Appl. Math. J. Chinese Univ.
2025, 40(4): 916-936

Fractional Milne-type inequalities by various function classes

Cihan Ünal[1] Fatih Hezenci[2,*] Hüseyin Budak[3,4]

Abstract. The manuscript's authors examine some Milne-type inequalities for various function classes. Firstly, some Milne-type inequalities are established for differentiable convex functions by using Riemann-Liouville integrals. Secondly, we provide some fractional Milne-type inequalities for bounded functions by fractional integrals. Afterwards, we offer several Milne-type inequalities for Lipschitzian functions. Likewise, we offers Milne-type inequalities by fractional integrals of bounded variation. Finally, we demonstrate the correctness of our results by using special cases and examples of the obtained theorems.

§1 Introduction

Numerical integration formulas and their error bounds using different techniques have been investigated by many mathematicians. In order to find the error bounds of numerical integration formulas, mathematical inequalities are studied with a variety of functions including convex, bounded, and Lipschitzian functions and so on. For instance, in papers [1,2], some error bounds have been established for the midpoint and trapezoidal inequalities of numerical integration applying convex functions. The error bounds of Simpson-type inequalities have been established utilizing the convex functions and some of these bounds can be found in papers [3–5]. The paper [6] presents Simpson-type inequalities and their application to quadrature inequalities in numerical analysis. A number of fractional Simpson-type inequalities are examined in the paper [7] for the case of functions whose second derivatives in absolute value are convex. Moreover, in paper [8], several variants of Simpson-type inequalities are studied for the case of differentiable convex functions by generalized fractional integrals. Please see references [9–18] and the cited sources therein for further details.

The three-point Newton-Cotes quadrature rule is followed by Simpson's second rule, which is why evaluations involving three-step quadratic kernels are frequently referred to as Newton-type results. The literature refers to these outcomes as Newton-type inequalities. For instance, in papers [19,20], some error bounds for Newton-type inequalities in numerical integration have also been proved by using the convex functions. For the case of functions whose first derivative

Received: 2024-04-03. Revised: 2024-08-08.
MR Subject Classification: 26D07, 26D10, 26D15, 65D32.
Keywords: Milne-type inequalities, quadrature formulae, open Newton-Cotes formulas.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-5194-9.
*Corresponding author.

in absolute value at a given power is arithmetically-harmonically convex, various Newton-type inequalities are investigated in paper [21]. Likewise, certain Newton-type inequalities based on convexity are presented and some applications for special cases of real functions are also given in paper [23]. Moreover, some of Riemann-Liouville fractional Newton-type inequalities for functions of bounded variation are considered in paper [22]. One may refer to [24-27] as well as the references listed in those sources. Several new estimates of Milne's quadrature rule are obtained by Djenaoui and Meftah [32], specifically for functions whose first derivative is s-convex. Moreover, in paper [29], some error estimations of Milne-type inequalities are presented for functions of bounded variation. Moreover, in papers [30–32], fractional versions of Milne-type inequalities are established by using the differentiable convex functions.

The main purpose of this paper is to establish several Milne-type inequalities for various function classes. The entire research structure takes eight sections including the introduction. In Section 2, there will be a few basic details about the paper. In Section 3, we will establish an essential equality involving Riemann–Liouville integrals. With the help of this equality, some Milne-type inequalities will be proved for differentiable convex functions. Afterwords, in Section 4, some fractional Milne-type inequalities will be investigated for bounded functions by fractional integrals. In Section 5, we will present some fractional Milne-type inequalities for Lipschitzian functions. Moreover, in Section 6, several Milne-type inequalities will be considered by fractional integrals of bounded variation. Furthermore, in Section 7, we will offer the correctness of our results by using special cases and examples of the obtained theorems. Finally, some conclusions of research will be given in Section 8.

§2 Preliminaries

Let's introduce some primary concepts that will be used in the following sections.

(i) The following is the expression for Simpson's quadrature formula, also referred to as Simpson's 1/3 rule

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\, dx \approx \frac{\delta-\sigma}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]; \tag{1}$$

(ii) The definition of Simpson's second formula, also referred to as the Newton-Cotes quadratic formula or Simpson's 3/8 rule (see [33]), is as follows:

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\, dx \approx \frac{\delta-\sigma}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]. \tag{2}$$

Formulas (1) and (2) are applicable to any function $\mathcal{F}$ that possesses a continuous fourth derivative on the interval $[\sigma,\delta]$. The following is one of the most famous Newton-Cotes quadrature techniques that uses a three-point Simpson-type inequality.

Theorem 1. *Note that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a four times continuously differentiable function on (σ,δ), and $\left\|\mathcal{F}^{(4)}\right\|_{\infty}=\sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$. Then, the following inequality holds*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right|\leq\frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\delta-\sigma)^{4}.$$

One of the classical closed-type quadrature rules is the Simpson 3/8 rule, which is founded on the Simpson 3/8 inequality, expressed as follows.

Theorem 2 (See [33]). *If $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a four times continuously differentiable function on (σ,δ), and $\left\|\mathcal{F}^{(4)}\right\|_\infty=\sup\limits_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$, then the following inequality holds*

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_\sigma^\delta\mathcal{F}(x)dx\right| \leq\frac{1}{6480}\left\|\mathcal{F}^{(4)}\right\|_\infty(\delta-\sigma)^4.$$

The well-known *Riemann-Liouville fractional integrals* that are defined as follows.

Definition 1 (See [34, 35]). The Riemann–Liouville integrals $J^\alpha_{\sigma+}\mathcal{F}$ and $J^\alpha_{\delta-}\mathcal{F}$ of order $\alpha>0$ with $\sigma\geq0$ are given by

$$J^\alpha_{\sigma+}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_\sigma^x(x-\xi)^{\alpha-1}\mathcal{F}(\xi)d\xi,\quad x>\sigma,$$

and

$$J^\alpha_{\delta-}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_x^\delta(\xi-x)^{\alpha-1}\mathcal{F}(\xi)d\xi,\quad x<\delta,$$

respectively. Here, $\mathcal{F}$ belongs to $L_1[\sigma,\delta]$ and $\Gamma(\alpha)$ is the Gamma function defining as

$$\Gamma(\alpha):=\int_0^\infty e^{-u}u^{\alpha-1}du.$$

The fractional integral equals to the classical integral for the case of $\alpha=1$.

§3 Fractional Milne-type inequalities for convex functions

In this section, we prove an crucial equality involving Riemann–Liouville integrals. Subsequently, some Milne-type inequalities are established for differentiable convex functions by taking the modulus of the newly established identity. Moreover, we establish some Milne-type inequalities with the help of Hölder and power-mean inequality.

