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Pseudo weak-demicompactness for 2×2 block operator matrices and some perturbation properties

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Abstract. In this paper, we give some properties for the so-called ε-pseudo weakly demicompact linear operators acting on Banach spaces with respect to a closed linear operator. Some sufficient conditions on the entries of an unbounded 2×2 block operator matrix $\mathcal{L}_0$ ensuring its ε-pseudo weak demicompactness are provided. In addition, we apply the obtained results to discuss the incidence of some perturbation results on the behavior of essential pseudospectra of $\mathcal{L}_0$. The results are formulated in terms of some denseness conditions on the topological dual space.

§1 Introduction

Many problems arising in mathematical physics can be first formulated by systems of partial or ordinary differential equations. In particular, systems of time evolution equations are governed by block operator matrices. When studying the asymptotic behavior of solutions to these systems, the spectral theory for the involved matrices plays a crucial role. Such studies have been discussed by different authors (see for instance, [13, 25, 26]).

This paper is devoted to some spectral properties related to the so-called ε-pseudo weak demicompactness, for a 2×2 block operator matrix (in short B.O.M) with domain $dom(\mathcal{L}_0) = (dom(A) \cap dom(C)) \times (dom(B) \cap dom(D))$ and represented in the following form

$$\mathcal{L}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The operator $\mathcal{L}_0$ acts on the product of Banach spaces $X \times Y$ with entries A, B, C and D. The operators A, B, C and D are linear densely defined and their domains are denoted by $dom(A)$, $dom(B)$, $dom(C)$ and $dom(D)$, respectively.

A pivotal focus lies on the concept of demicompactness, initially introduced by Petryshyn in 1966 [22, 23] to explore a novel approach to construct fixed points for this family of operators.

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In essence, a linear operator T on a Banach space X is called demicompact if, for every bounded sequence $(x_n)_n$ in $dom(T)$ such that the sequence $(Id_X - T)x_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$. The familly of linear demicompact operators on X is denoted by $\mathcal{DC}(X)$. In Fredholm theory, the first two old papers were developed by Petryshyn in 1972 [24] and by Akashi in 1984 [3]. Note that the demicompactness class plays an important role in the theory of perturbations since it contains compact and more general Fredholm perturbation operators. Recent research has furthered this direction. Noteworthy contributions include the work of Chaker, Jeribi and Krichen [6], who employed demicompact operators to probe the essential spectra of linear operators. In 2014, Krichen [14] extended the notion of demicompactness, introducing the class of relative demicompact linear operators with respect to a given linear operator S_0. This definition asserts that if $T : dom(T) \subset X \longrightarrow X$ and $S_0 : dom(S_0) \subset X \longrightarrow X$ are two linear operators with $dom(T) \subset dom(S_0)$, then T is said to be S_0-demicompact (or relatively demicompact with respect to S_0), if every bounded sequence $(x_n)_n$ in $dom(T)$ such that $(S_0 x_n - T x_n)_n$ converges in X, has a convergent subsequence. In 2018, Krichen and O'Regan [16] elaborated the class of relative weak demicompactness. If $T : dom(T) \subset X \longrightarrow X$ and $S_0 : dom(S_0) \subset X \longrightarrow X$ are two linear operators with $dom(T) \subset dom(S_0)$, T is said to be weakly S_0-demicompact (or weakly relatively demicompact with respect to S_0), if for every bounded sequence $(x_n)_n$ in $dom(T)$ such that $(S_0 x_n - T x_n)_n$ converges weakly in X, then there is a weakly convergent subsequence of $(x_n)_n$. The symbol $\mathcal{WDC}(S_0)(X)$ denotes the family of all weakly S_0-demicompact operators on X, and $\mathcal{WDC}(Id_X)(X) = \mathcal{WDC}(X)$. Note that, the class of demicompact operators acting on a Banach space contains the class of weakly compact operators. Lately, Ben Brahim, Jeribi and Krichen [5] developed the notion of pseudo demicompactnesss. For $\varepsilon > 0$, $T : dom(T) \subset X \longrightarrow X$ is said to be pseudo demicompact if for all bounded linear operator D acting on X such that $\|D\| < \varepsilon$ and for every bounded sequence $(x_n)_n$ in $dom(T)$ such that $((Id_X - T - D)x_n)_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$. Newly, Chtourou and Krichen [7] introduced the notion of a relatively ε-pseudo weakly demicompact operator as follows: Let $\varepsilon > 0$ and let $T : dom(T) \subset X \to X$, $S_0 : dom(S_0) \subset X \to X$ be two linear operators with $dom(T) \subset dom(S_0)$, then T is said to be ε-pseudo weakly S_0-demicompact (relative ε-pseudo weakly demicompact with respect to S_0), if for all bounded linear operator D acting on X such that $\|D\| < \varepsilon$ and for all bounded sequences $(x_n)_n$ in $dom(T)$ such that $(S_0 - T - D)x_n$ converges weakly in X, then $(x_n)_n$ has a weakly convergent subsequence. We denote by $\mathcal{WDC}_\varepsilon(S_0)(X)$ the family of ε-pseudo weakly S_0-demicompact operators on X. When $S_0 = Id_X$, T is simply said ε-pseudo weakly demicompact. This project aims to provide characterizations related to this concept, particularly focusing on describing this class through ε-pseudo Fredholm and upper ε-pseudo semi-Fredholm operators.

This paper also delves into the study of pseudo-spectra, which provide richer informations compared to spectra, particularly regarding transient behavior rather than just asymptotic behavior of dynamical systems. Historically, this concept was firstly introduced by Varah [27] in 1967 and has since been utilized by other mathematicians such as Landau [17], Trefethen [25] and Davies [8]. Specifically, the definition of pseudo-spectrum of a closed linear operator T is given for every $\varepsilon > 0$ by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\}.$$

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$ (spectrum of T). In [8], Davies has defined equivalently the pseudo-spectrum of any closed operator T as follows: for every $\varepsilon > 0$,

$$\sigma_\varepsilon(T) := \bigcup_{\|D\| < \varepsilon} \sigma(T + D).$$

Similarly to the Schechter essential spectrum, the authors in [4], studied some properties of the essential pseudo-spectrum of a densely defined, closed linear operator T acting on a Banach space X. This essential pseudo-spectrum is given by

$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(T + K)$, where $\mathcal{K}(X)$ is the ideal of compact linear operators acting on X. In the following, we recall some useful results related with $\sigma_{e5,\varepsilon}(T)$ and the class of Fredholm perturbation $\mathcal{F}(X)$.

Theorem 1.1. [18] Let X be a Banach space, T closed linear operator and $\varepsilon > 0$. Then,

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \mathcal{F}(X)} \sigma_\varepsilon(T + K). \qquad \diamondsuit$$

Proposition 1.1. [4] Let X be a Banach space, T closed linear operator and $\varepsilon > 0$. Then, $\lambda \notin \sigma_{e5,\varepsilon}(T)$ if and only if for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have

$$\lambda - T - D \in \Phi(X) \text{ and } i(\lambda - T - D) = 0.$$

This paper is organized as follows. In section 2, we recall some definitions and results which will be used in our work in subsequent sections. In section 3, we establish some results concerning the class of relatively ε-pseudo weakly demicompact operators. In section 4, we provide some sufficient conditions on the inputs of the block operator matrix $\mathcal{L}_0$ to ensure the ε-pseudo weak demicompactness. In section 5, we introduce some perturbation classes and determinate the stability of some essential pseudospectra involving the class of perturbations.

