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Relative controllability of Langevin delayed fractional system with multiple delays in control

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Abstract. A Langevin delayed fractional system with multiple delays in control, is a delayed fractional system that includes delay parameters in both state and control, is first introduced. This paper is devoted to investigating the relative controllability of the Langevin delayed fractional system with multiple delays in control. For linear systems to be relatively controllable, necessary and sufficient circumstances are identified by introducing and employing the Gramian matrix. The sufficient conditions for the relative controllability of semilinear systems are offered based on Schauder's fixed point theorem. As an unusual approach, the controllability results of the delayed system are built for the first time on the exact solution produced by the Mittag-Leffler type function although controllability ones in the literature are built on the Volterra integral equations or the mild solutions produced by resolvent families.

§1 Introduction

Undoubtedly, ordinary calculus has never reached the level of development it has reached today. While this shows us how much ordinary calculus contributes to science, it also shows how inadequate it represents the new world problems. We observe that this fundamental gap is filled by scientists with fractional calculus, which owes its existence only to an innocent sense of curiosity and is an expansion of ordinary calculus. The ability of fractional calculus to model real-world problems is better than ordinary calculus and pushes researchers to work in this field and to discover aspects that have not yet been discovered. This has led fractional calculus to be used in many areas such as mathematical physics, engineering, biophysics [1, 2, 7–11], etc; and in many kinds of applications as dynamics of interfaces between substrates and nanoparticles [12], signal processing [13], circuit theory [14], earthquakes [15], etc. Moreover, there has been so many kinds of different fractional definitions such as Riemman-Liouville, Caputo, Grünwald, Hadamard, cotangent fractional derivatives [68, 69], etc. In recent times, the conformable fractional derivatives has been defined and improved in many aspects [66,67,71].

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The Caputo fractional derivative is often considered superior in many applications due to its compatibility with classical initial conditions, making it more intuitive and practical for modeling real-world problems. Unlike the Riemann-Liouville derivative, which requires initial conditions to be defined in terms of fractional integrals, the Caputo derivative uses standard integer-order derivatives, aligning naturally with traditional physical interpretations. This simplifies the formulation and solution of fractional differential equations, particularly in engineering and physics, where initial states are typically specified in classical terms. Additionally, the Caputo derivative ensures a smooth transition between fractional and integer-order models, providing flexibility and consistency in analysis. Its numerical implementation is also more straightforward, facilitating its widespread use in applied fields. This combination of practical advantages makes the Caputo fractional derivative a preferred choice for many researchers and practitioners.

Control systems in which the notion of controllability plays an important role cause the production of control theory. Numerous researchers have substantially investigated both ordinary differential equations and fractional differential equations in terms of controllability concepts. Dauer and Gahl [17] managed to obtain controllability results for nonlinear equations incorporating delays. Balachandran and Dauer [18] investigated the controllability of linear and semilinear equations having delays. Balachandran [16, 19] examined the relative controllability results for nonlinear fractional equations with both distributional delays and multiple delays in control. Klamka [20, 21] determined controllability results for linear and nonlinear systems with time-varying delays in control. In recent times, Mur et al. [22] proved fractional-order linear systems with delays relatively controllable.

The Langevin delayed fractional system is a model that combines the Langevin equation with fractional calculus and time delays. It captures systems with memory effects, where past states influence future behavior, and is used to describe complex phenomena in various fields like physics and biology. The inclusion of fractional derivatives allows for more accurate modeling of anomalous diffusion and delayed feedback. Langevin delayed fractional system with multiple delays in control is applied in real-world systems where memory effects and time delays significantly influence dynamics. Examples include biological systems (e.g., population dynamics, neural networks), engineering (e.g., robotics, control systems), and finance (e.g., stock market modeling). The multiple delays account for the influence of past states at different time intervals, improving the accuracy of system predictions and control strategies in complex, real-world environments. More recently, Kaushik et al. [72] have offered new results on controllability analysis of nonlinear fractional order integrodifferential Langevin system with multiple delays. Controllability [73, 78] of Hilfer fractional Langevin evolution equations has been researched. Jothimani et al. [74] have worked on the controllability of the Hilfer-Langevin system via an integral contractor approach. Prabu et al. [75] have examined the controllability of nonlinear fractional Langevin systems using Ψ-Caputo fractional derivative. The controllability of fractional Langevin impulsive system [76], fractional delay integrodifferential Langevin systems [77], and fractal linear dynamical systems [70] have been studied.

A differential delayed equation is composed of the present state, the past state, and the derivatives of the present states. Minorsky and Volterra employed such differential delayed equations in their works such as viscoelasticity, predator-prey, automatic steering, and ship stabilization [23–25]. The common point of these equations is that the states only contain the delay parameters. Numerous works [26–51] exist about these sorts of differential delayed systems. But so few works exit differential delayed system with a delay or multiple delays in an admissible control. Such systems are mostly studied in terms of controllability [17,52–62]. It is significant for these equations to drive between the relative controllability in Euclidean space and function controllability concepts. This distinction emerges since the natural state space is originally a function space even though the solutions of such equations are trajectories in Euclidean space. For the aims of this paper, our discussion is limited to relative controllability.

