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Global attractivity of a rational difference equation with higher order and its application to several conjectures

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Abstract. We study the global dynamics of a rational difference equation with higher order, which includes many rational difference equations as its special cases. By some complicate computations and mathematical skills, we show that its unique nonnegative fixed point is globally attractive. As application, our results not only improve many known ones, but also solve several "Open Problems and Conjectures" given by Professors Ladas and Camouzis, et al.

§1 Introduction

Difference equation are one of the most powerful tools to describe the change rule of natural phenomena. There are many real applications for difference equation in various disciplines, such as cybernetics, biology, physics and other applied fields. Difference equation comes not only from the discretization of differential equations, but also from the modelling of real problems. Rational difference equation (for short, RDE) is a typical kind of nonlinear difference equations, whose research time is not long. Because the research of many core problems for difference equations is due to the prototype for the problems of RDEs, the investigations of RDEs have received much attention and developed rapidly in the past several decades. For example, refer to the monographs [1–3] and the papers [4–15] and the references therein.

Recently, Li and Zhu [16] considered the following RDE

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k} + sx_{n-l}}, \quad n = 0, 1, \cdots, \tag{1}$$

where the parameters $p, q, r, s \in [0, \infty)$, k and l are positive integers with $k < l$, and the initial conditions $x_{-m}, ..., x_{-1}, x_0 \in (0, \infty)$.

The results obtained in [16] not only include and improve many known ones [1–3, 17–22], but also solve several "Open Problemss and Conjectures" presented by famous professors Ladas,

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Kocic, Kulenovic and Camouzis, et al. in [1–3, 22]. Although those results in [16] are very pretty, we still find that there are some limitations for their results. For example, their results can not solve the attractivity for the following two difference equations [3, **Equation♯189**]

$$x_{n+1} = \frac{p + x_n}{A + x_n + sx_{n-1} + tx_{n-2}},\ n = 0, 1, ..., \tag{2}$$

with positive parameters p, A, s, t and arbitrary nonnegative initial conditions x_{-2}, x_{-1}, x_0, and [3, **Equation♯134**]

$$x_{n+1} = \frac{qx_n}{A + rx_n + sx_{n-1} + tx_{n-2}},\ n = 0, 1, ..., \tag{3}$$

with positive parameters q, A, r, s, t and arbitrary nonnegative initial conditions x_{-2}, x_{-1}, x_0. Certainly, there are still many other problems for higher order RDEs which need to be further investigated. These existing problems motivate us to consider in this paper the following RDE with higher order

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k} + sx_{n-l} + tx_{n-m}}, \quad n = 0, 1, \cdots. \tag{4}$$

where the parameters $p, q, r, s, t \in [0, \infty)$, k, l and m are positive integers with $k < l < m$, and the initial conditions $x_{-m}, ..., x_{-1}, x_0 \in (0, \infty)$. To avoid trivial cases, we suppose that $p+q > 0$ and $r + s + t > 0$.

Eq.(4), which is the main aim in this paper to be considered, obviously is a generalization of Eq.(1). Although the forms of Eqs.(1) and (4) look similar, they possess completely different recursive rules. So, Eq.(4) is worthy investigating. Our results in this paper not only include all of results in [16], but also generalize all of the corresponding results [17–21], and solves some new problems that can not be solved by known work.

§2 Main result and its proof

In this section, we formulate our main result in this paper and its proof. The main idea for the proof is to comprehensively use the three key lemmas cited in the appendix to transfer the higher order RDE (4) into one order difference eqution that is easily dealt with.

Eq.(4) has a unique nonnegative fixed point, denoted as $\bar{x}$, namely,

$$\bar{x} = \frac{q - 1 + \sqrt{(q-1)^2 + 4p(r + s + t)}}{2(r + s + t)}.$$

Our main result in this paper is as follows.

Theorem 2.1 Consider Eq.(4). Assume that the parameters $p, q, r, s, t \in [0, \infty)$ with $p + q > 0$ and $r + s + t > 0$, and the paramenters k, l and m are positive integers with $k < l < m$. Then the unique nonnegative fixed point $\bar{x}$ of Eq.(4) is a global attractor of all of its positive solutions.

Proof. When $p = 0$ and $q \in (0, 1]$, $\bar{x} = 0$. Then it follows from Eq.(4) that $x_{n+1} < qx_n$, so one can see x_n eventually monotonically approaches $\bar{x}$. Notice $\overline{\{p = 0\} \cap \{q \in [0, 1]\}} = \{p > 0\} \cup q \in (1, \infty)$. Hence, in the next, one assumes $p > 0$ or $q \in (1, \infty)$, namely, we only study the behavior of positive fixed point of Eq.(4).