Lemma 1. *Consider that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is an absolutely continuous function (σ,δ) so that $\mathcal{F}'\in L_1[\sigma,\delta]$. Then, the equality*

$$\begin{aligned}&\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\\&\qquad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^\alpha}\left[J^\alpha_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^\alpha_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\\&=\frac{\delta-\sigma}{4}\left[I_1+I_2\right]\end{aligned}$$

is valid. Here, Γ is Euler Gamma function and

$$\begin{cases}I_1=\int\limits_0^{\frac{1}{2}}\xi^\alpha\left[\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right]d\xi,\\ I_2=\int\limits_{\frac{1}{2}}^1\left(\xi^\alpha-\frac{4}{3}\right)\left[\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right]d\xi.\end{cases}$$

Proof. Utilizing the principles of integration by parts, we can easily obtain

$$I_1 = \int_0^{\frac{1}{2}} \xi^{\alpha} \left[\mathcal{F}'\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) - \mathcal{F}'\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] d\xi \tag{3}$$

$$= \frac{2\xi^{\alpha}}{\delta-\sigma} \left[\mathcal{F}\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) + \mathcal{F}\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] \Bigg|_0^{\frac{1}{2}}$$

$$- \frac{2\alpha}{\delta-\sigma} \int_0^{\frac{1}{2}} \xi^{\alpha-1} \left[\mathcal{F}\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) - \mathcal{F}\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] d\xi$$

$$= \frac{1}{(\delta-\sigma)} \frac{1}{2^{\alpha-1}} \left[\mathcal{F}\left(\frac{3\sigma+\delta}{4} \right) + \mathcal{F}\left(\frac{\sigma+3\delta}{4} \right) \right]$$

$$- \frac{2\alpha}{\delta-\sigma} \int_0^{\frac{1}{2}} \xi^{\alpha-1} \left[\mathcal{F}\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) + \mathcal{F}\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] d\xi.$$

In a similar way to the previous procedure, we have

$$I_2 = \int_{\frac{1}{2}}^{1} \left(\xi^{\alpha} - \frac{4}{3} \right) \left[\mathcal{F}'\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) - \mathcal{F}'\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] d\xi \tag{4}$$

$$= -\frac{2}{\delta-\sigma} \left(\frac{1}{2^{\alpha}} - \frac{4}{3} \right) \left[\mathcal{F}\left(\frac{\sigma+3\delta}{4} \right) + \mathcal{F}\left(\frac{3\sigma+\delta}{4} \right) \right] - \frac{4}{3(\delta-\sigma)} \mathcal{F}\left(\frac{\sigma+\delta}{2} \right)$$

$$- \frac{2\alpha}{\delta-\sigma} \int_{\frac{1}{2}}^{1} \xi^{\alpha-1} \left[\mathcal{F}\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) + \mathcal{F}\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] d\xi.$$

Combining (3) with (4) allows us to easily obtain

$$I_1 + I_2 = \frac{8}{3(\delta-\sigma)} \left[\mathcal{F}\left(\frac{\sigma+3\delta}{4} \right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4} \right) \right] - \frac{4}{3(\delta-\sigma)} \mathcal{F}\left(\frac{\sigma+\delta}{2} \right) \tag{5}$$

$$- \frac{2\alpha}{\delta-\sigma} \int_0^1 \xi^{\alpha-1} \left[\mathcal{F}\left(\frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma \right) + \mathcal{F}\left(\frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta \right) \right] d\xi.$$

Let us use the change of the variable $x = \frac{\xi}{2}\delta + \frac{2-\xi}{2}\sigma$ and $y = \frac{\xi}{2}\sigma + \frac{2-\xi}{2}\delta$ for $\xi \in [0,1]$. Then, the equality (5) can be rewritten as follows

$$I_1 + I_2 = \frac{8}{3(\delta-\sigma)} \left[\mathcal{F}\left(\frac{\sigma+3\delta}{4} \right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4} \right) \right] - \frac{4}{3(\delta-\sigma)} \mathcal{F}\left(\frac{\sigma+\delta}{2} \right) \tag{6}$$

$$- \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha+1}} \left[J^{\alpha}_{\frac{\sigma+\delta}{2}-} \mathcal{F}(\sigma) + J^{\alpha}_{\frac{\sigma+\delta}{2}+} \mathcal{F}(\delta) \right].$$

Consequently, multiplying both sides of (6) by $\frac{\delta-\sigma}{4}$ finishes the proof of Lemma 1. □

Theorem 3. *Let us consider that all assumptions of Lemma 1 hold and $|\mathcal{F}'|$ is a convex function on $[\sigma,\delta]$. Then, it yields*

$$\left| \frac{1}{3} \left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4} \right) - \mathcal{F}\left(\frac{\sigma+\delta}{2} \right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4} \right) \right] \right. \tag{7}$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[\ J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\ J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\Bigg|$$
$$\leq\frac{\delta-\sigma}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[|\mathcal{F}'(\sigma)|+|\mathcal{F}'(\delta)|\right].$$

Here,

$$\begin{cases}\Omega_1(\alpha)=\int\limits_0^{\frac{1}{2}}\xi^{\alpha}d\xi=\frac{1}{(\alpha+1)2^{\alpha+1}},\\ \Omega_2(\alpha)=\int\limits_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi=\frac{1}{(\alpha+1)}\left(\frac{1}{2^{\alpha+1}}-1\right)+\frac{2}{3}.\end{cases}$$

Proof. By Lemma 1 and convexity of $|\mathcal{F}'|$, we have

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.$$
$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[\ J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\ J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\Bigg|$$
$$\leq\frac{\delta-\sigma}{4}\left[\int\limits_0^{\frac{1}{2}}|\xi^{\alpha}|\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|d\xi\right.$$
$$\left.+\int\limits_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|d\xi\right]$$
$$\leq\frac{\delta-\sigma}{4}\left[\int\limits_0^{\frac{1}{2}}\xi^{\alpha}\left[\frac{\xi}{2}|\mathcal{F}'(\delta)|+\left(\frac{2-\xi}{2}\right)|\mathcal{F}'(\sigma)|+\frac{\xi}{2}|\mathcal{F}'(\sigma)|+\left(\frac{2-\xi}{2}\right)|\mathcal{F}'(\delta)|\right]d\xi\right.$$
$$\left.+\int\limits_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)\left[\frac{\xi}{2}|\mathcal{F}'(\delta)|+\left(\frac{2-\xi}{2}\right)|\mathcal{F}'(\sigma)|+\frac{\xi}{2}|\mathcal{F}'(\sigma)|+\left(\frac{2-\xi}{2}\right)|\mathcal{F}'(\delta)|\right]d\xi\right]$$
$$=\frac{\delta-\sigma}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[|\mathcal{F}'(\sigma)|+|\mathcal{F}'(\delta)|\right].$$