The Langevin differential equations have been employed to describe Brownian motions and quite a number of the stochastic processes in fluctuant environments [63, 64]. Langevin differential equations are inadequate in modeling some of today's sophisticated problems. For this reason, Langevin-type differential equations are in need of such various generalizations that they have the ability to describe physical processes [65] more appropriately. Undoubtedly, one of them is the Langevin-type fractional delayed equations with two different fractional orders, which include both the delay and the fractional derivatives. According to our observations, except for a few studies, there is almost no work on such equations. Considering the importance of the fractional differential equation and the delayed differential equation, as well as the inadequacy of studies on Langevin-type differential equations, we will dedicate this paper to the investigation of relative controllability of the following nonlinear Langevin delayed fractional system with multiple delays in control, for $\varsigma \in [0,T]$ with $T>0$,

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\rho(\varsigma) - \mu\,{}^{\mathfrak{C}}\mathfrak{D}^{\beta}_{0+}\rho(\varsigma) - \lambda\rho(\varsigma-\tau) = \sum_{k=0}^{d}\sigma_k \upsilon(\tau_k(\varsigma)) + \daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma)), \\ \rho(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-\tau,0], \quad \tau>0, \end{cases} \tag{1}$$

where ${}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}$ and ${}^{\mathfrak{C}}\mathfrak{D}^{\beta}_{0+}$ stand for the Caputo derivatives of fractional orders $1<\alpha\leq 2$ and $0<\beta\leq 1$. $\rho\in\mathbb{R}^n$, $\upsilon\in\mathbb{R}^m$, $\mu,\lambda,\sigma_k\in\mathbb{R}$, $k=0,1,2,\ldots,d$, d is the number of delays in control function, and $\tau_k:[0,T]\to\mathbb{R}$, $k=0,1,2,\ldots,d$, is a delayed function in control function. τ is a delay in the state function. The initial function $\phi:[-h,0]\to\mathbb{R}^n$ is continuous and the nonlinear function $\daleth:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is also continuous.

In addition, the controllability of Langevin delayed fractional systems with multiple delays in control is crucial for understanding and managing complex dynamic behaviors in various real-world applications. These systems, governed by fractional differential equations, incorporate memory and hereditary properties, making them highly suitable for modeling processes in fields such as physics, biology, finance, and engineering. The inclusion of delays further enhances the model's realism by accounting for the inherent time lags in real systems, such as response times in feedback mechanisms or propagation delays in communication networks. Controllability ensures that it is possible to steer the system from any initial state to a desired final state within a finite time using appropriate control inputs, even in the presence of these delays. This capability is fundamental for the effective design and implementation of control strategies, as it provides theoretical guarantees that the desired performance objectives can be achieved despite the system's complexity. Furthermore, the controllability analysis of such systems aids in optimizing control parameters, ensuring stability, and improving robustness against disturbances, making it an indispensable aspect of modern control theory and practical system design.

Unfortunately, the given information in the statement of system (1) remains incapable of proving the relative controllability of the linear or nonlinear version. So we need additional assumptions which are stated below:

A1) $\tau_k : [0,T] \to \mathbb{R}$ with $\tau_k(\varsigma) \le \varsigma$, $\varsigma \in [0,T]$, $k = 0,1,2,\dots,d$, are such functions that they are strictly increasing and continuously differentiable two times.

A2) The following information holds true

$$\tau_d(T) \le \tau_{d-1}(T) \le \dots \le \tau_{m+1}(T) \le 0 = \tau_m(T) < \tau_{m-1}(T),$$
$$\tau_{m-1}(T) = \tau_{m-2}(T) = \cdots = \tau_2(T) = \tau_1(T) = \tau_0(T) = T,$$
$$\tau_0(t) = t, \quad \varsigma \in [0,T].$$

A3) Time lead functions $h_k : [\tau_k(0), \tau_k(T)] \to [0,T]$, $k = 0,1,2,\dots,d$ are given $h_k(\tau_k(\varsigma)) = \varsigma$, $\varsigma \in [0,T]$, $k = 0,1,2,\dots,d$.

A4) With $\upsilon : [-\tau, T] \to \mathbb{R}^m$ being a function, the function υ_ς, $\varsigma \in [0,T]$ is given by $\upsilon_\varsigma(s) = \upsilon(\varsigma + s)$, $s \in [-\tau, 0]$.

In the current paper, the contributions are stated as follows:

i) We give representations of not only an exact solution of the linear version but also a global solution of the nonlinear version of system (1) in terms of determining functions.

ii) We offer necessary and sufficient circumstances for the relative controllability of the linear version of system (1) by introducing the Gramian matrix.

iii) We transform the relative controllability problem for the nonlinear version of system (1) to a fixed point problem, which allows us to exploit the Schauder fixed point theorem to show the accuracy of our main findings.

