer the questions that how the real factors cause the economic fluctuations and what the underlying transmission mechanism is, the real business cycle theory emphasizes on the importance of the supply shock and the intertemporal substitution between the leisure and the work. The latter explains why a small change in the wage can cause large and long-term variations of the output and employment.

During the debate of the schools of macroeconomic thought, the new Keynesian school borrowed the merits of the new classical macroeconomics and developed its own framework to support Keynes's economic thoughts. The representatives of the new Keynesian school macroeconomists include, but not limited to, N. Gregory Mankiw, Lawrence H. Summers, Olivier Blanchard, Julio Rotemberg, Edmund S. Phelps, George A. Akerlof, Janet L. Yellen, Joseph Stiglitz, Ben S. Bernanke, David H. Romer, and so on. In contrast to the new classical macroeconomics, the new Keynesian school does not assume that the market can clear once a shock hits the economy, since the supply of labor and products, the wage and the price adjust slowly. The new Keynesian school adopts the agents' optimization and rational expectation from the new classical macroeconomics, while following the Keynesian school to support that the fluctuations of the aggregate demand shift both the output and the price in the short-run. Therefore, the government's policy plays a key role to bring the aggregate demand back to the normal level during the economic recession.

As demonstrated in Gertler (2024), a typical macroeconomics model with micro-foundations is consisted of the sectors of households, firms, monetary, and fiscal authorities. The dynamic programming serves as an important method of solution for the optimization problems. Compared to the dynamic programming applied in microeconomics, the application in macroeconomics resides in a general equilibrium framework, which ultimately synthesizes the individual sectors' decisions into several types of equilibrium in macroeconomics, such as the competitive equilibrium, the stationary equilibrium, and the Markov equilibrium. We will compare these types of equilibrium later in Section 4.3.

2.1 Household's Utility Maximization

Suppose that a representative household chooses $\{C_t, L_t, M_t, P_t, B_{t+1}, K_{t+1}\}_{t\geq 0}$ to maximize its utility

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\gamma} C_t^{1-\gamma} + \frac{a_m}{1-\gamma_m} \left(\frac{M_t}{P_t} \right)^{1-\gamma_m} - \frac{1}{1+\varphi} L_t^{1+\varphi} \right] \right\}, \tag{1}$$

where $\mathbb{E}_0(\cdot)$ denotes the mathematical expectation operator, β is the discount factor, which is assumed to satisfy $0 < \beta < 1$, C_t is the household's consumption, M_t is the household's money holding, P_t is the price level, and L_t is the labor (hours worked). The parameters γ

and γ_m are the coefficients of the relative risk aversion, and φ^{-1} stands for the Frisch elasticity of labor supply. The parameter a_m stands for the weight of real money balance in the utility. For some special cases, γ, γ_m, φ, and a_m might take some specific values. For example, when $\gamma = \gamma_m = \varphi = 1$, the constant relative risk aversion separable utility becomes the log separable utility. For the case of cashless economy, $a_m \to 0$. The optimization problem in (1) is subject to the budget constraint given by

$$C_t = \frac{W_t}{P_t} L_t + Z_t K_t + \Pi_t + \mathrm{TR}_t - \frac{M_t - M_{t-1}}{P_t} - \frac{\left(\frac{1}{R_t^n}\right) B_{t+1} - B_t}{P_t} - Q_t \left(K_{t+1} - K_t\right),$$

where W_t is the nominal wage, Z_t is the rental cost of capital, K_t is the capital holding, Π_t is the profit from monopoly competitive firms, TR_t is the government transfer, B_t is the bond holding, R_t^n is the gross nominal interest rate, $1/R_t^n$ is the price of one period discount bond earning the gross nominal return R_t^n, and Q_t is the price of capital. Additionally, we assume that there is no Ponzi schemes. Note that the solution of the optimization problem yields the so-called control variable K_{t+1} at $t+1$ as a policy function of the state variable K_t at t.

2.2 Firms' Profit Maximization and Cost Minimization

There are three types of firms in the general equilibrium model: the final good firms, the intermediate good firms, and the capital producer.

2.2.1 Final Good Firms

Final good firms are competitive producers of a homogeneous good, Y_t, using intermediate goods, $Y_t(f)$. The production function that transforms intermediate goods into final output is given by

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $\varepsilon > 1$ is the (constant) elasticity of substitution between intermediate goods. Note that this production function also exhibits constant returns to scale and diminishing marginal product for each input with $(\varepsilon-1)/\varepsilon < 1$. Each firm chooses $Y_t(f)$ to minimize costs $\int_0^1 P_t(f) Y_t(f) df$ for a given level of output $Y_t = \left[\int_0^1 Y_t(f)^{\frac{\varepsilon-1}{\varepsilon}} df\right]^{\frac{\varepsilon}{\varepsilon-1}}$ and given $P_t(f)$. The result is the following demand function for each intermediate good f

$$Y_t(f) = \left[\frac{P_t(f)}{P_t}\right]^{-\varepsilon} Y_t.$$

Combining with the production function yields the following nominal price index for the final good

$$P_t = \left[\int_0^1 P_t(f)^{1-\varepsilon} df\right]^{\frac{1}{1-\varepsilon}}.$$

2.2.2 Intermediate Good Firms

There is a continuum of intermediate good firms, indexed by $f \in [0, 1]$. Each produces a differentiated good and is a monopolistic competitor. Each firm uses both labor $L_t(f)$ and capital $K_t(f)$ to produce output according to

$$Y_t(f) = A_t K_t(f)^\alpha L_t(f)^{1-\alpha},$$

where A_t is the level of productivity and $0 < \alpha < 1$ is the capital share. Firm f chooses inputs $K_t(f)$ and $L_t(f)$ to minimize the total cost given by

$$\frac{W_t}{P_t} L_t(f) + Z_t K_t(f),$$

subject to the output demand

$$A_t K_t(f)^\alpha L_t(f)^{1-\alpha} = \bar{Y},$$

where $\bar{Y}$ is a given output level. The first order conditions (FOC) for this optimization problem are as follows:

$$\frac{W_t/P_t}{(1-\alpha)Y_t(f)/L_t(f)} = \mathrm{MC}_t(f) \quad \text{and} \quad \frac{Z_t}{\alpha Y_t(f)/K_t(f)} = \mathrm{MC}_t(f),$$

where $\mathrm{MC}_t(f)$ is the Lagrange multiplier, interpretable as the marginal cost of producing output. By combining the two FOCs, it is easy to conclude that $L_t(f)/K_t(f) = (1-\alpha)Z_t P_t/(\alpha W_t)$. With this condition and the production function $Y_t(f) = A_t K_t(f)^\alpha L_t(f)^{1-\alpha}$, we could rewrite $\mathrm{MC}_t(f)$ as

$$\mathrm{MC}_t(f) = \frac{1}{A_t}\left(\frac{W_t/P_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{Z_t}{\alpha}\right)^{\alpha} \equiv \mathrm{MC}_t.$$

Note that the marginal cost and the gross markup are reciprocal,‘ that is, $1 + \mu_t = 1/\mathrm{MC}_t$, where μ_t is defined as the markup.

The intermediate good firms set prices on a staggered basis. Following Calvo (1983), each period a firm adjusts its price with probability $1-\theta$ and keeps it fixed with probability θ. All firms have the same likelihood of adjustment. The adjustment probability is independent over time and across firms. The average time for a price remaining fixed is given by

$$(1-\theta)\sum_{i=1}^{\infty}\theta^{i-1} i = \sum_{i=0}^{\infty}\theta^i = \frac{1}{1-\theta}.$$

Firms that are able to adjust their prices choose $P_t(f), Y_t(f), K_t(f)$ and $L_t(f)$. These firms maximize expected discounted profits given the production technology and the demand curve. They choose the optimal reset price $P_t^o(f)$ to maximize

$$\mathbb{E}_t\left\{\sum_{i=0}^{\infty}\theta^i\left[\Lambda_{t,i}\left(\frac{P_t^o(f)}{P_{t+i}} - \mathrm{MC}_{t+i}\right)Y_{t,t+i}(f)\right]\right\},$$

subject to

$$Y_{t,t+i}(f) = \left(\frac{P_t^o(f)}{P_{t+i}}\right)^{-\varepsilon} Y_{t+i},$$

where $\mathbb{E}_t(\cdot)$ is the mathematical expectation operator, $\Lambda_{t,i} = \beta^i C_{t+i}^{-\gamma}/C_t^{-\gamma}$ is the stochastic discount factor, ε is the elasticity of substitution between intermediate goods, and MC_{t+i} is the nominal marginal cost. Then, the FOC is

$$\mathbb{E}_t\left\{\sum_{i=0}^{\infty}\theta^i\Lambda_{t,i}Y_{t,t+i}(f)\left[\frac{P_t^o}{P_{t+i}} - (1+\mu)\mathrm{MC}_{t+i}\right]\right\} = 0,$$

where $1+\mu = \varepsilon/(\varepsilon-1)$ is the steady state gross markup. Given that (i) all firms that adjust in period t choose the same price P_t^0 and (ii) the average price of firms that do not adjust is simply last period's price level P_{t-1}, then, we can rewrite the price index as

$$P_t = \left[\theta\left(P_{t-1}\right)^{1-\varepsilon} + (1-\theta)\left(P_t^o\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

The entrance of P_{t-1} in P_t introduces nominal inertia.

2.2.3 Capital Producer

The capital producer's optimization problem is to choose the investment level I_t such that the profit from producing the capital is maximized

$$\max_{I_t} \ [Q_t J_t - I_t],$$

subject to $J_t = I_t - c\left(I_t/K_t - \delta\right)^2 K_t/2$, where J_t stands for the technology for producing new capital goods, and the capital producer invests I_t units of final output and rents K_t units of capital to produce J_t units of new capital. c is the adjustment cost parameter, δ is the capital depreciation, and $c\left(I_t/K_t - \delta\right)^2 K_t/2$ reflects increasing marginal costs of producing new capital goods after the depreciation. Then, the FOC yields $I_t/K_t = \delta + \left(1 - 1/Q_t\right)/c$.

2.3 Monetary and Fiscal Authorities

The central bank sets the nominal interest rate according to the following simple feedback rule

$$1 + r_t^n = (1+r)\left(\frac{P_t}{P_{t-1}}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_t^*}\right)^{\phi_y} e^{\upsilon_t},$$

where Y_t^* is the natural (i.e., flexible price equilibrium) level of output with $\phi_\pi > 1$ and $\phi_y > 0$, r_t^n is the nominal interest rate, r is the zero inflation steady state nominal interest rate, and υ_t is the monetary shock. The fiscal policy is given by $G_t = \bar{G}$, where G_t is the government expenditure, and $\bar{G}$ is an exogenously given level of the government spending. The government budget constraint is

$$G_t = T_t + \left(M_t - M_{t-1}\right)/P_t,$$

where T_t is the government's tax revenue, and M_t is the monetary supply.

2.4 Competitive Equilibrium

The resource constraints in the economy contain the income and expenditure constraint

$$Y_t = C_t + I_t + G_t,$$

and the evolution of capital constraint

$$K_{t+1} = I_t - \frac{1}{2}c\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t.$$

A competitive equilibrium is defined as an allocation $(Y_t, L_t, C_t, I_t, K_{t+1})$ and a price system $(Z_t, W_t, P_t, P_t^o, r_t^n, Q_t, \mu_t)$ such that all agents are maximizing subject to their respective constraints, all markets clear, and all resource constraints are satisfied, given P_{t-1}, A_t, and K_t.

