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Regular control surfaces of a toric patch and integer programming

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Abstract. Toric patch is a kind of rational multisided patch, which is associated with a finite integer lattice points set $\mathcal{A}$. A set of weights is defined which depend on a parameter according to regular decomposition of $\mathcal{A}$. When all weights of the patch tend to infinity, we obtain the limiting form of toric patch which is called its regular control surface. The different weights may induce the different regular control surfaces of the same toric patch. It prompts us to consider that how many regular control surfaces of a toric patch. In this paper, we study the regular decompositions of $\mathcal{A}$ by using integer programming method firstly, and then provide the relationship between all regular decompositions of $\mathcal{A}$ and corresponding state polytope. Moreover, we present that the number of regular control surfaces of a toric patch associated with $\mathcal{A}$ is equal to the number of regular decompositions of $\mathcal{A}$. An algorithm to calculate the number of regular control surfaces of toric patch is provided. The algorithm also presents a method to construct all of the regular control surfaces of a toric patch. At last, the application of proposed result in shape deformation is demonstrated by several examples.

§1 Introduction

In the early 1970s, toric varieties were introduced and developed in algebraic geometry. The theory of toric varieties plays an important role at the crossroads of geometry, algebra and combinatorics. It provides a fertile testing ground for general theories in algebraic geometry. So, toric varieties are an important area of research in algebraic geometry and feature in many applications [2]. The theory of toric varieties is associated with combinatorics of convex polytopes [9]. And the toric variety of convex polytope is the variety of its fan. Hence the geometry of a toric variety is fully determined by the combinatorics of its associated fan.

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In [13], Warren proposed the real toric variety, which can be applied in CAGD. In 2002, Krasauskas [6] defined toric patch, which is a kind of rational multisided patch. The classical rational Bézier curve,the classical Bézier triangle, tensor-product Bézier patch, and Warren's hexagonal patch [1,13] are also special cases of the toric patch, while the corresponding polygons are line segment, triangle, rectangle and hexagon [6]. Since the classical rational Bézier curve is the special case of toric patch, it is also called toric Bézier curve.

It is well known that the shape of patch is controlled by not only control structure but also the weights. If there exists an enough large weight, then the patch is pulled to the corresponding control point. We call it the geometric meaning of a single weight [7]. In 2011, García-Puente, Sottile and Zhu [3] explained the limiting surface of toric patch when all weights tend to infinity, which is called the regular control surface, and generalized the geometric meaning of a single weight of rational Bézier patch[7]. That is to say, there exists a sequence of weights, which depend on a parameter, pull the patch towards the corresponding control structure when the parameter tend to infinity [15,16]. For example, for a biquadratic rational Bézier patch (Fig. 1(a) is the points configuration and Fig. 1(b) is the original patch), the patch will be pulled to the central control point if its central weight tends to infinity (see Fig. 1(c)). This is also an explanation of the geometric meaning of single weight in [7]. And if different weights, which depend to a parameter, tends to infinity, the patch in Fig. 1(b) can also deform into different structures (see Fig. 1(d) to Fig. 1(f)).

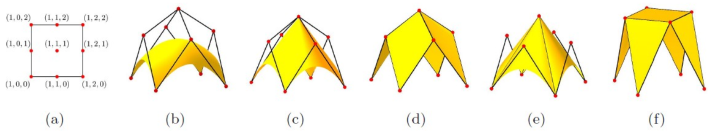


Figure 1. The geometric meaning of weights.

This phenomenon tells us that the different weights with a parameter may induce different regular control surfaces of the same toric patch. Thus a natural question is *how many regular control surfaces of a toric patch?* And how to *construct these regular control surfaces* is another interesting question. In fact, there are 4279 regular control surfaces for the biquadratic rational Bézier patch above mentioned. The specific calculation process and explanation are shown in Example 6.

Reference [14] presented a method to calculate the number of regular control surfaces of a toric patch and then answered the first question. They also provided the relationship between regular decompositions of $\mathcal{A}$ and corresponding secondary polytope. But the result for the patch is unsatisfied compared with the curve. Due to complexity of points configuration, the explicit formula only can calculate the number of regular control surfaces of toric patch for $\#(\mathcal{A}) \leq 9$ (i.e., the number of elements of $\mathcal{A}$ is less than or equal to 9). Not enough, we want to calculate the number of regular control surfaces of toric patch associated with arbitrary points configuration $\mathcal{A}$.