§2 Short preliminaries

In this section, we share a few necessary tools to be used in the forthcoming sections.

$\mathbb{R}$ represents the set of real numbers. $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers.

Definition 1. *[1], [2] The Caputo derivative* ${}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\rho(\varsigma)$ *of fractional order* $n-1 < \alpha < n$ *is defined by*

$$ {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\rho(\varsigma) = \int_0^{\varsigma} \frac{(\varsigma - s)^{n-\alpha-1}\rho^{(n)}(s)}{\Gamma(n-\alpha)} ds, \qquad \varsigma > 0,$$

where $\Gamma(\cdot)$ *is the famous Gamma function and the function* $\rho(\varsigma)$ *has absolutely continuous derivatives up to order* $(n-1)$.

Definition 2. *[3] A (control) function* $\upsilon(t) \in \mathbb{R}^m$ *is called admissible provided that it is both measurable and bounded on a finite interval.*

Definition 3. *The system (1) is called relatively controllable if, for an initial control function $\upsilon_0(\varsigma)$, $\varsigma \in [-\tau, 0]$, an initial function $\phi(\varsigma)$, $\varsigma \in [-\tau, 0]$, and the final state $\rho_T \in \mathbb{R}^n$ with time T, then there is such an admissible control $\upsilon(\varsigma)$, $\varsigma \in [0, T]$ that the corresponding solution $\rho(\varsigma)$, $\varsigma \in [-\tau, T]$ to system (1) fulfills $\rho(T) = \rho_T$ and $\rho(\varsigma) = \phi(\varsigma)$, $\varsigma \in [-\tau, 0]$.*

Lemma 4. *[4, Proposition 1] If the locally bounded function $\urcorner$ in $\mathbb{R}^n \times \mathbb{R}^m$ fulfills $\lim_{|(\rho,\upsilon)|\to\infty} \frac{|\urcorner(\varsigma,\rho,\upsilon)|}{|(\rho,\upsilon)|} = 0$ uniformly in $[0, T]$, then for every pair of constants δ and γ, there is a constant $\varepsilon > 0$ such that if $\|(\rho, \upsilon)\| \leq \varepsilon$, then $\delta |\urcorner(\varsigma, \rho, \upsilon)| + \gamma \leq \varepsilon$ for all $\varsigma \in [0, T]$.*

§3 Controllability of the linear version of system (1)

In this section, we first try to define the Gramian matrix. In the sequel, we will introduce such an admissible control function including the Grammian matrix that with the help of this control, one can easily prove the relative controllability of the linear version of system (1), which is expressed as follows:

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\rho(\varsigma) - \mu {}^{\mathfrak{C}}\mathfrak{D}^{\beta}_{0+}\rho(\varsigma) - \lambda\rho(\varsigma - \tau) = \sum_{k=0}^{d} \sigma_k \upsilon(\tau_k(\varsigma)), \quad \varsigma \in [0, T], \\ \rho(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-\tau, 0], \quad \tau > 0, \end{cases} \tag{2}$$

where all of the information is given in (1). Based on [5, Theorem 4.2.], the exact solution of system (2) can be expressed as

$$\rho(\varsigma) = \eta_1(\varsigma) + \sum_{k=0}^{d} \int_0^{\varsigma} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu, \lambda; \varsigma - s)\sigma_k \upsilon(\tau_k(s))ds,$$

where

$$\begin{aligned} \eta_1(\varsigma) &= \left(1 + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\mu, \lambda; \varsigma - \tau)\right)\phi(0) + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu, \lambda; \varsigma)\phi'(0) \\ &+ \lambda \int_{-\tau}^{\min\{\varsigma-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu, \lambda; \varsigma - \tau - s)\phi(s)ds, \end{aligned}$$

and the delayed analogue of M-L type function of three parameters [6] is given as follows:

$$\mathbb{E}^{\tau}_{\alpha,\beta,\gamma}(\mu, \lambda; \varsigma) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\lambda^i \mu^j}{\Gamma(i\alpha + j\beta + \gamma)} (\varsigma - i\tau)^{i\alpha+j\beta+\gamma-1} \mathcal{H}(\varsigma - i\tau),$$

where, $\mathcal{H}(\varsigma)$ is the known heaviside function. In order to introduce the so-called Gramian matrix, we will apply the following steps to make $\upsilon(\varsigma)$, $\varsigma \in [0, T]$ more visible in this solution.