That is the desired result. □

Theorem 4. *Suppose that all assumptions of Lemma 1 hold. If $|\mathcal{F}'|^q$ is convex on $[\sigma,\delta]$ where $q>1$, then we have the following Milne-type inequality*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right. \tag{8}$$
$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[\ J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\ J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\Bigg|$$
$$\leq\frac{\delta-\sigma}{4}\left\{\left(\frac{1}{\alpha p+1}\left(\frac{1}{2}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}'(\delta)|^q+7|\mathcal{F}'(\sigma)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^q+7|\mathcal{F}'(\delta)|^q}{16}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\int\limits_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\delta)|^q+5|\mathcal{F}'(\sigma)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+5|\mathcal{F}'(\delta)|^q}{16}\right)^{\frac{1}{q}}\right]\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking into account Lemma 1, we can easily get

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right. \\
&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \\
&\le\frac{\delta-\sigma}{4}\left[\int_0^{\frac{1}{2}}|\xi^{\alpha}|\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|d\xi\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|d\xi\right].
\end{aligned}\tag{9}$$

Now, we consider the integrals on the right side of (9). Using well-known Hölder inequality and the convexity of $|\mathcal{F}'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right. \\
&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \\
&\le\frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{1}{2}}|\xi^{\alpha}|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_0^{\frac{1}{2}}|\xi^{\alpha}|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{4}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha}-\frac{4}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|^q d\xi\right)^{\frac{1}{q}}\right\} \\
&\le\frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{1}{2}}\xi^{\alpha p}d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\xi}{2}|\mathcal{F}'(\delta)|^q+\frac{2-\xi}{2}|\mathcal{F}'(\sigma)|^q\right)d\xi\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_0^{\frac{1}{2}}\xi^{\alpha p}d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\xi}{2}|\mathcal{F}'(\sigma)|^q+\frac{2-\xi}{2}|\mathcal{F}'(\delta)|^q\right)d\xi\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{\xi}{2}\left|\mathcal{F}'(\delta)\right|^{q}+\frac{2-\xi}{2}\left|\mathcal{F}'(\sigma)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{\xi}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{2-\xi}{2}\left|\mathcal{F}'(\delta)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\delta-\sigma}{4}\left\{\left(\frac{1}{\alpha p+1}\left(\frac{1}{2}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|\mathcal{F}'(\delta)\right|^{q}+7\left|\mathcal{F}'(\sigma)\right|^{q}}{16}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+7\left|\mathcal{F}'(\delta)\right|^{q}}{16}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'(\delta)\right|^{q}+5\left|\mathcal{F}'(\sigma)\right|^{q}}{16}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+5\left|\mathcal{F}'(\delta)\right|^{q}}{16}\right)^{\frac{1}{q}}\right]\right\}.$$

This completes the proof of Theorem 4. □

Theorem 5. *Assume that all assumptions of Lemma 1 hold. If $|\mathcal{F}'|^{q}$ is convex on $[\sigma,\delta]$ where $q\geq 1$, then we have the following Milne-type inequality*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right. \\ &\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\frac{\sigma+\delta}{2}-}^{\alpha}\mathcal{F}(\sigma)+J_{\frac{\sigma+\delta}{2}+}^{\alpha}\mathcal{F}(\delta)\right]\right| \\ &\leq\frac{\delta-\sigma}{4}\left\{(\Omega_{1}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{3}(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}+\Omega_{4}(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\right. \\ &\quad\left.+\left(\Omega_{3}(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}+\Omega_{4}(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right] \\ &\quad+(\Omega_{2}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{5}(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}+\Omega_{6}(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right. \\ &\quad\left.\left.+\left(\Omega_{5}(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}+\Omega_{6}(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}.\end{aligned} \tag{10}$$

Here $\Omega_{1}(\alpha)$ and $\Omega_{2}(\alpha)$ are defined in Theorem 3 and

$$\begin{cases}\Omega_{3}(\alpha)=\int_{0}^{\frac{1}{2}}\frac{\xi^{\alpha+1}}{2}d\xi=\frac{1}{\alpha+2}\left(\frac{1}{2^{\alpha+3}}\right), \\ \Omega_{4}(\alpha)=\int_{0}^{\frac{1}{2}}\left(\frac{2-\xi}{2}\right)\xi^{\alpha}d\xi=\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{3\alpha+7}{2^{\alpha+3}}\right), \\ \Omega_{5}(\alpha)=\int_{\frac{1}{2}}^{1}\frac{\xi}{2}\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi=\frac{1}{\alpha+2}\left(\frac{1}{2^{\alpha+3}}-\frac{1}{2}\right)+\frac{1}{4}, \\ \Omega_{6}(\alpha)=\int_{\frac{1}{2}}^{1}\left(\frac{2-\xi}{2}\right)\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi=\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{3\alpha+7}{2^{\alpha+3}}-\frac{\alpha+3}{2}\right)+\frac{5}{12}.\end{cases}$$

Proof. Let us consider power mean inequality. Then, it follows

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right. \\ &\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\frac{\sigma+\delta}{2}-}^{\alpha}\mathcal{F}(\sigma)+J_{\frac{\sigma+\delta}{2}+}^{\alpha}\mathcal{F}(\delta)\right]\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{1}{2}}\left|\xi^\alpha\right| d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\xi^\alpha\right|\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^{\frac{1}{2}}\left|\xi^\alpha\right| d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\xi^\alpha\right|\left|\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left|\xi^\alpha-\frac{4}{3}\right| d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\xi^\alpha-\frac{4}{3}\right|\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\xi^\alpha-\frac{4}{3}\right| d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\xi^\alpha-\frac{4}{3}\right|\left|\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|^q d\xi\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Using the convexity of $|\mathcal{F}'|^q$, we have

