Appl. Math. J. Chinese Univ.
2025, 40(4): 901-915

Higher order Haar wavelet method for the numerical solution of second-order integro-differential equations

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Abstract. In this paper, we used higher order Haar wavelet method (HOHWM), introduced by Majak et al. [1], for approximate solution of second order integro-differential equations (IDEs) of second-kind. It is improvement of long-established Haar wavelet collocation method (HWCM) which has been much popular among researchers and has many applications in literature. Present study aims to improve the numerical results of second order IDEs from first order rate of convergence in case of HWCM to the second and fourth order rate of convergence using HOHWM, depending on parameter λ for values 1 and 2, respectively. Several problems available in the literature of both, Volterra and Fredholm type of IDEs, are tested and compared with HWCM to illustrate the performance of our proposed method.

§1 Introduction

An equation in which the unknown function occurs on one side as an ordinary derivative and appears on the other side under the integral sign is called an a integro-dfferential equation. And the equations involving second order as the highest derivative in the equation is known as second-order IDEs. Volterra and Fredholm are the two main types of IDEs. The upper limit of the integral in the Volterra type is variable, while in the Fredholm type it is constant, and general form of these equations are given below [2]. HOHWM will be developed for solution of second order Volterra and Fredholm IDEs. Consider Fredholm IDE of second order as

$$w''(t) + a_1(t)w'(t) + a_2(t)w(t) = \int_0^1 M(t,r)w(r)dr + f(t), \quad w(0) = w_0, w'(0) = w_1, \tag{1}$$

and Volterra as

$$w''(t) + a_1(t)w'(t) + a_2(t)w(t) = \int_0^t M(t,r)w(r)dr + f(t), \quad w(0) = w_0, w'(0) = w_1. \tag{2}$$

Second-order IDEs can model a large number of important problems in applications related to scientific and physical engineering [3]. A large range of initial and boundary value prob-

Received: 2024-02-20. Revised: 2024-05-21.
MR Subject Classification: 45Lxx
Keywords: IDEs, higher order Haar wavelet, Fredholm IDEs, Volterra IDEs.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-5165-1.
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lems can be resolved by using Volterra or Fredholm integral equations. The development of integral equations benefited most from the potential theory. Therefore, researchers in applied mathematics have focused a lot of attention on the solution to these equations. Here, certain numerical techniques are discussed in [4–7].

This paper is put in order as follows. Definition and applications of Haar wavelet are provided in Section 2. This section is extended with introduction to the higher order Haar wavelet method along with its recent work and applications. In Section 3, method is developed for second-order IDEs separately for both types, Voltera and Fredholm IDEs. In Section 4 , test problem from the literature are solved using HOHWM and results are displayed in the form of tables. Also comparison and brief discussion is given in the same section. Section 5 provides conclusions.

§2 Haar wavelets

The Haar Wavelet (HW) has an important role among all wavelets used for numerical approximations because of its good approximate properties as well as its simple representation. Alfred Haar defined HW for the first time in 1910, after then many researchers have used it in various domains. Literature contains number of HW applications for numerical approximations. IEs [8], differential equations [9], IDEs [10], partial differential equations [11], have all been numerically solved by using HW. Majak et al. [12, 13] presented theoretical results for HW convergence.

The following functions comprise HW family defined on $[0, 1)$

$$h_1(t) = \begin{cases} 1, & \text{if} \quad t \in [0,1), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$h_i(t) = \begin{cases} 1, & \text{if} \quad t \in [\alpha,\beta), \\ -1, & \text{if} \quad t \in [\beta,\gamma), \\ 0, & \text{otherwise}, \qquad\qquad i = 2, 3, \dots, \end{cases}$$

where

$$\alpha = \frac{k}{m}, \quad \beta = \frac{(0.5+k)}{m}, \quad \gamma = \frac{1+k}{m};$$
$$m = 2^j, \quad k = 0, 1, \dots, m-1, \quad j = 0, 1, \dots.$$

The translation parameter is k, while the wavelet's level is indicated by the number j. $i = k + m + 1$ is relation between i, m, and k. While $h_2(t)$ is mother wavelet for HW family, $h_1(t)$ is known as the scaling function.

We introduce the notations

$$R_{1,i}(t) = \int_0^t h_i(s)\,\mathrm{d}s,$$
$$R_{n+1,i}(t) = \int_0^t R_{n,i}(s)\,\mathrm{d}s, \quad n = 1, 2, \dots.$$

Above integrals are evaluated as

$$R_{n,i}(t) = \begin{cases} 0, & \text{if} \quad t \in [0,\alpha), \\ \frac{1}{n!}(t-\alpha)^n, & \text{if} \quad t \in [\alpha,\beta), \\ \frac{1}{n!}\left[(t-\alpha)^n - 2(t-\beta)^n\right], & \text{if} \quad t \in [\beta,\gamma), \\ \frac{1}{n!}\left[(t-\alpha)^n - 2(t-\beta)^n + (t-\gamma)^n\right], & \text{if} \quad t \in [\gamma,1), \quad n = 1,2,\ldots, \end{cases}$$

where $i = 2, 3, \ldots$. If $i = 1$, then

$$R_{n,1}(t) = \frac{t^n}{n!}, \quad n = 1,2,\ldots.$$

Chen and Hsiao methodology is mostly exploited in literature to implement numerical methods based on HW for various types of differential equations [14]. This methodology uses a HW expansion to determine the derivative with the highest order that occurs in the provided IDEs. Measures such as accuracy, convergence, efficiency, and stability are used to assess the effectiveness of numerical methods. Over the past 20 years, the Haar wavelet method (HWM), a collocation technique based on HW for numerical approximations, has been increasingly prominent. The theoretical convergence results were presented by Majak et al. [12, 13]. HWM was used by Lepik to solve partial and ordinary differential equations numerically [9, 15]. It was used by Aziz and Islam to solve IEs and IDEs numerically [16]. The slow convergence of the HWM is one of its drawbacks.