Firstly, one should apply the substitution $x = \tau_i(s)$, then the solution is transformed into the below form

$$\rho(\varsigma) = \eta_1(\varsigma) + \sum_{k=0}^{d} \int_{\tau_i(0)}^{\tau_i(\varsigma)} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu, \lambda; \varsigma - h_k(x))\sigma_k h_k'(x)\upsilon(x)dx.$$

Secondly, one should apply the inequalities and equalities in (A2), then the solution is transferred into the following form

$$\rho(T) = \eta_1(T) + \sum_{k=0}^{m} \int_{\tau_i(0)}^{\tau_i(T)} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu, \lambda; T - h_k(s))\sigma_k h_k'(s)\upsilon(s)ds$$

$$\begin{aligned}&+\sum_{k=m+1}^{d}\int_{\tau_i(0)}^{\tau_i(T)}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-h_k(s))\sigma_k h_k^{'}(s)\upsilon(s)ds\\
&=\eta_1(T)+\sum_{k=0}^{m}\int_{0}^{T}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-h_k(s))\sigma_k h_k^{'}(s)\upsilon(s)ds\\
&+\sum_{k=0}^{m}\int_{\tau_i(0)}^{0}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-h_k(s))\sigma_k h_k^{'}(s)\upsilon_0(s)ds\\
&+\sum_{k=m+1}^{d}\int_{\tau_i(0)}^{\tau_i(T)}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-h_k(s))\sigma_k h_k^{'}(s)\upsilon_0(s)ds.\end{aligned}$$

For simplicity,

$$\begin{aligned}\eta_2(\varsigma)=&\sum_{k=0}^{m}\int_{\tau_i(0)}^{0}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma-h_k(s))\sigma_k h_k^{'}(s)\upsilon_0(s)ds\\
&+\sum_{k=m+1}^{d}\int_{\tau_i(0)}^{\tau_i(\varsigma)}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma-h_k(s))\sigma_k h_k^{'}(s)\upsilon_0(s)ds,\end{aligned}$$

and

$$\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(\varsigma,s)=\sum_{k=0}^{m}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma-h_k(s))\sigma_k h_k^{'}(s).$$

Then, one can rewrite the solution as follows:

$$\rho(T)=\eta_3(T)+\int_0^T\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\upsilon(s)ds,$$

where $\eta_3(\varsigma)=\eta_1(\varsigma)+\eta_2(\varsigma)$. Now, we can introduce the Gramian matrix as noted below

$$G(0,T)=\int_0^T\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\right]^*ds,$$

where the star sign * stands for the transpose of a matrix.

The following theorem states necessary and sufficient circumstances for controllability of system (2).

Theorem 5. *The nonsingularity of the Gramian matrix requires that system (2) is relatively controllable, and vice versa.*

Proof. The nonsingularity of the Gramian matrix $G:=G(0,T)$ provides that the inverse of the Gramian matrix G^{-1} exists. Then one can introduce the well-defined control function as stated below

$$\upsilon(\varsigma)=\left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma)\right]^*G^{-1}\left[\rho_T-\eta_3(T)\right].$$

It is so easy to verify the relative controllability of system (2) as follows based on the Definition 3

$$\begin{aligned}\rho(T)&=\eta_3(T)+\int_0^T\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\right]^*\ G^{-1}\left[\rho_T-\eta_3(T)\right]ds\\
&=\eta_3(T)+GG^{-1}\left[\rho_T-\eta_3(T)\right]\\
&=\rho_T.\end{aligned}$$

The first part of the proof is finished. To prove the second part that the Gramian matrix is nonsingular by the method of reductio ad absurdum, assume that system (2) is relatively controllable and the Gramian matrix is singular. Due to the singularity of G, there is such a nonzero real vector $b \in \mathbb{R}^n$ that $Gb = 0$. Using this, one can get

$$b^* G b = 0 = \int_0^T b^* \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s) \left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\right]^* b ds,$$

and then

$$b^* \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma) = 0, \quad 0 \le \varsigma \le T. \tag{3}$$

Based on the Definition 3, under the initial functions there are such different admissible controls $\upsilon_1(\varsigma)$, $\upsilon_2(\varsigma)$, $\varsigma \in [0,T]$ that the corresponding solution $\rho(\varsigma)$, $\varsigma \in [-\tau, T]$ to system (1) fulfills $\rho(T) = 0$ and $\rho(T) = b$, respectively. This means that

$$\rho(T) = \eta_3(T) + \int_0^T \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\upsilon_1(s) ds = 0, \tag{4}$$

and

$$\rho(T) = \eta_3(T) + \int_0^T \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\upsilon_2(s) ds = b. \tag{5}$$

It can be acquired from equations (4) and (5) that

$$b^* b = \int_0^T \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\left(\upsilon_2(s) - \upsilon_1(s)\right) ds.$$

By means of the information (3), $b^* b = 0$ which gives $b = 0$. This obtained result contradicts with $b \neq 0$. □

§4 Controllability of the nonlinear version of system (1)

In the current section, we will offer sufficient conditions via a main theorem to prove the relative controllability of the nonlinear version of system (1) which is expressed as noted below

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\rho(\varsigma) - \mu {}^{\mathfrak{C}}\mathfrak{D}^{\beta}_{0+}\rho(\varsigma) - \lambda\rho(\varsigma - \tau) = \sum_{k=0}^{d} \sigma_k \upsilon(\tau_k(\varsigma)) + \urcorner(\varsigma, \rho(\varsigma), \upsilon(\varsigma)), \\ \rho(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-\tau, 0], \quad \tau > 0, \end{cases} \tag{6}$$

where all of the information is granted in (1).