§2 Preliminary results

In this section, we will give some notations, definitions and preliminary results that are necessary in the sequel.

First, let us recall some standard definitions and notations from Fredholm theory. Let X and Y be two Banach spaces. In what follows, we denote $\to$ for the strong convergence (i.e. norm convergence in X) and $\rightharpoonup$ for the weak convergence (with respect to the weak topology of X). Throughout this paper, we consider $V : dom(V) \subset X \longrightarrow Y$ as a linear operator with domain $dom(V)$ and range $\mathcal{R}(V) \subset Y$. If the graph of V is a closed subset of $X \times Y$, then V is closed. The set of all closed (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$). We denote by $\mathcal{K}(X, Y)$ the subset of compact operators of $\mathcal{L}(X,Y)$. For $V \in \mathcal{C}(X,Y)$, we use notations $\alpha(V)$ for the dimension of the kernel $\mathcal{N}(V)$ and $\beta(V)$ for the codimension of the range $\mathcal{R}(V)$ in Y. The graph norm of $x \in dom(V)$ is defined by

$$\|x\|_V := \|x\| + \|Vx\|.$$

It follows from the closedness of V that $X_V := (dom(V), \|.\|_V)$ is a Banach space. Clearly, we have

$$\|Vx\| \leq \|x\|_V, \text{ for every } x \in dom(V),$$

and consequently,

$$V \in \mathcal{L}(X_V, X).$$

Definition 2.1. Let X, Y and Z be three Banach spaces. Let $V : dom(V) \subset X \longrightarrow Y$ and $U : dom(U) \subset X \longrightarrow Z$ be two linear operators. U is said to be V-bounded, if $dom(V) \subset dom(U)$ and there exist constants $a, b \geq 0$ such that

$$\|Ux\| \leq a\|x\| + b\|Vx\|, \text{ for all } x \in dom(V).$$

The greatest lower bound of all possible values $b \geq 0$ is called the relative bound of U with respect to V or the V-bound of U.

A linear operator $U : X \longrightarrow Y$ is said to be V-defined if $dom(V) \subset dom(U)$. We denote by $\widehat{U}$ the restriction of U to $dom(V)$. Besides, if $\widehat{U}$ is bounded from X_V into Y, we say that U is V-bounded. We can see that, if U is closed, then U is V-bounded. Therefore, we have the obvious relations:
(i) $\alpha(\widehat{V}) = \alpha(V)$, $\beta(\widehat{V}) = \beta(V)$, $\mathcal{R}(\widehat{U}) = \mathcal{R}(U)$,
(ii) $\alpha(\widehat{V} + \widehat{U}) = \alpha(T + U)$, $\beta(\widehat{V} + \widehat{U}) = \beta(V + U)$, $\mathcal{R}(\widehat{U} + \widehat{V}) = \mathcal{R}(U + V)$.

Definition 2.2. Let X be a Banach space. An operator $V \in \mathcal{L}(X,Y)$ is said to be weakly compact if $V(B)$ is relatively weakly compact in Y for every bounded set $B \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, the family of weakly compact operators on X is simply denoted by $\mathcal{W}(X) := \mathcal{W}(X,X)$. The set $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [9,11]).

Now, we define the sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y, respectively, by

$$\Phi_+(X,Y) = \{V \in \mathcal{C}(X,Y) \text{ such that } \alpha(V) < \infty \text{ and } \mathcal{R}(V) \text{ closed in } Y\},$$
$$\Phi_-(X,Y) = \{V \in \mathcal{C}(X,Y) \text{ such that } \beta(V) < \infty \text{ and } \mathcal{R}(V) \text{ closed in } Y\},$$
$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y),$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

For $V \in \Phi_\pm(X,Y)$, we define the index of V by the following difference

$$i(V) := \alpha(V) - \beta(V).$$

By the index theorem we have

$$i(UV) = i(U) + i(V).$$

If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. If $V \in \mathcal{C}(X)$, $\rho(V)$ denotes the resolvent set of V, $\sigma(V)$ the spectrum of V.

Definition 2.3. [12] Let X and Y be two Banach spaces and let $U \in \mathcal{L}(X,Y)$. The operator U is called:
(i) Fredholm perturbation if $V + U \in \Phi(X,Y)$, whenever $V \in \Phi(X,Y)$;
(ii) Upper semi-Fredholm perturbation if $V + U \in \Phi_+(X,Y)$, whenever $V \in \Phi_+(X,Y)$;
(iii) Lower semi-Fredholm perturbation if $V + U \in \Phi_-(X,Y)$, whenever $V \in \Phi_-(X,Y)$.

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.
In general, we have

$$\mathcal{K}(X,Y) \subset \mathcal{F}_+(X,Y) \subset \mathcal{F}(X,Y),$$
$$\mathcal{K}(X,Y) \subset \mathcal{F}_-(X,Y) \subset \mathcal{F}(X,Y).$$

If $X = Y$, $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ are replaced by $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$, respectively.

Definition 2.4. Let X and Y be two Banach spaces and let $V \in \mathcal{C}(X,Y)$.
(i) An operator V is said to have a left Fredholm inverse if there exists $V_l \in \mathcal{L}(Y, X_V)$ such that $Id_{X_V} - V_l\widehat{V} \in \mathcal{K}(X_V)$. The operators V_l is called left Fredholm inverse of V.
(ii) An operator V is said to have a right Fredholm inverse if there exists $V_r \in \mathcal{L}(Y, X_V)$ such that $Id_Y - \widehat{V}V_r \in \mathcal{K}(Y)$. The operators V_r is called right Fredholm inverse of V.

The sets of all left and right Fredholm inverse are, respectively, denoted by $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$.
If $X = Y$, the sets $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ are replaced by $\Phi_l(X)$ and $\Phi_r(X)$, respectively.
According to [20], it can be inferred that

$$\Phi_l(X,Y) = \{V \in \Phi_+(X,Y) \text{ such that } \mathcal{R}(V) \text{ is complemented}\},$$

and

$$\Phi_r(X,Y) = \{V \in \Phi_-(X,Y) \text{ such that } \mathcal{N}(V) \text{ is complemented}\}.$$

Recall that a subspace $N \subset X$ is called complemented if there exists a closed subspace $M \subset X$ such that $N \oplus M = X$.
We have the following inclusions:

$$\Phi(X,Y) \subset \Phi_l(X,Y) \subset \Phi_+(X,Y),$$
$$\Phi(X,Y) \subset \Phi_r(X,Y) \subset \Phi_-(X,Y).$$

Definition 2.5. [20] Let X and Y be two Banach spaces. We define by

$$\mathcal{F}_l(X,Y) = \{V \in \mathcal{L}(X,Y) \text{ such that } V+U \in \Phi_l(X,Y), \text{ for all } U \in \Phi_l(X,Y)\},$$

and

$$\mathcal{F}_r(X,Y) = \{V \in \mathcal{L}(X,Y) \text{ such that } V+U \in \Phi_r(X,Y), \text{ for all } U \in \Phi_r(X,Y)\}.$$

If $X = Y$, the sets $\mathcal{F}_l(X,Y)$ and $\mathcal{F}_r(X,Y)$ are replaced by $\mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$, respectively.

Remark 2.1. $\mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$ are two-sided ideals of $\mathcal{L}(X)$.

Proposition 2.1. Let X and Y be two Banach spaces. If the set $\Phi(X,Y)$ is not empty, then:
(i) $\mathcal{K}(X,Y) \subset \mathcal{F}_l(X,Y) \subset \mathcal{F}(X,Y)$;
(ii) $\mathcal{K}(X,Y) \subset \mathcal{F}_r(X,Y) \subset \mathcal{F}(X,Y)$.