2.1 Higher order Haar wavelet method

Majak et al. presented a new HOHWM [1] to increase the convergence rate of HWM. Following that, it was effectively utilized to solve nonlinear evolution equations [17], nonlinear PDEs [18], vibration analysis of nanobeams [19], ordinary differential equations [20], an efficient technique based on higher order Haar wavelet method for Lane-Emden equations [21], dynamics of flight of the fragments with higher order Haar wavelet method [22] and IEs [23]. Compared to the traditional HW approach, the convergence order increases from 2 to 4 in HOHWM. When using the usual HW technique, the number of boundary conditions and number of constants that result from integration are the same. But in new method, number of unknown constants exceeds the boundary conditions as the HW series is expended with higher derivative than the order of derivative in the problem. Increment in the number of derivatives is denoted by the parameter λ. Let us assume the highest derivative of order n exists in the equation

$$\frac{d^{n+2\lambda}w(t)}{dt^{n+2\lambda}} = \sum_{i=1}^{\infty} a_i h_i(t), \tag{3}$$

where $\lambda = 1, 2, \ldots$. We take the even increment 2λ due to symmetry and more accurate results [1]. By integrating the preceding expression, values for all of its derivatives that are included in the equation and unknown function are obtained. The formula obtained by integrating $n + 2\lambda$ times Eq. (3) is as follows

$$w(t) = \frac{a_1 t^{n+2\lambda}}{(n+2\lambda)!} + \sum_{k=0}^{\infty} \sum_{l=0}^{2^k-1} a_{2^k+l+1} R_{n+2\lambda,2^k+l+1}(t) + S(t) + H(t), \tag{4}$$

where $l = 0, 1, \ldots, t-1$, $t = 2^k$ and $T = 2^K$. In this case, T represents the highest degree of resolution and t indicates the HW resolution. In Eq.(4), the words $S(t)$ and $H(t)$ are defined as follows

$$S(t) = \sum_{r=0}^{n-1} c_r \frac{t^r}{r!}, \qquad H(t) = \sum_{r=n}^{n+2\lambda-1} c_r \frac{t^r}{r!}, \tag{5}$$

and the $n+2\lambda$ constants $c_r, r = 0, 1, \ldots, n+2\lambda-1$ are involved in aforementioned procedure. While values of remaining 2λ constants can be found by using the original equation, the values of n integration constants may be computed using the boundary conditions. To produce 2λ more equations, we particularly take into account 2λ points and put these points in discretized equation. The collocation points (CPs) and these 2λ points should be different. There are an infinite number of ways to take the 2λ constants, which implies there are infinite number of methods to calculate the remaining 2λ constants.

This method [1] considers higher order derivatives of the unknown function that are higher by an even increment than the highest ordered derivative found in the provided equation. However, we take into account all derivatives that are greater than the highest ordered derivative found in provided equation in order to apply the approach to integral and IDEs. As a result, the unknown function $w(t)$ will be approximated as

$$\frac{d^\lambda w(t)}{dt^\lambda} = \sum_{i=1}^{N} a_i h_i(t), \quad \lambda = 1, 2, \ldots, \tag{6}$$

where the unknown constants N are $a_i, i = 1, 2, \ldots, N$. By integrating the aforementioned expression λ times, the function $w(t)$ is obtained. Through this process, we obtain an extra λ constant, making a total of $N + \lambda$ constants. We will use CPs listed below for

$$t_j = \frac{j-0.5}{N}, \qquad j = 1, 2, \ldots, N. \tag{7}$$

We establish $N \times (N+\lambda)$ linear system by discretizing utilizing the above collocation points and substituting the HW expressions in the provided IDEs. The HOHWM [1] indicates that any λ points from the domain other than CPs in the original equation can be substituted to produce the additional λ equations. We consider only two values $\lambda = 1$, and $\lambda = 2$.

§3 Numerical method

This section presents numerical solution of second-order IDEs with HOHWM. The first step in developing the second-order IDE is to approximate $w'''(t)$ by using Haar wavelet series. The unknown function $w(t)$ will be obtained by integrating the obtained formula three times. The method for linear Volterra and Fredholm IDEs of second-order is developed.

3.1 Fredholm IDEs

Let us consider second-order Fredholm IDEs as defined in Eq. (1). The procedures for $\lambda = 1$ and $\lambda = 2$ will be discussed.

Case 1 ($\lambda = 1$).