Since the definition of toric patch is associated with toric variety and toric ideal, and the methods in integer programming can be applied to study the toric varieties and toric ideals [4,9]. In references [3,14], the regular control surface is defined by the lifting function. It is interesting that the lifting function is the cost function of integer programming. Hence, we aim to present another new method to study the regular control surfaces of a toric patch for the arbitrary finitely integer lattice points set $\mathcal{A}$ in this paper. Unlike reference [14], we present a method to calculate the number of regular decompositions of $\mathcal{A}$ by using the theories of integer programming and universal Gröbner bases. And all regular decompositions of $\mathcal{A}$ can be constructed at the same time. We get the conclusion that each regular control surface of toric patch defined by $\mathcal{A}$ is associated to a regular decomposition by a rational map, which means the number of regular control surfaces of a toric patch is equal to the number of regular decompositions of $\mathcal{A}$. An algorithm is also provided to calculate the number of regular control surfaces of a toric patch and all of these regular control surfaces can be constructed too. So,we can answer two questions raised above accurately. At last, the application of proposed result in shape deformation is demonstrated by several examples.

§2 Cost functions of linear programming in regular decomposition

Let $cone(\mathcal{A}) = cone(\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\})$ replace the closed convex polyhedral d-cone $\{\mathcal{A}\mathbf{x} : \mathbf{x} \in \mathbb{R}^n_{\geq 0}\}$. A *polyhedral subdivision* of $cone(\mathcal{A})$ is a collection of subsets $cone(\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\})$, where these subsets are called *cells* (or *faces*) of the subdivision. These cells construct a *polyhedral fan* covering $cone(\mathcal{A})$. If $dim(cone(\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\})) = k$, then $cone(\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\})$ denotes a k-cell of the subdivision of $\mathcal{A}$. A subdivision of $cone(\mathcal{A})$ is a *triangulation* $\triangle$ if each d-cell of the complex is simplicial. In 1969, Walkup and Wets [12] provided the Basis Decomposition Theorem for Linear Programming as follows: the general parametric linear programming problem is

$$LP_{\mathcal{A},\mu}(\mathbf{b}) = min\{\mu \cdot \mathbf{x} : \mathcal{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}, \tag{1}$$

where cost function $\mu \in \mathbb{R}^n$ is fixed and $\mathcal{A}$ is a fixed $d \times n$-matrix of rank d. $LP_{\mathcal{A},\mu}(\mathbf{b})$ is feasible if and only if $\mathbf{b}$ lies in $cone(\mathcal{A})$; $LP_{\mathcal{A},\mu}(\mathbf{b})$ is bounded for all $\mathbf{b} \in cone(\mathcal{A})$ and all $\mu \in \mathbb{R}^n$ if and only if $ker(\mathcal{A}) \cap \mathbb{R}^n_{\geq 0} = \{0\}$; If $LP_{\mathcal{A},\mu}(\mathbf{b})$ is bounded, then there exist a triangulation $\triangle$ of $cone(\mathcal{A})$ such that the d-dimensional cells of $\triangle$ is $\mathcal{C} = cone(\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_d}\})$, and the column $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_d}$ construct an optimal basis for arbitrary b in the cell $\mathcal{C}$.

Denote $LP_{\mathcal{A},\mu}$ be the family of $LP_{\mathcal{A},\mu}(\mathbf{b})$ obtained by varying $\mathbf{b} \in cone(\mathcal{A})$ and fixed $\mathcal{A}, \mu$, and $LP_{\mathcal{A}}$ be the family obtained by keeping only $\mathcal{A}$ fixed.

It is well known that every sufficiently generic vector $\mu \in \mathbb{R}^n$ defines a triangulation $\triangle_\mu$ as follows: a $cone(\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\})$ is a cell of $\triangle_\mu$, if there exists a vector $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d$ such that $\mathbf{w} \cdot \mathbf{a}_j = \mu_j$ if $j \in \{i_1, \ldots, i_k\}$ and $\mathbf{w} \cdot \mathbf{a}_j < \mu_j$ if $j \in \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. Then the triangulation obtained in this way are called *regular*. We call μ is *generic*, if $\triangle_\mu$ is regular triangulation. If we find $\mathbf{x} \in \mathbb{R}^n$ such that $\mathcal{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$, and $supp(\mathbf{x})$ is a subset of a cell of $\triangle_\mu$, then the optimal solutions $\mathbf{x}$ of $LP_{\mathcal{A},\mu}(\mathbf{b})$ are the solutions to the problem. The set of feasible solutions to $LP_{\mathcal{A},\mu}(\mathbf{b})$ is the polyhedron $P_{\mathbf{b}} = conv\{\mathbf{x} \geq 0 : \mathcal{A}\mathbf{x} = \mathbf{b}\}$. $P_{\mathbf{b}}$ is non-empty if and only if $\mathbf{b} \in cone(\mathcal{A})$. Consider linear map $\pi_{\mathcal{A}} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^d, \mathbf{x} \mapsto \mathcal{A}\mathbf{x}$, we have $P_{\mathbf{b}} = \pi_{\mathcal{A}}^{-1}(\mathbf{b})$. $P_{\mathbf{b}}$ is the $\mathbf{b}$-fiber of $\pi_{\mathcal{A}}$. If μ is generic, then μ supports a vertex in each fiber $P_{\mathbf{b}}$ of $LP_{\mathcal{A}}$.