Evidently, Eq.(4) may be written as

$$x_{n+1} = x_n \frac{\frac{p}{x_n} + q}{1 + rx_{n-k} + sx_{n-l} + tx_{n-m}}. \tag{5}$$

Set

$$f(u_0, u_1, .., u_k, .., u_l, .., u_m) = \frac{\frac{p}{u_0} + q}{1 + ru_k + su_l + tu_m}.$$

The function f may be verified to satisfy the conditions (H1)-(H4) of Lemma 4.1 in the appendix. So, the function G defined by (23) may be derived as

$$\begin{aligned} G(x,y) &= y\frac{\frac{p}{y} + q}{1 + (r+s+t)x} \frac{\frac{p}{\bar{x}} + q}{1 + (r+s)\bar{x} + ty} \left(\frac{\frac{p}{\bar{x}} + q}{1 + (r+s+t)x}\right)^{m-1} \\ &= \left[\frac{1 + (r+s+t)\bar{x}}{1 + (r+s+t)x}\right]^m \frac{p + qy}{1 + (r+s)\bar{x} + ty}. \end{aligned}$$

Moreover,

$$\frac{\partial G(x,y)}{\partial y} = \left[\frac{1 + (r+s+t)\bar{x}}{1 + (r+s+t)x}\right]^m \frac{q(1 + (r+s)\bar{x}) - pt}{(1 + (r+s)\bar{x} + ty)^2}. \tag{6}$$

In order to apply Lemma 4.3, one has to calculate the function F defined by (22). Consider the following two cases.

Case I: $0 < q \leq \frac{pt}{1+(r+s)\bar{x}}$.

In this case, it follows from (6) that the function F can be given by

$$\begin{aligned} F(x) &= \left[\frac{1 + (r+s+t)\bar{x}}{1 + (r+s+t)x}\right]^m \frac{p + qx}{1 + (r+s)\bar{x} + tx} \\ &= \frac{A(p + qx)}{[1 + (r+s+t)x]^m (1 + (r+s)\bar{x} + tx)}, \end{aligned} \tag{7}$$

where $\bar{x}, x \in (0, \infty), A = [1 + (r+s+t)\bar{x}]^m$.

Now in order to show that $\overline{x}$ is a global attractor of all positive solutions of Eq.(4), by Lemma 4.1 (b), it suffices for us to verify that the function F has no periodic points of prime 2 except $\overline{x}$. According to Lemma 4.2, one must prove that $\overline{x}$ is a global attractor of positive solutions of the difference equation (24) with $F(x)$ defined by Eq.(7) and $x_0 \in [0, \infty)$. Accordingly, in view of Lemma 4.3, one has to verify that F has a negative Schwarzian derivative.

To do this, notice

$$\begin{aligned} F'(x) &= A\frac{q[(1 + (r+s+t)x)^m(1 + (r+s)\bar{x} + tx)]}{(1 + (r+s+t)x)^{2m}(1 + (r+s)\bar{x} + tx)^2} \\ &\quad - A\frac{(p+qx)[m(r+s+t)(1 + (r+s+t)x)^{m-1}(1 + (r+s)\bar{x} + tx)]}{(1 + (r+s+t)x)^{2m}(1 + (r+s)\bar{x} + tx)^2} \\ &\quad - A\frac{(p+qx)[t(1 + (r+s+t)x)^m]}{(1 + (r+s+t)x)^{2m}(1 + (r+s)\bar{x} + tx)^2} \\ &= A\frac{q[(1 + (r+s+t)x)(1 + (r+s)\bar{x} + tx)]}{(1 + (r+s+t)x)^{m+1}(1 + (r+s)\bar{x} + tx)^2} \\ &\quad - A\frac{(p+qx)[m(r+s+t)(1 + (r+s)\bar{x} + tx) + t(1 + (r+s+t)x)]}{(1 + (r+s+t)x)^{m+1}(1 + (r+s)\bar{x} + tx)^2} \\ &= \frac{A\Delta}{(1 + (r+s+t)x)^{m+1}(1 + (r+s)\bar{x} + tx)^2}, \end{aligned}$$

where

$$\begin{aligned}\Delta =: & [q(1+(r+s)\bar{x})-pt](1+(r+s+t)x) \\ & -m(r+s+t)(1+(r+s)\bar{x}+tx)(p+qx) \\ = & -T(1+(r+s+t)x)-m(r+s+t)(1+(r+s)\bar{x}+tx)(p+qx), \\ T =: & pt-q(1+(r+s)\bar{x}).\end{aligned} \tag{8}$$

The condition $0<q\leq \frac{pt}{(1+(r+s)\bar{x})}$ implies $T\geq 0$. So $\Delta<0$ and hence, $F'(x)<0$.