Let $C([0,T],\mathbb{R}^n)$ be the Banach space of all continuous function endowed with $\|\rho\| = \sup\{|\rho(\varsigma)| : \varsigma \in [0,T]\}$ and $U = C([0,T],\mathbb{R}^n) \times C([0,T],\mathbb{R}^m)$ be the Banach space given by the uniform norm $\|(\rho,\upsilon)\| = \|\rho\| + \|\upsilon\|$.

Based on [5, Theorem 4.2.], the global solution of system (6) can be expressed as

$$\begin{aligned} \rho(\varsigma) = \eta_1(\varsigma) &+ \sum_{k=0}^{d} \int_0^{\varsigma} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma - s)\sigma_k \upsilon(\tau_k(s)) ds \\ &+ \int_0^{\varsigma} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma - s)\urcorner(s,\rho(s),\upsilon(s)) ds. \end{aligned}$$

As done in the previous section, the substitution $x = \tau_i(s)$ and (A2) transform the global solution to the below form

$$\rho(T)=\eta_3(T)+\int_0^T \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\upsilon(s)ds + \int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s)\urcorner(s,\rho(s),\upsilon(s))ds.$$

Assume that the function pair (ρ,υ) generates a solution pair to the set of the below nonlinear equations

$$\upsilon(\varsigma)=\left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma)\right]^* G^{-1}\left[\rho_T-\eta_3(T)-\int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s) \times \urcorner(s,\rho(s),\upsilon(s))ds\right],$$

$$\rho(\varsigma)=\eta_3(\varsigma)+\int_0^\varsigma \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(\varsigma,s)\upsilon(s)ds+\int_0^\varsigma \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma-s) \times \urcorner(s,\rho(s),\upsilon(s))ds,$$

where it is supposed that ρ is a solution to the system (6) corresponding to the control function υ. So, it can be easily confirmed that

$$\begin{aligned}\rho(T)&=\eta_3(T)+\int_0^T \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,s)\left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,T)\right]^* ds \\ &\quad\times G^{-1}\left[\rho_T-\eta_3(T)-\int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s)\urcorner(s,\rho(s),\upsilon(s))ds\right] \\ &\quad+\int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s)\urcorner(s,\rho(s),\upsilon(s))ds \\ &=\eta_3(T)+GG^{-1}\left[\rho_T-\eta_3(T)-\int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s)\urcorner(s,\rho(s),\upsilon(s))ds\right] \\ &\quad+\int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s)\urcorner(s,\rho(s),\upsilon(s))ds \\ &=\rho_T.\end{aligned}$$

We will identify sufficient circumstances which are stated in the following theorem to guarantee the existence of a solution pair to the set of the just-above equations.

Theorem 6. *Let* $\urcorner$ *be such a continuous function such that it satisfies uniformly in* $[0,T]$

$$\lim_{|(\rho,\upsilon)|\to\infty}\frac{|\urcorner(\varsigma,\rho,\upsilon)|}{|(\rho,\upsilon)|}=0.$$

The nonlinear system (6) is relatively controllable provided that the linear system (2) is relatively controllable.

Proof. In order to transfer the relative controllability problem into the Schauder fixed point problem, we need to define the following operator $\mathcal{P}:U\to U$ by $\mathcal{P}(\rho,\upsilon)=(z,\nu)$ where

$$\nu(\varsigma)=\left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma)\right]^* G^{-1}\left[\rho_T-\eta_3(T)-\int_0^T \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s) \times \urcorner(s,\rho(s),\upsilon(s))ds\right],$$

$$z(\varsigma) = \eta_3(\varsigma) + \int_0^\varsigma \mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(\varsigma,s)\nu(s)ds + \int_0^\varsigma \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma-s)$$
$$\times \daleth(s,\rho(s),\upsilon(s))ds.$$