Let

$$w'''(t) = \sum_{i=1}^{N} a_i h_i(t), \tag{8}$$

integrating three times and using initial conditions, we have

$$w''(t) = E_1 + \sum_{i=1}^{N} a_i R_{1,i}(t), \tag{9}$$

$$w'(t) = w_1 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t), \tag{10}$$

and

$$w(t) = w_0 + w_1 t + E_1 \frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t). \tag{11}$$

Equation (1) becomes, with this assumption as

$$\begin{aligned} E_1 + \sum_{i=1}^{N} a_i R_{1,i}(t) + a_1(t) \left(w_1 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t) \right) \\ + a_2(t) \left(w_0 + w_1 t + E_1 \frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t) \right) \\ = f(t) + \int_0^1 M(t,r) \left(w_0 + w_1 r + E_1 \frac{r^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(r) \right) dr, \end{aligned} \tag{12}$$

which implies

$$\begin{aligned} &\sum_{i=1}^{N} a_i \left(R_{1,i}(t) + a_1(t) R_{2,i}(t) + a_2(t) R_{3,i}(t) - \int_0^1 M(t,r) R_{3,i}(r) dr \right) \\ &= f(t) - w_1(a_1(t) + t a_2(t)) - w_0 a_2(t) - E_1 \left(1 + t a_1(t) + \frac{t^2}{2} a_2(t) \right) \\ &+ \int_0^1 \left(w_0 + w_1 r + E_1 \frac{r^2}{2} \right) M(t,r) dr, \end{aligned} \tag{13}$$

after simplification, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} a_i \left(R_{1,i}(t) + a_1(t) R_{2,i}(t) + a_2(t) R_{3,i}(t) - \int_0^1 M(t,r) R_{3,i}(r) dr \right) \\ &= f(t) - w_0 \left(a_2(t) - \int_0^1 M(t,r) dr \right) - w_1 \left(a_1(t) + t a_2(t) - \int_0^1 r M(t,r) dr \right) \\ &- E_1 \left(1 + t a_1(t) + \frac{t^2}{2} a_2(t) - \int_0^1 \frac{r^2}{2} M(t,r) dr \right). \end{aligned} \tag{14}$$

Substituting $t = 0$ into the above equation to obtain extra condition,

$$\begin{aligned} -\sum_{i=1}^{N} a_i \int_0^1 M(0,r) R_{3,i}(r) dr = f(0) - w_0 \left(a_2(0) - \int_0^1 M(0,r) dr \right) \\ - w_1 \left(a_1(0) - \int_0^1 r M(0,r) dr \right) - E_1 \left(1 - \int_0^1 \frac{r^2}{2} M(0,r) dr \right), \end{aligned} \tag{15}$$

which reduces to

$$E_1\left(1-\int_0^1 \frac{r^2}{2}M(0,r)dr\right)=f(0)-w_0\left(a_2(0)-\int_0^1 M(0,r)dr\right) - w_1\left(a_1(0)-\int_0^1 rM(0,r)dr\right)+\sum_{i=1}^N a_i\int_0^1 M(0,r)R_{3,i}(r)dr. \quad (16)$$

Let us introduce the notation

$$\mathcal{K}_0^2=\int_0^1 M(0,r)R_{2,i}(r)dr,\quad \mathcal{K}_{00}=\int_0^1 M(0,r)dr,\quad \mathcal{K}_{10}=\int_0^1 sM(0,r)dr. \quad (17)$$

Using above notations, we obtain

$$E_1\left(1-\mathcal{K}_{10}\right)=\phi(0)-w_0\left(a(0)-\mathcal{K}_{00}\right)+\sum_{i=1}^N a_i\mathcal{K}_0^2. \quad (18)$$

Thus

$$E_1=\frac{1}{(1-\mathcal{K}_{10})}\left(\phi(0)-w_0\left(a(0)-\mathcal{K}_{00}\right)+\sum_{i=1}^N a_i\mathcal{K}_0^2\right), \quad (19)$$

$$\mathcal{K}_0^3=\int_0^1 M(0,r)R_{3,i}(r)dr,\quad \mathcal{K}_{20}=\int_0^1 \frac{s^2}{2}M(0,r)dr, \quad (20)$$

and $\mathcal{K}_{00}$, $\mathcal{K}_{10}$ are given in (17). Furthermore, for $t=1$, we have

$$\sum_{i=1}^N a_i\left(R_{2,i}(1)+a(1)R_{3,i}(1)-\mathcal{K}_1^3\right)=\phi(1)-w_0(a(1)-\mathcal{K}_{01})-E_1\left(1+a(1)-\mathcal{K}_5\right) - E_2\left(1+\frac{a(1)}{2}-\mathcal{K}_{21}\right), \quad (21)$$

where

$$\mathcal{K}_1^3=\int_0^1 M(1,r)R_{3,i}(r)dr,\ \mathcal{K}_{01}=\int_0^1 M(1,r)dr,\ \mathcal{K}_{11}=\int_0^1 sM(1,r)dr,\ \mathcal{K}_{21}=\int_0^1 \frac{s^2}{2}M(1,r)dr. \quad (22)$$

We compute E_1 and E_2 as

$$\begin{aligned}E_1=\frac{1}{S}\Bigg(&\left(1+\frac{a(1)}{2}-\mathcal{K}_{21}\right)\phi(0)\\ &+\mathcal{K}_{20}\phi(1)-\left(\left(a(0)-\mathcal{K}_{00}\right)\left(1+\frac{a(1)}{2}-\mathcal{K}_{21}\right)+\mathcal{K}_{20}(a(1)-\mathcal{K}_{01})\right)w_0\\ &+\left(1+\frac{a(1)}{2}-\mathcal{K}_{21}\right)\sum_{i=1}^N a_i\mathcal{K}_0^3-\mathcal{K}_{20}\sum_{i=1}^N a_i\left(R_{2,i}(1)+a(1)R_{3,i}(1)-\mathcal{K}_1^3\right)\Bigg), \quad (23)\end{aligned}$$