Now, we have a question *if μ is not generic, are these conclusions still valid?* If μ is non-generic, then the cell $cone(\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\})$ of $\diamondsuit_\mu$ is polygon. And the *decomposition* of $\mathcal{A}$ is a collection of subsets $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}$ of $\mathcal{A}$. We call μ is *non-generic*, if subdivision of $cone(\mathcal{A})$ is polyhedral subdivision denoted by $\diamondsuit_\mu$. We mainly discuss the problem of non-generic cost functions in linear programming.

Definition 1. *We define a new linear programming problem:*

$$LP^{new}_{\mathcal{A},\mu}(\boldsymbol{b}) = min\{\mu \cdot \boldsymbol{y}_\mu : \mathcal{A} \cdot \boldsymbol{y}_\mu = \boldsymbol{b}, \boldsymbol{y}_\mu = \boldsymbol{X} \cdot \boldsymbol{c}, \boldsymbol{\xi} \cdot \boldsymbol{y}_\mu = 0, \sum_{i=1}^{r} c_i = 1, \quad \boldsymbol{y}_\mu \geq 0, 1 \geq c_i \geq 0\},$$

where non-generic cost function $\mu \in \mathbb{R}^n$ is fixed and $\boldsymbol{\xi} \in \mathbb{R}^n$ is a row vector. $\boldsymbol{X} = \{\boldsymbol{x}^{op}_{\mu^{(1)}}, \ldots, \boldsymbol{x}^{op}_{\mu^{(r)}}\}$ be an $n \times r$ matrix, where $\boldsymbol{x}^{op}_{\mu^{(i)}}$ is the optimal solution of $LP_{\mathcal{A},\mu^{(i)}}(\boldsymbol{b})$. What's more, $\mu^{(i)}$ is generic and $\diamondsuit_\mu$ is subdivided by $\triangle_{\mu^{(i)}}$ $(i = 1, \ldots, r)$.

Remark 1. *Because of $\mu^{(i)}$ is generic and $\diamondsuit_\mu$ is subdivided by $\triangle_{\mu^{(i)}}$ $(i = 1, \ldots, r)$, we can obtain the form of $\diamondsuit_\mu$. If we substitute $\boldsymbol{y}_\mu = \boldsymbol{X} \cdot \boldsymbol{c}$ into $\mathcal{A} \cdot \boldsymbol{y}_\mu = \boldsymbol{b}$ and $\sum_{i=1}^{r} c_i = 1$, then $\mathcal{A} \cdot (\boldsymbol{X} \cdot \boldsymbol{c}) = \mathcal{A} \cdot (\boldsymbol{x}^{op}_{\mu^{(1)}}, \ldots, \boldsymbol{x}^{op}_{\mu^{(r)}}) \cdot \boldsymbol{c} = (\mathcal{A} \cdot \boldsymbol{x}^{op}_{\mu^{(1)}}, \ldots, \mathcal{A} \cdot \boldsymbol{x}^{op}_{\mu^{(r)}}) \cdot \boldsymbol{c} = \boldsymbol{b}$. The support of $\boldsymbol{y}_\mu$, $supp(\boldsymbol{y}_\mu)$, is a subset of index set $\{1, 2, \ldots, n\}$ of $\mathcal{A}$. Besides, the points of this subset are linear dependence. Hence, there exists a row vector $\boldsymbol{\xi}$ such that $\boldsymbol{\xi} \cdot \boldsymbol{y}_\mu = 0$. We have $(\boldsymbol{\xi} \cdot \boldsymbol{X}) \cdot \boldsymbol{c} = 0$ after substituting $\boldsymbol{y}_\mu = \boldsymbol{X} \cdot \boldsymbol{c}$ into $\boldsymbol{\xi} \cdot \boldsymbol{y}_\mu = 0$. So, we construct a coefficient integer matrix $\mathcal{M} = \begin{bmatrix} \boldsymbol{1}_{1 \times r} \\ (\boldsymbol{\xi} \cdot \boldsymbol{X})_{l \times r} \end{bmatrix}$.*