Theorem 2.1. [2] Let X, Y and Z be three Banach spaces, $V \in \mathcal{L}(Y,Z)$ and $U \in \mathcal{L}(X,Y)$.
(i) If $U \in \Phi_l(Y,Z)$ and $V \in \Phi_l(X,Y)$, then $UV \in \Phi_l(X,Z)$;
(ii) If $U \in \Phi_r(Y,Z)$ and $V \in \Phi_r(X,Y)$, then $UV \in \Phi_r(X,Z)$;
(iii) If $UV \in \Phi_l(X,Z)$, then $V \in \Phi_l(X,Y)$;
(iv) If $UV \in \Phi_r(X,Z)$, then $U \in \Phi_r(Y,Z)$.

Lemma 2.1. Let X, Y be two Banach spaces, $U \in \mathcal{L}(X)$, $V \in \mathcal{L}(Y)$ and let the 2×2 operator matrix $M_T = \begin{pmatrix} U & T \\ 0 & V \end{pmatrix}$ for all $T \in \mathcal{L}(Y,X)$. Then:
(i) If $U \in \Phi_l(X)$ and $V \in \Phi_l(Y)$, then $M_T \in \Phi_l(X\times Y)$;
(ii) If $U \in \Phi_r(X)$ and $V \in \Phi_r(Y)$, then $M_T \in \Phi_r(X\times Y)$;
(iii) If $M_T \in \Phi_l(X\times Y)$, then $U \in \Phi_l(X)$;
(iv) If $M_T \in \Phi_r(X\times Y)$, then $V \in \Phi_r(Y)$.

Proof. (i) We can write M_T in the following form

$$M_T = \begin{pmatrix} Id_X & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} Id_X & T \\ 0 & Id_Y \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & Id_Y \end{pmatrix}. \tag{1}$$

Since $U \in \Phi_l(X)$ and $V \in \Phi_l(Y)$, then $\begin{pmatrix} U & 0 \\ 0 & Id_Y \end{pmatrix} \in \Phi_l(X\times Y)$ and $\begin{pmatrix} Id_X & 0 \\ 0 & V \end{pmatrix} \in \Phi_l(X\times Y)$. Since $\begin{pmatrix} Id_X & T \\ 0 & Id_Y \end{pmatrix}$ is invertible, then $M_T \in \Phi_l(X\times Y)$.
(ii) can be checked in the same way as (i).
(iii) Using Theorem 2.1 (iii) in Eq. (1), we have $\begin{pmatrix} U & 0 \\ 0 & Id_Y \end{pmatrix} \in \Phi_l(X\times Y)$ and so $U \in \Phi_l(X)$.
(iv) can be checked in the same way as (iii). ☐

Definition 2.6. Let X and Y be two Banach spaces and let $V \in \mathcal{C}(X,Y)$ and $\varepsilon > 0$.
(i) V is called a ε-pseudo upper (resp. lower) semi-Fredholm operator if $V + D$ is an upper (resp. lower) semi-Fredholm operator for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$;

(ii) V is called a ε-pseudo semi-Fredholm operator if $V+D$ is a semi-Fredholm operator for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$;
(iii) V is called a ε-pseudo Fredholm operator if $V+D$ is a Fredholm operator for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$.

The sets of all ε-pseudo Fredholm, ε-pseudo upper Fredholm and ε-pseudo lower Fredholm are, respectively, denoted by $\Phi^{\varepsilon}(X,Y)$, $\Phi^{\varepsilon}_{+}(X,Y)$ and $\Phi^{\varepsilon}_{-}(X,Y)$.
If $X = Y$, the sets $\Phi^{\varepsilon}(X,Y)$, $\Phi^{\varepsilon}_{+}(X,Y)$ and $\Phi^{\varepsilon}_{-}(X,Y)$ are replaced by $\Phi^{\varepsilon}(X)$, $\Phi^{\varepsilon}_{+}(X)$ and $\Phi^{\varepsilon}_{-}(X)$, respectively.
Moreover, we have the following inclusions

$$\Phi^{\varepsilon}_{+}(X,Y) \subsetneq \Phi_{+}(X,Y),$$
$$\Phi^{\varepsilon}_{-}(X,Y) \subsetneq \Phi_{-}(X,Y), \text{ and}$$
$$\Phi^{\varepsilon}(X,Y) \subsetneq \Phi(X,Y).$$

Lemma 2.2. Let X be a Banach space and $\varepsilon > 0$. Let $V \in \mathcal{L}(X)$ and $U \in \mathcal{L}(X)$.
(i) If $V \in \Phi(X)$, $U \in \Phi^{\varepsilon}(X)$ and $(Id_X - V) \in \mathcal{F}(X)$, then $VU \in \Phi^{\varepsilon}(X)$ and $i(VU+D) = i(V) + i(U+D)$ for all $D \in \mathcal{L}(X)$ satisfying $\|D\| < \varepsilon$;
(ii) If $V \in \Phi_{+}(X)$, $U \in \Phi^{\varepsilon}_{+}(X)$ and $(Id_X - V) \in \mathcal{F}_{+}(X)$, then $VU \in \Phi^{\varepsilon}_{+}(X)$.

Proof. (i) For each $D \in \mathcal{L}(X)$ satisfying $\|D\| < \varepsilon$, we have

$$VU + D = V(U+D) + (Id_X - V)D. \tag{2}$$

Since $V \in \Phi(X)$ and $U + D \in \Phi(X)$, then using [21] and the fact that $(Id_X - V)D \in \mathcal{F}(X)$, we get $VU \in \Phi^{\varepsilon}(X)$ and $i(VU+D) = i(V) + i(U+D)$.
(ii) We reason in the same way as the proof of (i). □

Definition 2.7. Let X and Y be two Banach spaces and let $U \in \mathcal{L}(X,Y)$ and $\varepsilon > 0$.
(i) U is said to have an ε-pseudo left Fredholm inverse if there exists $U \in \mathcal{L}(Y,X)$ and $K \in \mathcal{K}(X)$ such that $U_l(U+D) = Id_X - K$, for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$. The operator U_l is called ε-pseudo left Fredholm inverse of U;
(ii) U is said to have an ε-pseudo right Fredholm inverse if there exists $U_r \in \mathcal{L}(Y,X)$ and $K \in \mathcal{K}(Y)$ such that $(U+D)U_r = Id_Y - K$, for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$. The operator U_r is said an ε-pseudo right Fredholm inverse of U.

The sets of all ε-pseudo left and ε-pseudo right Fredholm inverse are, respectively, denoted by $\Phi^{\varepsilon}_{l}(X,Y)$ and $\Phi^{\varepsilon}_{r}(X,Y)$.
If $X = Y$, the sets $\Phi^{\varepsilon}_{l}(X,Y)$ and $\Phi^{\varepsilon}_{r}(X,Y)$ are replaced by $\Phi^{\varepsilon}_{l}(X)$ and $\Phi^{\varepsilon}_{r}(X)$, respectively.

Lemma 2.3. Let X be a Banach space and $\varepsilon > 0$. Let $V \in \mathcal{L}(X)$ and $U \in \mathcal{L}(X)$.
(i) If $V \in \Phi_l(X)$, $U \in \Phi^{\varepsilon}_{l}(X)$ and $(Id_X - V) \in \mathcal{F}_l(X)$, then $VU \in \Phi^{\varepsilon}_{l}(X)$;
(ii) If $V \in \Phi_r(X)$, $U \in \Phi^{\varepsilon}_{r}(X)$ and $(Id_X - V) \in \mathcal{F}_r(X)$, then $VU \in \Phi^{\varepsilon}_{r}(X)$.