In practice, it is convenient to express the equilibrium as a system of 10 equations for $(Y_t, C_t, I_t, L_t, P_t, P_t^o, r_t^n, Q_t, \mu_t, K_{t+1})$, given the predetermined states P_{t-1}, A_t, K_t. It is useful to group the equations into aggregate demand, aggregate supply and policy blocks as follows. First, the aggregate demand block has the following 4 equations:

$$Y_t = C_t + I_t + \bar{G}, \qquad \text{(resource constraint)}$$

$$C_t = \mathbb{E}_t\left\{\left[(1+r_t^n)\frac{P_t}{P_{t+1}}\beta\right]^{-\sigma} C_{t+1}\right\}, \qquad \text{(consumption Euler equation)}$$

$$\frac{I_t}{K_t} = \delta + \frac{1}{c}\left(1 - \frac{1}{Q_t}\right), \qquad \text{(link between asset prices and investment)}$$

and

$$\mathbb{E}_t\left\{\Lambda_{t,+1}(1+r_t^n)\frac{P_t}{P_{t+1}}\right\} = \mathbb{E}_t\left\{\Lambda_{t,+1}\left[\frac{Z_t + (1-\delta)Q_{t+1}}{Q_t}\right]\right\},$$

(marginal cost of funds = marginal return to capital)

with $Z_t = \alpha Y_t/[(1+\mu_t)K_t]$, $\Lambda_{t,+1} = \beta C_{t+1}^{-\gamma}/C_t^{-\gamma}$, and $\sigma = 1/\gamma$. These equations define an investment-savings curve that relates spending inversely to the real rate $(1+r_t^n)P_t/P_{t+1}$ and expectations of the future. Note that in equilibrium, $G_t = \bar{G}$ is substituted in the resource constraint $Y_t = C_t + I_t + G_t$. Second, the aggregate supply block has the following 5 equations:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} V_t, \qquad \text{(production function)}$$

$$(1-\alpha)\frac{Y_t}{L_t} = (1+\mu_t)\frac{L_t^{\varphi}}{C_t^{-\gamma}}, \qquad \text{(labor market equilibrium)}$$

$$P_t = \left[\theta(P_{t-1})^{1-\varepsilon} + (1-\theta)(P_t^o)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \qquad \text{(price adjustment)}$$

$$\mathbb{E}_t\left\{\sum_{i=0}^{\infty}\theta^i\Lambda_{t,i}\left(\frac{P_t^o}{P_{t+i}}\right)^{-\varepsilon} Y_{t+i}\left(\frac{P_t^o}{P_{t+i}} - \frac{1+\mu}{1+\mu_{t+i}}\right)\right\} = 0, \qquad \text{(Phillips curve)}$$

and

$$K_{t+1} = I_t - \frac{1}{2}c\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t, \qquad \text{(evolution of capital)}$$

with $V_t = \left[\int_0^1\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} df\right]^{-1}$ and $[1+\mu_{t+i}]^{-1} = \mathrm{MC}_{t+i}$. V_t reflects the misallocation of intermediate inputs due to the relative price dispersion. Note that $V_t = 1$ in the zero inflation steady state. Finally, the policy block is given by

$$1 + r_t^n = (1+r)\left(\frac{P_t}{P_{t-1}}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_t^*}\right)^{\phi_y} e^{\upsilon_t}, \qquad \text{(interest rate rule)}$$

where Y_t^* denotes the level of output in the flexible price equilibrium (the natural output). Given that the exogenous process of the log of productivity A_t the AR(1) model

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{at},$$

and the monetary shock υ_t satisfy the AR(1) model

$$\upsilon_t = \rho_m \upsilon_{t-1} + \varepsilon_{mt}.$$

Therefore, the equilibrium system with 10 unknown variables $(Y_t, C_t, I_t, L_t, P_t, P_t^o, r_t^n, Q_t, \mu_t, K_{t+1})$ in the above 10 equations is complete.

Note that for the extension to models in the open economy, there are generally two types of settings. The first one is a small open economy model and the second one is a two-country open economy model. Compared to the closed economy model, the exchange rate, the foreign price level, the consumption of foreign goods and other foreign country variables are added to the model. In particular, the open economy model produces the Backus-Smith condition, the uncovered/covered interest parity, the purchasing power parity condition, the law of one price, etc. For readers who have further interests in the DSGE models of the open economy, please refer to Uribe and Schmitt-Grohé (2017), Céspedes, Chang, and Velasco (2004), Andrea, Gertler, and Svensson (2008), Xavier and Maggiori (2015), Geanakoplos and Wang (2020), Liu, Spiegel, and Zhang (2021), and so on.

§3 Computational Challenges for the Discrete-Time DSGE Models

3.1 Log-linearization

To solve this typical nonlinear dynamic stochastic general equilibrium models, the most common practice since Kydland and Prescott (1982) and King et al. (1988) is to approximate the solutions using linear methods, especially the log-linearization method. As an illustration, we follow the model that has been set up above and show the procedure of the log-linearization as below.

Let X with no time subscript, no star superscript, and no tilde denote the level of a variable at the zero-inflation steady state, where $P_t/P_{t-1} = 1$[1]. Let $\widetilde{X}_t \equiv \ln X_t - \ln X$ stand for the log-linear deviation of a variable from its zero-inflation steady state. Specially, since the interest rate has already been in a percentage, its log-linearization follows $\widetilde{R_t^n} \equiv \ln R_t^n$, where $R_t^n = 1 + r_t^n$ denotes the gross nominal interest rate. Further, we have the net interest rate $r_t^n \approx \widetilde{R_t^n}$ and $\widetilde{r_t^n} = dr_t^n = r_t^n - r$. Let $\widehat{\mu}_t = \mu_t - \mu$ denote the deviation of the markup from its steady state level. Let X_t^* denote the level of a variable in the flexible price equilibrium (the natural level). Let $\rho \equiv -\log\beta$ and $r \approx \beta^{-1} - 1$. Log-linearize around the steady state with zero inflation and we can write the log-linearized equilibrium into three blocks as follows.

The first block is for the aggregate demand given by

$$\widetilde{Y}_t = \frac{C}{Y}\widetilde{C}_t + \frac{I}{Y}\widetilde{I}_t, \qquad \text{(resource constraint)}$$

$$\widetilde{C_t} = \mathbb{E}_t\left\{-\sigma\left(r_t^n - \pi_{t+1} - \rho\right) + \widetilde{C_{t+1}}\right\}, \qquad \text{(consumption Euler equation)}$$

$$\widetilde{I}_t - \widetilde{K_t} = \frac{1}{c\delta}\widetilde{Q_t}, \qquad \text{(link between asset prices and investment)}$$

$$r_t^n - \pi_{t+1} - \rho = (1-\tau)\left(-\widehat{\mu}_t + \widetilde{Y}_t - \widetilde{K_t}\right) + \tau\widetilde{Q_{t+1}} - \widetilde{Q_t},$$

(marginal cost of funds = marginal return to capital)

where $\widetilde{Z}_t = -\widehat{\mu}_t + \widetilde{Y}_t - \widetilde{K}_t$, $\pi_t = \widetilde{P}_t - \widetilde{P_{t-1}}$, and $\tau = 1 - \delta\left[\frac{\alpha Y}{(1+\mu)K} + (1-\delta)\right]^{-1} = (1-\delta)/[Z + (1-\delta)]$.

The second block is for the aggregate supply formulated as

$$\widetilde{Y}_t = \widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{L}_t, \qquad \text{(production function)}$$

$$\widetilde{Y}_t - \widetilde{L_t} = \widehat{\mu}_t + \varphi\widetilde{L_t} + \gamma\widetilde{C_t}, \qquad \text{(labor market equilibrium)}$$

$$\pi_t = -\lambda\widehat{\mu}_t + \beta\mathbb{E}_t\left\{\pi_{t+1}\right\}, \qquad \text{(price adjustment and Phillips curve)}$$

[1]Note that in the zero-inflation steady state $r^n = r = \rho$.

where $\lambda = (1-\theta)(1-\theta\beta)/\theta$, and

$$\widetilde{K_{t+1}} = \delta\widetilde{I}_t + (1-\delta)\widetilde{K}_t. \qquad \text{(evolution of capital)}$$

Finally, the last one is for the monetary policy provided by

$$r_t^n = \rho + \phi_\pi \pi_t + \phi_y \left(\widetilde{Y}_t - \widetilde{Y_t^*}\right) + \upsilon_t. \qquad \text{(interest rate rule)}$$

For readers with further interests in the details of the log-linearization method, we provide the detailed technical derivations in the Appendix for the above results.

The log-linearization method is a simple and convenient tool to study the equilibrium dynamics. If the shocks driving aggregate fluctuations are small and an interior stationary solution exists, the first-order approximations provide adequate answers to questions such as local existence, determinacy of equilibrium and the size of the second moments of endogenous variables; see, for example, Schmitt-Grohé and Uribe (2004) for details. However, there are several problems that exist in the process of log-linearization. First, the log-linearization omits the higher-order terms. Therefore, the topics that are closely related with the higher-order terms, such as the risk terms, are not able to be fully studied. Due to this shortage, the model's connection with the asset prices and the financial risk is very limited. In addition, the first-order approximation techniques are not well suited to handle questions such as welfare comparisons across alternative stochastic or policy environments.

Second, the log-linearization is not able to provide the global solution for nonlinear models. It is only able to solve the model locally, at the cost of missing the most optimal solution due to missing the true global optimum[2]. This becomes a relevant issue not only for its qualitative and quantitative economic implications but also from an econometric and statistical perspective. When concerned about the estimation of the structural parameters of the model, an econometrician/statistician is more interested in studying the global shape of the approximated likelihood function. This will not be possible if the solution of the model is built from a local approximation. Furthermore, as shown in Fernández-Villaverde and Rubio-Ramírez (2005), it is possible to obtain a better fit of the model to the data as well as more accurate point estimates of the moments of the model by exploiting the nonlinear structure of the economic and mathematical model, which can only be achieved through the use of global methods; see, for instance, Parra-Alvarez (2018) for details.

Finally, we mark that even if the performance of linear methods is disappointing along a number of dimensions, linearization in levels is preferred to the log-linearization for both the benchmark calibration and the highly nonlinear cases in some real applications, as argued in Aruoba et al. (2006).

3.2 Nonlinear Methods of Solutions

The ability to find global solutions and estimate highly nonlinear DSGE models is of critical importance for central banks and policy makers with their interests in quantifying the impacts of

[2]One exception is that the equivalent linearization representation of a nonlinear problem generates the same global optimum but more efficiently.

economic policies in a DSGE model. Since Taylor and Uhlig (1990) and Coleman (1990, 1991), a number of nonlinear solution methods in discrete time have been proposed including the perturbation methods proposed by Judd and Guu (1997) and the projection methods initiated by Judd (1992), as alternatives to the linear approaches and to the value function iteration. The perturbation method, formally introduced by Fleming (1971), has been applied extensively to economic models by Judd and co-authors. Note that a first order perturbation is equivalent to linearization when performed in levels, and is equivalent to log-linearization when performed in logs. Aruoba et al. (2006) found that higher order perturbations display a much superior performance over linear methods for a trivial marginal cost. These findings are based on the computations in first, second, and fifth order, both in levels and in logs. The projection method, on the other hand, is found to be more stable and accurate than the perturbation method. For the projection method, Aruoba et al. (2006) found that finite elements perform very well for all parameterizations. It is extremely stable and accurate over the range of the state space even for high values of the risk aversion and the variance of the shock. This property is crucial in estimation procedures, where accuracy is required to obtain unbiased estimates. Chebyshev polynomials share all the good results of the finite elements method and are easier to implement. However, in a model where policy functions have complicated local behavior, finite elements might outperform Chebyshev polynomials.

Other methods to solve the DSGE models globally include the state space based approach, as in Krusell and Smith (1997, 1998) with a parametric law of motion and as in Den Haan and Rendahl (2010) with a nonlinear law of motion, and the simulation based approach as in Judd et al. (2011) and Maliar et al. (2011), which solve a model only in the realized ergodic state space in the equilibrium. For more details, the reader is referred to the aforementioned papers.