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right. \\
&\left.\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^\alpha}\left[J^\alpha_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^\alpha_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \\
&\leq \frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{1}{2}}\xi^\alpha d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\xi^\alpha\left[\frac{\xi}{2}\left|\mathcal{F}'(\delta)\right|^q+\frac{2-\xi}{2}\left|\mathcal{F}'(\sigma)\right|^q\right] d\xi\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^{\frac{1}{2}}\xi^\alpha d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\xi^\alpha\left[\frac{\xi}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\xi}{2}\left|\mathcal{F}'(\delta)\right|^q\right] d\xi\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\xi^\alpha\right) d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\xi^\alpha\right)\left[\frac{\xi}{2}\left|\mathcal{F}'(\delta)\right|^q+\frac{2-\xi}{2}\left|\mathcal{F}'(\sigma)\right|^q\right] d\xi\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\xi^\alpha\right) d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\xi^\alpha\right)\left[\frac{\xi}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\xi}{2}\left|\mathcal{F}'(\delta)\right|^q\right] d\xi\right)^{\frac{1}{q}}\right\} \\
&=\frac{\delta-\sigma}{4}\left\{\left(\Omega_1(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Omega_3(\alpha)\left|\mathcal{F}'(\delta)\right|^q+\Omega_4(\alpha)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\Omega_3(\alpha)\left|\mathcal{F}'(\sigma)\right|^q+\Omega_4(\alpha)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right] \\
&+\left(\Omega_2(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Omega_5(\alpha)\left|\mathcal{F}'(\delta)\right|^q+\Omega_6(\alpha)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\Omega_5(\alpha)\left|\mathcal{F}'(\sigma)\right|^q+\Omega_6(\alpha)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

□

§4 Fractional Milne-type inequalities for bounded functions

In this section, we present a fractional Milne-type inequality for bounded functions.

Theorem 6. *Let us consider that the conditions of Lemma 1 hold. If there exist $m, M \in \mathbb{R}$ such that $m \leq \mathcal{F}'(\xi) \leq M$ for $\xi \in [\sigma, \delta]$, then it yields*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.\\ &\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|\\ &\leq\frac{\delta-\sigma}{4}\left\{\frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha}}-1\right)+\frac{2}{3}\right\}(M-m).\end{aligned}\tag{11}$$

Proof. By using Lemma 1, we have

$$\begin{aligned}&\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\\ &\qquad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\\ &=\frac{\delta-\sigma}{4}\left\{\int_0^{\frac{1}{2}}\xi^{\alpha}\left[\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\frac{m+M}{2}\right]d\xi\right.\\ &\quad+\int_0^{\frac{1}{2}}\xi^{\alpha}\left[\frac{m+M}{2}-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right]d\xi\\ &\quad+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha}-\frac{4}{3}\right)\left[\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\frac{m+M}{2}\right]d\xi\\ &\quad\left.+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha}-\frac{4}{3}\right)\left[\frac{m+M}{2}-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right]d\xi\right\}.\end{aligned}\tag{12}$$

Using the absolute value of (12), we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.\\ &\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|\\ &\leq\frac{\delta-\sigma}{4}\left\{\int_0^{\frac{1}{2}}\xi^{\alpha}\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\frac{m+M}{2}\right|d\xi\right.\\ &\quad+\int_0^{\frac{1}{2}}\xi^{\alpha}\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|d\xi\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\frac{m+M}{2}\right|d\xi$$
$$\left.+\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|d\xi\right\}.$$

It is known that $m\leq\mathcal{F}'(\xi)\leq M$ for $\xi\in[\sigma,\delta]$. Thus, we get

$$\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2},\tag{13}$$

and

$$\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right|\leq\frac{M-m}{2}.\tag{14}$$

If we consider (13) and (14), then we get

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\delta-\sigma}{4}(M-m)\left\{\int_{0}^{\frac{1}{2}}\xi^{\alpha}d\xi+\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi\right\}$$
$$=\frac{\delta-\sigma}{4}\left\{\frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha}}-1\right)+\frac{2}{3}\right\}(M-m).$$

□

§5 Fractional Milne-type inequalities for Lipschitzian functions

Now, we give a fractional Milne's rule for the case of Lipschitzian functions.

Theorem 7. *Note that the assumptions of Lemma 1 are valid. If $\mathcal{F}'$ is a L-Lipschitzian function on $[\sigma,\delta]$, then the following inequality holds*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{(\delta-\sigma)^{2}}{4}L\left\{\frac{1}{6}+\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha+3}{2^{\alpha+1}}-1\right)\right\}.$$

Proof. Using the fact that $\mathcal{F}'$ is L-Lipschitzian function, by Lemma 1, we have

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\delta-\sigma}{4}\left\{\int_0^{\frac{1}{2}} \xi^\alpha \left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right| d\xi\right.$$

$$\left.+\int_{\frac{1}{2}}^1 \left(\frac{4}{3}-\xi^\alpha\right)\left|\mathcal{F}'\left(\frac{\xi}{2}\delta+\frac{2-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{\xi}{2}\sigma+\frac{2-\xi}{2}\delta\right)\right| d\xi\right\}$$

$$\leq \frac{\delta-\sigma}{4}\left\{\int_0^{\frac{1}{2}} \xi^\alpha L(1-\xi)(\delta-\sigma)\,d\xi+\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\xi^\alpha\right)L(1-\xi)(\delta-\sigma)\,d\xi\right\}$$

$$=\frac{(\delta-\sigma)^2}{4}L\left\{\frac{1}{6}+\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha+3}{2^{\alpha+1}}-1\right)\right\}.$$

□

§6 Fractional Milne-type inequalities for functions of bounded variation

Now, a Milne-type inequality is offered by fractional integrals of bounded variation.

Theorem 8. *Suppose that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a function of bounded variation on $[\sigma,\delta]$. Then, we obtain*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^\alpha}\left[J^\alpha_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^\alpha_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$
$$\leq \frac{1}{2}\max\left\{\frac{1}{2^\alpha},\frac{4}{3}-\frac{1}{2^\alpha}\right\}\bigvee_\sigma^\delta(\mathcal{F}).$$

Here, $\bigvee_c^d(\mathcal{F})$ denotes the total variation of $\mathcal{F}$ on $[c,d]$.