Proof. (i) For each $D \in \mathcal{L}(X)$ satisfying $\|D\| < \varepsilon$, we have

$$VU + D = V(U+D) + (Id_X - V)D. \tag{3}$$

Since $V \in \Phi_l(X)$ and $U + D \in \Phi_l(X)$, then applying Theorem 2.1 (i) on Eq. (3) and using the fact that $(Id_X - V)D \in \mathcal{F}_l(X)$, we get $VU \in \Phi_l^{\varepsilon}(X)$.
(ii) We reason in the same way as the proof of (i). □

Definition 2.8. Let X and Y be two Banach spaces and let $U \in \mathcal{C}(X,Y)$ and $\varepsilon > 0$.
(i) U is said to have an ε-pseudo left weak-Fredholm inverse if there exists $U_l^w \in \mathcal{L}(Y, X_U)$ and $W \in \mathcal{W}(X_U)$ such that $U_l^w(U + D) = Id_{X_U} - W$, for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$. The operator U_l^w is called ε-pseudo left weak-Fredholm inverse of U;
(ii) U is said to have an ε-pseudo right weak-Fredholm inverse if there exists $U_r^w \in \mathcal{L}(Y, X_U)$ and $W \in \mathcal{W}(Y)$ such that $(U + D)U_r^w = Id_Y - W$, for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$. The operator U_r^w is said an ε-pseudo right weak-Fredholm inverse of U.

In this research work, we are basically interested in the following essential pseudo-spectra

$$\sigma_{e1,\varepsilon}(V) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_+^{\varepsilon}(X)\},$$
$$\sigma_{e2,\varepsilon}(V) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_-^{\varepsilon}(X)\},$$
$$\sigma_{e3,\varepsilon}(V) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_\pm^{\varepsilon}(X)\},$$
$$\sigma_{e4,\varepsilon}(V) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi^{\varepsilon}(X)\},$$
$$\sigma_{e5,\varepsilon}(V) := \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(V + K),$$
$$\sigma_{e6,\varepsilon}(V) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_l^{\varepsilon}(X)\},$$
$$\sigma_{e7,\varepsilon}(V) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_r^{\varepsilon}(X)\}.$$

Note that if ε tends to 0, we recover the well-known definitions of essential spectra of V (see, for instance [10, 15, 19, 21, 28]).

§3 Main result

Definition 3.1. Let $(Y, \|.\|_Y)$ be a Banach space and let X be a subspace of Y endowed with a norm $\|.\|_X$ such $(X, \|.\|_X)$ is a Banach space. Let $T : dom(T) \subset X \to Y$ be a closed linear operator, $S_0 : X \to Y$ be a bounded linear operator and $\varepsilon > 0$. Then, T is called ε-pseudo weakly S_0-demicompact if for every sequence $(x_n)_n$ in $dom(T)$ and $D \in \mathcal{L}(Y)$ with $\|D\|_Y < \varepsilon$ such that $(S_0x_n - Tx_n - Dx_n)_n$ converges weakly in Y, then there exists a weakly convergent subsequence of $(x_n)_n$ in X.

We denote by $\mathcal{WDC}_\varepsilon(S_0)(X,Y)$, the set of all ε-pseudo weakly S_0-demicompact operators from X into Y. If $S_0 = Id_X$, we simply denote by $\mathcal{WDC}_\varepsilon(X,Y)$. If $(X, \|.\|_X) = (Y, \|.\|_Y)$, we simply denote by $\mathcal{WDC}_\varepsilon(S_0)(X)$.

Theorem 3.1. Let X be a Banach space and let $T \in \mathcal{C}(X)$ and $S_0 \in \mathcal{L}(X)$ such that $S_0 \neq 0$. Assume that $X^* + X^* \circ T$ is dense in $(X_T)^*$, where X^* and $(X_T)^*$ denote the topological dual spaces of X and $X_T = (dom(T), \|.\|_T)$, respectively. Then, for every $\varepsilon > 0$ the following equivalence holds

$$T \in \mathcal{WDC}_\varepsilon(S_0)(X) \text{ if and only if } \widehat{T} \in \mathcal{WDC}_\varepsilon(S_0)(X_T, X).$$

Proof. Let $\varepsilon > 0$, $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and let $(x_n)_n$ be a bounded sequence of X_T such that $S_0x_n - \widehat{T}x_n - Dx_n \rightharpoonup y$, in X. Clearly, $(x_n)_n$ is bounded in X and $S_0x_n - Tx_n - Dx_n \rightharpoonup y$. Since $T \in \mathcal{WDC}_\varepsilon(S_0)(X)$, then there exists a subsequence $(x_{\varphi(n)})_n \subset dom(T)$ such that $x_{\varphi(n)} \rightharpoonup x,\ x \in X$. We have to show that $x_{\varphi(n)} \rightharpoonup x$ in X_T. For this purpose, let $f \in (X_T)^*$, it follows that there exists $(f_m)_m$ with $f_m = g_m + h_m \circ T,\ m \in \mathbb{N}$. Where $(g_m)_m \subset X^*$, $(h_m)_m \subset X^*$ and $\|f_m - f\|_{(X_T)^*} \to 0$, as $m \to +\infty$. Clearly, $g_m(x_{\varphi(n)}) \to g_m(x)$ for all $m \in \mathbb{N}$. Now,

$$Tx_{\varphi(n)} = Tx_{\varphi(n)} + Dx_{\varphi(n)} - S_0x_{\varphi(n)} - Dx_{\varphi(n)} + S_0x_{\varphi(n)} \rightharpoonup -y + S_0x - Dx.$$

It follows from the closedness of T that $x \in dom(T)$ and $S_0x - Dx - y = Tx$. Consequently, $Tx_{\varphi(n)} \rightharpoonup Tx$ in X. Which implies that $h_m(Tx_{\varphi(n)}) \rightharpoonup h_m(Tx)$ for all $m \in \mathbb{N}$.
It follows that $f_m(x_{\varphi(n)}) \to f_m(x)$, for all $m \in \mathbb{N}$. Now, write

$$\begin{aligned}|f(x_{\varphi(n)}) - f(x)| &\le |f(x_{\varphi(n)}) - f_m(x_{\varphi(n)})| + |f_m(x_{\varphi(n)}) - f_m(x)| + |f_m(x) - f(x)| \\ &\le \|f - f_m\|_{X_T^*}\|x_{\varphi(n)}\|_T + |f_m(x_{\varphi(n)}) - f_m(x)| + \|f_m - f\|_{X_T^*}\|x\|_T.\end{aligned}$$

Since $(x_n)_n$ is a bounded sequence of X_T, then there exists $M > 0$ such that $\|x_{\varphi(n)}\| \le M$ and $\|Tx_{\varphi(n)}\| \le M$. Let $\delta > 0$ then there exists $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$,

$$\|f - f_m\|_{X_T^*} < \frac{\delta}{3M}.$$

It follows that

$$|f_m(x_{\varphi(n)}) - f(x)| \le \frac{\delta}{3} + |f_{m_0}(x_{\varphi(n)}) - f_{m_0}(x)| + \frac{\delta}{3}.$$