Even though the nonlinear methods as mentioned above can overcome some shortages of the linear methods by improving the approximation, there exist some limitations as noted by several papers. For example, Taylor and Uhlig (1990) found that the nonlinear solution methods for solving the stochastic growth models are not satisfactory in answering the volatility related questions, such as the relative volatility of investment and consumption. Schmitt-Grohè and Uribe (2004) derived a second-order approximation to the policy function of a general class of dynamic, discrete time, rational expectations models, and showed that the coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. Therefore, up to the second order, the presence of uncertainty affects only the constant term of the decision rules.

In parallel, a desire to understand the economic phenomena that can not be easily captured by linear models makes nonlinear models more relevant for empirical macroeconomics, especially since the end of the great moderation, such as the financial crisis and the COVID-19 pandemic. As noted by Aruoba et al. (2013), there are several types of nonlinearity that can appear in a nonlinear DSGE model. The first is for the case that decision rules display curvature and possibly asymmetries such as non-convex adjustment costs, and the second is for kinks in decision rules, such as the zero lower bound on interest rates, credit constraints, borrowing constraints, and defaults. In addition to this categorization, we add a third type of nonlinearity, that is the nonlinearity related with the uncertainty, such as the stochastic volatility, time-varying risk premia, rare disasters, Poisson jumps, and Markov switches; see Fernández-Villaverde et al. (2011) and Rudebusch and Swanson (2012) for examples.

More recently, Fernández-Villaverde and Levintal (2018) used a mixture of projection and perturbation methods for computing the equilibrium of DSGE models with rare disasters. They found that the Taylor projection delivers the best accuracy/speed tradeoff compared to the third-order perturbations and the Smolyak collocation. Cao et al. (2023) introduced the global DSGE (GDSGE) framework and a novel global solution method, called the simultaneous transition and policy function iterations, for solving the DSGE models. In the GDSGE, the state variables and their global domain need to be specified. The algorithm solves jointly for policy and transition functions over the iterations and is a pure projection method using wealth share as an endogenous state variable with an implicit law of motion, different from the standard policy function iteration algorithms as in Coleman (1990, 1991) and Judd (1992). Auclert et al. (2021) and Lee (2024) solved the DSGE models globally in the sequence space. Additionally, Lee (2024) developed the repeated transition method to accurately compute the sequence of the conditional expectation of economic agents utilizing the ergodicity of DSGE models. Neither a parametric law of motion nor parameterized expectation is necessary for the implementation. The method is flexibly applicable to standard macroeconomics models with and without micro-level heterogeneity, especially for solving models with substantial nonlinearity in aggregate fluctuations, as the method does not rely on a (potentially misspecified) parametric form of the aggregate law of motion. This method provides a novel angle that a nonlinear model with complex endogenous aggregate states (e.g., HAMs) can be solved using the sufficient statistic approach, and the validity of the approach can be tested based on some theory. For further explorations, there are approaches which adopt the machine learning and deep learning techniques, as in Han et al. (2021), Azinovic et al. (2022), and Fernández-Villaverde et al. (2023), as well as the adaptive sparse grids, as in Winschel and Krätzig (2010) and Brumm and Scheidegger (2017).

The estimation of the DSGE models can be categorized into likelihood-based approaches and moments-based approaches; see DeJong and Dave (2007) and Canova (2007). The likelihood-based approaches use nonlinear filters for the construction of the likelihood function, such as the particle filter, the extended Kalman filter, and the unscented Kalman filter. Farmer (2021) developed the discretization filter for approximating the likelihood of nonlinear, non-Gaussian state space models. The major difficulty that arises when studying nonlinear state space models is that the likelihood cannot be evaluated recursively in closed form as it can in linear models with the Kalman filter. The discretization filter solves this problem by constructing a discrete-valued Markov chain that approximates the dynamics of the state variables.

The moments-based approaches for estimating the nonlinear DSGE models include the generalized method of moments (GMM); reviewed as in Ruge-Murcia (2013), the instrumental variables approach; see Canova (2007), the simulated method of moments (SMM); see Duffie and Singleton (1993), Ruge-Murcia (2007), and Ruge-Murcia (2012), and the indirect inference; see Smith (1993), Dridi et al. (2007), and Creel and Kristensen (2011).

Finally, as noted in Andreasen et al. (2017), the higher-order approximations often generate explosive sample paths because of the resulting unstable steady states in the approximated system. The presence of explosive behavior complicates any model evaluation because no unconditional moments exist in this approximation. Any estimation method using unconditional moments, such as GMM or SMM, is inapplicable because it relies on finite moments from stationary and ergodic probability distributions. Non-explosive sample paths are also required for likelihood methods, for instance, when using the particle filter outlined in Fernández-Villaverde and Rubio-Ramírez (2007). To overcome this issue, Andreasen et al. (2017) applied pruning to perturbation approximations of any order and showed how pruning greatly facilitates the inference of DSGE models.

§4 Dynamic Programming Methods for Continuous-Time Macro-Finance Models

The continuous-time models can be dated back to the finance literature since the seminal works by Robert C. Merton and others in 1970s. During 1990s, the continuous-time models were successfully applied to the growth and the neoclassical investment theories. Since the financial crisis in 2008, there has been a boom of continuous-time methods in macroeconomics, especially in the fields of business cycles and financial markets. These literature connect the areas that are seemingly disconnected in the past: finance, macroeconomics, and mathematics as well as statistics. As a bridge, the continuous-time method can provide a promising framework to integrate asset pricing theories studied in the finance literature to the real side of the economy studied in macroeconomics, together with mathematical and statistical tools.

Theoretically, macroeconomics models in continuous time are preferred over the discrete-time models because of their analytical tractability. The continuous-time methods transform optimal control problems into stochastic differential equations, such as the Hamilton-Jacobi-Bellman (HJB) equation, the Kolmogorov forward (KF) equation, and the Black-Scholes model. Solving these SDEs is much simpler than solving the Bellman or the Chapman-Kolmogorov equations in discrete time. Compared with the discrete-time framework, the elegant and powerful mathematics such as differential equations and stochastic processes can be well applied in the continuous-time framework. In fact, it is possible to derive closed-form solutions for a wider class of models in continuous time without the need for strong parametric restrictions. On the handling of borrowing constraints, the continuous-time framework is advantageous in the presence of occasionally binding constraints, as these are dealt with using boundary conditions rather than inequalities of the optimality conditions. Finally, continuous-time methods are suited to studying optimal stopping problems and situations where actions are taken infrequently because they entail a fixed cost impulse control problem as described in Stokey (2009), such as a country's decision to default on its sovereign debt as discussed in Parra-Alvarez (2018).

In particular, the continuous-time framework is well applied in the HAMs as in the work of Achdou et al. (2022). When recast in continuous time, HAMs boil down to systems of two coupled SDEs. The first SDE is the HJB equation for the optimal choices of a single individual who takes the evolution of the distribution and hence prices as given. An individual's consumption saving decision depends on the evolution of the interest rate which is in turn determined by the evolution of the distribution. And the second SDE is the KF equation characterizing the evolution of the distribution, given optimal choices of individuals. The evolution of the distribution depends on individuals' saving decisions. More generally, this approach is to cast HAMs in terms of the mathematical theory of the mean field games (MFG) initiated by Huang et al. (2003, 2007) and Lasry and Lions (2007). The system of coupled HJB and KF equations is known as the backward-forward MFG system. The two equations run in opposite directions in time: the HJB equation runs backward and looks forward. It answers the question "given an individual's valuation of income and wealth tomorrow, how much will she save today and what is the corresponding value function today?" In contrast, the KF equation runs forward and looks backward. It answers the question "given the wealth distribution, savings decisions and the random evolution of income today, what is the wealth distribution tomorrow?"

Computationally, continuous time has resurfaced as a popular environment for economic models because of its efficiency in numerical analysis. Solving a workhorse incomplete markets model in continuous time is much faster compared to its discrete-time counterpart as argued in Rendahl (2022). Financial frictions in macroeconomics require the nonlinear techniques. This is especially important for the macro-finance models which include the financial intermediary in the continuous-time framework, such as Chen (2010), Brunnermeier and Sannikov (2014), Phelan (2016), Drechsler, Savov, and Schnabl (2018), He and Krishnamurthy (2019), Hansen et al. (2024) and D'Avernas and Vandeweyer (2024). Modeling this class of problems rarely leads to analytical solutions and needs to resort to the numerical techniques that provide accurate and fast solution methods.

For the continuous-time macro-finance models in the open economy, new research is fast-growing. Literature along this direction includes but is not limited to Grinols and Turnovsky (1994), Zapatero (1995), Kumhof and Nieuwerburgh (2007), Pavlova and Rigobon (2010), Végh (2013), Brunnermeier and Sannikov (2015), Nuño and Thomas(2015), Maggiori (2017), and Oskolkov (2023).

Continuous time imparts a number of computational advantages relative to the discrete time. First, in the continuous-time models, the optimality conditions (the first-order conditions) that describe the equilibrium allocations of a stochastic economy are deterministic; see, Chang (2010). Since the HJB equation does not contain future values of the value function and the optimal policies only depend on the current value function, there is no need to approximate expected values numerically. Hence, the computational cost and the numerical errors can be reduced. This feature can also tame the "curse of dimensionality", with which the standard dynamic programming in discrete-time models often struggle with, given that the dynamic programming equation does not include any composition of functions or expectation operators in the continuous-time framework; see, for example, Doraszelski and Judd (2012). The efficiency of the perturbation methods and the projection methods can be improved in continuous time as well. Since there is no need to approximate the composition of unknown functions, neither to numerically approximate the integrals associated with expected values, the approximations use much less computing time in both perturbation and projection methods. Specifically, Parra-Alvarez (2018) assessed the performance of the first- and second-order perturbation and the projection methods to compute an approximated solution of continuous-time DSGE models based on the maximized HJB equation and the first-order conditions. It is found that the fit of perturbation deteriorates when the degree of nonlinearity increases and the approximated value is different from that obtained by global methods. Despite the perturbation being only locally accurate, the increase in the order of approximation improves the goodness of fit substantially. Aruoba (2006) pointed that the projection methods are more accurate and robust than perturbations for a wide range of values of the state-space centered around the deterministic steady state, similarly to the discrete-time case. Their accuracy extends to different degrees of nonlinearity.

Second, the discretized state space in continuous-time problems is very sparse. The sparse structure of the implicit method (with finite differences) is by many considered the most important practical implication of continuous time, such as the sparsity of transition matrices as discussed in Rendahl (2022).

Third, the viscosity solutions and finite difference methods can handle the non-differentiable and non-convex problems in continuous time without the need to change the algorithm, while these problems are difficult to handle in the standard discrete time methods.

To compare and contrast the performance of the continuous-time methods and the discrete-time methods further, Rendahl (2022) compared the value function iteration for discrete time, and the explicit and implicit finite difference methods for continuous time. The implicit finite difference method is the continuous-time equivalent to Howard's improvement algorithm in discrete time, the implicit method is faster. Since the HJB equation does not contain future values of the value function, the update of the value function can be formulated as the solution to a system of linear equations. Through the sparse matrix operations, this system can be solved efficiently. The explicit finite difference method in continuous time is where the value function iteration in discrete time converges to. Since the value function iteration is less efficient, the explicit method generally has slow convergence.

Mathematically or statistically, the continuous-time models do not impose a priori and perfect synchronization of decisions among economic agents, since the decision interval of the model is not tied to the observation interval in the data. They also allow for a clear distinction between flows and stocks in the economy; see, the paper by Parra-Alvarez (2018) for more discussions.