Proof. Define the function $K_\alpha(x)$ by

$$K_\alpha(x)=\begin{cases}(x-\sigma)^\alpha, & \sigma\leq x<\frac{3\sigma+\delta}{4},\\ (x-\sigma)^\alpha-\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^\alpha, & \frac{3\sigma+\delta}{4}\leq x<\frac{\sigma+\delta}{2},\\ -(\delta-x)^\alpha+\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^\alpha, & \frac{\sigma+\delta}{2}\leq x<\frac{\sigma+3\delta}{4},\\ -(\delta-x)^\alpha, & \frac{\sigma+3\delta}{4}\leq x\leq\delta.\end{cases}$$

With the help of the integrating by parts, we obtain

$$\int_\sigma^\delta K_\alpha(x)\,d\mathcal{F}(x)$$

$$
\begin{aligned}
&= \int_{\sigma}^{\frac{3\sigma+\delta}{4}} (x-\sigma)^{\alpha}\, d\mathcal{F}(x) + \int_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}} \left[(x-\sigma)^{\alpha} - \frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right] d\mathcal{F}(x) \\
&\quad + \int_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}} \left[-(\delta-x)^{\alpha} + \frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right] d\mathcal{F}(x) + \int_{\frac{\sigma+3\delta}{4}}^{\delta} \left[-(\delta-x)^{\alpha}\right] d\mathcal{F}(x) \\
&= (x-\sigma)^{\alpha}\,\mathcal{F}(x)\Big|_{\sigma}^{\frac{3\sigma+\delta}{4}} - \alpha \int_{\sigma}^{\frac{3\sigma+\delta}{4}} (x-\sigma)^{\alpha-1}\,\mathcal{F}(x)dx \\
&\quad + \left[(x-\sigma)^{\alpha} - \frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]\mathcal{F}(x)\Big|_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}} - \alpha \int_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}} (x-\sigma)^{\alpha-1}\,\mathcal{F}(x)dx \\
&\quad + \left[-(\delta-x)^{\alpha} + \frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]\mathcal{F}(x)\Big|_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}} - \alpha \int_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}} (\delta-x)^{\alpha-1}\,\mathcal{F}(x)dx \\
&\quad + \left[-(\delta-x)^{\alpha}\right]\mathcal{F}(x)\Big|_{\frac{\sigma+3\delta}{4}}^{\delta} - \alpha \int_{\frac{\sigma+3\delta}{4}}^{\delta} (\delta-x)^{\alpha-1}\,\mathcal{F}(x)dx \\
&= \frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right) + \frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right) - \frac{2}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \\
&\quad - \alpha \int_{\sigma}^{\frac{\sigma+\delta}{2}} (x-\sigma)^{\alpha-1}\,\mathcal{F}(x)dx - \alpha \int_{\frac{\sigma+\delta}{2}}^{\delta} (\delta-x)^{\alpha-1}\,\mathcal{F}(x)dx \\
&= \frac{(\delta-\sigma)^{\alpha}}{3\cdot 2^{\alpha-1}}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right) - \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right] \\
&\quad - \Gamma(\alpha+1)\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma) + J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right].
\end{aligned}
$$

This follows

$$
\begin{aligned}
&\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right) - \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right] \\
&\qquad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma) + J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right] \\
&= \frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}} \int_{\sigma}^{\delta} K_{\alpha}(x)d\mathcal{F}(x).
\end{aligned}
$$

It is known that if $\mathcal{G}, \mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ are such that $\mathcal{G}$ is continuous on $[\sigma, \delta]$ and $\mathcal{F}$ is of bounded variation on $[\sigma,\delta]$, then $\int\limits_{\sigma}^{\delta}\mathcal{G}(\xi)d\mathcal{F}(\xi)$ exist and

$$\left|\int\limits_{\sigma}^{\delta}\mathcal{G}(\xi)d\mathcal{F}(\xi)\right|\leq\sup_{\xi\in[\sigma,\delta]}|\mathcal{G}(\xi)|\bigvee_{\sigma}^{\delta}(\mathcal{F}).\tag{15}$$

By using (15), we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.\\
&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|\\
&=\frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left|\int\limits_{\sigma}^{\delta}K_{\alpha}(x)d\mathcal{F}(x)\right|\leq\frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left\{\left|\int\limits_{\sigma}^{\frac{3\sigma+\delta}{4}}(x-\sigma)^{\alpha}d\mathcal{F}(x)\right|\right.\\
&\quad+\left|\int\limits_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}}\left[(x-\sigma)^{\alpha}-\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]d\mathcal{F}(x)\right|+\left|\int\limits_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}}\left[-(\delta-x)^{\alpha}+\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]d\mathcal{F}(x)\right|\\
&\quad\left.+\left|\int\limits_{\frac{\sigma+3\delta}{4}}^{\delta}\left[-(\delta-x)^{\alpha}\right]d\mathcal{F}(x)\right|\right\}\leq\frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left\{\sup_{x\in\left[\sigma,\frac{3\sigma+\delta}{4}\right]}|(x-\sigma)^{\alpha}|\bigvee_{\sigma}^{\frac{3\sigma+\delta}{4}}(\mathcal{F})\right.\\
&\quad+\sup_{x\in\left[\frac{3\sigma+\delta}{4},\frac{\sigma+\delta}{2}\right]}\left|(x-\sigma)^{\alpha}-\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right|\bigvee_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}}(\mathcal{F})\\
&\quad\left.+\sup_{x\in\left[\frac{\sigma+\delta}{2},\frac{\sigma+3\delta}{4}\right]}\left|\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-(\delta-x)^{\alpha}\right|\bigvee_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}}(\mathcal{F})+\sup_{x\in\left[\frac{\sigma+3\delta}{4},\delta\right]}|(\delta-x)^{\alpha}|\bigvee_{\frac{\sigma+3\delta}{4}}^{\delta}(\mathcal{F})\right\}\\
&=\frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left\{\left(\frac{\delta-\sigma}{4}\right)^{\alpha}\bigvee_{\sigma}^{\frac{3\sigma+\delta}{4}}(\mathcal{F})+\left[\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{4}\right)^{\alpha}\right]\bigvee_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}}(\mathcal{F})\right.\\
&\quad\left.+\left[\frac{4}{3}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{4}\right)^{\alpha}\right]\bigvee_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}}(\mathcal{F})+\left(\frac{\delta-\sigma}{4}\right)^{\alpha}\bigvee_{\frac{\sigma+3\delta}{4}}^{\delta}(\mathcal{F})\right\}\\
&=\frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\\
&\quad\times\left\{\frac{1}{2^{\alpha}}\bigvee_{\sigma}^{\frac{3\sigma+\delta}{4}}(\mathcal{F})+\left[\frac{4}{3}-\frac{1}{2^{\alpha}}\right]\bigvee_{\frac{3\sigma+\delta}{4}}^{\frac{\sigma+\delta}{2}}(\mathcal{F})+\left[\frac{4}{3}-\frac{1}{2^{\alpha}}\right]\bigvee_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+3\delta}{4}}(\mathcal{F})+\frac{1}{2^{\alpha}}\bigvee_{\frac{\sigma+3\delta}{4}}^{\delta}(\mathcal{F})\right\}\\
&\leq\frac{1}{2}\max\left\{\frac{1}{2^{\alpha}},\frac{4}{3}-\frac{1}{2^{\alpha}}\right\}\bigvee_{\sigma}^{\delta}(\mathcal{F}).
\end{aligned}$$

□

§7 Special cases and examples of main results

In this section, the correctness of our results are established by using special cases and examples of the obtained theorems.