It is known from [5, Lemma 5.1.] that $\|\mathbb{E}^{\tau}_{\alpha,\beta,\gamma}(\mu,\lambda;\varsigma)\| \leq \varsigma^{\gamma-1}e^{|\lambda|\varsigma^{\alpha}+|\mu|\varsigma^{\beta}}$. For simplicity, set:

$$c := \max\{T\|\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,0)\|, 1\},$$
$$\delta_1 := 4c\|[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,0)]^*\|\|G^{-1}\|T^{\alpha}e^{|\lambda|T^{\alpha}+|\mu|T^{\alpha-\beta}},$$
$$\gamma_1 := 4c\|[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,0)]^*\|\|G^{-1}\|(\|\rho_T\| + 2\|\eta_3(T)\|),$$
$$\delta_2 := 4\max\{T\|\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,0)\|, T^{\alpha}e^{|\lambda|T^{\alpha}+|\mu|T^{\alpha-\beta}}\},$$
$$\gamma_2 := 4\|\eta_3(T)\|,\ \delta := \max\{\delta_1,\delta_2\},\ \gamma := \{\gamma_1,\gamma_2\}.$$

By means of Lemma 4, there is such $\varepsilon > 0$ that $\delta\,|\daleth(\varsigma,\rho,\upsilon)| + \gamma \leq \varepsilon$ for all $\varsigma \in [0,T]$ provided that $\|(\rho,\upsilon)\| \leq \varepsilon$. First of all, it should be proved that $\mathcal{P}(B_\varepsilon) \subset B_\varepsilon$ where $B_\varepsilon = \{(\rho,\upsilon) \in U : \|(\rho,\upsilon)\| \leq \varepsilon\}$. Assume that $(\rho,\upsilon) \in U$. Then

$$\begin{aligned}
\|\nu(\varsigma)\| \leq & \left\| \left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma)\right]^* \right\| \|G^{-1}\|(\|\rho_T\| + \|\eta_3(T)\|) \\
& + \|[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,0)]^*\|\|G^{-1}\|T^{\alpha}e^{|\lambda|T^{\alpha}+|\mu|T^{\alpha-\beta}} \sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))| \\
& \leq \gamma_1(4c)^{-1} + \delta_1(4c)^{-1}\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))| \\
& \leq (4c)^{-1}(\gamma + \delta\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))|) \\
& \leq (4c)^{-1}\varepsilon \leq \frac{\varepsilon}{4},
\end{aligned}$$

and

$$\begin{aligned}
\|z(\varsigma)\| = & \|\eta_3(\varsigma)\| + T\|\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,0)\|\|\nu\| \\
& + T^{\alpha}e^{|\lambda|T^{\alpha}+|\mu|T^{\alpha-\beta}}\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))|) \\
& \leq \frac{\gamma_2}{4} + c\|\nu\| + \frac{\delta_2}{4}\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))|) \\
& \leq \frac{1}{4}(\gamma + \delta\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))|) + \frac{\varepsilon}{4} \\
& \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

Due to $\|(z,\nu)\| = \|z(\varsigma)\| + \|\nu(\varsigma)\| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \frac{3\varepsilon}{4} \leq \varepsilon$, the desired result $\mathcal{P}(B_\varepsilon) \subset B_\varepsilon$ is obtained. Now, we will prove that $\mathcal{P}(B_\varepsilon)$ is equicontinuous. Take arbitrary elements $\varsigma_1,\varsigma_2 \in [0,T]$ and for all $(z,\nu) \in B_\varepsilon$, consider

$$\begin{aligned}
& \|\nu(\varsigma_1) - \nu(\varsigma_2)\| \\
& \leq \left\| \left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma_1)\right]^* - \left[\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(T,\varsigma_2)\right]^* \right\| \|G^{-1}\| \Big[\|\rho_T\| \\
& \quad + \|\eta_3(T)\| + \int_0^T \|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;T-s)\|\|\daleth(s,\rho(s),\upsilon(s))\|ds\Big],
\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
&\|z(\varsigma_1)-z(\varsigma_2)\| \\
&\le |\lambda|\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\mu,\lambda;\varsigma_1-\tau)-\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\mu,\lambda;\varsigma_2-\tau)\|\|\phi(0)\| \\
&\quad+\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_1)-\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2)\|\|\phi^{'}(0)\| \\
&\quad+|\lambda|\int_{\min\{\varsigma_1-\tau,0\}}^{\min\{\varsigma_2-\tau,0\}}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-\tau-s)\|\|\phi(s)\|ds \\
&\quad+|\lambda|\int_{-\tau}^{\min\{\varsigma_1-\tau,0\}}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_1-\tau-s)-\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-\tau-s)\| \\
&\quad\times\|\phi(s)\|ds \\
&\quad+\sum_{k=0}^{m}\int_{\tau_i(0)}^{0}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_1-h_k(s))-\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-h_k(s))\| \\
&\quad\times\|\sigma_k\|\|h_k^{'}(s)\|\|\upsilon_0(s)\|ds \\
&\quad+\sum_{k=m+1}^{d}\int_{\tau_i(\varsigma_1)}^{\tau_i(\varsigma_2)}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-h_k(s))\|\|\sigma_k\|\|h_k^{'}(s)\|\|\upsilon_0(s)\|ds \\
&\quad+\sum_{k=m+1}^{d}\int_{\tau_i(0)}^{\tau_i(\varsigma_1)}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_1-h_k(s))-\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-h_k(s))\| \\
&\quad\times\|\sigma_k\|\|h_k^{'}(s)\|\|\upsilon_0(s)\|ds \\
&\quad+\int_{\varsigma_1}^{\varsigma_2}\|\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(\varsigma_2,s)\|\|\nu(s)\|ds \\
&\quad+\int_{0}^{\varsigma_1}\|\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(\varsigma_1,s)-\mathcal{X}^{\mu,\lambda,\tau}_{\alpha,\alpha-\beta,\alpha}(\varsigma_2,s)\|\|\nu(s)\|ds \\
&\quad+\int_{0}^{\varsigma_1}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_1-s)-\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-s)\|ds\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))| \\
&\quad+\int_{\varsigma_1}^{\varsigma_2}\|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\mu,\lambda;\varsigma_2-s)\|ds\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\rho(\varsigma),\upsilon(\varsigma))|. \qquad (8)
\end{aligned}
$$