$$\begin{aligned}E_2=\frac{1}{S}\Bigg(&-(1+a(1)-\mathcal{K}_{11})\phi(0)+(1-\mathcal{K}_{10})\phi(1)+(a(0)-a(1)+a(0)a(1)-\mathcal{K}_{00}-a(1)\mathcal{K}_{00}\\ &+a(1)\mathcal{K}_{10}+\mathcal{K}_{01}-\mathcal{K}_{10}\mathcal{K}_{01}-a(0)\mathcal{K}_{11}+\mathcal{K}_{00}\mathcal{K}_{11})w_0-(1+a(1)-\mathcal{K}_{11})\sum_{i=1}^N a_i\mathcal{K}_0^3\\ &-(1-\mathcal{K}_{10})\sum_{i=1}^N a_i\left(R_{2,i}(1)+a(1)R_{3,i}(1)-\mathcal{K}_1^3\right)\Bigg), \quad (24)\end{aligned}$$

where

$$S = (1-\mathcal{K}_{10})\left(1+\frac{a(1)}{2}-\mathcal{K}_{21}\right)+\mathcal{K}_{20}\left(1+a(1)-\mathcal{K}_{11}\right). \tag{25}$$

Using the notations $\mathcal{K}_0^3$, $\mathcal{K}_{00}$, $\mathcal{K}_{10}$ and $\mathcal{K}_{20}$ defined in Eqs. (17) and (20), we have

$$E_1\left(1-\mathcal{K}_{20}\right) = f(0) - w_0\left(a_2(0)-\mathcal{K}_{00}\right) - w_1\left(a_1(0)-\mathcal{K}_{10}\right) + \sum_{i=1}^{N} a_i \mathcal{K}_0^3. \tag{26}$$

Thus the value of E_1 is given as

$$E_1 = \frac{1}{(1-\mathcal{K}_{20})}\left(f(0) - w_0\left(a_2(0)-\mathcal{K}_{00}\right) - w_1\left(a_1(0)-\mathcal{K}_{10}\right) + \sum_{i=1}^{N} a_i \mathcal{K}_0^3\right). \tag{27}$$

Hence, we have

$$\begin{aligned}
&\sum_{i=1}^{N} a_i\left(R_{1,i}(t)+a_1(t)R_{2,i}(t)+a_2(t)R_{3,i}(t)-\int_0^1 M(t,r)R_{3,i}(r)dr\right)\\
&= f(t) - w_0\left(a_2(t)-\int_0^1 M(t,r)dr\right) - w_1\left(a_1(t)+ta_2(t)-\int_0^1 rM(t,r)dr\right)\\
&\quad -\frac{1}{(1-\mathcal{K}_{20})}\left(f(0) - w_0\left(a_2(0)-\mathcal{K}_{00}\right) - w_1\left(a_1(0)-\mathcal{K}_{10}\right) + \sum_{i=1}^{N} a_i \mathcal{K}_0^3\right)\\
&\quad \times\left(1+ta_1(t)+\frac{t^2}{2}a_2(t)-\int_0^1 \frac{r^2}{2}M(t,r)dr\right).
\end{aligned} \tag{28}$$

Putting the CPs, we get a linear system which can be solved by any linear solver.

Case 2 ($\lambda = 2$)**.**

Here, it is assumed that

$$w^{iv}(t) = \sum_{i=1}^{N} a_i h_i(t), \tag{29}$$

integrating four times and using initial conditions, we have

$$w'''(t) = E_1 + \sum_{i=1}^{N} a_i R_{1,i}(t), \tag{30}$$

$$w''(t) = E_2 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t), \tag{31}$$

and

$$w'(t) = w_1 + E_2 t + E_1\frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t), \tag{32}$$

$$w(t) = w_0 + w_1 t + E_2\frac{t^2}{2} + E_1\frac{t^3}{6} + \sum_{i=1}^{N} a_i R_{4,i}(t). \tag{33}$$

With this assumption, Eq. (1) becomes

$$E_1 t + E_2 + \sum_{i=1}^{N} a_i R_{2,i}(t) + a_1(t)\left(w_1 + E_1\frac{t^2}{2} + E_2 t + \sum_{i=1}^{N} a_i R_{3,i}(t)\right)$$

$$+ a_2(t)\left(w_0 + w_1 t + E_1\frac{t^3}{6} + E_2\frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{4,i}\right)$$
$$= f(t) + \int_0^1 M(t,r)\left(w_0 + w_1 s + E_1\frac{s^3}{6} + E_2\frac{s^2}{2} + \sum_{i=1}^{N} a_i R_{4,i}(s)\right)dr, \quad (34)$$

which implies

$$\sum_{i=1}^{N} a_i\left(R_{2,i}(t) + a_1(t)R_{3,i}(t) + a_2(t)R_{4,i} - \int_0^1 M(t,r)R_{4,i}(r)dr\right)$$
$$= f(t) + \int_0^1 M(t,r)\left(w_0 + w_1 s + E_1\frac{s^3}{6} + E_2\frac{s^2}{2}\right)dr - E_2 - tE_1 - a_1(t)\left(w_1 + \frac{t^2}{2}E_1 + tE_2\right)$$
$$- a_2(t)\left(w_0 + tw_1 + E_1\frac{t^3}{6} + E_2\frac{t^2}{2}\right). \quad (35)$$