Take $I=(0,\frac{pA}{1+(r+s)\bar{x}}]$. For any given $x\in I$, one has $0<F(x)<F(0)=\lim_{x\to 0^+}F(x)=\frac{pA}{1+(r+s)\bar{x}}$. So, $F(I)\subset I$. In addition,

$$F''(x)=A\frac{\Delta'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)]-\Delta[U+2t(1+(r+s+t)x)]}{(1+(r+s+t)x)^{m+2}(1+(r+s)\bar{x}+tx)^3},$$

where $U=:(m+1)(r+s+t)(1+(r+s)\bar{x}+tx)>0$. From (2.4), one has

$$\Delta'=(r+s+t)[-T-ptm-qm(1+(r+s)\bar{x}+2tx)]<0. \tag{9}$$

Denote

$$\Gamma=\Delta'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)]-\Delta[U+2t(1+(r+s+t)x)]. \tag{10}$$

Then

$$F''(x)=\frac{A\Gamma}{(1+(r+s+t)x)^{l+2}(1+(r+s)\bar{x}+tx)^3}, \tag{11}$$

Let $V=:(r+s+t)(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)$. Then $V>0$. Now let us determine the sign of Γ. Obviously,

$$\begin{aligned}\Gamma = & V[-T-ptm-qm(1+(r+s)\bar{x}+2tx)]+(m+1)TV+2t(1+(r+s+t)x)^2T \\ & +m(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)+2tm(p+qx)V \\ = & V[-T-ptm-qm(1+(r+s)\bar{x}+2tx)+T(m+1)+2tm(p+qx)] \\ & +lU(r+s+t)(1+(r+s)\bar{x}+tx)(p+qx)+2t(1+(r+s+t)x)^2T \\ = & 2mTV+m(r+s+t)(1+(r+s)\bar{x}+tx)(p+qx)U+2t(1+(r+s+t)x)^2T.\end{aligned}$$

So, $\Gamma>0$, and from (11), one can see $F''(x)>0$. Furthermore,

$$\frac{F''(x)}{F'(x)}=\frac{\Gamma}{(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta}$$

and

$$\begin{aligned}(\frac{F''(x)}{F'(x)})' = & \frac{\Gamma'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta]}{(1+(r+s+t)x)^2(1+(r+s)\bar{x}+tx)^2\Delta^2} \\ & -\frac{\Gamma[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta]'}{(1+(r+s+t)x)^2(1+(r+s)\bar{x}+tx)^2\Delta^2}.\end{aligned}$$

Set

$$\begin{aligned}\Omega = & \Gamma'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta] \\ & -\Gamma[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta]'.\end{aligned} \tag{12}$$

Accordingly,

$$\begin{aligned}SF(x) = & [\frac{F''(x)}{F'(x)}]'-\frac{1}{2}[\frac{F''(x)}{F'(x)}]^2 \\ = & \frac{\Omega-1/2\Gamma^2}{(1+(r+s+t)x)^2(1+(r+s)\bar{x}+tx)^2\Delta^2}.\end{aligned} \tag{13}$$

According to the definition (10) of Γ, we have

$$\begin{aligned}\Gamma' = & 2m(r+s+t)T[(r+s+t)(1+(r+s)\bar{x}+tx)+t(1+(r+s+t)x)] \\ & + 2mt(m+1)(r+s+t)^2(p+qx)(1+(r+s)\bar{x}+tx) \\ & + qm(ml+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2 \\ & + 4tT(r+s+t)(1+(r+s+t)x).\end{aligned} \tag{14}$$

Denote $I =: (r+s+t)(1+(r+s)\bar{x}+tx)+t(1+(r+s+t)x) > 0$. Then

$$\begin{aligned}\Gamma' = & 2m(r+s+t)TI + 2mt(m+1)(r+s+t)^2(p+qx)(1+(r+s)\bar{x}+tx) \\ & + qm(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2 \\ & + 4tT(r+s+t)(1+(r+s+t)x) > 0.\end{aligned}$$

Put $\Phi = \Omega - 1/2\Gamma^2$ and $H = \Gamma'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta]$. Now simplify Φ to see whether $\Phi < 0$ or not.