At last, Equation (1) with unknown $\mathbf{c}$ is simplified as

$$LP_{\mathcal{M},\mu}(\mathbf{b}) = min\{\mu \cdot (\mathbf{X} \cdot \mathbf{c}) : \mathcal{M} \cdot \mathbf{c} = (1, \mathbf{0}_{1 \times l})^{\mathrm{T}}, 1 \geq c_i \geq 0\}. \tag{2}$$

In conclusion, for every $\mathbf{b} \in cone(\mathcal{A})$, $LP_{\mathcal{M}}$ has a unique optimal solution. Moveover, non-generic μ supports a point $\mathbf{y}_\mu = \mathbf{X} \cdot \mathbf{c}$ in each fiber $P_{\mathbf{b}}$ of $LP_{\mathcal{M}}$ and $\mathbf{y}_\mu \in conv(\mathbf{x}^{op}_{\mu^{(1)}}, \ldots, \mathbf{x}^{op}_{\mu^{(r)}})$.

According to Definition 1 and Remark 1, we have two definitions as follows:

Definition 2. *Polyhedral subdivision $\diamondsuit_\mu$ is called regular, if $LP_{\mathcal{M},\mu}(\boldsymbol{b})$ have a unique optimal solution.*

Definition 3. *A decomposition $\mathcal{D}_\mu$ of $\mathcal{A}$ is called regular, if each subset of $\mathcal{D}_\mu$ lies in the cell of $\diamondsuit_\mu$.*

To be sure, a regular polyhedral subdivision $\diamondsuit_\mu$ is a decomposition of $cone(\mathcal{A})$, and a regular decomposition $\mathcal{D}_\mu$ is collection of subsets of $\mathcal{A}$. They have an essential difference.

Example 1. *This example shows how to find a unique optimal solution of $LP_{\mathcal{M},\mu}(\boldsymbol{b})$. Let $\mathcal{A} = \{(1, 0), (1, 1), (1, 2), (1, 3)\}$ be affine points set with index set $\{1, 2, 3, 4\}$. In fact, the set $\mathcal{A}$ has four regular triangulations [8]. It needs to be emphasized that every regular triangulation are unique, but the generic cost function is not unique. So, for every regular triangulation,* *we list a representative generic cost function* $\mu^{(i)} \in \mathbb{R}^4$ $(i = 1, 2, 3, 4)$ *respectively as* $\triangle_{\mu^{(1)}} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$, $\mu^{(1)} = (2, 1, 1, 2)$; $\triangle_{\mu^{(2)}} = \{\{1, 3\}, \{3, 4\}\}$, $\mu^{(2)} = (3, 3, 1, 2)$; $\triangle_{\mu^{(3)}} = \{\{1, 4\}\}$, $\mu^{(3)} = (1, 2, 2, 1)$; $\triangle_{\mu^{(4)}} = \{\{1, 2\}, \{2, 4\}\}$, $\mu^{(4)} = (2, 1, 3, 3)$.