A more precise representation of this equation is as follows

$$\sum_{i=1}^{N} a_i\left(R_{2,i}(t) + a_1(t)R_{3,i}(t) + a_2(t)R_{4,i} - \int_0^1 M(t,r)R_{4,i}(s)dr\right)$$
$$= f(t) - w_0\left(a_2(t) - \int_0^1 M(t,r)dr\right) - w_1\left(a_1(t) + ta_2(t) - \int_0^1 sM(t,r)dr\right)$$
$$- E_1\left(t + \frac{t^2}{2}a_1(t) + \frac{t^3}{6}a_2(t) - \int_0^1 \frac{s^3}{6}M(t,r)dr\right)$$
$$- E_2\left(1 + ta_1(t) + \frac{t^2}{2}a_2(t) - \int_0^1 \frac{s^2}{2}M(t,r)dr\right), \quad (36)$$

and we substitute $t = 0$ and $t = 1$ in the above equation to obtain the extra condition,

$$-\sum_{i=1}^{N} a_i \int_0^1 M(0,r)R_{4,i}(s)dr = f(0) - w_0\left(a_2(0) - \int_0^1 M(0,r)dr\right)$$
$$- w_1\left(a_1(0) - \int_0^1 sM(0,r)dr\right) + E_1\left(\int_0^1 \frac{s^3}{6}M(0,r)dr\right) - E_2\left(1 - \int_0^1 \frac{s^2}{2}M(0,r)dr\right). \quad (37)$$

Let

$$\mathcal{K}_0^4 = \int_0^1 M(0,r)R_{4,i}(s)dr, \quad \mathcal{K}_{30} = \int_0^1 \frac{s^3}{6}M(0,r)dr. \quad (38)$$

Combining with the notations previously presented in Eqs. (17) and (20), we have

$$- E_1\mathcal{K}_{30} + E_2\left(1 - \mathcal{K}_{20}\right) = f(0) - w_0\left(a_2(0) - \mathcal{K}_{00}\right) - w_1\left(a_1(0) - \mathcal{K}_{10}\right) + \sum_{i=1}^{N} a_i\mathcal{K}_0^4, \quad (39)$$

also

$$\sum_{i=1}^{N} a_i\left(R_{2,i}(1) + a_1(1)R_{3,i}(1) + a_2(1)R_{4,i}(1) - \int_0^1 M(1,s)R_{4,i}(s)dr\right)$$
$$= f(1) - w_0\left(a_2(1) - \int_0^1 M(1,s)dr\right) - w_1\left(a_1(1) + a_2(1) - \int_0^1 sM(1,s)dr\right)$$

$$- E_1\left(1+\frac{1}{2}a_1(1)+\frac{1}{6}a_2(1)-\int_0^1\frac{s^3}{6}M(1,s)dr\right)$$
$$- E_2\left(1+a_1(1)+\frac{1}{2}a_2(1)-\int_0^1\frac{s^2}{2}M(1,s)dr\right). \tag{40}$$

Using the notations

$$\mathcal{K}_1^4=\int_0^1 M(1,s)R_{4,i}(s)dr,\quad \mathcal{K}_{31}=\int_0^1\frac{s^3}{6}M(1,s)dr. \tag{41}$$

Using these notations along with the ones provided in Eq. (22), we get

$$E_1\left(1+\frac{1}{2}a_1(1)+\frac{1}{6}a_2(1)-\mathcal{K}_{31}\right)+E_2\left(1+a_1(1)+\frac{1}{2}a_2(1)-\mathcal{K}_{21}\right)$$
$$= f(1)-w_0\left(a_2(1)-\mathcal{K}_{01}\right)-w_1\left(a_1(1)+a_2(1)-\mathcal{K}_{11}\right)$$
$$-\sum_{i=1}^{N}a_i\left(R_{2,i}(1)+a_1(1)R_{3,i}(1)+a_2(1)R_{4,i}(1)-\mathcal{K}_1^4\right). \tag{42}$$

From this, we derive the values of E_1 and E_2 as

$$E_1=\frac{1}{D_1}\Big((1-\mathcal{K}_{20})f(1)-S_2f(0)-w_0\big((1-\mathcal{K}_{20})\left(a_2(1)-\mathcal{K}_{01}\right)-S_2\left(a_2(0)-\mathcal{K}_{00}\right)\big)$$
$$-w_1\big((1-\mathcal{K}_{20})\left(a_1(1)+a_2(1)-\mathcal{K}_{11}\right)-S_2\left(a_1(0)-\mathcal{K}_{10}\right)\big)$$
$$-\sum_{i=1}^{N}a_i\left((1-\mathcal{K}_{20})\left(R_{2,i}(1)+a_1(1)R_{3,i}(1)+a_2(1)R_{4,i}(1)-\mathcal{K}_1^4\right)+S_2\mathcal{K}_0^4\right)\Big), \tag{43}$$

and

$$E_2=\frac{1}{D_1}\Big((S_1f(0)-\mathcal{K}_{30}f(1)-w_0\left(S_1\left(a_2(0)-\mathcal{K}_{00}\right)-\mathcal{K}_{30}\left(a_2(1)-\mathcal{K}_{01}\right)\right)$$
$$-w_1\left(S_1\left(a_1(0)-\mathcal{K}_{10}\right)-\mathcal{K}_{30}\left(a_1(1)+a_2(1)-\mathcal{K}_{11}\right)\right)$$
$$+\sum_{i=1}^{N}a_i\left(S_1\mathcal{K}_0^4+\mathcal{K}_{30}(R_{2,i}(1)+a_1(1)R_{3,i}(1)+a_2(1)R_{4,i}(1)-\mathcal{K}_1^4)\right)\Big), \tag{44}$$

where

$$S_1=\left(1+\frac{1}{2}a_1(1)+\frac{1}{6}a_2(1)-\mathcal{K}_{31}\right), S_2=\left(1+a_1(1)+\frac{1}{2}a_2(1)-\mathcal{K}_{21}\right),$$
$$D_1=S_1\left(1-\mathcal{K}_{20}\right)+S_2\mathcal{K}_{30}. \tag{45}$$

The remaining steps follow the same procedure as the IDEs section previously.