Now, from the fact that $f_{m_0}(x_{\varphi(n)}) \to f_{m_0}(x)$, as $n \to +\infty$, we deduce that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$|f_{n_0}(x_{\varphi(n)}) - f_{n_0}(x)| \le \frac{\delta}{3}.$$

Consequently,

$$|f(x_{\varphi(n)}) - f(x)| < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} < \delta.$$

Hence, $x_{\varphi(n)} \rightharpoonup x$ in X_T.
Conversely, let $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, $S_0 \in \mathcal{L}(X)$ and $(x_n)_n$ be a bounded sequence of X such that $S_0x_n - Tx_n - Dx_n \rightharpoonup y$ in X. Then, there exists $M > 0$ such that $\|x_n\| \le M$, $\|S_0x_n\| \le M$ and $\|Tx_n - S_0x_n + Dx_n\| \le M$ for all $n \ge 0$. It follows that

$$\|x_n\|_T = \|Tx_n\| + \|x_n\| \le (3 + \varepsilon)M.$$

Then $(x_n)_n$ is bounded in X_T. Since $S_0x_n - \widehat{T}x_n - Dx_n \rightharpoonup y$ in X and $\widehat{T} \in \mathcal{WDC}_\varepsilon(X_T, X)$, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ and $x \in X$ such that $x_{\varphi(n)} \rightharpoonup x$ in X_T, which achieves the proof. □

Theorem 3.2. Let X be a Banach space and let $T \in \mathcal{C}(X)$ and $S_0 \in \mathcal{L}(X)$ such that $S_0 \neq 0$. Assume that $X^* + X^* \circ T$ is dense in $(X_T)^*$. Fix $\varepsilon > 0$ and $S \in \mathcal{L}(X)$. If $T \in \mathcal{WDC}_\varepsilon(X)$ and the operator $S_0 - T$ has a left (resp. right) ε-pseudo weakly Fredholm inverse T_l (resp. T_r)

such that ST_l (resp. T_rS) $\in \mathcal{WDC}(X)$, then $T+S \in \mathcal{WDC}_\varepsilon(S_0)(X)$.

Proof. Let $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, then there exist $T_l \in \mathcal{L}(X, X_T)$ (resp. $T_r \in \mathcal{L}(X, X_T)$) and $K \in \mathcal{W}(X_T)$ (resp. $K' \in \mathcal{W}(X)$) such that

$$T_l(S_0 - \widehat{T} - D) = Id_X - K \text{ on } X_T.$$

$$(\text{resp. } (S_0 - \widehat{T} - D)T_r = Id_Y - K', \text{ on } Y).$$

Then, the operator $S_0 - \widehat{T} - S - D$ can be written as follows

$$S_0 - \widehat{T} - S - D = (Id_X - ST_l)(S_0 - \widehat{T} - D) - SK. \tag{4}$$

$$(\text{resp. } S_0 - \widehat{T} - S - D = (S_0 - \widehat{T} - D)(Id_Y - T_rS) - K'S). \tag{5}$$

Now, let $(x_n)_n$ be a bounded sequence of X_T satisfying $(S_0 - \widehat{T} - S - D)x_n$ converges weakly to an element of X. It follows from Eq. (4) (resp. Eq. (5)) together with the weak compactness of SK (resp. $K'S$), the weak demicompactness of ST_l (resp. T_rS) and the boundedness of $(S_0 - \widehat{T} - D)x_n$ that $(S_0 - \widehat{T} - D)x_n$ has a weakly convergent subsequence. Since T is ε-pseudo weakly S_0-demicompact, according to Theorem 3.1, $\widehat{T}$ is ε-pseudo weakly demicompact. Therefore, $(x_n)_n$ admits a weakly convergent subsequence in X_T and this shows that $\widehat{T} + S$ is ε-pseudo weakly S_0-demicompact operator and consequently, $T + S$ is ε-pseudo weakly demicompact operator. □

§4 ε-Pseudo weak-demicompactness for B.O.M

Throughout this section, we denote by $\mathcal{I} := \begin{pmatrix} Id_X & 0 \\ 0 & Id_X \end{pmatrix}$ the identity matrix.

Proposition 4.1. Let X be a Banach space, $\varepsilon > 0$ and A: $dom(A) \subset X \longrightarrow X$ be a closed linear operator and D: $X \longrightarrow X$ be a bounded linear operator. Let $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} := \widetilde{\mathcal{A}} + \widetilde{\mathcal{D}}$ with $\widetilde{\mathcal{A}} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\widetilde{\mathcal{D}} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$. Assume that $X^* + X^* \circ A$ is dense in X_A^*. If $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact matrix and $\mathcal{I} - \widetilde{\mathcal{A}}$ has a left (resp. right) ε-pseudo weakly Fredholm inverse $\widetilde{\mathcal{A}}_l$ (resp. $\widetilde{\mathcal{A}}_r$) such that $\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l \in \mathcal{DC}(X \times X)$. Then $\mathcal{A} \in \mathcal{WDC}_\varepsilon(X \times X)$.

Proof. First, let us prove that $(X \times X)^* + (X \times X)^* \circ \widetilde{\mathcal{A}}$ is dense in $(X \times X)^*_{\widetilde{\mathcal{A}}}$. Let $f : (dom(A) \times X, \|.\|_{\widetilde{\mathcal{A}}}) \longrightarrow \mathbb{R}$ a bounded linear form. Then there exist $f_1 : X_A \longrightarrow \mathbb{R}$ and $f_2 : X \longrightarrow \mathbb{R}$ two bounded linear forms such that $f(x,y) = f_1(x) + f_2(y)$ (put $f_1(x) = f(x,0)$ and $f_2(y) = f(0,y)$). Since $X^* + X^* \circ A$ is dense in X_A^*, there exist two sequence $(h_{1n})_n, (k_{1n})_n$ in X^* such that $h_{1n} + k_{1n} \circ A \to f_1$. Set $H_n(x,y) = h_{1n}(x) + f_2(y)$ and $K_n(x,y) = k_{1n}(x)$ for all $(x,y) \in X \times X$. Observe that H_n and K_n are linear. Moreover,

$$\begin{aligned} |H_n(x,y)| &\leq \|h_{1n}\|\|x\| + \|f_2\|\|y\| \\ &\leq (\|h_{1n}\| + \|f_2\|)\|(x,y)\|, \end{aligned}$$

and

$$|K_n(x,y)| \leq \|k_{1n}\| \|x\|.$$

Since $H_n(x,y) + K_n \circ \widetilde{\mathcal{A}}(x,y) = h_{1n}(x) + f_2(y) + k_{1n}(Ax)$. Then,

$$H_n + K_n \circ \widetilde{\mathcal{A}} \to f.$$

Consequently, $(X \times X)^* + (X \times X)^* \circ \widetilde{\mathcal{A}}$ is dense in $(X \times X)^*_{\widetilde{\mathcal{A}}}$.