The non-generic cost function $\mu_1 = (2, 1, 2, 3)$ *and* $\boldsymbol{b} = (18, 11)$ *are fixed. We obtain a fiber* $P_{\boldsymbol{b}}$ *which is a quadrangle. The optimal solutions* $\boldsymbol{x}^{op}_{\mu^{(1)}} = (0, 4, 7, 0)$ *of* $LP_{\mathcal{A},\mu^{(1)}}(\boldsymbol{b})$ *and* $\boldsymbol{x}^{op}_{\mu^{(4)}} = (0, 15/2, 0, 7/2)$ *of* $LP_{\mathcal{A},\mu^{(4)}}(\boldsymbol{b})$ *can be obtained by using simplex algorithm, respectively. And the solutions* $\boldsymbol{x}^{op}_{\mu^{(1)}}$ *and* $\boldsymbol{x}^{op}_{\mu^{(4)}}$ *are the vertices of the fiber* $P_{\boldsymbol{b}}$*. Because of* $\Diamond_{\mu_1}$ *is subdivided by* $\triangle_{\mu^{(1)}}$ *and* $\triangle_{\mu^{(4)}}$*, we have* $\Diamond_{\mu_1} = \{\{1, 2\}, \{2, 4\}\}$ *and* $\mathcal{D}_{\mu_1} = \{\{1, 2\}, \{2, 3, 4\}\}$*. According to Equation (1),* $\boldsymbol{y}_{\mu_1} = \boldsymbol{x}^{op}_{\mu^{(1)}} c_1 + \boldsymbol{x}^{op}_{\mu^{(4)}} c_2 = (0, 4c_1 + \frac{15}{2}c_2, 7c_1, \frac{7}{2}c_2)$*. Obviously, index set* $\{2, 3, 4\}$ *is a subset of index set* $\{1, 2, \ldots, n\}$ *of* $\mathcal{A}$*. Besides, the points of this subset are linear dependence as follows:* $2 \times 7c_1 - 4c_1 - \frac{15}{2}c_2 - \frac{7}{2}c_2 = 10c_1 - 11c_2 = 0$*. That is to say, we take* $\boldsymbol{\xi} = \{0, -1, 2, -1\}$*, then we have* $\mathcal{M} = \begin{bmatrix} 1 & 1 \\ 10 & -11 \end{bmatrix}$*. Therefore* $LP_{\mathcal{M},\mu_1}(\boldsymbol{b}) = min\{(0, 4c_1 + \frac{15}{2}c_2,\ 14c_1, \frac{21}{2}c_2) : \mathcal{M} \cdot \boldsymbol{c} = (1, 0)^{\mathrm{T}}, 1 \geq c_i \geq 0\}$*. We have the optimal solution* $(\frac{11}{21}, \frac{10}{21})$ *and* $\boldsymbol{y}_{\mu_1} = (0, \frac{17}{3}, \frac{11}{3}, \frac{5}{3}) \in face_{\mu_1}[(0, 4, 7, 0), (0, 15/2, 0, 7/2)]$*. Similarly, given* $\mu_2 = (3, 2, 1, 3)$*, we have* $\mathcal{D}_{\mu_2} = \{\{1, 2, 3\}, \{3, 4\}\}$ *and* $\boldsymbol{y}_{\mu_2} = (\frac{9}{2}, 5, \frac{11}{2}, 0)$*. Given* $\mu_3 = (1, 3, 1, 1)$*, we have* $\mathcal{D}_{\mu_3} = \{1, 3, 4\}$ *and* $\boldsymbol{y}_{\mu_3} = (\frac{53}{9}, 0, \frac{13}{3}, \frac{25}{9})$*. Given* $\mu_4 = (1, 1, 3, 1)$*, we have* $\mathcal{D}_{\mu_4} = \{1, 2, 4\}$ *and* $\boldsymbol{y}_{\mu_4} = (\frac{34}{9}, \frac{13}{3}, 0, \frac{44}{9})$*.*

If $\mu^{(i)}$ is generic, then a *test set* [10,11] for the family $LP_{\mathcal{A},\mu^{(i)}}$ is any finite subset $\mathcal{V}_{\mu^{(i)}}$ of $ker(\mathcal{A})$ such that $\mu^{(i)} \cdot \mathbf{v} > 0$ for all $\mathbf{v} \in \mathcal{V}_{\mu^{(i)}}$, and for every $\mathbf{b} \in cone(\mathcal{A})$ and every $\mathbf{x} \in P_{\mathbf{b}}$, either $\mathbf{x}$ is the optimal solution of $LP_{\mathcal{A},\mu^{(i)}}(b)$ or there exists $\mathbf{v} \in \mathcal{V}_{\mu^{(i)}}$ and $\epsilon > 0$ such that $\mathbf{x} - \epsilon\mathbf{v} \geq 0$. Moreover, the definition of *minimal non-face* of $\triangle_{\mu^{(i)}}$: if $\mathcal{F} \subset \{1, \ldots, n\}$ is not a face of $\triangle_{\mu^{(i)}}$ but every proper subset of $\mathcal{F}$ is a face of $\triangle_{\mu^{(i)}}$, then $\mathcal{F}$ is a minimal non-face of $\triangle_{\mu^{(i)}}$. For non-generic μ, we introduce three definitions as follows:

Definition 4. $\mathcal{F}$ *is called minimal non-face of* $\Diamond_\mu$*, if* $\mathcal{F} \subset \{1, \ldots, n\}$ *is not a face of* $\Diamond_\mu$ *but every proper subset of* $\mathcal{F}$ *is a face of* $\Diamond_\mu$*, and* $\mathcal{F}$ *is not included in any face of* $\Diamond_\mu$*.*

Definition 5. $\mathcal{V}_\mu$ *is called a test set for the family* $LP_{\mathcal{A},\mu}$*, if* $\Diamond_\mu$ *is subdivided by* $\triangle_{\mu^{(i)}}(i = 1, \ldots, r)$*.*

Definition 6. *For non-generic* μ*, a finite subset* $\mathcal{V}'_\mu$ *is called regular set if* $\mathcal{V}'_\mu = \{\bigcup_{i=1}^r \mathcal{V}_{\mu^{(i)}}\} \setminus \mathcal{V}_\mu$*.*

The vector $\mathbf{v} \in \mathcal{V}'_\mu$ can be written as coordinate representation, denoted by $\mathbf{u}$. It is convenient to consider $\mathbf{u}$ as a line segment $[\mathbf{u}^+, \mathbf{u}^-]$, where $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. We define that $[\mathbf{u}^+, \mathbf{u}^-]$ is directed from $\mathbf{u}^+$ to $\mathbf{u}^-$ since $\mu \cdot \mathbf{u}^+ \geq \mu \cdot \mathbf{u}^-$.