3.2 Volterra IDEs

The Volterra IDEs of second order, as defined by (2), will be studied. In this section, we will develop the suggested approach for the two cases where $\lambda = 1$ and $\lambda = 2$.

Case 1 ($\lambda = 1$)**.**

Let

$$w'''(t)=\sum_{i=1}^{N}a_ih_i(t), \tag{46}$$

integrating three times and using initial conditions, we have

$$w''(t) = E_1 + \sum_{i=1}^{N} a_i R_{1,i}(t), \tag{47}$$

$$w'(t) = w_1 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t), \tag{48}$$

$$w(t) = w_0 + w_1 t + E_1 \frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t). \tag{49}$$

Substitute the above assumption in Eq. (2) to get the following equation,

$$\begin{aligned} E_1 + \sum_{i=1}^{N} a_i R_{1,i}(t) + a_1(t) \left(w_1 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t) \right) \\ + a_2(t) \left(w_0 + w_1 t + E_1 \frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t) \right) \\ = f(t) + \int_0^t M(t,r) \left(w_0 + w_1 s + E_1 \frac{s^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(s) \right) dr, \end{aligned} \tag{50}$$

so

$$\begin{aligned} &\sum_{i=1}^{N} a_i \left(R_{1,i}(t) + a_1(t) R_{2,i}(t) + a_2(t) R_{3,i}(t) - \int_0^t M(t,r) R_{3,i}(s) dr \right) \\ &= f(t) - E_1 \left(1 + a_1(t) t + a_2(t) \frac{t^2}{2} - \int_0^t M(t,r) \frac{s^2}{2} dr \right) \\ &\quad - w_0 \left(a_2(t) - \int_0^t M(t,r) dr \right) \\ &\quad - w_1 \left(a_1(t) + a_2(t) t - \int_0^t s M(t,r) dr \right), \end{aligned} \tag{51}$$

and we substitute $t = 0$ in the above equation to obtain extra condition and value of E_1 as

$$E_1 = f(0) - a_1(0) w_1 - a_2(0) w_0. \tag{52}$$

Case 2 ($\lambda = 2$)**.**

Consider

$$w^{iv}(t) = \sum_{i=1}^{N} a_i h_i(t), \tag{53}$$

integrating four times and using initial conditions, we have

$$w'''(t) = E_1 + \sum_{i=1}^{N} a_i R_{1,i}(t), \tag{54}$$

$$w''(t) = E_2 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t), \tag{55}$$

$$w'(t) = w_1 + E_2 t + E_1 \frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t), \tag{56}$$

and

$$w(t) = w_0 + w_1 t + E_2 \frac{t^2}{2} + E_1 \frac{t^3}{6} + \sum_{i=1}^{N} a_i R_{4,i}. \tag{57}$$

By substituting all of this in Eq. (2), we have

$$\begin{aligned} E_2 + E_1 t + \sum_{i=1}^{N} a_i R_{2,i}(t) + a_1(t) \left(w_1 + E_2 t + E_1 \frac{t^2}{2} + \sum_{i=1}^{N} a_i R_{3,i}(t) \right) \\ + a_2(t) \left(w_0 + w_1 t + E_2 \frac{t^2}{2} + E_1 \frac{t^3}{6} + \sum_{i=1}^{N} a_i R_{4,i} \right) \\ = f(t) + \int_0^t M(t,r) \left(w_0 + w_1 r + E_2 \frac{r^2}{2} + E_1 \frac{r^3}{6} + \sum_{i=1}^{N} a_i R_{4,i}(r) \right) dr. \end{aligned} \tag{58}$$

It can be simplified as

$$\begin{aligned} & \sum_{i=1}^{N} a_i \Big(R_{2,i}(t) + a_1(t) R_{3,i}(t) + a_2(t) R_{4,i} - \int_0^t M(t,r) R_{4,i}(r) dr \Big) \\ & = f(t) - E_2 \left(1 + a_1(t) t + a_2(t) \frac{t^2}{2} - \int_0^t M(t,r) \frac{r^2}{2} dr \right) \\ & \quad - E_1 \left(t + a_1(t) \frac{t^2}{2} + a_2(t) \frac{t^3}{6} - \int_0^t M(t,r) \frac{r^3}{6} dr \right) \\ & \quad - w_0 \left(a_2(t) - \int_0^t M(t,r) dr \right) - w_1 \left(a_1(t) + a_2(t) t - \int_0^t r M(t,r) dr \right). \end{aligned} \tag{59}$$