$$\begin{aligned}\Phi = & \Omega - 1/2\Gamma^2 \\ = & \Gamma'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta] \\ & -\Gamma[(I\Delta + (1+(r+s)\bar{x}+tx)(1+(r+s+t)x)\Delta' + 1/2\Gamma] \\ = & H - \Gamma\{I[-T(1+(r+s+t)x) - m(r+s+t)(1+(r+s)\bar{x}+tx)(p+qx)] \\ & + V[-T - ptm\ddot{} - qm(1+(r+s)\bar{x}+2tx)] + t(1+(r+s+t)x)^2T \\ & + mTV + 1/2m(r+s+t)(1+(r+s)\bar{x}+tx)(p+qx)U\} \\ = & H - \Gamma\{-TV - t(1+(r+s+t)x)^2T - m(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) \\ & - mt(p+qx)V + V[-T - m(pt + q(1+(r+s)\bar{x}+2tx)] + mTV \\ & + 1/2m(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) + t(1+(r+s+t)x)^2T\} \\ = & H - \Gamma\{1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) + mTV \\ & + V[-T - T - mt(p+qx) - m(pt + 2q(1+(r+s)\bar{x}) + 2qtx - q(1+(r+s)\bar{x})]\} \\ = & H - \Gamma\{1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) + mTV \\ & + V[-2T - mT - m(pt + qtx + 2q(1+(r+s)\bar{x}+tx))]\} \\ = & H - \Gamma\{1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) + mTV \\ & + V[-(m+2)T - m(pt - q(1+(r+s)\bar{x}) + qtx + 3q(1+(r+s)\bar{x}) + 2qtx]\} \\ = & H - \Gamma[1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) \\ & -(m+2)TV - 3qm(1+(r+s)\bar{x}+tx)V] \\ =: & H + Q,\end{aligned}$$

where $Q = -\Gamma[1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) - (m+2)TV - 3qm(1+(r+s)\bar{x}+tx)V]$.

Now further simply H and Q.

$$\begin{aligned}H = & \Gamma'[(1+(r+s+t)x)(1+(r+s)\bar{x}+tx)\Delta] \\ = & \Gamma'[-T(1+(r+s+t)x)^2(1+(r+s)\bar{x}+tx) - m(p+qx)(1+(r+s)\bar{x}+tx)V] \\ = & 2m(r+s+t)^2(1+(r+s)\bar{x}+tx)T\cdot(-T(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2)\end{aligned}$$

$$
\begin{aligned}
&+2m(r+s+t)^2(1+(r+s)\bar{x}+tx)T\cdot(-m(p+qx)(1+(r+s)\bar{x}+tx)V)\\
&+2mt(r+s+t)(1+(r+s+t)x)T\cdot(-T(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2)\\
&+2mt(r+s+t)(1+(r+s+t)x)T\cdot(-m(p+qx)(1+(r+s)\bar{x}+tx)V)\\
&+2mt(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)T\cdot(-T(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2)\\
&+2mt(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)T\cdot(-m(p+qx)(1+(r+s)\bar{x}+tx)V)\\
&+qm(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2\cdot(-T(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2)\\
&+qm(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2\cdot(-m(p+qx)(1+(r+s)\bar{x}+tx)V)\\
&+4t(r+s+t)(1+(r+s+t)x)T\cdot(-T(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2)\\
&+4t(r+s+t)(1+(r+s+t)x)T\cdot(-m(p+qx)(1+(r+s)\bar{x}+tx)V)\\
=&-2mT^2V^2-2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV\\
&-2mt(1+(r+s+t)x)^2T^2V-2m^2t(p+qx)TV^2-qm(l+1)(1+(r+s)\bar{x}+tx)TV^2\\
&-qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)V-2tm(m+1)(p+qx)TV^2\\
&-2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2V\\
&-4t(1+(r+s+t)x)^2T^2V-4mt(p+qx)TV^2\\
=&-2mT^2V^2-TV^2[2mt(m+1)(p+qx)+qm(m+1)(1+(r+s)\bar{x}+tx)]\\
&-TV^2[4mt(p+qx)+2m^2t(p+qx)]-T^2V[2mt(1+(r+s+t)x)^2\\
&+4t(1+(r+s+t)x)^2]-2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV\\
&-V[qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)\\
&+2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2]\\
=&-2mT^2V^2-TV^2[m^2(2pt+2qtx+q(1+(r+s)\bar{x}+tx))\\
&+m(2pt+2qtx+q(1+(r+s)\bar{x}+tx))]-TV^2[4mt(p+qx)+2m^2t(p+qx)]\\
&-T^2V[2mt(1+(r+s+t)x)^2+4t(1+(r+s+t)x)^2]\\
&-2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV\\
&-V[qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)\\
&+2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2]\\
&=-2mT^2V^2-TV^2[m^2(2pt-2q(1+(r+s)\bar{x})+3q(1+(r+s)\bar{x})+3qtx)\\
&+m(2pt-2q(1+(r+s)\bar{x})+3q(1+(r+s)\bar{x})+3qtx)]\\
&-TV^2[4mt(p+qx)+2m^2t(p+qx)]-T^2V[2mt(1+(r+s+t)x)^2\\
&+4t(1+(r+s+t)x)^2]-2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV\\
&-V[qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)\\
&+2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2]\\
=&-2mT^2V^2-TV^2[m^2(2T+3q(1+(r+s)\bar{x}+tx))\\
&+m(2T+3q(1+(r+s)\bar{x}+tx))]-TV^2[4mt(p+qx)+2m^2t(p+qx)]
\end{aligned}
$$