4.1 Hamilton-Jacobi-Bellman Equations

The HJB equation is one of the most popular mathematical and statistical models in macroeconomics to characterize the macroeconomic activities; see, for example, Fernández-Villaverde and Nuño (2021) for details. The basic setup is that an agent maximizes the following

$$\max_{\{\alpha_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} u\left(x_t, \alpha_t\right) dt \right],$$

subject to the law of motion for the state

$$dx_t = \mu_t\left(x_t, \alpha_t\right) dt,$$

where $x_t \in \mathbb{X} \subset \mathbb{R}^N$ is the state variable (such as K_t in the new Keynesian model), $\alpha_t \in \mathbb{A} \subset \mathbb{R}^M$ is the control variable (such as C_t, L_t, and K_{t+1} in the new Keynesian model), $\alpha_t = \alpha_t\left(x_t\right)$ is the policy function, $\rho > 0$ is the discount factor (recall that in the notation of new Keynesian model: $\beta = e^{-\rho}$), $\mu(\cdot)$ is the drift function, and $u(\cdot)$ is the instantaneous utility function. It satisfies the HJB equation as follows:

$$\rho \mathbf{V}(\mathbf{x}) = \frac{\partial \mathbf{V}}{\partial t} + \max_{\boldsymbol{\alpha}} \left\{ \mathbf{u}(\mathbf{x}, \boldsymbol{\alpha}) + \boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\alpha})^{\top} \nabla_x \mathbf{V}(\mathbf{x}) \right\},$$

with a transversality condition $\lim_{T\to\infty} e^{-\rho T} \mathbf{V}_T(\mathbf{x}) = \mathbf{0}$, where $\mathbf{V}(\mathbf{x})$ stands for the value function at time t with the time subscript dropped, which might abuse notation. Here, $\mathbf{A}^{\top}$ denotes the transpose of a vector or matrix $\mathbf{A}$ and $\operatorname{tr}(\cdot)$ stands for the trace of a matrix. Note that in the HJB equation, we use $\mathbf{x}$ to denote the vector of the state variables (dimension $N \times 1$), $\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\alpha})$ as the vector of the drift terms (dimension $N \times 1$), and $\boldsymbol{\sigma}(\mathbf{x}, \boldsymbol{\alpha})$ as the vector of the risk terms (dimension $N \times 1$), and $\boldsymbol{\sigma}^2(\mathbf{x}, \boldsymbol{\alpha})$ stands for the $N \times N$ variance-covariance matrix. Finally, we use $\nabla_x \mathbf{V}(\mathbf{x})$ to denote the gradient of $\mathbf{V}(\mathbf{x})$ (dimension $N \times 1$) and $\Delta_x \mathbf{V}(\mathbf{x})$ to denote the Hessian matrix of $\mathbf{V}(\mathbf{x})$ (dimension $N \times N$).

In a diffusion format, popular in the finance literature, the state is now

$$dx_t = \mu_t\left(x_t, \alpha_t\right) dt + \sigma_t\left(x_t, \alpha_t\right) d\mathbb{W}_t,$$

where $\mathbb{W}_t$ is the standard Brownian motion. Then, the HJB equation is given by

$$\rho \mathbf{V}(\mathbf{x}) = \frac{\partial \mathbf{V}}{\partial t} + \max_{\boldsymbol{\alpha}} \left\{ \mathbf{u}(\mathbf{x}, \boldsymbol{\alpha}) + \boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\alpha})^{\top} \nabla_x \mathbf{V}(\mathbf{x}) + \frac{1}{2} \operatorname{tr}\left(\boldsymbol{\sigma}^2(\mathbf{x}, \boldsymbol{\alpha}) \Delta_x \mathbf{V}(\mathbf{x}) \right) \right\}.$$

To characterize a possible extension, one can use the following jump model

$$dx_t = \mu_t\left(x_t, \alpha_t, s_i\right) dt,$$

where $s_i \in \{s_1, s_2\}$ is a two-state continuous-time Markov chain $s_i \in \{s_1, s_2\}$ and the Poisson process jumps from state 1 to state 2 with intensity λ_1 and vice-versa with intensity λ_2, which is the so-called famous Markov switching (diffusion) model in the finance literature. Thus, the HJB equation for this case is

$$\rho \mathbf{V}_i(\mathbf{x}) = \frac{\partial \mathbf{V}_i}{\partial t} + \max_{\boldsymbol{\alpha}} \left\{ \mathbf{u}(\mathbf{x}, \boldsymbol{\alpha}) + \boldsymbol{\mu}\left(\mathbf{x}, \boldsymbol{\alpha}, s_i\right)^{\top} \nabla_x \mathbf{V}_i(\mathbf{x}) \right\} + \lambda_i \left[\mathbf{V}_j(\mathbf{x}) - \mathbf{V}_i(\mathbf{x}) \right],$$

for $i, j = 1, 2$, $i \neq j$, where $\mathbf{V}_i(\mathbf{x}) \equiv \mathbf{V}\left(\mathbf{x}, s_i\right)$ denotes the value function at time t and state s_i, with the time subscript dropped. When the HJB equation includes both the volatility and jumps, we have the following jump-diffusion process

$$dx_t = \mu_t\left(x_t, \alpha_t, s_i\right) dt + \sigma_t\left(x_t, \alpha_t\right) d\mathbb{W}_t,$$

where s_i and $\sigma_t(x_t, \alpha_t)\, d\mathbb{W}_t$ constitute a Lévy process. Hence, the HJB equation that we want to solve numerically is given by

$$\rho \mathbf{V}_i(\mathbf{x}) = \frac{\partial \mathbf{V}_i}{\partial t} + \max_{\boldsymbol{\alpha}} \left\{ \mathbf{u}(\mathbf{x}, \boldsymbol{\alpha}) + \boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\alpha}, s_i)^\top \nabla_x \mathbf{V}_i(\mathbf{x}) + \frac{1}{2} \operatorname{tr}\left(\boldsymbol{\sigma}^2(\mathbf{x}, \boldsymbol{\alpha}) \Delta_x \mathbf{V}_i(\mathbf{x})\right) \right\}$$
$$+ \lambda_i \left[\mathbf{V}_j(\mathbf{x}) - \mathbf{V}_i(\mathbf{x})\right],$$

with a transversality condition $\lim_{T \to \infty} e^{-\rho T} \mathbf{V}_T(\mathbf{x}) = \mathbf{0}$, and some boundary conditions defined by the dynamics of x_t.

4.2 Fokker-Plank Equations

Given a stochastic process x_t with an associated infinitesimal generator $\mathcal{A}$, its probability density function $g(x)$ is defined as $\mathbb{P}_{t_0}[x_t \in \Omega] = \int_\Omega g(x) dx$ for any $\Omega \subset \mathbb{X}$ following the dynamics $\partial g / \partial t = \mathcal{A}^* g$, where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$. Note that we omit the time subscript of the function $g(x)$. Let x_t be a stochastic process given by the SDE as follows

$$dx_t = \mu_t(x_t)\, dt + \sigma_t(x_t)\, d\mathbb{W}_t,$$

which is a diffusion model or the so-called Black-Scholes model, widely used in the finance literature. The evolution of the associated density is given by

$$\frac{\partial \mathbf{g}}{\partial t} = \mathcal{A}^* \mathbf{g} = -\frac{\partial}{\partial x}\left[\boldsymbol{\mu}(\mathbf{x}) \mathbf{g}(\mathbf{x})\right] + \frac{1}{2} \frac{\partial^2}{\partial x^2}\left[\boldsymbol{\sigma}^2(\mathbf{x}) \mathbf{g}(\mathbf{x})\right]$$

with the initial value $g_0(x_t) = \delta(x_t - x_0)$, which is the well known Kolmogorov forward equation.

Now, consider the case with a Poisson jump. To do so, let x_t be a stochastic process given by

$$dx_t = \mu_t(x_t, s_i)\, dt,$$

where s_i is a two-state continuous-time Markov chain $s_i \in \{s_1, s_2\}$ with intensities λ_1 and λ_2, respectively. Then, the evolution of the density is given by:

$$\partial \mathbf{g}_i / \partial t = \mathcal{A}^* \mathbf{g} = -\frac{\partial}{\partial x}\left[\boldsymbol{\mu}(\mathbf{x}, s_i)\, \mathbf{g}_i(\mathbf{x})\right] - \lambda_i \mathbf{g}_i(\mathbf{x}) + \lambda_j \mathbf{g}_j(\mathbf{x})$$

for $i, j = 1, 2$, $i \neq j$ with the initial value $g_{1,0}(x_t) = \delta(x_t - x_0)$ and $g_{2,0}(x_t) = 0$. Next, for x_t following a Lévy process

$$dx_t = \mu_t(x_t, s_i)\, dt + \sigma_t(x_t)\, d\mathbb{W}_t,$$

the evolution of the density is given by

$$\partial \mathbf{g}_i / \partial t = \mathcal{A}^* \mathbf{g} = -\frac{\partial}{\partial x}\left[\boldsymbol{\mu}(\mathbf{x}, s_i)\, \mathbf{g}_i(\mathbf{x})\right] - \lambda_i \mathbf{g}_i(\mathbf{x}) + \lambda_j \mathbf{g}_j(\mathbf{x}) + \frac{1}{2} \frac{\partial^2}{\partial x^2}\left[\boldsymbol{\sigma}^2(\mathbf{x}) \mathbf{g}_i(\mathbf{x})\right].$$

4.3 Markov Equilibrium

Similar to the discrete-time model, a competitive equilibrium in the continuous time model is characterized by the market prices together with the allocations, such that given prices, agents optimize and markets clear. From the perspective of the macroeconomic dynamics, the continuous-time framework can be used to further study the stationary equilibrium and the Markov equilibrium due to its advanced stochastic features. In particular, the three types of equilibrium in macroeconomics models are summarized in Table 1 below.

Table 1. Types of equilibrium in macroeconomics models.

Type of equilibrium	Property	Steady state	Dynamics of state variables
Competitive equilibrium	Deterministic (no shock)	Converge to a deterministic steady state (stationary value); Time-invariant competitive equilibrium	Deterministic evolution
Stationary equilibrium	Stochastic (subject to shocks, but volatilities are at constant levels); Time-invariant Markov equilibrium	Converge to a stochastic steady state (the value of the state variable with the highest probability from the stationary distribution)	Evolution with constant volatilities eg, $\sigma_t = \sigma$
Markov equilibrium	Stochastic (subject to shocks and volatilities are time-varying)	Converge to a stationary distribution	Evolution with time-varying volatilities eg, $\sigma_t \neq \sigma$

The stationary equilibrium is reached when the volatilities in the economic system are at constant levels, even though the economy is still subject to the fundamental shocks. A Markov equilibrium is a set of functions for the control variables, the monetary and fiscal policies, the drifts and diffusions such that the agents' optimal controls solve their respective HJB equations given the law of motion of the state variable. In contrast to the stationary equilibrium, the Markov equilibrium is where there is uncertainty in the value of the state variable and its law of motion is endogenously solved by other structural variables in the model. The state variable reaches a stationary distribution in the Markov equilibrium, in contrast to the case where the state variable reaches a steady state value in the competitive equilibrium. A stochastic steady state under the stationary distribution is the value of the state variable with the highest probability from the stationary distribution[3].

To solve the Markov equilibrium in continuous time, firstly we need to solve the optimization problems and derive the allocations and prices as smooth functions of the state variable. Then, define the stochastic process for the state variable and derive the law of motion of the state variable, which is determined by the structural variables in the model using Ito's lemma. Finally, solve for the equilibrium by converting the stochastic differential equations into a system of ODEs in the asset prices to find a numerical solution if there is no explicit solution; see, for example, the book by Zhu et al. (2013) for more details. The ODEs can be solved using appropriate boundary conditions.

[3]Note that the stationary equilibrium in our paper is not equivalent to the stationary value, which stands for the deterministic steady state under the competitive equilibrium. Indeed, Achdou et al. (2022) defined the stationary equilibrium as the time-invariant competitive equilibrium. However, we refer this concept as the stationary value in this paper.