The right hand side of the inequalities (7) and (8) is independent of $(\rho,\upsilon)\in B_\varepsilon$ and goes to zero as $\varsigma_1\to\varsigma_2$. This implies that $\mathcal{P}$ is equicontinuous, and hence is completely continuous by the application of Arzela-Ascoli's theorem. Because B_ε is nonempty, closed, bounded, and convex, the Schauder fixed point theorem gives that $\mathcal{P}$ has a fixed point in B_ε. □

§5 Numerical and simulated exemplifications

In this section, we illustrate our theoretical results by means of the following examples. The numerical computations were performed and the graphics were drawn using [Mathematica] on a [Computer Specifications: Intel Core i7 processor with 16 GB RAM] running [Operating System: Windows 10].

Example 7. *We will consider the following nonlinear Langevin delayed fractional system with* *multiple delays in control, for* $\varsigma \in [0,3]$,

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{1.5}_{0^+}\rho(\varsigma) - 3\,{}^{\mathfrak{C}}\mathfrak{D}^{0.8}_{0^+}\rho(\varsigma) - 2\rho(\varsigma - 0.2) = 3\upsilon(\varsigma) + 4\upsilon(\varsigma - 2), \\ \rho(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-0.2, 0], \end{cases} \tag{9}$$

where the initial function $\phi(\varsigma) = 0$, *the nonlinear function* $\daleth(\varsigma, \rho(\varsigma), \upsilon(\varsigma)) = 0$. *Here,*

$$\mathcal{X}^{3,2,0.2}_{1.5,0.7,1.5}(\varsigma, s) = 3\mathbb{E}^{0.2}_{1.5,0.7,1.5}(3,2;\varsigma) + 4\mathbb{E}^{0.2}_{1.5,0.7,1.5}(3,2;\varsigma - (s+2)).$$

The Gramian matrix is as noted below

$$G(0,3) = \int_0^3 \mathcal{X}^{3,2,0.2}_{1.5,0.7,1.5}(3,s)\left[\mathcal{X}^{3,2,0.2}_{1.5,0.7,1.5}(3,s)\right]^* ds = 398.162,$$

which is nonzero, so it is nonsingular. Based on Theorem 5, system (9) is relatively controllable. The graphs of the control function $\upsilon(\varsigma)$ *and the solution function* $\rho(\varsigma)$ *corresponding to the control function are given in Figure 1.*

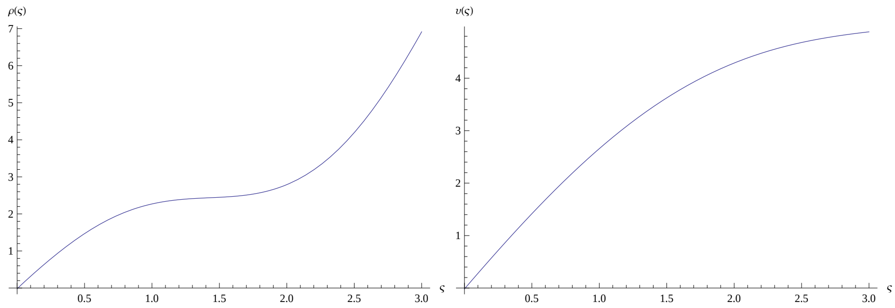

Figure 1. Graphs of the control $\upsilon(\varsigma)$ and the solution $\rho(\varsigma)$.