Lemma 1. $\boldsymbol{x}^{op}_{\mu^{(1)}}, \ldots, \boldsymbol{x}^{op}_{\mu^{(r)}}$ *move to the unique point* $\boldsymbol{y}_\mu$ *in* $P_{\boldsymbol{b}}$ *along directed segments of regular set* $\mathcal{V}'_\mu$*, respectively.*

Proof. The optimal solution $\mathbf{c}$ is obtained by solving $LP_{\mathcal{M},\mu}(\mathbf{b})$ and $\mathbf{y}_\mu = \{\mathbf{x}^{op}_{\mu^{(1)}}, \ldots, \mathbf{x}^{op}_{\mu^{(r)}}\} \cdot \mathbf{c}$. The directed segments of regular set $\mathcal{V}'_\mu$ trace r monotone paths from the $\mathbf{x}^{op}_{\mu^{(1)}}, \ldots, \mathbf{x}^{op}_{\mu^{(r)}}$ to $\mathbf{y}_\mu$. Due to optimal solution $\mathbf{c}$ is unique, the point $\mathbf{y}_\mu$ on the $P_{\mathbf{b}}$ is unique. □

Example 1 cont. *According to regular polyhedral subdivisions of* $\mathcal{A}$ *in Example 1, the regular set can be obtained. We obtain* $\mathcal{F}_{\triangle_{\mu(1)}} = \{1,3\},\{1,4\},\{2,4\}$, $\mathcal{F}_{\triangle_{\mu(2)}} = \{2\},\{1,4\}$, $\mathcal{F}_{\triangle_{\mu(3)}} = \{2\},\{3\}$, *and* $\mathcal{F}_{\triangle_{\mu(4)}} = \{3\},\{1,4\}$. *According to Definition 4, we obtain* $\mathcal{F}_{\diamondsuit_{\mu_1}} = \{1,3\},\{1,4\}$. *So,* $\mathcal{V}'_{\mu_1} = \{\mathcal{V}_{\mu^{(1)}} \bigcup \mathcal{V}_{\mu^{(4)}}\} \setminus \mathcal{V}_{\mu_1} = \{2e_3 - e_2 - e_4, e_2 + e_4 - 2e_3\}$. *According to Lemma 1, we have* $x_2^4 x_3^7 \xrightarrow[\frac{5}{3}\ times]{x_3^2 - x_2x_4} x_2^{\frac{17}{3}} x_3^{\frac{11}{3}} x_4^{\frac{5}{3}}$, $\quad x_2^{\frac{15}{2}} x_4^{\frac{7}{2}} \xrightarrow[\frac{11}{6}\ times]{x_2x_4 - x_3^2} x_2^{\frac{17}{3}} x_3^{\frac{11}{3}} x_4^{\frac{5}{3}}$. *Similarly,* $\mathcal{V}'_{\mu_2} = \{2e_2 - e_1 - e_3, e_1 + e_3 - 2e_2\}$, $\mathcal{V}'_{\mu_3} = \{3e_3 - e_1 - 2e_4, e_1 + 2e_4 - 3e_3\}$, *and* $\mathcal{V}'_{\mu_4} = \{3e_2 - 2e_1 - e_4, 2e_1 + e_4 - 3e_2\}$.

§3 Integer programming and universal Gröbner bases

We recall the general integer programming problem [5]

$$IP_{\mathcal{A},\mu}(\mathbf{b}) = min\{\mu \cdot \mathbf{x} : \mathcal{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{N}^n\}. \tag{3}$$