4.4 Solution Methods

There are several methods to solve the SDEs (the HJB equations and the KF equations) in the continuous-time models. First, the finite difference methods can approximate the derivatives by differences; see, for instance, the book by Zhu et al. (2013) for details. Second, the perturbation method can use a Taylor expansion of order q to solve the SDEs around the deterministic steady state. Third, the projection method can project the value function over a subspace of functions.

As an illustrative example, an implicit upwind finite difference scheme is used to solve a simple HAM, the so-called Huggett (1993) model. This scheme converges to the viscosity solution of the problem, as long as it satisfies three properties: monotonicity, stability, and consistency. To be specific, an agent maximizes

$$\max_{\{c_t\}_{i=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} u\left(c_t\right) dt \right],$$

subject to $da_t = \left(s_i + R_t a_t - c_t\right) dt$, which yields a global solution, where the agent's idiosyncratic endowment $s_i \in \{s_1, s_2\}$ is a Markovian chain with intensities $s_1 \to s_2 : \lambda_1$ and $s_2 \to s_1 : \lambda_2$, a_t is the agent's wealth at the period t, c_t is the consumption, $\rho > 0$, $u(\cdot)$ is the momentary utility function, and R_t is the gross real interest rate. Then the HJB equation is as follows:

$$\rho V_i(a) = \max_c \left\{u(c) + \mu_i(a) V_i'(a)\right\} + \lambda_i \left(V_j(a) - V_i(a)\right),$$

for $i = 1$ and 2, where $\mu_i(a)$ is the drift with $\mu_i(a) = s_i + Ra - c(a)$ and $V_i'(a)$ is the derivative of $V_i(a)$. When the market clears, the aggregate income normalizes to one, that is, $\mathbb{E}\left[s_i\right] = 1$. Total assets in zero net supply is $\sum_{i=1}^{2} \int a g_t\left(a, s_i\right) da = 0$, where $g_t\left(a, s_i\right)$ is the income wealth density. There are two ways that the value function can be updated, the explicit method and the implicit method. Here we introduce the implicit method, since it is more efficient and more stable/reliable. In addition, the step size in the implicit method can be arbitrarily large. Using the implicit finite difference scheme, we can write the finite difference approximation of the HJB equation as:

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = u\left(c_{i,j}^{n}\right) + \left(V_{i,j}^{n+1}\right)' \left(s_j + Ra_i - c_{i,j}^{n}\right) + \lambda_j \left(V_{i,-j}^{n+1} - V_{i,j}^{n+1}\right),$$

where the parameter Δ denotes the step size of the implicit method. $V_{i,j}^{n}$ denotes the n'th iteration value of the value function $V_{i,j}$. For $j = 1, 2$, assume that when $j = 1$, $-j = 2$ and vice versa, following Achdou et al. (2022). The derivative $V_{i,j}' = V_j'\left(a_i\right)$ is approximated with either a forward or a backward difference approximation

$$V_{i,j,F}' \equiv \frac{V_{i+1,j} - V_{i,j}}{\Delta a}, \quad \text{and} \quad V_{i,j,B}' \equiv \frac{V_{i,j} - V_{i-1,j}}{\Delta a}.$$

Under the upwind scheme, using the forward difference approximation whenever the drift of the state variable is positive and the backward difference approximation whenever it is negative, we have

$$\mu_{i,j,F} = s_j + Ra_i - c_{i,j,F}, \quad \mu_{i,j,B} = s_j + Ra_i - c_{i,j,B},$$

and

$$V_{i,j}' = V_{i,j,F}' \mathbf{1}_{\{\mu_{i,j,F} > 0\}} + V_{i,j,B}' \mathbf{1}_{\{\mu_{i,j,B} < 0\}} + \bar{V}_{i,j}' \mathbf{1}_{\{\mu_{i,j,F} \leqslant 0 \leqslant \mu_{i,j,B}\}},$$

where $\bar{V}'_{i,j} = u'(s_j + Ra_i)$. Thus, the HJB equation can be written as:

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = u\left(c_{i,j}^n\right) + \left(V_{i,j,F}^{n+1}\right)'\left[s_j + Ra_i - c_{i,j,F}^n\right]^+ + \left(V_{i,j,B}^{n+1}\right)'\left[s_j + Ra_i - c_{i,j,B}^n\right]^- + \lambda_j\left[V_{i,-j}^{n+1} - V_{i,j}^{n+1}\right],$$

This equation constitutes a system of $2 \times I$ [4] linear equations, and it can be written in matrix notation using the following steps

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = u\left(c_{i,j}^n\right) + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{\Delta a}\left(\mu_{i,j,F}^n\right)^+ + \frac{V_{i,j}^{n+1} - V_{i-1,j}^{n+1}}{\Delta a}\left(\mu_{i,j,B}^n\right)^- + \lambda_j\left[V_{i,-j}^{n+1} - V_{i,j}^{n+1}\right].$$

Collecting terms with the same subscripts on the right-hand side, it leads to

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^{n+1} = u\left(c_{i,j}^n\right) + V_{i-1,j}^{n+1}x_{i,j} + V_{i,j}^{n+1}y_{i,j} + V_{i+1,j}^{n+1}z_{i,j} + V_{i,-j}^{n+1}\lambda_j,$$

where $x_{i,j} = -\left(\mu_{i,j,B}^n\right)^- / \Delta a$, $y_{i,j} = -\left[\left(\mu_{i,j,F}^n\right)^+ - \left(\mu_{i,j,B}^n\right)^-\right] / \Delta a - \lambda_j$, and $z_{i,j} = \left(\mu_{i,j,F}^n\right)^+ / \Delta a$. Thus, this equation can be written in the matrix notation as $\frac{1}{\Delta}\left(\mathbf{V}^{n+1} - \mathbf{V}^n\right) + \rho\mathbf{V}^{n+1} = \mathbf{u}^n + \mathbf{A}^n\mathbf{V}^{n+1}$, where

$$\mathbf{A}^n = \begin{pmatrix} y_{1,1} & z_{1,1} & 0 & \cdots & 0 & \lambda_1 & 0 & 0 & \cdots & 0 \\ x_{2,1} & y_{2,1} & z_{2,1} & 0 & \cdots & 0 & \lambda_1 & 0 & 0 & \cdots \\ 0 & x_{3,1} & y_{3,1} & z_{3,1} & 0 & \cdots & 0 & \lambda_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & x_{I,1} & y_{I,1} & 0 & 0 & 0 & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 & 0 & 0 & y_{1,2} & z_{1,2} & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & x_{2,2} & y_{2,2} & z_{2,2} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & x_{3,2} & y_{3,2} & z_{3,2} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & 0 & x_{I,2} & y_{I,2} \end{pmatrix}_{2I \times 2I},$$

which is from the final HJB iteration, and $\mathbf{u}^n = \left(u\left(c_{1,1}^n\right), \ldots, u\left(c_{I,1}^n\right), u\left(c_{1,2}^n\right), \ldots, u\left(c_{I,2}^n\right)\right)^\top$. This system can be written as $\mathbf{B}^n\mathbf{V}^{n+1} = \mathbf{b}^n$, $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n$, and $\mathbf{b}^n = \mathbf{u}^n + \frac{1}{\Delta}\mathbf{V}^n$, which can be solved very efficiently using sparse matrix (since $\mathbf{A}^n$ is a sparse matrix) routines. The matrix $\mathbf{A}^n$ is a transition matrix that summarizes the Poisson intensities when the process is approximated by the finite difference method with a discrete Poisson process. All rows in $\mathbf{A}^n$ sum to zero, diagonal elements are non-positive and off-diagonal elements are non-negative. All entries in a row being zero imply that the state remains fixed over time.

To solve the KF equation, we have to solve the following ODE using the finite difference method

$$0 = -\frac{d}{da}\left[\mu_j(a)g_j(a)\right] - \lambda_j g_j(a) + \lambda_{-j}g_{-j}(a),$$

where $g(\cdot)$ is the density function. With the Poisson jumps, $g_j(\cdot)$ denotes the density function under state j, and $g_{-j}(\cdot)$ is the density function under state $-j$. Then, the ODE can be

[4] We assume that $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

approximated by

$$-\frac{g_{i,j}\left(\mu_{i,j,F}^{n}\right)^{+}-g_{i-1,j}\left(\mu_{i-1,j,F}^{n}\right)^{+}}{\Delta a}-\frac{g_{i+1,j}\left(\mu_{i+1,j,B}^{n}\right)^{-}-g_{i,j}\left(\mu_{i,j,B}^{n}\right)^{-}}{\Delta a}-g_{i,j}\lambda_j+g_{i,-j}\lambda_{-j}=0.$$

Collecting terms, we obtain

$$g_{i-1,j}z_{i-1,j}+g_{i,j}y_{i,j}+g_{i+1,j}x_{i+1,j}+g_{i,-j}\lambda_{-j}=0,$$

where $x_{i+1,j} = -\left(\mu_{i,j+1,B}^{n}\right)^{-}/\Delta a$, $y_{i,j} = -\left[\left(\mu_{i,j,F}^{n}\right)^{+}-\left(\mu_{i,j,B}^{n}\right)^{-}\right]/\Delta a-\lambda_j$, and $z_{i-1,j} = \left(\mu_{i,j-1,F}^{n}\right)^{+}/\Delta a$. This approximation can be written in the matrix form $\mathbf{A}^{\top}\mathbf{g}=\mathbf{0}$ where $\mathbf{A}^{\top}$ is the transpose of the intensity matrix $\mathbf{A}$ ($\mathbf{A}=\lim_{n\to\infty}\mathbf{A}^n$) from the HJB equation.

To find the stationary distribution, one solves the eigenvalue problem $\mathbf{A}^{\mathrm{T}}\mathbf{g}=\mathbf{0}$, a system of $2\times$J linear equations. The reason for using the transpose of the intensity matrix $\mathbf{A}$ can be made more precise by the differential operators, so that one can write the HJB equation in terms of a differential operator $\mathcal{A}$, the infinitesimal generator of the process. Similarly, the Kolmogorov forward equation can be written in terms of an operator $\mathcal{A}^*$, the adjoint of the operator $\mathcal{A}$ in the HJB equation, which is the infinite-dimensional analogue of a matrix transpose. $\mathbf{A}$ is simply the discretized infinitesimal generator, whereas $\mathbf{A}^{\top}$ is the discretized version of its adjoint, the Kolmogorov forward operator.

4.5 Transition Dynamics and Impulse Responses

In this section, we provide several recent methods in the existing literature on computing the transition dynamics and impulse response functions in the continuous-time models. The computational method for solving transition dynamics from an arbitrary initial condition can also be used to compute nonlinear impulse responses to unanticipated aggregate shocks, the so-called MIT shocks, i.e., an unanticipated (zero probability) shock followed by a deterministic transition as in Krusell and Smith (1998). Recently, Fernández-Villaverde et al. (2023) further extended the model in Krusell and Smith (1998) by proposing a nonparametric perceived law of motion (PLM) and updated with machine learning, and Boppart et al. (2018) and Auclert et al. (2021) used the linearized counterpart to compute linear impulse responses to small MIT shocks in order to obtain further speed gains. Additionally, both Auclert et al. (2021) and Oskolkov (2023) used the sequence-space Jacobians to compute the impulse responses. For the type of the representative agent model, He and Krishnamurthy (2019) provided a method to compute the impulse responses in a continuous-time framework. Now, let us introduce these methods in detail.