$$
\begin{aligned}
&- T^2V[2mt(1+(r+s+t)x)^2 + 4t(1+(r+s+t)x)^2] \\
&- 2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV \\
&- V[qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx) \\
&+ 2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2] \\
=& -2mT^2V^2 - 2m^2T^2V^2 - 2mT^2V^2 - 3qm(1+(r+s)\bar{x}+tx)(m+1)TV^2 \\
&- TV^2[4mt(p+qx) + 2m^2t(p+qx)] - T^2V[2mt(1+(r+s+t)x)^2 \\
&+ 4t(1+(r+s+t)x)^2] - 2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV \\
&- V[qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx) \\
&+ 2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2] \\
=& -2m(m+2)T^2V^2 - T^2V[2mt(1+(r+s+t)x)^2 + 4t(1+(r+s+t)x)^2] \\
&- TV^2[3qm(1+(r+s)\bar{x}+tx)(m+1) + 4mt(p+qx) + 2m^2t(p+qx)] \\
&- 2m^2(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV \\
&- V[qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx) \\
&+ 2tm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2]
\end{aligned}
$$

and

$$
\begin{aligned}
Q =& -1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)\cdot 2mTV \\
&- 1/2m(m-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2\cdot m(m+1) \\
&- 1/2m(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)\cdot 2t(1+(r+s+t)x)^2T \\
&+ (m+2)TV\cdot 2mTV + (m+2)TV\cdot 2t(1+(r+s+t)x)^2T \\
&+ (m+2)TV\cdot m(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) \\
&+ 3qm(1+(r+s)\bar{x}+tx)V\cdot 2mTV + 3qm(1+(r+s)\bar{x}+tx)V\cdot 2t(1+(r+s+t)x)^2T \\
&+ 3qm(1+(r+s)\bar{x}+tx)V\cdot m(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx) \\
=& -m^2(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV \\
&- 1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2 \\
&- mt(m-1)(p+qx)TV^2 + 2m(m+2)T^2V^2 \\
&+ m(m+1)(m+2)(p+qx)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2TV \\
&+ 2t(m+2)(1+(r+s+t)x)^2T^2V + 6qm^2(1+(r+s)\bar{x}+tx)TV^2 \\
&+ 3qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)V \\
&+ 6qmt(1+(r+s+t)x)^2(1+(r+s)\bar{x}+tx)TV.
\end{aligned}
$$

Combine H and Q to obtain

$$
\begin{aligned}
\Phi =& T^2V^2[2m(m+2) - 2m(m+2)] + T^2V[2t(m+2)(1+(r+s+t)x)^2 \\
&- 2mt(1+(r+s+t)x)^2 - 4t(1+(r+s+t)x)^2] \\
&+ TV^2[6qm^2(1+(r+s)\bar{x}+tx) - mt(m-1)(p+qx) \\
&- 3qm(1+(r+s)\bar{x}+tx)(m+1) - 4mt(p+qx) - 2m^2t(p+qx)]
\end{aligned}
$$

$$
\begin{aligned}
&+TV[m(m+1)(m+2)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)\\
&+6qmt(1+(r+s+t)x)^2(1+(r+s)\bar{x}+tx)\\
&-2m^2(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)\\
&-m^2(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)]\\
&+V[3qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)\\
&-qm^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^3(p+qx)\\
&-2m^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)^2]\\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2\\
=\ &TV^2[3qm^2(1+(r+s)\bar{x}+tx)-3qm(1+(r+s)\bar{x}+tx)\\
&-m^2pt-m^2qtx+mpt+mqtx-2m^2(p+qx)-4mt(p+qx)]\\
&+TV\cdot 2m(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)\\
&+TV\cdot 6qmt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2\\
&+V[2m^2(m+1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)[q(1+(r+s)\bar{x}+tx)-t(p+qx)]]\\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2\\
=\ &TV^2[m^2(3q(1+(r+s)\bar{x}+tx)-pt-qtx)-m(3q(1+(r+s)\bar{x}+tx)-pt-qtx)\\
&-2m^2(p+qx)-4mt(p+qx)]+6qmt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2TV\\
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV\\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2\\
=\ &TV^2[m^2(2q(1+(r+s)\bar{x}+tx)+qtx+q(1+(r+s)\bar{x})-pt-qtx)\\
&-m(2q(1+(r+s)\bar{x}+tx)+qtx+q(1+(r+s)\bar{x})-pt-qtx)\\
&-2m^2(p+qx)-4mt(p+qx)]+6qmt(1+(r+s)\bar{x}+tx)(1+(r+s)x)^2TV\\
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV\\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2.\\
=\ &TV^2[-m^2T+mT+2m^2[q(1+(r+s)\bar{x}+tx)-t(p+qx)]\\
&-2m[q(1+(r+s)\bar{x}+tx)+2t(p+qx)]]\\
&+6qmt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2TV\\
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV\\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2\\
=\ &(-m^2+m)T^2V^2+TV^2[2m^2(-T)-2m(q(1+(r+s)\bar{x})+qtx-pt+3pt+2qtx)]\\
&+6qmt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2TV\\
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV\\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2\\
=\ &-3m(m-1)T^2V^2-6mt(p+qx)TV^2+6qmt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)^2TV
\end{aligned}
$$