Let $\mathcal{P} \in \mathcal{L}(X \times X)$ such that $\|\mathcal{P}\| < \varepsilon$, then there exist $\mathcal{K} \in \mathcal{W}((X \times X)_{\widetilde{\mathcal{A}}})$ and $\widetilde{\mathcal{A}}_l \in \mathcal{L}(X \times X, (X \times X)_{\widetilde{\mathcal{A}}})$ such that:

$$\widetilde{\mathcal{A}}_l(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P}) = I - \mathcal{K}.$$

Then, the matrix $I - \widehat{\mathcal{A}} - \mathcal{P}$ can be written as follows

$$I - \widehat{\mathcal{A}} - \mathcal{P} = (I - \widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l)(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P}) - \widetilde{\mathcal{D}}\mathcal{K}. \tag{6}$$

Now, let $(x_n, y_n)_n$ be a bounded sequence of $(X \times X)_{\widetilde{\mathcal{A}}}$ such that $(I - \mathcal{A} - \mathcal{P})(x_n, y_n)_n$ converges weakly to an element of $X \times X$. It follows from Eq. 6 together with the weak compactness of $\widetilde{\mathcal{D}}\mathcal{K}$, the weak demicompactness of $\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l$ and the boundedness of $(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P})(x_n, y_n)$ that $(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P})(x_n, y_n)_n$ admits a weakly convergent subsequence. Since $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact, then by applying Theorem 3.1, we infer that $\widehat{\widetilde{\mathcal{A}}}$ is ε-pseudo weakly demicompact. Therefore, $(x_n, y_n)_n$ admits a weakly convergent subsequence in $(X \times X)_{\widetilde{\mathcal{A}}}$ and this shows the ε-pseudo weak demicompactness of $\widehat{\mathcal{A}}$. So, $\mathcal{A} \in \mathcal{WDC}_\varepsilon(X \times X)$. □

Proposition 4.2. Let X be a Banach space, $\varepsilon > 0$. Let A: $dom(A) \subset X \longrightarrow X$ and D: $dom(D) \subset X \longrightarrow X$ be two closed linear operators. Let B: $X \longrightarrow X$ and C: $X \longrightarrow X$ be two bounded linear operators.

Let $\mathcal{B} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \widetilde{\mathcal{A}} + \widetilde{\mathcal{B}}$ with $\widetilde{\mathcal{A}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, $\widetilde{\mathcal{B}} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Assume that $X^* + X^* \circ A$ is dense in X_A^* and $X^* + X^* \circ D$ is dense in X_D^*. If $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact matrix and $\mathcal{I} - \widetilde{\mathcal{A}}$ has a left (resp. right) ε-pseudo weakly Fredholm inverse $\widetilde{\mathcal{A}}_l$ (resp. $\widetilde{\mathcal{A}}_r$) such that $\widetilde{\mathcal{B}}\widetilde{\mathcal{A}}_l \in \mathcal{DC}(X \times X)$. Then $\mathcal{B} \in \mathcal{WDC}_\varepsilon(X \times X)$.

Proof. First, let us prove that $(X \times X)^* + (X \times X)^* \circ \widetilde{\mathcal{A}}$ is dense in $(X \times X)^*_{\widetilde{\mathcal{A}}}$. Let $f : (dom(A) \times X, \|.\|_{\widetilde{\mathcal{A}}}) \longrightarrow \mathbb{R}$ a bounded linear form. Then there exist $f_1 : X_A \longrightarrow \mathbb{R}$ and $f_2 : X \longrightarrow \mathbb{R}$ two bounded linear forms such that $f(x,y) = f_1(x) + f_2(y)$ (put $f_1(x) = f(x,0)$ and $f_2(y) = f(0,y)$). Since $X^* + X^* \circ A$ is dense in X_A^*, there exist two sequence $(h_{1n})_n, (k_{1n})_n$ in X^* such that $h_{1n} + k_{1n} \circ A \to f_1$ and $X^* + X^* \circ D$ is dense in X_D^*, there exist two sequences h_{2n}, k_{2n} in X^* such that $h_{2n} + k_{2n} \circ D \to f_2$. Set $W_n(x,y) = H_n(x,y) + K_n \circ \widetilde{\mathcal{A}}(x,y)$ where $H_n(x,y) = h_{1n}(x) + h_{2n}(y)$ and $K_n(x,y) = k_{1n}(x) + k_{2n}(y)$ for all $(x,y) \in X \times X$. Observe that W_n, H_n and k_n are linear. Moreover,

$$\begin{aligned} |H_n(x,y)| &\leq \|h_{1n}\| \|x\| + \|h_{2n}\| \|y\| \\ &\leq (\|h_{1n}\| + \|h_{2n}\|) \|(x,y)\|, \end{aligned}$$

and

$$\begin{aligned} |K_n(x,y)| &\leq \|k_{1n}\| \|x\| + \|k_{2n}\| \|y\| \\ &\leq (\|k_{1n}\| + \|k_{2n}\|) \|(x,y)\|, \end{aligned}$$

Therefore,

$$W_n(x,y) = h_{1n}(x) + h_{2n}(y) + k_{1n}(Ax) + k_{2n}(Dx) \to f_1(x) + f_2(y) = f(x,y).$$

Consequently, $(X \times X)^* + (X \times X)^* \circ \widetilde{\mathcal{A}}$ is dense in $(X \times X)^*_{\widetilde{\mathcal{A}}}$.

Let $\mathcal{P} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times X)$ such that $\|\mathcal{P}\| < \varepsilon$, then there exist $\mathcal{K} \in \mathcal{W}((X \times X)_{\widetilde{\mathcal{A}}})$ and $\widetilde{\mathcal{A}}_l \in \mathcal{L}(X \times X, (X \times X)_{\widetilde{\mathcal{A}}})$ such that

$$\mathcal{A}_l(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P}) = I - \mathcal{K}.$$

Then, the matrix $I - \widehat{\mathcal{B}} - \mathcal{P}$ can be written as follows

$$I - \widehat{\mathcal{B}} - \mathcal{P} = (I - \widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l)(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P}) - \widetilde{\mathcal{D}}\mathcal{K}. \tag{7}$$

Now, let $(x_n, y_n)_n$ be a bounded sequence of $(X \times X)_{\widetilde{\mathcal{A}}}$ such that $(I - \widehat{\mathcal{B}} - \mathcal{P})(x_n, y_n)_n$ converges weakly to an element of $X \times X$. It follows from Eq. (7) together with the weak compactness of $\widetilde{\mathcal{D}}\mathcal{K}$, the weak demicompactness of $\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l$ and the boundedness of $(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P})(x_n, y_n)$ we infer that $(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P})(x_n, y_n)_n$ admits a weakly convergent subsequence. Since $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact, then by applying Theorem 3.1, we infer that $\widehat{\widetilde{\mathcal{A}}}$ is ε-pseudo weakly demicompact. Therefore, $(x_n, y_n)_n$ admits a weakly convergent subsequence in $(X \times X)_{\widetilde{\mathcal{A}}}$ and this shows the ε-pseudo weak demicompactness of $\widehat{\mathcal{B}}$. So, $\mathcal{B} \in \mathcal{WDC}_\varepsilon(X \times X)$. □

§5 Some perturbation properties

Definition 5.1. Let X and Y be two Banach spaces. We define by

$$\mathcal{F}_l^\varepsilon(X,Y) = \{V \in \mathcal{L}(X,Y) : V + D \in \mathcal{F}_l^\varepsilon(X,Y), \text{ for all } D \in \mathcal{L}(X,Y) \text{ such that } \|D\| < \varepsilon\}$$

and

$$\mathcal{F}_r^\varepsilon(X,Y) = \{V \in \mathcal{L}(X,Y) : V + D \in \mathcal{F}_r^\varepsilon(X,Y), \text{ for all } D \in \mathcal{L}(X,Y) \text{ such that } \|D\| < \varepsilon\}.$$

If $X = Y$, the sets $\mathcal{F}_l^\varepsilon(X,Y)$ and $\mathcal{F}_r^\varepsilon(X,Y)$ are replaced by $\mathcal{F}_l^\varepsilon(X)$ and $\mathcal{F}_r^\varepsilon(X)$, respectively. Moreover, we have the following inclusions

$$\mathcal{F}_l^\varepsilon(X,Y) \subsetneq \mathcal{F}_l(X,Y) \text{ and}$$
$$\mathcal{F}_r^\varepsilon(X,Y) \subsetneq \mathcal{F}_r(X,Y).$$

Theorem 5.1. Let X and Y be two Banach spaces and let $T \in \mathcal{C}(X,Y)$ and $F : X \to Y$ be a linear operator. Then

(i) $T + F \in \Phi_l^\varepsilon(X,Y)$ whenever $T \in \Phi_l^\varepsilon(X,Y)$ and $F \in \mathcal{F}_l^\varepsilon(X,Y)$;

(ii) $T + F \in \Phi_r^\varepsilon(X,Y)$ whenever $T \in \Phi_r^\varepsilon(X,Y)$ and $F \in \mathcal{F}_r^\varepsilon(X,Y)$.