4.5.1 Aggregate Uncertainty and Linear Dynamics

When there is aggregate uncertainty, Krusell and Smith (1998) proposed a bounded rationality method. Households in the model approximate the distribution by a number of its moments, e.g., the mean $\int_0^\infty\int_{\underline{s}}^{\bar{s}} ag(a,s)dads = K_t$, where s_t is the labor productivity following the Ornstein-Uhlenbeck process as $ds_t=\theta\left(\hat{s}-s_t\right)dt+\sigma dB_t$. The production function is given by $Y_t=F(A_t,K_t,L_t)=A_tK_t^{\alpha}L_t^{1-\alpha}$ with the aggregate total factor productivity (TFP) A_t, following a diffusion process as $dA_t=\mu_A\left(A_t\right)dt+\sigma_A\left(A_t\right)dW_t$. The capital K_t evolves according to $dK_t = \mu_K(K_t, A_t) K_t dt$. Then, the HJB simplifies to

$$\rho V(a,s,A,K) = \max_{c\geq 0} u(c) + [ws + Ra - c]\frac{\partial V}{\partial a} + \theta(\hat{s} - s)\frac{\partial V}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial s^2} + \mu_A(A)\frac{\partial V}{\partial A} + \frac{\sigma_A^2(A)}{2}\frac{\partial^2 V}{\partial A^2} + K\mu_K(K,A)\frac{\partial V}{\partial K}.$$

Suppose the perceived law of motion (PLM) has a simple parametric (linear) form

$$\mu_K(K, A; \boldsymbol{\theta}) = \theta_0 + \theta_1 K + \theta_2 KA + \theta_3 A. \tag{PLM}$$

Then, begin with an initial guess of $\boldsymbol{\theta}^0 = \left(\theta_0^0, \theta_1^0, \theta_2^0, \theta_3^0\right)$, and set $n := 0$.

(1) Given $\mu_K\left(K, A; \boldsymbol{\theta}^0\right)$, solve the HJB equation and obtain the transition matrix $\mathbf{A}$.

(2) Conduct the Monte Carlo simulation to obtain the simulated data of the aggregate TFP $\{A_m\}_{m=0}^M$ based on the model $\Delta A_m = \mu_A(A_{m-1})\Delta t + \sigma_A(A_{m-1})\sqrt{\Delta t}\varepsilon_m$, where $\varepsilon_m \sim \mathcal{N}(0,1)$.

(3) Compute the dynamics of the distribution using the KF equation and use it to obtain aggregate capital $\int_0^\infty \int_{\underline{s}}^{\bar{s}} a g(a,s) da ds = K_t$.

(4) Run an ordinary least squares $\Delta K_m / K_m = \mu_K(K_m, A_m; \boldsymbol{\theta})\Delta t$ over the simulated sample $\{A_m, K_m\}_{m=0}^M$ to update coefficients $\boldsymbol{\theta}^{n+1}$. If $\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^n$ or they are very close, stop, otherwise go back to step 1 .

In a linearized system, Boppart et al. (2018) employed the MIT shock and obtained the first-order perturbation solution just by computing transitional dynamics. The initial state is the deterministic steady state $f_{ss}(\cdot)$. The aggregate TFP evolves with time according to

$$\Delta A_0 = \mu_A(A_0)\Delta t + \sigma_A(A_0)\sqrt{\Delta t}, \quad \text{and} \quad \Delta A_t = \mu_A(A_t)\Delta t, \;\; t > 0,$$

where $A_0 = A_{ss}$. If the model is approximately linear, the response to an MIT shock is the impulse response function of the model. The method can be extended to the case with n shocks $d\mathbf{A} \equiv \left(dA^0, dA^1, \ldots, dA^n\right)^\top$.

4.5.2 Aggregate Uncertainty and Nonlinear Dynamics

For models with aggregate nonlinear dynamics, which is a general form of PLM, Fernández-Villaverde et al. (2023) extended the Krusell and Smith (1998) methodology and proposed a nonparametric perceived law of motion to globally compute and estimate the HAM, updated using machine learning such as a neural network. As claimed by Fernández-Villaverde et al. (2023), their algorithm can approximate the PLM arbitrarily well; see, for instance, the paper by Fernández-Villaverde et al. (2023) for details.

Households consider a PLM of aggregate debt B_t

$$dB_t = h(B_t, N_t)\, dt,$$

where $h(B_t, N_t) = \mathbb{E}[dB_t \mid B_t, N_t]/dt$ and $N_t = K_t - B_t$, the net wealth (i.e., inside equity) of the expert, which is the difference between his assets (capital) and liabilities (debt). Given the

PLM, the household's HJB equation becomes

$$\rho V_i(a,B,N) = \max_c \frac{c^{1-\gamma}-1}{1-\gamma} + \mu_i(a,B,N)\frac{\partial V_i}{\partial a} + h(B,N)\frac{\partial V_i}{\partial B} + \mu^N(B,N)\frac{\partial V_i}{\partial N} + \lambda_i\left[V_j(a,B,N) - V_i(a,B,N)\right] + \frac{\left[\sigma^N(B,N)\right]^2}{2}\frac{\partial^2 V_i}{\partial N^2},$$

where $i \neq j = 1,2$, the net wealth N_t evolves as $dN_t = \mu^N(B_t,N_t)\,dt + \sigma^N(B_t,N_t)\,dW_t$, and the household's saving a_t follows $da_t = (w_t s_t + R_t a_t - c_t)\,dt = \mu_i(a_t,B_t,N_t)\,dt$ and $B_t \equiv \int a dG_t(a,s)$.

Instead of using the projection method to approximate the PLM: $h(\mathbf{x};\boldsymbol{\theta}) = \theta_0 + \sum_{q=1}^{Q}\theta_q\psi_q(\mathbf{x})$, Fernández-Villaverde et al. (2023) approximated the PLM with a neural network as $h(\mathbf{x};\boldsymbol{\theta}) = \theta_0^1 + \sum_{q=1}^{Q}\theta_q^1\phi\left(\theta_{0,q}^2 + \sum_{i=1}^{2}\theta_{i,q}^2 x^i\right)$, where $\mathbf{X} = \{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_J\}$ and $\mathbf{x}_j = \left\{x_j^1, x_j^2\right\} = \left\{B_{t_j}, N_{t_j}\right\}$ is a two-dimensional input. $\boldsymbol{\theta} = \left(\theta_0^1,\theta_1^1,\ldots,\theta_Q^1,\theta_{0,1}^2,\theta_{1,1}^2,\theta_{2,1}^2,\ldots,\theta_{0,Q}^2,\theta_{1,Q}^2,\theta_{2,Q}^2\right)$ denotes the vector of weights. This is a neural network with one hidden layer, with a linear combination of Q activation functions. $\phi(\cdot)$ is an activation function, such as $\phi(x) = \log(1+e^x)$. For the approximation of a two-dimensional function, one single layer is enough. This neural network can also be extended to include multiple hidden layers, which is the case of deep neural network. To train the neural network, $\boldsymbol{\theta}$ is selected to minimize the quadratic error function $\mathcal{E}(\boldsymbol{\theta};\mathbf{X},\widehat{\mathbf{h}})$ given a simulation $(\mathbf{X},\widehat{\mathbf{h}})$, where $\widehat{\mathbf{h}} = \left\{\widehat{h}_1,\widehat{h}_2\ldots,\widehat{h}_J\right\}$ and $\widehat{h}_j \equiv \frac{B_{t_j+\Delta t} - B_{t_j}}{\Delta t}$, that is,

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}}\mathcal{E}(\boldsymbol{\theta};\mathbf{X},\widehat{\mathbf{h}}) = \arg\min_{\boldsymbol{\theta}}\sum_{j=1}^{J}\mathcal{E}\left(\boldsymbol{\theta};\mathbf{x}_j,\widehat{h}_j\right) = \arg\min_{\boldsymbol{\theta}}\frac{1}{2}\sum_{j=1}^{J}\left\|h(\mathbf{x}_j;\boldsymbol{\theta}) - \widehat{h}_j\right\|^2.$$

Fernández-Villaverde et al. (2023) solved this problem using the batch gradient descent algorithm. Note that some other machine learning or deep learning methods can be applied to solve the above system too.

Auclert et al. (2021) proposed a general and highly efficient method for solving and estimating the general equilibrium HAMs with aggregate shocks in discrete time. The model is set up in the sequence space by assuming perfect foresight with respect to aggregates. The approach relies on the rapid computation of sequence space Jacobians, the derivatives of perfect-foresight equilibrium mappings between aggregate sequences around the steady state. This algorithm can be combined with a systematic approach of composing and inverting Jacobians to solve for general equilibrium impulse responses. A rapid procedure is obtained for likelihood-based estimation and computation of nonlinear perfect-foresight transitions.

Equilibrium in the sequence space can always be expressed as a solution to a nonlinear system $\mathbf{F}(\mathbf{X},\mathbf{Z}) = 0$, where $\mathbf{X}$ represents the time path of endogenous variables (usually aggregate prices and quantities) and $\mathbf{Z}$ represents the time path of exogenous shocks. Obtaining the impulse responses of unknowns to shocks, $d\mathbf{X} = -\mathbf{F}_{\mathbf{X}}^{-1}\mathbf{F}_{\mathbf{Z}}d\mathbf{Z}$, requires computing the Jacobians $\mathbf{F}_{\mathbf{X}}$ and $\mathbf{F}_{\mathbf{Z}}$, which are formed by combining Jacobians from different parts of the model.

Similarly, Oskolkov (2023) employed the methods from Kaplan et al. (2018) and Auclert et al. (2021) to analyze nonlinear solutions for aggregate one-time unanticipated shocks. In particular, this paper works in the sequence space and computes the sequence-space Jacobians by solving the linearized version of the coupled system of equations. For estimation, a sample path of $x(t)$ is integrated from a simulated sequence of shocks dW given the parameters, which come from the calibration and determine the steady state. To estimate the parameters of the processes, the simulated method of moments (SMM) of McFadden (1989) is used, which is to compute the moments of these series, to compare them to the moments of simulated sequences, and to look for a combination of parameters that minimize a quadratic distance.

Finally, in a representative agent model, He and Krishnamurthy (2019) studied the effect of -1% shock in σdW_t, which means the fundamental shock leading capital to fall exogenously by 1%. The computational algorithm to calculate the impulse response functions is as follows, where the focus is on the mean path by shutting future shocks to zero.

(1) Compute the benchmark path of these variables without any shocks, but still subject to the endogenous drift of the state variable in the model. In other words, calculate the benchmark path for the realizations of $dW_{t+m} = 0$ for $m \geq 0$.

(2) Compute the shocked path of these variables given the initial shock $\sigma dW_t = -1\%$, but setting future realizations of shocks to be zero, i.e., $dW_{t+m} = 0$ for $m > 0$.

(3) Calculate and plot the (log) difference between the path with the shock and the mean path without any shock. This computation is meant to mimic a deviation-from-steady-state computation that is typically plotted in impulse-response functions in the linear-non-stochastic models.

Note that in traditional linear models, the impulse-response functions are independent of future shocks. However, the impact of a shock depends on future shocks in nonlinear models. For more on the difference between impulse responses in linear models with a non-stochastic steady state and those nonlinear models with a stochastic steady state; see, for example, the papers by Koop et al. (1996) and Borovička et al. (2011). An alternative method to calculate the impulse response functions in the stochastic nonlinear models is to calculate the expected impact of the initial shock $\sigma dW_t = -1\%$ on the variable at $t+m$ by integrating over all possible future paths.

4.6 Estimation and Dynamic Programming: A Data Driven Approach

The sections above introduced the solution methods and the numerical analysis of the transition dynamics and the impulse responses. To see how the methods above can be connected with the empirical analysis using the real-world data[5], we propose a method that combines the empirical estimation and the numerical computation in order to acquire the parameter values and the solutions of the optimization problem simultaneously. Our current working paper tries to address this issue; see, for example, Cai and Hu (2025) for details. The novelty of this approach is its data driven feature, where the empirical estimation is embedded into the algorithm that finds the optimal controls of the optimization problem, and then the parameters and the

[5] We appreciate the anonymous referee for bringing our attention to this issue.

optimal policies update recursively. We provide a brief description of the algorithm below, and this method can be further applied in various adaptations according to the specific problems.