Remark 8. *It is clear that system (9) is linear. So we prove that the corresponding Gramian matrix is invertible, and then by Theorem 5, we show system (9) relatively controllable. The admissible control steering system (9) from the initial control function* $\upsilon_0(\varsigma) = 0$, $\varsigma \in [-0.2, 0]$, *an initial function* $\phi(\varsigma) = 0$, $\varsigma \in [-0.2, 0]$ *to the final state* $\rho_3 = 1 + \eta_3(3) \in \mathbb{R}$ *is given by the following formula*

$$\upsilon(\varsigma) = \left[\mathcal{X}^{3,2,0.2}_{1.5,0.7,1.5}(3,\varsigma)\right]^* G^{-1}.$$

Moreover, the solution function $\rho(\varsigma)$ *corresponding to the control function* $\upsilon(\varsigma)$ *is offered by*

$$\rho(\varsigma) = \eta_3(\varsigma) + \int_0^\varsigma \mathcal{X}^{3,2,0.2}_{1.5,0.7,1.5}(\varsigma, s)\upsilon(s)ds.$$

The solution pair (ρ, υ) *to system (9) is drawn in Figure 1.*

Example 9. *We will investigate the following nonlinear Langevin delayed fractional system with multiple delays in control, for* $\varsigma \in [0,5]$,

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\rho(\varsigma) - \mu\,{}^{\mathfrak{C}}\mathfrak{D}^{\beta}_{0^+}\rho(\varsigma) - \lambda\rho(\varsigma - \tau) = \sum_{k=0}^{1}\sigma_k\upsilon(\tau_k(\varsigma)) + \daleth(\varsigma, \rho(\varsigma), \upsilon(\varsigma)), \\ \rho(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-\tau, 0], \quad \tau > 0, \end{cases} \tag{10}$$

where $\alpha = 1.2$ *and* $\beta = 0.3$. $\rho(\varsigma) = [\rho_1(\varsigma) \quad \rho_2(\varsigma)]^*$, $\upsilon(\varsigma) = [\upsilon_1(\varsigma) \quad \upsilon_2(\varsigma)]^*$, $\mu = 3, \lambda = -5, \sigma_1 = 6, \sigma_2 = 1$, $\tau = 1$ $\tau_0(\varsigma) = \varsigma$, $\tau_1(\varsigma) = \varsigma - 1$. *The initial function* $\phi(\varsigma) = [\varsigma + 2 \quad 2\varsigma^2 + 1]^*$ *is* *continuous and the nonlinear function* $\urcorner(\varsigma, \rho(\varsigma), \upsilon(\varsigma)) = [0 \quad \frac{\upsilon+\rho}{1+\upsilon^2+\rho^2}]^*$ *is also continuous. It is easy to compute*

$$\mathcal{X}^{3,-5,1}_{1.2,0.9,1.2}(\varsigma, s) = 6\mathbb{E}^{1}_{1.2,0.9,1.2}(3, -5; \varsigma - s) + \mathbb{E}^{1}_{1.2,0.9,1.2}(3, -5; \varsigma - (s + 1)).$$

The corresponding Gramian matrix is given by

$$G(0, 5) = \int_0^5 \mathcal{X}^{3,-5,1}_{1.2,0.9,1.2}(5, s) \left[\mathcal{X}^{3,-5,1}_{1.2,0.9,1.2}(5, s)\right]^* ds = 481502,$$

which is nonzero, so it is nonsingular. Based on Theorem 5, the linear version of system (10) is relatively controllable. $\urcorner(\varsigma, \rho(\varsigma), \upsilon(\varsigma)) = [0 \quad \frac{\upsilon+\rho}{1+\upsilon^2+\rho^2}]^*$ *is continuous and satisfies the conditions on Theorem 6. As a result, Theorem 6 ensures that the nonlinear system (10) is relatively controllable.*

Remark 10. *The system (10) is nonlinear. It is more complicated than the previous one. According to Theorem 6, there are two main conditions to guarantee that the nonlinear system (10) is relatively controllable. One of them is that the linear part of it is controllable, and the other is the limit condition. We take the help of Theorem 5 to ensure that the linear part of it is controllable as in Example 1. In the sequel, it is demonstrated that the limit requirement built on the nonlinear function is satisfied. In light of Theorem 6, it has emerged that the nonlinear system (10) is relatively controllable.*

§6 Conclusion

In the current paper, we shared the exact and global solutions of the linear and nonlinear system (1), respectively. In the sequel, we defined the Gramian matrix to control the linear version of system (1) relatively and proved the nonlinear system (1) relatively controllable by means of the Schauder fixed point theorem.

This paper is quite comprehensive because all obtained results are also new in the cases of both $\tau = 0$ and $\lambda = 0$ individually. For $\mu = \tau = 0$, the findings in this paper also coincide with those of [16] providing $\lambda = A \in \mathbb{R}$.

As a future study, the obtained results may be extended to the semilinear Langevin delayed fractional systems with distributed delays in an admissible control, also to the semilinear Langevin Sobolev-type evolution equations with multiple delays or distributed delays in an admissible control, and to the semilinear Langevin delayed Sobolev-type evolution equations with multiple delays or distributed delays in an admissible control. Additionally, introducing nonlinearities or stochastic components could improve its application to real-world systems like robotics, ecological models, or financial markets, where delays and uncertainties play a significant role.

Declarations

Conflict of interest The authors declare no conflict of interest.

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