$$
\begin{aligned}
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV \\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2. \\
=\ &6mt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)TV[q(1+(r+s+t)x-(p+qx)(r+s+t)] \\
&-3m(m-1)T^2V^2-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV \\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2. \\
=\ &-3m(m-1)T^2V^2 \\
&-6mt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)(q(r+s)\bar{x}+T+p(r+s))TV \\
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV \\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2.
\end{aligned}
$$

Namely, eventually, one has

$$
\begin{aligned}
\Phi =\ &-3m(m-1)T^2V^2 \\
&-2m(m+1)(m-1)(r+s+t)^2(1+(r+s)\bar{x}+tx)^2(p+qx)TV \\
&-6mt(1+(r+s)\bar{x}+tx)(1+(r+s+t)x)(q(r+s)\bar{x}+T+p(r+s))TV \\
&-1/2m^2(m^2-1)(r+s+t)^4(1+(r+s)\bar{x}+tx)^4(p+qx)^2.
\end{aligned}
$$

Noticing that $m>1$ is a positive integer, it is easy to see $\Phi<0$. Therefore, in lihgt of (13), one sees $SF(x)<0$. Accordingly, Lemma 4.3 reads that $\bar{x}$ is a global attractor of all positive solutions of Eq.(24). In turn, according to Lemma 4.2, $\bar{x}$ is the only fixed point of F^2 in $(0,\infty)$. Then, using Lemma 4.1 (b), it has been shown that $\bar{x}$ is a global attractor for all positive solutions of Eq.(21), hence Eq.(5) and so Eq.(4).

Case II: $q>\frac{pt}{1+(r+s)\bar{x}}$.

In this case, in view of (22), one can easily derive

$$
F(x)=[\frac{1+(r+s+t)\bar{x}}{1+(r+s+t)x}]^m\frac{p+q\bar{x}}{1+(r+s)\bar{x}+t\bar{x}}=\frac{B}{[1+(r+s+t)x]^m}, \tag{15}
$$

where $\bar{x},x\in(0,\infty),B=:(p+q\bar{x})(1+(r+s+t)\bar{x})^{m-1}$.

Now verify that the function F has no nontrivial periodic points of prime period 2. In order to arrive at this, let $L=F(M)$ and $M>0$ be the fixed point of $F^2(x)$, i.e., $M=F^2(M)=F(L)$. From $L=\frac{B}{[1+(r+s+t)M]^m}$ and $M=\frac{B}{[1+(r+s+t)L]^m}$, one has

$$
\frac{[1+(r+s+t)M]^m}{M}=\frac{[1+(r+s+t)L]^m}{L}. \tag{16}
$$

It is easy to see from (15) that $0<F(x)<F(0)=lim_{x\to0^+}F(x)=B$. Take $I=[0,\ B]$, then, $F(I)\subset I$. In order to apply Lemma 4.3, it next suffices to show that the Schwarzian derivative of the function F is negative.

Some calculations display

$$
F'(x)=-\frac{Bm(r+s+t)}{[1+(r+s+t)x]^{m+1}},\ F''(x)=\frac{Bm(m+1)(r+s+t)^2}{[1+(r+s+t)x]^{m+2}}.
$$

Accordingly,

$$
\frac{F''(x)}{F'(x)}=-\frac{(m+1)(r+s+t)}{1+(r+s+t)x},\ \left[\frac{F''(x)}{F'(x)}\right]'=\frac{(m+1)(r+s+t)^2}{[1+(r+s+t)x]^2}.
$$

Thus,

$$SF(x) = \left[\frac{F''(x)}{F'(x)}\right]' - \frac{1}{2}\left[\frac{F''(x)}{F'(x)}\right]^2$$

$$\begin{aligned} &= \frac{(m+1)(r+s+t)^2}{[1+(r+s+t)x]^2} - \frac{1}{2}\frac{(m+1)^2(r+s+t)^2}{[1+(r+s+t)x]^2} \\ &= \frac{(m+1)(1-m)(r+s+t)^2}{2[1+(r+s+t)x]^2}. \end{aligned} \tag{17}$$

Because $m > 1$ is a positive integer, it follows from (17) that $SF(x) < 0$. By Lemma 4.3, $\bar{x}$ is a global attractor of all positive solutions of Eq.(21). Thereout, in view of Lemma 4.2, $\bar{x}$ is the only fixed point of F^2 in $(0, \infty)$. Then Lemma 4.1 (b) says that the unique nonnegative fixed point $\bar{x}$ of Eq.(5), hence Eq.(4), is a global attractor of all of its positive solutions.