Step 1: Use the maximum likelihood estimation (MLE) method to estimate the drift and the diffusion terms in the law of motion of the state variable. This step uses the data from the financial market, such as the firms' net worth value. For a reference on the empirical macro-finance analysis, please see Gilchrist and Zakrajšek (2012).

Step 2: Based on the theoretical derivation, make a guess of the parametric form of the drift function and the diffusion function of the state variable, where the control variable serves as a key factor. Use the real-world data of the control variable to estimate the parameters in the parametric functions of the drift and the diffusion.

Step 3: Substitute the estimated drift function and the diffusion function of the state variable in the HJB equation. Use the finite difference method introduced in Section 4.4 to compute the value function in the HJB equation.

Step 4: Substitute the computed value function in the FOCs, and solve the optimal policy: the optimal control variable as a function of the state variable. Then use the estimated law of motion (the drift and the diffusion) of the state variable and the Ito's lemma, in addition to the optimal policy function, to calculate the stochastic process (the drift and the diffusion) that the control variable follows.

Step 5: Conduct the Monte Carlo simulation to obtain a series of simulated data of the control variable based on the computed process in Step 4. Use the simulated data of the control variable to update the parametric estimation of the drift function and the diffusion function of the state variable in Step 2.

Step 6: Begin again from Step 3, substitute the updated drift function and the diffusion function of the state variable in the HJB equation. Recompute the value function in the HJB equation, and then update Steps 4 and 5. Iterate until the parameters of the drift function and the diffusion function of the state variable converge.

To describe the process above in a nutshell, we aim to evaluate the parameters in the drift function and the diffusion function of the state variable using the real-world data combined with the optimization solution. We start from the real-world data of the state variable and the control variable to acquire an initial estimation, then use the HJB equation, the FOC, and the Ito's lemma to derive the values of the control variable's drift and diffusion from the state variable's drift and diffusion. Lastly we use the simulated data from the control variable's stochastic process to update the parametric estimation of the state variable's drift function and diffusion function.

Alternatively, the parametric method can be substituted by the nonparametric method, such as the neural network, when the number of states and controls grows. When training the neural network, first make a guess of the initial values of the parameters and then construct a time series of data through the simulation. Then train the parameters on the simulated data and obtain the updated values of parameters by minimizing a quadratic error function. Iterate until the value of the parameters converge.

Note that there are other approaches that can handle the combination of the parameter estimation and the dynamic programming in addition to the methods introduced above, one of such extension is the reinforcement learning.

4.7 A More General Framework

Achdou et al. (2022) extended the study to the backward-forward MFGs system in n dimensions, which is a natural generalization of the equations for the Bewley-Huggett-Aiyagari (BHA) models, proposed by Bewley (1987), Hugget (1993), and Aiyagari (1994), respectively. The mathematical MFG literature typically writes this system using the language of the modern theory of SDEs, especially the vector calculus notation, described as follows. For more details about modeling BHA type models, the reader is referred to the papers by Kirkby (2018) and Hansak (2023).

The mathematics literature typically only considers the case where the state variables follow diffusion processes rather than processes featuring jumps. Under this assumption, a general backward-forward MFG system in n dimensions is

$$\rho V = \max_{\alpha} \left\{ r(x, \alpha, g) + \sum_{i=1}^{n} \alpha_i \partial_i V \right\} + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(x) \partial_{ii} V + \partial_t V,$$

in $\mathbb{R}^n \times (0, T)$, where we use the short-hand notation $\partial_a v = \partial v / \partial a$, and so on, and

$$\partial_t g = -\sum_{i=1}^{n} \partial_i \left(\alpha_i^*(x, g) g \right) + \frac{1}{2} \sum_{i=1}^{n} \partial_{ii} \left(\sigma_i^2(x) g \right),$$

in $\mathbb{R}^n \times (0, T)$, with

$$g_0 = g(0) \quad \text{and} \quad V_T = V(x, g(T)),$$

in $\mathbb{R}^n$, where $x \in \mathbb{R}^n$ is an n-dimensional state vector, $\alpha \in \mathbb{R}^n$ is a control vector and α^* its optimally chosen policy function, $V(x, t)$ is the value function, $g(x, t)$ the density, $r(x, \alpha, g)$ a period return function, and $\sigma_i^2(x)$ a diffusion coefficient. The first equation is the HJB equation, the second equation is the KF equation and the equations in the third line are the initial condition on the density and the terminal condition on the value function. The system iterates backward-forward in the sense that given the steady state value function, the system updates backward using the HJB equation to obtain the policies. Given the initial distribution, the system updates forward using the KF equation to propagate the distribution. For a two-dimensional special case, in the Huggett model with a diffusion process (close to but different from the Huggett model discussed in Section 4.4),

$$\begin{aligned} ds_t &= \mu(s_t)\, dt + \sigma(s_t)\, dW_t, \\ \rho v(a, s) &= \max_c u(c) + \partial_1 v(a, s) \alpha_1 + \partial_2 v(a, s) \alpha_2 + \frac{1}{2} \partial_{22} v(a, s) \sigma^2(s), \\ 0 &= -\partial_1 \left(\alpha_1 g(a, s) \right) - \partial_2 \left(\alpha_2 g(a, s) \right) + \frac{1}{2} \partial_{22} \left(\sigma^2(s) g(a, s) \right), \end{aligned}$$

where $x \in \mathbb{R}^2$ with $x_1 = a$ and $x_2 = s$.

Now, we define three useful operators: the gradient ∇, the Laplacian Δ and the divergence. First, for a function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient vector is the vector of first derivatives $\nabla f := [\partial f/\partial x_1, \ldots, \partial f/\partial x_n]^\top$. Second, for a function $f : \mathbb{R}^n \to \mathbb{R}$, the Laplacian is the sum of all the unmixed second derivatives $\Delta f := \sum_{i=1}^n \partial^2 f/\partial x_i^2$. Third, for a vector-valued function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\mathbf{v}(x_1, \ldots, x_n) = [v_1(x_1, \ldots, x_n), \ldots, v_n(x_1, \ldots, x_n)]^\top$, the divergence of $\mathbf{v}$ is $\mathrm{div}(\mathbf{v}) := \sum_{i=1}^n \partial v_i/\partial x_i$. Note that $\Delta f = \mathrm{div}(\nabla f)$.

The MFG literature typically assumes that $\sigma_i^2(x) = 2\nu$ for all x and all i which implies that the second-order terms simplify. Assume $H(x,p,g) := \max_\alpha \{r(x,\alpha,g) + \sum_{i=1}^n \alpha_i p_i\}$, where $H(x,p,g)$ denotes the Hamiltonian. The optimal drift of each state variable equals $\alpha_i^*(x,g) = \partial_{p_i} H(x, \nabla v, g)$. Using the Laplacian and the divergence just defined, the backward-forward MFG system can be written into the standard mathematical formulation

$$\begin{aligned}
\rho v &= H(x,\nabla v, g) + \nu \Delta v + \partial_t v \quad \text{in } \mathbb{R}^n \times (0,T),\\
\partial_t g &= -\,\mathrm{div}\left(\nabla_p H(x,\nabla v, g) g\right) + \nu \Delta g \quad \text{in } \mathbb{R}^n \times (0,T),\\
g(0) &= g_0, \quad v(x,T) = V(x, g(T)) \quad \text{in } \mathbb{R}^n.
\end{aligned}$$

Note that the MFG literature typically sets $\rho = 0$ for simplicity, i.e., it ignores discounting.

The backward-forward MFG system above describes general HAMs without aggregate uncertainty. However, in many economically interesting situations, it is important to allow for aggregate risk in addition to idiosyncratic risk as in Den Haan (1997) and Krusell and Smith (1998). Fortunately, the theory of MFGs has also studied that case, with mathematicians referring to aggregate uncertainty as "common noise". In the most general case, such MFGs can be written in terms of the so-called "Master equation" as in Cardaliaguet et al. (2019). This Master equation is an equation on the space of measures, i.e., it is an equation that is set in infinite-dimensional space. In the case without aggregate uncertainty, the Master equation reduces to the backward-forward MFG system.

§5 Conclusion

This selective review outlined the mathematical/statistical tools and the computational methods in mathematics for solving both discrete-time and continuous-time models in macroeconomics. As the mathematical tools and computational methods are more advanced in continuous time, we see a bright future for the macroeconomics modeling in continuous time. For example, in addition to a neural network as employed by Fernández-Villaverde et al. (2023), some advanced machine learning such as deep learning methods or AI methods can be applied to this field too, especially for nonlinear models, which can attract some young scholars and Ph.D. students in economics, mathematics and statistics, to find their own interesting research topics for continuous-time models in macroeconomics. In addition, the well-developed asset pricing theories in continuous time also shed light on the inclusion of financial risk analysis in the macroeconomics models. Looking forward, more work can be for sure done to study the connection between the real economy's business cycles and the financial market's fluctuations within a continuous-time general equilibrium framework. Even though the discrete-time model and the continuous-time model may share similar computational accuracy, we still have much confidence on the scope of the analysis that can be done only in continuous time which can not be substituted by the discrete-time model, such as the formulation of problems related with uncertainty. Finally, similar to the model specification problem for conventional stochastic diffusions, well studied in the literature such as the pioneer work by Aït-Sahalia (1996), it would be interesting to consider some possible model specification tests for HAMs with aggregate shocks under full or partial information as addressed in Cai, Mei and Wang (2024), which is definitely warranted as future research. The reader is referred to the paper by Cai, Mei and Wang (2024) for more discussions.

Appendix. The details of log-linearization

In the following detailed derivation of the equations in Section 3.1, we use the general formula of the log-linearization as: $\widetilde{X}_t \equiv \ln X_t - \ln X \approx (X_t - X)/X$, so that $X_t = Xe^{\widetilde{X}_t}$, $X_t \approx X\left(1+\widetilde{X}_t\right)$, and $e^{\widetilde{X}_t} \approx 1+\widetilde{X}_t$. Also, $\ln(1+X_t) \approx X_t$ and $d\ln X_t = \ln X_t - \ln X \approx (X_t - X)/X = dX_t/X \approx \widetilde{X}_t$. Next, we derive the equations listed in Section 3.1.

1. The resource constraint can be written as follows:
   $Y_t = C_t + I_t + G$ and $Y\left(1+\widetilde{Y}_t\right) = C\left(1+\widetilde{C}_t\right) + I\left(1+\widetilde{I}_t\right) + G$.
   At the steady state, we have that $Y = C + I + G$. Thus, $Y\widetilde{Y}_t = C\widetilde{C}_t + I\widetilde{I}_t$ and $\widetilde{Y}_t = C\widetilde{C}_t/Y + I\widetilde{I}_t/Y$.