Example 1. Let us take a function $\mathcal{F} : [\sigma, \delta] = [0, 4] \longrightarrow \mathbb{R}$ given by $\mathcal{F}(x) = x^2$ with $\alpha \in (0, 10]$. Thus, the left-hand side of (7) reduces to

$$\left| \frac{1}{3}\left[2\mathcal{F}(3) - \mathcal{F}(2) + 2\mathcal{F}(1)\right] - \frac{\Gamma(\alpha+1)}{2^{\alpha+1}}\left[\ J^{\alpha}_{2-}\mathcal{F}(0) + \ J^{\alpha}_{2+}\mathcal{F}(4)\right] \right| \tag{16}$$

$$= \left| \frac{16}{3} - \frac{\alpha}{2^{\alpha+1}} \left[\int_0^2 \xi^{\alpha-1}\xi^2 d\xi + \int_2^4 (4-\xi)^{\alpha-1}\xi^2 d\xi \right] \right|$$

$$= \left| \frac{4(\alpha-1)(\alpha+4)}{3(\alpha+1)(\alpha+2)} \right|.$$

Moreover, the right-hand side of (7) becomes to

$$8\left[\frac{1}{(\alpha+1)2^{\alpha+1}} + \frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha+1}} - 1\right) + \frac{2}{3} \right].$$

This follows that

$$\left| \frac{(\alpha-1)(\alpha+4)}{6(\alpha+1)(\alpha+2)} \right| \leq \frac{1}{(\alpha+1)2^{\alpha+1}} + \frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha+1}} - 1\right) + \frac{2}{3}.$$

As one can see in Figure 1, the left-hand side of (7) in Example 1 is always below the right-hand side of this equation, for all values of $\alpha \in (0, 10]$.

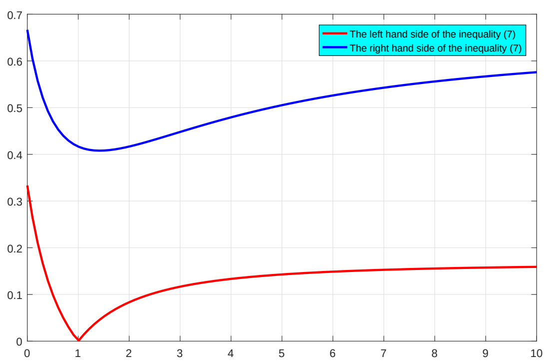

Figure 1. MATLAB was used to compute and represent the graph of both sides of (7) in Example 1.

Remark 1. If we choose $\alpha = 1$ in Theorem 3, then we can get the Milne-type inequality

$$\left| \frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right) - \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right) \right] - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathcal{F}(\xi)\, d\xi \right|$$

$$\leq \frac{5(\delta-\sigma)}{48}\left[|\mathcal{F}'(\sigma)| + |\mathcal{F}'(\delta)| \right],$$

which is proved by Ali et al. in paper [31, Corollary 1]. This inequality helps us find the error bound of Milne-type inequality.

Example 2. Let us take a function $\mathcal{F}:[\sigma,\delta]=[0,4]\longrightarrow\mathbb{R}$ given by $\mathcal{F}(x)=x^2$ with $\alpha\in(0,10]$. From Theorem 4 with $p=q=2$, the left-hand side of (8) becomes to (16) and the right-hand side of (8) coincides with

$$\left(\frac{1}{(2\alpha+1)\,2^{2\alpha+1}}\right)^{\frac{1}{2}}\left(2\left(1+\sqrt{7}\right)\right)$$
$$+\left(\frac{8}{9}+\frac{8}{3(\alpha+1)}\left[\frac{1}{2^{\alpha+1}}-1\right]-\frac{1}{2\alpha+1}\left[\frac{1}{2^{2\alpha+1}}-1\right]\right)^{\frac{1}{2}}\left(2\left(\sqrt{3}+\sqrt{5}\right)\right).$$

This yields

$$\left|\frac{2(\alpha-1)(\alpha+4)}{3(\alpha+1)(\alpha+2)}\right|\leq\left(\frac{1}{(2\alpha+1)\,2^{2\alpha+1}}\right)^{\frac{1}{2}}\left(1+\sqrt{7}\right)$$
$$+\left(\frac{8}{9}+\frac{8}{3(\alpha+1)}\left[\frac{1}{2^{\alpha+1}}-1\right]-\frac{1}{2\alpha+1}\left[\frac{1}{2^{2\alpha+1}}-1\right]\right)^{\frac{1}{2}}\left(\sqrt{3}+\sqrt{5}\right).$$

Using MATLAB software, one can see in Figure 2, the left-hand side of (8) in Example 2 is always below the right-hand side of this equation, for all values of $\alpha\in(0,10]$.

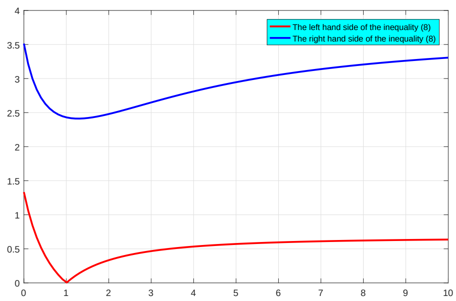

Figure 2. Evaluated and plotted using MATLAB, the graph of both sides of (8) in Example 2 depends on α.

Remark 2. Let us note that $\alpha=1$ in Theorem 4. Then, we obtain the following Milne's formula

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)\,d\xi\right|$$
$$\leq\frac{\delta-\sigma}{4}\left\{\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}'(\delta)|^q+7|\mathcal{F}'(\sigma)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^q+7|\mathcal{F}'(\delta)|^q}{16}\right)^{\frac{1}{q}}\right]\right.$$
$$+\left(\frac{5^{p+1}}{6^{p+1}(p+1)}-\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}$$
$$\left.\times\left[\left(\frac{3|\mathcal{F}'(\delta)|^q+5|\mathcal{F}'(\sigma)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+5|\mathcal{F}'(\delta)|^q}{16}\right)^{\frac{1}{q}}\right]\right\},$$

which is given by Ali et al. in paper [31, Corollary 2]. We can find the error bound of Milne-type inequality with the aid of this inequality.