Up to here, Cases I and II together finish the proof of the theorem.

Remark 2.1 Eq.(4) is equivalent to the following difference equation

$$x_{n+1} = \frac{p + qx_n}{A + rx_{n-k} + sx_{n-l} + tx_{n-m}}, \ n = 0, 1, \cdots, \tag{18}$$

where the parameters $A > 0$, $p, q, r, s, t \in [0, \infty)$ with $p + q > 0$ and $r + s + t > 0$, k, l and m are positive integers with $k < l < m$, and the initial conditions $x_{-m}, ..., x_{-1}, x_0 \in (0, \infty)$.

In fact, when $A > 0$, the parameter changes $(p, q, r, s, t) \to (Ap, Aq, Ar, As, At)$ in Eq.(18) transform Eq.(18) into Eq.(4). So, according to Theorem 2.1, we have the following result.

Corollary 2.1 Consider Eq.(18). Assume that the parameters $A > 0$, $p, q, r, s, t \in [0, \infty)$ with $p + q > 0$ and $r + s + t > 0$, and the paramenters k, l and m are positive integers with $k < l < m$. Then the unique nonnegative fixed point $\bar{x}$ of Eq.(18) is a global attractor of all of its positive solutions.

Remark 2.2 Because the results derived in [16] are special cases of our result, namely, Theorem 2.1 with $t = 0$, those "Open Problems and Conjectures" (nubered as Conjecture 1.1 and Conjecture 1.2 in [16] and "Open Problemss and Conjectures" given by Ladas, Kocic, Kulenovic and Camzious, et al. in [1–3, 22]) are also solved by our Theorem 2.1. Moreover, our results also improve and generalize corresponding results in [1–3, 17–22].

§3 Applications

In the section we formulate some new applications of our results, namely, Theorem 2.1 and Corollary 2.1.

Example 1. Consider the following different equation [3, **Equation**♯189]

$$x_{n+1} = \frac{p + x_n}{A + x_n + sx_{n-1} + tx_{n-2}}, \ n = 0, 1, ... \tag{19}$$

with positive parameters p, A, s, t and arbitrary nonnegative initial conditions x_{-2}, x_{-1}, x_0.

Professors Camouzis and Ladas in [3] stated, for $A \geq 1$, the equilibrium $\bar{x}(= \frac{1-A+\sqrt{(1-A)^2+4p(1+s+t)}}{2(1+s+t)})$ of Eq.(3.1) is globally asymptotically stable.

But, by our Corollary 2.1, one can see that the fixed point $\bar{x}$ of Eq.(19) is globally asymptotically stable for $A > 0$. So our result improves the corresponding result in [3].

In addition, the authors of [3] presented the following Conjecture.

Conjecture 3.1 [3, Conjecture 5.189.2, P_{399}] Show that Eq.(19) has solutions that do not converge to the equilibrium point $\bar{x}$ or to a periodic solution when $A < 1$.

In view of our Corollary 2.1, the unique fixed point $\bar{x}$ of Eq.(19) is a global attractor of all of its positive solutions. That is to say, all solutions of Eq.(19) converge to $\bar{x}$ for $A > 0$. So the Conjecture 3.1 is incorrect.

Example 2. Consider the following different equation [3, **Equation**♯134]

$$x_{n+1} = \frac{qx_n}{A + rx_n + sx_{n-1} + tx_{n-2}},\ n = 0, 1, ... \tag{20}$$

with positive parameters q, A, r, s, t and arbitrary nonnegative initial conditions x_{-2}, x_{-1}, x_0.

The authors of [3] argued, for $A \geq 1$, the fixed point $\bar{x}(= \frac{q-A}{r+s+t})$ of Eq.(20) is globally asymptotically stable.

But, according to our Corollary 2.1, when $A > 0$, the equilibrium $\bar{x}$ of Eq.(20) is globally asymptotically stable. So our result improves the corresponding result in [3].

In addition, Professors Camouzis and Ladas in [3] presented the following Conjecture.

Conjecture 3.2 [3, Conjecture 5.134.2, P_{320}] Show that Eq.(20) has solutions that do not converge to the equilibrium point $\bar{x}$ or to a periodic solution.

According to our Corollary 2.1, the unique fixed point $\bar{x}$ of Eq.(20) is a global attractor of all of its positive solutions. In other words, all solutions of Eq.(20) tend to the equilibrium point $\bar{x}$ of Eq.(20) for $A > 0$. So the Conjecture 3.2 is incorrect.