Using $t = 0$ and $t = 1$ in Eq. (59), we can obtain the extra constants E_1 and E_2. We obtain the value of E_2 for $t = 0$ as

$$E_2 = f(0) - a_1(0) w_1 - a_2(0) w_0. \tag{60}$$

For $t = 1$, we have

$$\begin{aligned} & E_1 \left(1 + \frac{1}{2} a_1(1) + \frac{1}{6} a_2(1) - \int_0^1 \frac{r^3}{6} M(1,r) dr \right) \\ & = f(1) - E_2 \left(1 + a_1(1) + a_2(1) \frac{1}{2} - \int_0^1 M(1,s) \frac{r^2}{2} dr \right) \\ & \quad - w_0 \left(a_2(1) - \int_0^1 M(1,r) dr \right) - w_1 \left(a_1(1) + a_2(1) - \int_0^1 r M(1,r) dr \right) \\ & \quad - \sum_{i=1}^{N} a_i \Big(R_{2,i}(1) + a_1(1) R_{3,i}(1) + a_2(1) R_{4,i}(1) - \int_0^1 M(1,r) R_{4,i}(r) dr \Big). \end{aligned} \tag{61}$$

Using the notations introduced in Eqs. (22) and (41), we can write

$$\begin{aligned} E_1 \left(1 + \frac{1}{2} a_1(1) + \frac{1}{6} a_2(1) - \mathcal{K}_{31} \right) = f(1) - E_2 \left(1 + a_1(1) + a_2(1) \frac{1}{2} - \mathcal{K}_{21} \right) - w_0 \left(a_2(1) - \mathcal{K}_{01} \right) \\ - w_1 \left(a_1(1) + a_2(1) - \mathcal{K}_{11} \right) - \sum_{i=1}^{N} a_i \big(R_{2,i}(1) + a_1(1) R_{3,i}(1) + a_2(1) R_{4,i}(1) - \mathcal{K}_1^4 \big). \end{aligned} \tag{62}$$

Table 1. Numerical results for Example 1 at $p = 0.5$.

		HOHWM ($\lambda = 1$)		HOHWM ($\lambda = 2$)	
J	N	$E_p(N)$	$R_p(N)$	$E_p(N)$	$R_p(N)$
1	4	$3.81e-04$	-	$4.32e-06$	-
2	8	$9.82e-05$	1.96	$2.91e-07$	3.89
3	16	$2.47e-05$	1.99	$1.87e-08$	3.96
4	32	$6.19e-06$	1.99	$1.19e-09$	3.98
5	64	$1.55e-06$	1.99	$7.46e-011$	3.99

The value of E_1 is

$$E_1 = \frac{1}{D_2}\Big(f(1) - \big(f(0) - a_1(0)w_1 - a_2(0)w_0\big)\big(1 + a_1(1) + a_2(1)\frac{1}{2} - \mathcal{K}_{21}\big) - w_0\left(a_2(1) - \mathcal{K}_{01}\right)$$

$$- w_1\left(a_1(1) + a_2(1) - \mathcal{K}_{11}\right) - \sum_{i=1}^{N} a_i\big(R_{2,i}(1) + a_1(1)R_{3,i}(1) + a_2(1)R_{4,i}(1) - \mathcal{K}_1^4\big)\Big), \quad (63)$$

where

$$D_2 = 1 + \frac{1}{2}a_1(1) + \frac{1}{6}a_2(1) - \mathcal{K}_{31}. \quad (64)$$

§4 Numerical experiments

This section includes the computation of the rate of convergence and absolute point-wise errors at random points in order to assess the effectiveness of the suggested method. $E_p(N)$, point-wise absolute error at a point p using N number of CPs is defined as

$$E_p(N) = \left|w(p) - w^*(p)\right|, \quad (65)$$

where approximate solution at point p is $w^*(p)$, and the exact solution is $w(p)$. Additionally, $R_p(N)$ will represent the experimental rate of convergence

$$R_p(N) = \frac{\ln\left(E_p(N/2)/E_p(N)\right)}{\ln 2}. \quad (66)$$

Example 1. Consider the Fredholm IDE [24]

$$w''(t) = e^t - t + \int_0^1 trw(r)dr, \quad w(0) = 1, \quad w'(0) = 1, \quad (67)$$

whose exact solution is $w(t) = e^t$.

HOHWM technique is applied to solve test problem 1. The problem was examined at $t = 0.5$. The results for various HOHWM resolution levels are shown in Table 1, in columns 3 and 4 for $\lambda = 1$ and $\lambda = 2$, respectively. Particularly, an improvement in absolute error can be seen by comparing the results for $\lambda = 1$ and $\lambda = 2$. The final column of this table shows the rate of convergence, which reduces to 4.

Example 2. Consider the Volterra IDE [25]

$$w''(t) = 1 + \int_0^t (t - r)w(r)dr, \quad w(0) = 1, \quad w'(0) = 0, \quad (68)$$

Table 2. Numerical results for Example 2 at $p = 0.5$.

J	N	HOHWM ($\lambda = 1$)		HOHWM ($\lambda = 2$)	
		$E_p(N)$	$R_p(N)$	$E_p(N)$	$R_p(N)$
1	4	$3.29e-04$	-	$4.32e-6$	-
2	8	$8.29e-05$	1.98890	$2.9e-07$	3.89
3	16	$2.08e-05$	1.99717	$1.87e-08$	3.95
4	32	$5.19e-06$	1.99929	$1.19e-09$	3.98
5	64	$1.30e-06$	1.99982	$7.46e-011$	3.99

whose exact solution is $w(t) = \cosh(t)$.