Lemma 5.1. Let X_1 and X_2 be two Banach spaces. Let

$$\mathcal{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where $F_{ij} \in \mathcal{L}(X_i, X_j)$, with $i, j = 1, 2$. Then
(i) $\mathcal{F} \in \mathcal{F}_l^\varepsilon(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_l^\varepsilon(X_i, X_j)$, with $i, j = 1, 2$;
(ii) $\mathcal{F} \in \mathcal{F}_r^\varepsilon(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_r^\varepsilon(X_i, X_j)$ with $i, j = 1, 2$.

Proof. (i) Suppose that $F_{ij} \in \mathcal{F}_l^\varepsilon(X_i, X_j)$ with $i, j = 1, 2$ and we will prove that $\mathcal{F} \in \mathcal{F}_l^\varepsilon(X_1 \times X_2)$.

Let $\mathcal{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{L}(X_1 \times X_2)$ such that $\|\mathcal{P}\| < \varepsilon$. First, let us consider the following decomposition

$$\mathcal{F}+\mathcal{P} = \begin{pmatrix} F_{11}+P_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & F_{12}+P_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F_{21}+P_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_{22}+P_{22} \end{pmatrix}.$$

It is sufficient to prove that $F_{ij} \in \mathcal{F}_l^\varepsilon(X_i, X_j)$ with $i, j = 1, 2$, then each operator in the right side of the previous equality is ε-pseudo Fredholm perturbation on $X_1 \times X_2$. For example, we will prove the result for the first operator. The proof for the other operators will be similarly achieved. Let $\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi(X_1 \times X_2)$ and let us denote $\widetilde{F} := \begin{pmatrix} F_{11}+P_{11} & 0 \\ 0 & 0 \end{pmatrix}$. From Atkinson Theorem [20] , it follows that there exist

$$\mathcal{L}_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2)$$

and

$$\mathcal{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \in \mathcal{K}(X_1 \times X_2),$$

such that

$$\mathcal{L}_0\mathcal{L} = I - \mathcal{K} \text{ on } X_1 \times X_2.$$

Then

$$\mathcal{L}_0(\mathcal{L}+\widetilde{F}) = I - \mathcal{K} + \mathcal{L}_0\widetilde{F} = \begin{pmatrix} I - K_{11} + A_0(F_{11}+P_{11}) & K_{12} + B_0(F_{11}+P_{11}) \\ -K_{21} & I - K_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}_l^\varepsilon(X_1)$, then $F_{11}+P_{11} \in \mathcal{F}_l(X_1)$. Using the fact that $\mathcal{F}_l(X_1)$ is two-sided ideal of $\mathcal{L}(X_1)$, we have

$$Id_{X_1} - K_{11} + A_0(F_{11}+P_{11}) \in \Phi_l(X_1).$$

So, there exist an operator $S \in \mathcal{L}(X_1)$ and $K_0 \in \mathcal{K}(X_1)$ such that

$$S(Id_{X_1} - K_{11} + A_0(F_{11}+P_{11})) = Id_{X_1} - K_0.$$

Therefore,

$$\begin{pmatrix} S & 0 \\ 0 & Id_{X_1} \end{pmatrix}(\mathcal{L}+\widetilde{F})\mathcal{L}_0 = I - \begin{pmatrix} K_0 & K_{12} \\ K_{21} & K_{22} \end{pmatrix} + \begin{pmatrix} 0 & S(F_{11}+P_{11}) \\ 0 & 0 \end{pmatrix}.$$

Using Remark 2.1, Proposition 2.1 and Theorem 2.4 in [1], we deduce that $\begin{pmatrix} 0 & S(F_{11}+P_{11}) \\ 0 & 0 \end{pmatrix} \in \mathcal{F}(X_1 \times X_2)$, and so, $\begin{pmatrix} S & 0 \\ 0 & Id_{X_1} \end{pmatrix}(\mathcal{L}+\widetilde{F})\mathcal{L}_0 \in \Phi(X_1 \times X_2)$, then there exist $\mathcal{S} \in \mathcal{L}(X_1 \times X_2)$

and $\widetilde{\mathcal{K}} \in \mathcal{K}(X_1 \times X_2)$ such that

$$\mathcal{S}\begin{pmatrix} S & 0 \\ 0 & Id_{X_1} \end{pmatrix}(\mathcal{L}+\widetilde{F})\mathcal{L}_0 = I - \widetilde{\mathcal{K}},$$

which implies $\mathcal{L}+\widetilde{F} \in \Phi_l(X_1 \times X_2)$.

Conversely, assume that $\mathcal{F} \in \mathcal{F}^{\varepsilon}(X_1 \times X_2)$, so $\mathcal{F}+\mathcal{P} \in \mathcal{F}(X_1 \times X_2)$ for all $\mathcal{P} := \begin{pmatrix} P_{12} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$ and we will prove that $F_{11}+P_{11} \in \mathcal{F}(X_1)$. Let $A \in \Phi(X)$ and let define the operator

$$\mathcal{L}_1 := \begin{pmatrix} A & -F_{12}-P_{12} \\ 0 & Id_{X_2} \end{pmatrix}.$$

From Proposition 2.1 (i) in [1], it follows that

$$\mathcal{L}_1 \in \Phi(X_1 \times X_2).$$

Hence,

$$\mathcal{F}+\mathcal{P}+\mathcal{L}_1 = \begin{pmatrix} A+F_{11}+P_{11} & 0 \\ F_{21}+P_{21} & Id_{X_2}+F_{22}+P_{22} \end{pmatrix} \in \Phi(X_1 \times X_2).$$

Using Proposition 2.2 (iii) in [1], we have

$$A+F_{11}+P_{11} \in \Phi_-(X_1). \tag{8}$$

In the same way, we may consider the Fredholm operator

$$\begin{pmatrix} A & 0 \\ -F_{21}-P_{21} & Id_{X_2} \end{pmatrix} \in \Phi(X_1 \times X_2).$$

Using Proposition 2.1 and 2.2 in [1], it is easy to deduce that

$$A+F_{11}+P_{11} \in \Phi_+(X_1). \tag{9}$$

From Eqs. (8) and (9), it follows that $A+F_{11}+P_{11} \in \Phi(X_1) \subset \Phi_l(X_1)$ and consequently, $F_{11}+P_{11} \in \mathcal{F}_l(X_1)$. In the same way, we can prove that

$$F_{22} \in \mathcal{F}_l^{\varepsilon}(X_2).$$

Now, we have to prove that $F_{12} \in \mathcal{F}^{\varepsilon}(X_2, X_1)$ and $F_{21} \in \mathcal{F}^{\varepsilon}(X_1, X_2)$. For this, let us consider $A \in \Phi(X_2, X_1)$ and $B \in \Phi(X_1, X_2)$. Then,

$$\begin{pmatrix} 0 & A+P_{12} \\ B+P_{21} & 0 \end{pmatrix} \in \Phi(X_1 \times X_2), \text{ for all } \mathcal{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{L}(X \times Y) \text{ such that } \|\mathcal{P}\| < \varepsilon.$$