§4 Conclusion and discussion

We investigate in this note the global attractity of unique nonnegative equilibrium solution for a rational difference equation with higher order. By some lengthy and difficult computations, we eventually demonstrate that the unique nonnegative fixed point of the rational difference equation is globally attractive. As application, our results not only improve and generalize many known results [1–3, 17–22], but also solve many "Open Problems and Conjectures" given in [1–3, 22] by famous professors Ladas, Kocic, Kulenovic and Camzious, et al.

Although the forms of RDEs look very simple, generally speaking, it is extremely difficult for one to derive a complete result for some characters in their entire parameter space because such RDEs generally contain many parameters, such as 8 paramerters in Eq.(1.4), and some calculations to derive such properties which look vey simle are actually very complex and fritter one's patience. Hence, one often only obtains part results in the entire parameter space. However, RDEs possess many fascinating properties, such as dichotomy [23,24], trichotomy [25], bifurcation [26–31], and chaos [26,32]. Up to now, one has not found any effective methods or ways to deal with such problems yet. One always tries to look for effective methods or ways to deal with such problems. This just is the charm for one to like investigating RDEs.

§5 Appendix

For readers' convenience, several key lemmas used in this paper to prove our main result are presented here.

Lemma 4.1 [1, Theorem 2.3.1, P_{40}] Consider the difference equation

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, \cdots, x_{n-k_r}), \tag{21}$$

where $k_1, k_2, \cdots, k_r$ are positive integers. Denote by k the maximum of $k_1, k_2, \cdots, k_r$. Also, assume that the function f satisfies the following hypotheses:

(H1) $f \in C[(0,\infty)\times[0,\infty)^r,\ (0,\infty)]$ and $g \in C[[0,\infty)^{r+1},\ (0,\infty)]$, where $g(u_0, u_1, \cdots, u_r) = u_0 f(u_0, u_1, \cdots, u_r)$ for $u_0 \in (0,\infty)$ and $u_1, \cdots, u_r \in [0,\infty)$, $g(0, u_1, \cdots, u_r) = \lim_{u_0\to 0^+} g(u_0, u_1, \cdots, u_r)$;

(H2) $f(u_0, u_1, \cdots, u_r)$ is nonincreasing in $u_1, \cdots, u_r$;

(H3) The equation $f(x, x, \cdots, x) = 1$ has a unique positive solution $\overline{x}$;

(H4) Either the function $f(u_0, u_1, \cdots, u_r)$ does not depend on u_0 or for every $x > 0$ and $u \geq 0$,

$$[f(x, u, \cdots, u) - f(\overline{x}, u, \cdots, u)](x - \overline{x}) \leq 0$$

with

$$[f(x, \overline{x}, \cdots, \overline{x}) - f(\overline{x}, \overline{x}, \cdots, \overline{x})](x - \overline{x}) < 0 \quad for \quad x \neq \overline{x}.$$

Define a new function F given by

$$F(x) = \begin{cases} max_{x\leq y\leq \overline{x}} G(x, y), & for \quad 0 \leq x \leq \overline{x}, \\ min_{\overline{x}\leq y\leq x} G(x, y), & for \quad x > \overline{x}, \end{cases} \tag{22}$$

where

$$G(x, y) = y f(y, x, \cdots, x) f(\overline{x}, \overline{x}, \cdots, \overline{x}, y)[f(\overline{x}, x, \cdots, x)]^{k-1}. \tag{23}$$

Then

(a) $F \in C[(0,\infty),(0,\infty)]$ and F is nonincreasing in $[0,\infty)$;

(b) Assume that the function F has no periodic points of prime period 2. Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(21).

Lemma 4.2 [1, Lemma 1.6.3 (a) and (d)] Let $F \in [[0,\infty),(0,\infty)]$ be a nonincreasing function and let $\overline{x}$ denote the unique fixed point of F, then the following statements are equivalent:

(a) $\overline{x}$ is the only fixed point of F^2 in $(0,\infty)$;

(b) $\overline{x}$ is a global attractor of all positive solutions of the difference equation

$$x_{n+1} = F(x_n), \quad n = 0, 1, \cdots \tag{24}$$

with $x_0 \in [0,\infty)$.

Lemma 4.3 [1,5] Consider Eq.(24), where F is a nonincreasing function which maps some interval I into itself. Assume that F has a negative Schwarzian derivative

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2 = \left[\frac{F''(x)}{F'(x)}\right]' - \frac{1}{2}\left(\frac{F''(x)}{F'(x)}\right)^2 < 0,$$

everywhere on I, except for point x, where $F'(x) = 0$. Then the positive equilibrium $\overline{x}$ of Eq.(24) is a global attractor of all positive solutions of Eq.(24).

Declarations

Conflict of interest The authors declare no conflict of interest.

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