$IP_{\mathcal{A},\mu}(\mathbf{b})$ is feasible if and only if $\mathbf{b} \in cone_{\mathbb{N}}(\mathcal{A})$. And $P_{\mathbf{b}}^I$ $conv\{\mathbf{x} \in \mathbb{N}^n : \mathcal{A}\mathbf{x} = \mathbf{b}\}$ is a polytope for each $\mathbf{b} \in cone_{\mathbb{N}}(\mathcal{A})$. $P_{\mathbf{b}}^I$ is the fiber of $\mathbf{b}$ where $\pi_{\mathcal{A}}^I : \mathbb{N}^n \to \mathbb{Z}^d$, $\mathbf{x} \mapsto \mathcal{A}\mathbf{x}$. We call $P_{\mathbf{b}}^I$ the **b**-fiber of $\pi_{\mathcal{A}}^I$. Denote $IP_{\mathcal{A},\mu}$ be the family of $IP_{\mathcal{A},\mu}(\mathbf{b})$ which obtained by varying $\mathbf{b} \in cone_{\mathbb{N}}(\mathcal{A})$ and fixed $\mathcal{A}, \mu$, and $IP_{\mathcal{A}}$ be the family obtained by keeping only $\mathcal{A}$ fixed.

Given $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, we define the *initial form* $in_\mu(f)$ to be the sum of all terms $\zeta_i \cdot \mathbf{x}^{\mathbf{a}_i}$ such that $\mu \cdot \mathbf{a}_i$ is maximal for any polynomials $f = \sum \zeta_i \cdot \mathbf{x}^{\mathbf{a}_i}$. For an ideal I, we define the *initial ideal* to be the ideal generated by all initial forms $in_\mu(I) = \langle in_\mu(f) : f \in I\rangle$. A finite subset $\mathcal{G}_\mu \subset I$ is a *Gröbner basis* for I with respect to μ if $in_\mu(I) = \langle in_\mu(g) : g \in \mathcal{G}_\mu\rangle$. It is called *reduced* if, for any two distinct elements $g_i, g_j \in \mathcal{G}_\mu$, no term of g_j is divisible by $in_\mu(g_i)$.

It is clear that toric ideal $I_{\mathcal{A}}$ needs not be a monomial ideal. If μ is non-generic cost function for $IP_{\mathcal{A},\mu}$, then the ideal $in_\mu(I_{\mathcal{A}})$ is a binomial ideal, and $\mathbf{x}^{\mathbf{u}} \notin in_\mu(I_{\mathcal{A}})$ for each $\mathbf{b} \in cone_{\mathbb{N}}(\mathcal{A})$. So, if cost function $\mu \in \mathbb{R}^n$ is non-generic with respect to $IP_{\mathcal{A},\mu}$, then $\mu \cdot \mathbf{u}^+ = \mu \cdot \mathbf{u}^-$ which implies that the form $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in in_\mu(I_{\mathcal{A}})$, and vice versa. Thus initial ideal $in_\mu(I_{\mathcal{A}})$ is a binomial ideal. And the radical of the binomial ideal $in_\mu(I_{\mathcal{A}})$ contains binomial whose index set is a subset of $\mathcal{D}_\mu$:

$$\begin{aligned} rad(in_\mu(I_{\mathcal{A}})) =&\langle x_{i_1} x_{i_2} \cdots x_{i_s}, \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \{i_1, i_2, \ldots, i_s\}\ is\ a\ minimal \\ & non-face\ of\ \diamondsuit_\mu\ and\ binomial\ \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in in_\mu(I_{\mathcal{A}})\rangle. \end{aligned} \tag{4}$$

The monomial $x_{i_1} x_{i_2} \cdots x_{i_s}$ is a square-free monomial, and the minimal non-face of $\diamondsuit_\mu$ is $\{i_1, i_2, \ldots, i_s\}$. So, the decomposition form of $\mathcal{A}$ obtained in this way are called *regular decomposition* $\mathcal{D}_\mu$.

Example 1 cont. *We continue with the previous example. Regular decomposition form of* $\mathcal{A}$ *can be obtained by the theories of integer programming. Using software Macaulay 2, we obtain the toric ideal* $I_{\mathcal{A}} = \langle x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3\rangle$. *The cost function* $\mu_1 = (2,1,2,3)$ *is fixed. We obtain initial ideal* $in_{\mu_1}(I_{\mathcal{A}}) = \langle x_1x_3, x_1x_4, x_1x_4^2, x_1^2x_4, x_2x_4 - x_3^2\rangle$ *and* $rad(in_{\mu_1}(I_{\mathcal{A}})) = \langle x_1x_3, x_1x_4, x_2x_4 - x_3^2\rangle$. *By Equation (4), the minimal non-faces of* $\diamondsuit_{\mu_1}$ *are* $\{1,3\}$, $\{1,4\}$, *and* $\{2,3,4\}$ *is a subset of* $\mathcal{D}_{\mu_1}$. *So,* $\mathcal{D}_{\mu_1} = \{\{1,2\},\{2,3,4\}\}$.