Example 3. Choose a function $\mathcal{F}:[\sigma,\delta]=[0,4]\longrightarrow\mathbb{R}$ given by $\mathcal{F}(x)=x^2$. By Theorem 5 with $\alpha\in(0,10]$ and $q=2$, the left-hand side of (10) reduces to equality (16) and the right-side of (10) coincides with

$$8\left\{(\Omega_1(\alpha))^{\frac{1}{2}}\left[[\Omega_3(\alpha)]^{\frac{1}{2}}+[\Omega_4(\alpha)]^{\frac{1}{2}}\right]\right.$$
$$\left.+(\Omega_2(\alpha))^{\frac{1}{2}}\left[[\Omega_5(\alpha)]^{\frac{1}{2}}+[\Omega_6(\alpha)]^{\frac{1}{2}}\right]\right\}.$$

Finally, we have

$$\left|\frac{(\alpha-1)(\alpha+4)}{6(\alpha+1)(\alpha+2)}\right|\le(\Omega_1(\alpha))^{\frac{1}{2}}\left[[\Omega_3(\alpha)]^{\frac{1}{2}}+[\Omega_4(\alpha)]^{\frac{1}{2}}\right]$$
$$+(\Omega_2(\alpha))^{\frac{1}{2}}\left[[\Omega_5(\alpha)]^{\frac{1}{2}}+[\Omega_6(\alpha)]^{\frac{1}{2}}\right].$$

Figure 3 presents that in Example 3, the left-hand side of (10) is constantly below the right-hand side by using MATLAB software for all values of $\alpha\in(0,10]$.

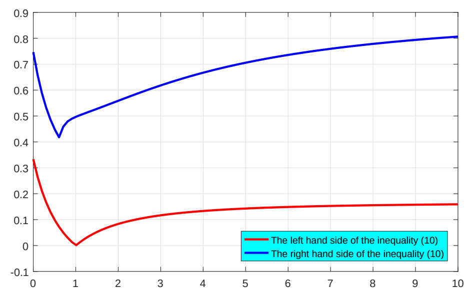

Figure 3. The graphs of both sides of (10) in Example 3, as functions of α, were examined and drawn using MATLAB.

Corollary 1. *If we take $\alpha=1$ in Theorem 5, then the following Milne's formula holds*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)\,d\xi\right|$$
$$\le\frac{\delta-\sigma}{4}\left\{\frac{1}{8}\left[\left(\frac{5|\mathcal{F}'(\delta)|^q+|\mathcal{F}'(\sigma)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{5|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\frac{7}{24}\right)^{1-\frac{1}{q}}\left[\left(\frac{9|\mathcal{F}'(\delta)|^q+5|\mathcal{F}'(\sigma)|^q}{48}\right)^{\frac{1}{q}}+\left(\frac{9|\mathcal{F}'(\sigma)|^q+5|\mathcal{F}'(\delta)|^q}{48}\right)^{\frac{1}{q}}\right]\right\}.$$

This inequality helps us find the error bound of Milne's rule.

Example 4. Consider a function $\mathcal{F}:[\sigma,\delta]=[0,4]\longrightarrow\mathbb{R}$ given by $\mathcal{F}(x)=x^2$. By Theorem 6 with $\alpha\in(0,10]$ and $0\le\mathcal{F}'(x)\le 8$, the left-hand side of (11) reduces to (16) and the right-side of (11) coincides with

$$8\left\{\frac{1}{\alpha+1}\left(\frac{1}{2^\alpha}-1\right)+\frac{2}{3}\right\}.$$

Consequently, we get

$$\left|\frac{(\alpha-1)(\alpha+4)}{6(\alpha+1)(\alpha+2)}\right| \leq \frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha}}-1\right)+\frac{2}{3}.$$

Hence, the left-hand side of (11) in Example 4 continuously stays below the right-hand side.

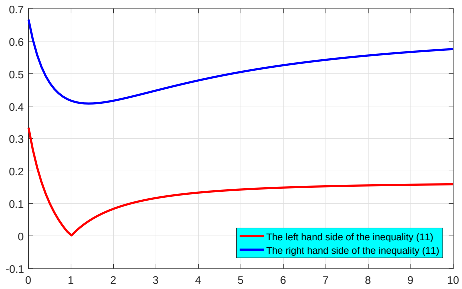

Figure 4. The graphs of both sides of (11) in Example 4, as functions of $\alpha \in (0,10]$, were evaluated and plotted using MATLAB.

Corollary 2. *If we select $\alpha = 1$ in Theorem 6, then we have*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int\limits_{\sigma}^{\delta}\mathcal{F}(\xi)\,d\xi\right|$$
$$\leq \frac{5(\delta-\sigma)}{48}(M-m).$$

Corollary 3. *Under assumption of Theorem 6, if there exist $M \in \mathbb{R}^{+}$ such that $|\mathcal{F}'(\xi)| \leq M$ for all $\xi \in [\sigma,\delta]$, then we get*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$
$$\leq \frac{\delta-\sigma}{2}\left\{\frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha}}-1\right)+\frac{2}{3}\right\}M.$$

Corollary 4. *If we assign $\alpha = 1$ in Corollary 3, then we get*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right|$$
$$\leq \frac{5(\delta-\sigma)}{24}M.$$

Corollary 5. *Take $\alpha = 1$ in Theorem 7. Then, the following Milne-type inequality holds:*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right|$$
$$\leq \frac{(\delta-\sigma)^{2}}{24}L.$$

Corollary 6. *Let us consider $\alpha = 1$ in Theorem 8. Then, we have the following Milne-type inequality*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\sigma+3\delta}{4}\right)-\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+2\mathcal{F}\left(\frac{3\sigma+\delta}{4}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right| \leq \frac{5}{12}\bigvee_{\sigma}^{\delta}(\mathcal{F}).$$

§8 Concluding remarks

This paper represents several Milne-type inequalities for various function classes. More precisely, several new versions of Milne-type inequalities are given for the case of differentiable convex functions by using Riemann-Liouville fractional integrals. Namely, Milne-type inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Afterwards, some Milne-type inequalities are given for bounded functions by fractional integrals. Subsequently, we present some fractional Milne-type inequalities for the case of Lipschitzian functions. Furthermore, we investigate Milne-type inequalities by fractional integrals of bounded variation. Finally, the correctness of our results are proved by using special cases and examples of the obtained theorems.