Table 2 provides results for this example at various resolution levels, which were solved by using recently introduced method HOHWM. The performance of suggested technique improves with higher resolution levels, however for $\lambda = 2$ absolute error, it decreases quite quickly.

§5 Conclusion

This article presents a numerical technique for solving second-order Volterra and Fredholm IDEs of second-kind by using HOHWM. The second-order fredholm and Voltera IDE were treated with this technique. The results of the table confirm that the accuracy of the solution is improved compared to the classical HWM and the convergence rate approaches the proven convergence rate for HOHWM, which is 2, for $\lambda = 1$ and 4, for $\lambda = 2$. Based on its successful performance in achieving the goal, this work can be expanded to higher order IDEs.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] J Majak, M Pohlak, K Karjust, et al. *New higher order Haar wavelet method: Application to FGM structures*, Composite Structures, 2018, 201: 72-78.

[2] M Ghasemi, K Mohammadi, A Alipanah. *Numerical solution of system of second-order integro-differential equations using nonclassical Sinc collocation method*, Boundary Value Problems, 2023, 2023: 38.

[3] M Fathy, M El-Gamel, M S El-Azab. *Legendre-Galerkin method for the linear Fredholm integro-differential equations*, Applied Mathematics and Computation, 2014, 243: 789-800.

[4] S Yeganeh, Y Ordokhani, A Saadatmandi. *A Sinc-Collocation Method for Second-Order Boundary Value Problems of Nonlinear Integro-Differential Equation*, Journal of Information and Computing Science, 2012, 7: 151-160.

[5] Z Wang, L Liu, Y Wu. *The unique solution of boundary value problems for nonlinear second-order integral-differential equations of mixed type in Banach spaces*, Computers & Mathematics with Applications, 2007, 54: 1293-1301.

[6] S H Behiry, H Hashish. *Wavelet methods for the numerical solution of Fredholm integro-differential equations*, International Journal of Applied Mathematics, 2002, 11: 27-35.

[7] R Saadeh. *An Iterative Approach to Solve Volterra Nonlinear Integral Equations*, European Journal of Pure and Applied Mathematics, 2023, 16: 1491-1507.

[8] I Aziz. *New algorithms for numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets*, Journal of Computational and Applied Mathematics, 2013, 239: 333-345.

[9] Ü Lepik. *Numerical solution of differential equations using Haar wavelets*, Mathematics and Computers in Simulation, 2005, 68: 127-143.

[10] Ü Lepik. *Haar wavelet method for nonlinear integro-differential equations*, Applied Mathematics and Computation, 2006, 176: 324-333.

[11] Ü Lepik. *Solving PDEs with the aid of two-dimensional Haar wavelets*, Computers & Mathematics with Applications, 2011, 61: 1873-1879.

[12] J Majak, B Shvartsman, M Kirs, et al. *Convergence theorem for the Haar wavelet based discretization method*, Composite Structures, 2015, 126: 227-232.

[13] J Majak, B Shvartsman, K Karjust, et al. *On the accuracy of the Haar wavelet discretization method*, Composites Part B: Engineering, 2015, 80: 321-327.

[14] C F Chen, C H Hsiao. *Haar wavelet method for solving lumped and distributed-parameter systems*, IEE Proceedings-Control Theory and Applications, 1997, 144: 87-94.

[15] Ü Lepik. *Numerical solution of evolution equations by the Haar wavelet method*, Applied Mathematics and Computation, 2007, 185: 695-704.

[16] I Aziz, B Šarler. *Wavelets collocation methods for the numerical solution of elliptic BV problems*, Applied Mathematical Modelling, 2013, 37: 676-694.

[17] M Ratas, A Salupere. *Application of Higher Order Haar Wavelet Method for Solving Nonlinear Evolution Equations*, Mathematical Modelling and Analysis, 2020, 25: 271-288.

[18] M Ratas, A Salupere, J Majak. *Solving Nonlinear PDEs Using the Higher Order Haar Wavelet Method on Nonuniform and Adaptive Grids*, Mathematical Modelling and Analysis, 2021, 26: 147-169.

[19] J Majak, B Shvartsman, M Ratas, et al. *Higher-order Haar wavelet method for vibration analysis of nanobeams*, Materials Today Communications, 2020, 25: 101290.

[20] J Majak, M Pohlak, M Eerme, et al. *Solving ordinary differential equations with higher order Haar wavelet method*, AIP Conference Proceedings, 2019, 2116(1): 330002.

[21] M Singh, K Singh. *An efficient technique based on higher order Haar wavelet method for Lane-Emden equations*, Mathematics and Computers in Simulation, 2023, 206: 21-39.

[22] L Kivistik, M Mehrparvar, M Eerme, et al. *Dynamics of flight of the fragments with higher order Haar wavelet method*, Proceedings of the Estonian Academy of Sciences, 2024, 73: 108-115.

[23] S Yasmeen, S Islam, R Amin. *Higher order Haar wavelet method for numerical solution of integral equations*, Computational and Applied Mathematics, 2023, 42: 147.

[24] C Jalius, Z A Majid. *Numerical solution of Second-Order Fredholm integrodifferential equations with boundary conditions by Quadrature-Difference method*, Journal of Applied Mathematics, 2017, 2017: 2645097.

[25] R Amin, I Mahariq, K Shah, et al. *Numerical solution of the second order linear and nonlinear integro-differential equations using Haar wavelet method*, Arab Journal of Basic and Applied Sciences, 2021, 28(1): 12-20.

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