Using the fact that $F_{11}+P_{11} \in \mathcal{F}(X_1)$ and $F_{22}+P_{22} \in \mathcal{F}(X_2)$, we can deduce that

$$\mathcal{F}+\mathcal{P}+\begin{pmatrix} -F_{11}-P_{11} & 0 \\ 0 & -F_{22}-P_{22} \end{pmatrix} \in \mathcal{F}(X_1 \times X_2).$$

Hence,

$$\begin{pmatrix} 0 & A+F_{12}+P_{12} \\ B+F_{21}+P_{21} & 0 \end{pmatrix} \in \Phi(X_1 \times X_2).$$

So,

$$A+F_{12} \in \Phi^{\varepsilon}(X_2, X_1) \subset \Phi_l^{\varepsilon}(X_2, X_1)$$

and

$$B + F_{21} \in \Phi^{\varepsilon}(X_1, X_2) \subset \Phi_l^{\varepsilon}(X_2, X_1). \qquad \square$$

Theorem 5.2. Let X be a Banach space, $\varepsilon > 0$ and A: $X \longrightarrow X$, B: $X \longrightarrow X$, C: $X \longrightarrow X$ and D: $X \longrightarrow X$ are four bounded linear operators. Let $L := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

(i) If $CA \in \mathcal{F}_l^{\varepsilon}(X), CB \in \mathcal{F}_l^{\varepsilon}(X)$ and $C \in \mathcal{F}_l(X)$ then,

$$\sigma_{e6,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\};$$

(ii) If $CA \in \mathcal{F}_r^{\varepsilon}(X), CB \in \mathcal{F}_r^{\varepsilon}(X)$ and $C \in \mathcal{F}_r(X)$ then,

$$\sigma_{e7,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e7,\varepsilon}(A) \cup \sigma_{e7,\varepsilon}(D)]\backslash\{0\}.$$

Proof. (i) Let $\lambda \in \mathbb{C}\backslash\{0\}$. Then, we have

$$\begin{aligned} \lambda - L &= \begin{pmatrix} \lambda - A & -B \\ -C & \lambda - D \end{pmatrix} \\ &= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ -CA & -CB \end{pmatrix} + \begin{pmatrix} Id_X & 0 \\ \frac{-C}{\lambda} & Id_X \end{pmatrix}\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix}. \end{aligned}$$

Suppose $\lambda \notin [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\}$, then by Lemma 2.1,

$$\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix} \in \Phi_l^{\varepsilon}(X \times X).$$

Since $\begin{pmatrix} Id_X & 0 \\ \frac{-C}{\lambda} & Id_X \end{pmatrix}$ is invertible then the operator matrix

$$\begin{pmatrix} Id_X & 0 \\ \frac{-C}{\lambda} & Id_X \end{pmatrix} \text{ is left Fredholm inverse.}$$

Moreover by hypothesis and by applying Theorem 2.4 in [1], we get

$$\begin{pmatrix} Id_X & 0 \\ 0 & Id_X \end{pmatrix} - \begin{pmatrix} Id_X & 0 \\ \frac{-C}{\lambda} & Id_X \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{-C}{\lambda} & 0 \end{pmatrix} \in \mathcal{F}_l(X).$$

Consequently, by using Lemma 2.3, we get

$$\begin{pmatrix} Id_X & 0 \\ \frac{-C}{\lambda} & Id_X \end{pmatrix}\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix} \in \Phi_l^{\varepsilon}(X \times X).$$

On the other hand, since $CA \in \mathcal{F}_l^{\varepsilon}(X), CB \in \mathcal{F}_l^{\varepsilon}(X)$, it follows from the Lemma 5.1 that

$$\begin{pmatrix} 0 & 0 \\ -CA & -CB \end{pmatrix} \in \mathcal{F}_l^{\varepsilon}(X \times X).$$

So, applying Theorem 5.1, we get

$$\lambda - L \in \Phi_l^{\varepsilon}(X \times X).$$

Thus,

$$\lambda \notin \sigma_{e6,\varepsilon}(L)\backslash\{0\}.$$

Hence,

$$\sigma_{e6,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\}.$$

The proof of (ii) may be checked in the same way as the proof of (i). $\square$

Theorem 5.3. Let X, Y be two Banach spaces, $\varepsilon > 0$ and A:$X \longrightarrow X$, B: $Y \longrightarrow X$, C: $X \longrightarrow Y$ and D: $Y \longrightarrow Y$ are four bounded operators. Let $L := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

(i) If $C \in \mathcal{F}_l^{\varepsilon}(X,Y)$ then,

$$\sigma_{e6,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\};$$

(ii) If $C \in \mathcal{F}_r^{\varepsilon}(X,Y)$ then,

$$\sigma_{e6,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\}.$$

Proof. (i) Let $\mathcal{P} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$. Then, for all $\lambda \in \mathbb{C}\backslash\{0\}$, we have

$$\begin{aligned} \lambda - L - \mathcal{P} &= \begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ -C - P_3 & \lambda - D - P_4 \end{pmatrix} \\ &= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ -(C+P_3)(A+P_1) & -(C+P_3)(B+P_2) \end{pmatrix} \\ &+ \begin{pmatrix} Id_X & 0 \\ \frac{-(C+P_3)}{\lambda} & Id_Y \end{pmatrix}\begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ 0 & \lambda - D - P_4 \end{pmatrix}. \end{aligned}$$

Suppose $\lambda \notin [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\}$, then by Lemma 2.1,

$$\begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ 0 & \lambda - D - P_4 \end{pmatrix} \in \Phi_l(X \times Y).$$

Since $\begin{pmatrix} Id_X & 0 \\ \frac{-(C+P_3)}{\lambda} & Id_Y \end{pmatrix}$ is invertible then $\begin{pmatrix} Id_X & 0 \\ \frac{-(C+P_3)}{\lambda} & Id_Y \end{pmatrix}\begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ 0 & \lambda - D - P_4 \end{pmatrix}$ is a left Fredholm inverse matrix.

On the other hand, it follows from the hypothesis that $(C + P_3)(B + P_2) \in \mathcal{F}_l(Y)$ and $(C + P_3)(A + P_1) \in \mathcal{F}_l(X, Y)$ and so, by using Lemma 5.1

$$\begin{pmatrix} 0 & 0 \\ -(C+P_3)(A+P_1) & -(C+P_3)(B+P_2) \end{pmatrix} \in \mathcal{F}_l(X \times Y).$$

So, $\lambda - L - \mathcal{P} \in \Phi_l(X \times Y)$. Thus, $\lambda \notin \sigma_{e6,\varepsilon}(L)\backslash\{0\}$. Hence,

$$\sigma_{e6,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e6,\varepsilon}(A) \cup \sigma_{e6,\varepsilon}(D)]\backslash\{0\}.$$

(ii) The proof of (ii) may be checked in the same way as the proof of (i). $\square$

Declarations

Conflict of interest The authors declare no conflict of interest.

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