In future papers, the concepts and strategies related to our results associated with Milne-type inequalities by Riemann-Liouville fractional integrals may open the way for new avenues in mathematics. Moreover, one may consider generalizing our findings by exploring alternative classes of convex functions or different types of fractional integral operators. Furthermore, one can obtain Milne-type inequalities with the help of the various function classes by using quantum calculus.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] S S Dragomir, R Agarwal. *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl Math Lett, 1998, 11: 91-95.

[2] U Kirmaci. *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl Math Comput, 2004, 147: 137-146.

[3] J Chen, X Huang. *Some new inequalities of Simpson's type for s-convex functions via fractional integrals*, Filomat, 2017, 31: 4989-4997.

[4] T Du, Y Li, Z Yang. *A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions*, Appl Math Comput, 2017, 293: 358-369.

[5] E Set, A O Akdemir, M E Ozdemir. *Simpson type integral inequalities for convex functions via Riemann-Liouville integrals*, Filomat, 2017, 31: 4415-4420.

[6] S S Dragomir, R P Agarwal, P Cerone. *On Simpson's inequality and applications*, J Inequal Appl, 2000, 5: 533-579.

[7] F Hezenci, H Budak, H Kara. *New version of Fractional Simpson type inequalities for twice differentiable functions*, Adv Differ Equ, 2021, 2021: 460.

[8] H Budak, F Hezenci, H Kara. *On parametrized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral*, Math Methods Appl Sci, 2021, 44(30): 12522-12536.

[9] H Budak, F Hezenci, H Kara. *On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co-ordinated convex functions via generalized fractional integrals*, Adv Differ Equ, 2021, 2021: 312.

[10] S K Sahoo, Y S Hamed, P O Mohammed, et al. *New midpoint type Hermite-Hadamard-Mercer inequalities pertaining to Caputo-Fabrizio fractional operators*, Alexandria Eng J, 2023, 65: 689-698.

[11] H M Srivastava, M Tariq, P Mohammed, et al. *On modified integral inequalities for a generalized class of convexity and applications*, Axioms, 2023, 12(2): 162.

[12] A Kashuri, S K Sahoo, P O Mohammed, et al. *Some New Hermite-Hadamard Type Inequalities Pertaining to Fractional Integrals with an Exponential Kernel for Subadditive Functions*, Symmetry, 2023, 15(3): 748.

[13] A Kashuri, S K Sahoo, P O Mohammed, et al. *Novel inequalities for subadditive functions via tempered fractional integrals and their numerical investigations* AIMS Mathematics, 2024, 9(5): 13195-13210.

[14] H M Srivastava, S K Sahoo, P O Mohammed, et al. *Results on Minkowski-Type Inequalities for Weighted Fractional Integral Operators*, Symmetry, 2023, 15(8): 1522.

[15] B Bin-Mohsin, M Z Javed, M U Awan, et al. *Exploration of Quantum MilneCMercer-Type Inequalities with Applications*, Symmetry, 2023, 15(5): 1096.

[16] Y M Chu, M U Awan, M Z Javad, et al. *Bounds for the Remainder in Simpson's Inequality via n-Polynomial Convex Functions of Higher Order Using Katugampola Fractional Integrals*, J Math, 2020, 2020(1): 4189036.

[17] M U Awan, M Z Javed, M T Rassias, et al. *Simpson type inequalities and applications*, J Anal, 2021, 29: 1403-1419.

[18] M U Awan, A Kashuri, K S Nisar, et al. *New fractional identities, associated novel fractional inequalities with applications to means and error estimations for quadrature formulas*, J Inequal Appl, 2022, 2022: 3.

[19] S Iftikhar, P Kumam, S Erden. *Newton's-type integral inequalities via local fractional integrals*, Fractals, 2020, 28: 2050037.

[20] Y M Li, S Rashid, Z Hammouch, et al. *New Newton's type estimates pertaining to local fractional integral via generalized p-convexity with applications*, Fractals, 2021, 29: 2140018.

[21] S Erden, S Iftikhar, P Kumam, et al. *Some Newton's like inequalities with applications*, Rev R Acad Cienc Exactas Fís Nat Ser A Mat, 2020, 114: 195.

[22] F Hezenci, H Budak, P Kosem. *A new version of Newton's inequalities for Riemann-Liouville fractional integrals*, Rocky Mt J Math, 2023, 53(1): 49-64.

[23] S Gao, W Shi. *On new inequalities of Newton's type for functions whose second derivatives absolute values are convex*, Int J Pure Appl Math, 2012, 74(1): 33-41.

[24] F Hezenci, H Budak. *Some Perturbed Newton type inequalities for Riemann-Liouville fractional integrals*, Rocky Mt J Math, 2023, 53(4): 1117-1127.

[25] S Iftikhar, S Erden, P Kumam, et al. *Local fractional Newton's inequalities involving generalized harmonic convex functions*, Adv Differ Equ, 2020, 2020: 185.

[26] M A Noor, K I Noor, S Iftikhar. *Some Newton's type inequalities for harmonic convex functions*, J Adv Math Stud, 2016, 9(1): 7-16.

[27] T Sitthiwirattham, K Nonlaopon, M A Ali, et al. *Riemann-Liouville fractional Newton's type inequalities for differentiable convex functions*, Fractal Fract, 2022, 6(3): 175.

[28] M Djenaoui, B Meftah. *Milne type inequalities for differentiable s-convex functions*, Honam Math J, 2022, 44: 325-338.

[29] M Alomari. *New error estimations for the Milne's quadrature formula in terms of at most first derivatives*, Konuralp J Math, 2013, 1(1): 17-23.

[30] H Budak, P Kösem, H Kara. *On new Milne-type inequalities for fractional integrals*, J Inequal Appl, 2023, 2023: 10.

[31] M A Ali, Z Zhang, M Fečkan. *On Some Error Bounds for Milne's Formula in Fractional Calculus*, Mathematics, 2023, 11(1): 146.

[32] B Meftah, D Benchettah, A Lakhdari. *Some New Local Fractional Newton-type Inequalities*, J Inequal Math Anal, 2025 1(2): 79-96.

[33] P J Davis, P Rabinowitz. *Methods of numerical integration*, New York: Academic Press, 1975.

[34] R Gorenflo, F Mainardi. *Fractional calculus: Integral and differential equations of fractional order*, Wien: Springer Verlag, 1997.

[35] A A Kilbas, H M Srivastava, J J Trujillo. *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.

[1]Department of Mathematics, Faculty of Science and Arts, Sinop University, Sinop 57000, Turkey.
[2]Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce 81620, Turkey.
Email: fatihezenci@gmail.com
[3]Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India.
[4]Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli 41001, Turkey.