2. The consumption Euler equations become the following equations
$$C_t = E_t\left\{\left[(1+r_t^n)\frac{P_t}{P_{t+1}}\beta\right]^{-\sigma} C_{t+1}\right\},$$
$$Ce^{\widetilde{C}_t} = E_t\left\{\left[e^{\widetilde{1+r_t^n}} \times \frac{e^{\widetilde{P}_t}}{e^{\widetilde{P_{t+1}}}} \times \beta\right]^{-\sigma} Ce^{\widetilde{C_{t+1}}}\right\},$$
$$1+\widetilde{C}_t = E_t\left\{\left[\left(1+\widetilde{1+r_t^n}\right)\left(1+\widetilde{P}_t - \widetilde{P_{t+1}}\right)\beta\right]^{-\sigma}\left(1+\widetilde{C_{t+1}}\right)\right\},$$
$$\ln\left(1+\widetilde{C}_t\right) = E_t\left\{-\sigma\left[\ln\left(1+\widetilde{1+r_t^n}\right) + \ln(1-\pi_{t+1}) + \ln\beta\right] + \ln\left(1+\widetilde{C_{t+1}}\right)\right\},$$
   and
$$\widetilde{C}_t = E_t\left\{-\sigma\left(\widetilde{1+r_t^n} - \pi_{t+1} - \rho\right) + \widetilde{C_{t+1}}\right\},$$
   where $\pi_t = \widetilde{P}_t - \widetilde{P_{t-1}}$ and $\ln\beta = -\rho$. Since $1+r_t^n = e^{\widetilde{1+r_t^n}}$, then, $\ln(1+r_t^n) = \widetilde{1+r_t^n}$, so that $r_t^n \approx \widetilde{1+r_t^n}$ and
$$\widetilde{C}_t = E_t\left\{-\sigma\left(r_t^n - \pi_{t+1} - \rho\right) + \widetilde{C_{t+1}}\right\}.$$

3. The link between asset prices and investment is given by

$$\frac{I_t}{K_t} = \delta + \frac{1}{c}\left(1 - \frac{1}{Q_t}\right),$$

$$\frac{Ie^{\widetilde{I}_t}}{Ke^{\widetilde{K}_t}} = \delta + \frac{1}{c}\left(1 - \frac{1}{Qe^{\widetilde{Q}_t}}\right),$$

$$IQe^{\widetilde{I}_t}e^{\widetilde{Q}_t} = \delta QKe^{\widetilde{Q}_t}e^{\widetilde{K}_t} + \frac{1}{c}\left(QKe^{\widetilde{Q}_t}e^{\widetilde{K}_t} - Ke^{\widetilde{K}_t}\right),$$

and

$$IQ\left(1+\widetilde{I}_t\right)\left(1+\widetilde{Q}_t\right) = \delta QK\left(1+\widetilde{Q}_t\right)\left(1+\widetilde{K}_t\right) + \frac{1}{c}\left[QK\left(1+\widetilde{Q}_t\right)\left(1+\widetilde{K}_t\right) - K\left(1+\widetilde{K}_t\right)\right].$$

At the steady state, we have

$$\frac{I}{K} = \delta + \frac{1}{c}\left(1 - \frac{1}{Q}\right).$$

Therefore,

$$IQ\widetilde{I}_t - \frac{1}{c}K\widetilde{Q}_t = IQ\widetilde{K}_t \quad \text{and} \quad \widetilde{I}_t - \widetilde{K}_t = \frac{1}{c}\frac{K}{IQ}\widetilde{Q}_t.$$

In the equilibrium, we have

$$\frac{I}{K} - \delta = 0,$$

which also implies $Q = 1$ with the steady state condition. Thus,

$$\widetilde{I}_t - \widetilde{K}_t = \frac{1}{c\delta}\widetilde{Q}_t.$$

4. Marginal cost of funds and marginal return to capital are given by

$$E_t\left\{\Lambda_{t,+1}\left(1+r_t^n\right)\frac{P_t}{P_{t+1}}\right\} = E_t\left\{\Lambda_{t,+1}\left(\frac{Z_t + (1-\delta)Q_{t+1}}{Q_t}\right)\right\},$$

and

$$\ln\left(1+r_t^n\right) + \ln P_t - \ln P_{t+1} = \ln\left[Z_t + (1-\delta)Q_{t+1}\right] - \ln Q_t.$$

Take total differential to obtain the following

$$d\ln\left(1+r_t^n\right) + d\ln P_t - d\ln P_{t+1} = d\ln\left[Z_t + (1-\delta)Q_{t+1}\right] - d\ln Q_t,$$

and

$$dr_t^n + \widetilde{P}_t - \widetilde{P_{t+1}} = \frac{[dZ_t + (1-\delta)dQ_{t+1}]}{Z + (1-\delta)} - \widetilde{Q}_t r_t^n - \pi_{t+1} - \rho = (1-\tau)\widetilde{Z}_t + \tau\widetilde{Q_{t+1}} - \widetilde{Q}_t.$$

Since $dr_t^n = r_t^n - r = \widetilde{r_t^n}$, $Z_t = \alpha Y_t/[(1+\mu_t)K_t]$, and at the steady state $r = \rho$, it is easy to see that

$$r_t^n - \rho - \pi_{t+1} = \frac{Z}{Z + (1-\delta)}\widetilde{Z}_t + \frac{1-\delta}{Z+(1-\delta)}\widetilde{Q_{t+1}} - \widetilde{Q}_t.$$

Now, define $\tau = (1-\delta)\left[\frac{\alpha Y}{(1+\mu)K} + (1-\delta)\right]^{-1} = (1-\delta)/[Z + (1-\delta)]$. Then, $1 - \tau = Z/[Z + (1-\delta)]$. Thus, $r_t^n - \pi_{t+1} - \rho = (1-\tau)\widetilde{Z_t} + \tau\widetilde{Q_{t+1}} - \widetilde{Q_t}$. Since $\widetilde{Z}_t = d\ln Z_t = d\ln\left(\frac{1}{1+\mu_t}\alpha\frac{Y_t}{K_t}\right) = -d\ln(1+\mu_t) + d\ln Y_t - d\ln K_t = -\widehat{\mu_t} + \widetilde{Y}_t - \widetilde{K_t}$, and $d\ln(1+\mu_t) = d\mu_t = \widehat{\mu_t}$, in particular, we arrive at

$$r_t^n - \pi_{t+1} - \rho = (1-\tau)\left(-\widehat{\mu_t} + \widetilde{Y}_t - \widetilde{K_t}\right) + \tau\widetilde{Q_{t+1}} - \widetilde{Q_t}.$$

5. Production function is $Y_t = A_t K_t^\alpha L_t^{1-\alpha} V_t$, so that $\ln Y_t = \ln A_t + \alpha \ln K_t + (1-\alpha)\ln L_t + \ln V_t$. Taking total differential leads to
$$\frac{dY_t}{Y} = \frac{dA_t}{A} + \alpha\frac{dK_t}{K} + (1-\alpha)\frac{dL_t}{L}.$$
Since $V_t = \left[\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon} df\right]^{-1}$, it vanishes in a first-order log linearization around a zero inflation steady state due to the reason that the deviation of $\ln(P_t(f)/P_t)$ must average to zero. Thus,
$$\widetilde{Y_t} = \widetilde{A_t} + \alpha\widetilde{K_t} + (1-\alpha)\widetilde{L_t}.$$

6. Labor market equilibrium is $(1-\alpha)\frac{Y_t}{L_t} = (1+\mu_t)\frac{L_t^\varphi}{C_t^{-\gamma}}$, then, $\ln(1-\alpha) + \ln Y_t - \ln L_t = \ln(1+\mu_t) + \varphi \ln L_t + \gamma \ln C_t$. Taking total differential, one obtains $\widetilde{Y_t} - \widetilde{L_t} = \widehat{\mu_t} + \varphi\widetilde{L_t} + \gamma\widetilde{C_t}$.

7. Price adjustment and Phillips curve can be simplified as follows. It follows from $P_t = \left[\theta(P_{t-1})^{1-\varepsilon} + (1-\theta)(P_t^o)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$ that $\widetilde{P_t} = \theta\widetilde{P_{t-1}} + (1-\theta)\widetilde{P_t^o}$, which can be transformed into $\pi_t = \widetilde{P_t} - \widetilde{P_{t-1}} = (1-\theta)\left(\widetilde{P_t^o} - \widetilde{P_t}\right)/\theta$. Also, it is easy to see from $E_t\sum_{i=0}^\infty \theta^i \Lambda_{t,i}(P_t^o/P_{t+i})^{-\varepsilon} Y_{t+i}\left[P_t^o/P_{t+i} - (1+\mu)/(1+\mu_{t+i})\right] = 0$ that
$$\widetilde{P_t^o} = (1-\theta\beta)E_t\sum_{i=0}^\infty(\theta\beta)^i\left(\widetilde{\mathrm{MC}_{t+i}} + \widetilde{P_{t+i}}\right) = (1-\theta\beta)\left(\widetilde{\mathrm{MC}_t} + \widetilde{P_t}\right) + \theta\beta E_t\left\{\widetilde{P_{t+1}^o}\right\}.$$
Therefore,
$$\widetilde{P_t^o} - \widetilde{P_t} = (1-\theta\beta)\widetilde{\mathrm{MC}_t} + \theta\beta E_t\left\{\widetilde{P_{t+1}^o} - \widetilde{P_{t+1}} + \widetilde{P_{t+1}} - \widetilde{P_t}\right\}.$$
Thus,
$$\frac{\theta}{1-\theta}\pi_t = (1-\theta\beta)\widetilde{\mathrm{MC}_t} + \theta\beta E_t\left\{\frac{\theta}{1-\theta}\pi_{t+1} + \pi_{t+1}\right\},$$
and
$$\pi_t = \frac{(1-\theta)(1-\theta\beta)}{\theta}\widetilde{\mathrm{MC}_t} + \beta E_t\{\pi_{t+1}\} = -\lambda\widehat{\mu}_t + \beta E_t\{\pi_{t+1}\},$$
where $\lambda = (1-\theta)(1-\theta\beta)/\theta$.

8. Evolution of capital is approximated by
$$K_{t+1} = I_t - \frac{1}{2}c\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t,$$
$$Ke^{\widetilde{K_{t+1}}} = Ie^{\widetilde{I_t}} - \frac{1}{2}c\left(\frac{I}{K}e^{\widetilde{I_t}\widetilde{K_t}} - \delta\right)^2 Ke^{\widetilde{K_t}} + (1-\delta)Ke^{\widetilde{K_t}},$$
and
$$K\left(1 + \widetilde{K_{t+1}}\right) = I\left(1 + \widetilde{I_t}\right) - \frac{1}{2}c\left[\frac{I}{K}\left(1 + \widetilde{I_t} - \widetilde{K_t}\right) - \delta\right]^2 K\left(1 + \widetilde{K_t}\right) + (1-\delta)K\left(1 + \widetilde{K_t}\right).$$
At the steady state, the fact that $K = I - \frac{1}{2}c\left(\frac{I}{K} - \delta\right)^2 K + (1-\delta)K$ implies that $K\widetilde{K_{t+1}} = I\widetilde{I_t} - cI\left(\frac{I}{K} - \delta\right)\left(\widetilde{I_t} - \widetilde{K_t}\right) + [K - I - (1-\delta)K]\widetilde{K_t} + (1-\delta)K\widetilde{K_t}$ and
$$\widetilde{K_{t+1}} - \widetilde{K_t} = \left[\frac{I}{K} - c\frac{I}{K}\left(\frac{I}{K} - \delta\right)\right]\left(\widetilde{I_t} - \widetilde{K_t}\right).$$

In the equilibrium, $I/K = \delta$, so that $\widetilde{K_{t+1}} = \delta \widetilde{I_t} + (1-\delta)\widetilde{K_t}$.

9. Interest rate rule is given by $1 + r_t^n = (1+r)\left(\frac{P_t}{P_{t-1}}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_t^*}\right)^{\phi_y} e^{v_t}$. Then, $\ln(1+r_t^n) = \ln(1+r) + \phi_\pi (\ln P_t - \ln P_{t-1}) + \phi_y (\ln Y_t - \ln Y_t^*) + v_t$, which yields $r_t^n = r + \phi_\pi \left(\widetilde{P_t} - \widetilde{P_{t-1}}\right) + \phi_y \left(\widetilde{Y_t} - \widetilde{Y_t^*}\right) + v_t$. Since in the equilibrium, $r = \rho$, thus, $r_t^n = \rho + \phi_\pi \pi_t + \phi_y \left(\widetilde{Y_t} - \widetilde{Y_t^*}\right) + v_t$.

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