According to Definition 6, it is clear that the elements of $\mathcal{V}'_\mu$ are directed segments. Besides, all of directed segments come in pairs, with each pair in the opposite direction (i.e., $[\mathbf{u}^+, \mathbf{u}^-] =$

$-[\mathbf{u}^-, \mathbf{u}^+]$). Now, if we ignore the direction, directed segments become common line segments. Hence, binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is equal to $\mathbf{x}^{\mathbf{u}^-} - \mathbf{x}^{\mathbf{u}^+}$. We introduce a new term, named *reduced path* RP_μ.

Definition 7. *Reduced path* RP_μ *is a finite set of binomials* $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-}$, *where the binomials without direction are supported by* $\mathcal{V}'_\mu$.

Example 1 cont. *According to regular sets obtained above, it is easy to get reduced path. We have obtained* $\mathcal{V}'_{\mu_1} = \{2e_3 - e_2 - e_4, e_2 + e_4 - 2e_3\}$, $\mathcal{V}'_{\mu_2} = \{2e_2 - e_1 - e_3, e_1 + e_3 - 2e_2\}$, $\mathcal{V}'_{\mu_3} = \{3e_3 - e_1 - 2e_4, e_1 + 2e_4 - 3e_3\}$, *and* $\mathcal{V}'_{\mu_4} = \{3e_2 - 2e_1 - e_4, 2e_1 + e_4 - 3e_2\}$. *According to Definition 7, we can easily obtain* $RP_{\mu_1} = \{x_3^2 - x_2x_4\}$, $RP_{\mu_2} = \{x_2^2 - x_1x_3\}$, $RP_{\mu_3} = \{x_1x_4^2 - x_3^3\}$, *and* $RP_{\mu_4} = \{x_1^2x_4 - x_2^3\}$. *For* $\mu_5 = (1,1,1,1)$, *we have* $\mathcal{V}_{\mu_5} = \emptyset$. *Hence* $\mathcal{V}'_{\mu_5} = \bigcup_{i=1}^4 \mathcal{V}'_{\mu_i}$. *Obviously,* $RP_{\mu_5} = \bigcup_{i=1}^4 RP_{\mu_i} = \{x_3^2 - x_2x_4, x_2^2 - x_1x_3, x_1x_4^2 - x_3^3, x_1^2x_4 - x_2^3\}$ *is a combined reduced path, and* $\mathcal{D}_{\mu_5} = \{1,2,3,4\}$. *Besides, regular decomposition* $\mathcal{D}_{\mu_5}$ *which is* $\mathcal{A}$ *itself is subdivided by regular decompositions* $\mathcal{D}_{\mu_1}, \mathcal{D}_{\mu_2}, \mathcal{D}_{\mu_3}$, *and* $\mathcal{D}_{\mu_4}$.

According to Lemma 1, we have a conclusion that the optimal solutions $\mathbf{x}^{op}_{\mu^{(1)}} \ldots, \mathbf{x}^{op}_{\mu^{(r)}}$ for $LP_{\mathcal{A},\mathbf{b}}$ move towards the unique point $\mathbf{y}_\mu$ in $P_\mathbf{b}$ along directed segments of regular set $\mathcal{V}'_\mu$. Hence, we hope that the integer programming can have similar result. Now, we define that the integer solution $\mathbf{x}^{\mu^{(i)}}$ is the optimum of $IP_{\mathcal{A},\mu^{(i)}}(\mathbf{b})$ for all integer points on the $conv(\mathbf{x}^{op}_{\mu^{(1)}}, \ldots, \mathbf{x}^{op}_{\mu^{(r)}})$. That is to say, the solution $\mathbf{x}^{\mu^{(i)}}$ of $IP_{\mathcal{A},\mu^{(i)}}(\mathbf{b})$ depends on the optimal solution $\mathbf{x}^{op}_{\mu^{(i)}}$ of $LP_{\mathcal{A},\mu^{(i)}}(\mathbf{b})$. Moreover, we have $conv(\mathbf{x}^{\mu^{(1)}}, \ldots, \mathbf{x}^{\mu^{(r)}}) \subset conv(\mathbf{x}^{op}_{\mu^{(1)}}, \ldots, \mathbf{x}^{op}_{\mu^{(r)}})$. Hence, we obtain following lemma.

Lemma 2. $\boldsymbol{x}^{\mu^{(1)}}, \ldots, \boldsymbol{x}^{\mu^{(r)}}$ *move towards the unique point* $\boldsymbol{y}^I_\mu$ *in* $P^I_{\boldsymbol{b}}$ *along reduced paths* RP_μ, *respectively.*

Proof. According to Lemma 1 and Definition 7, there must exist monotone paths from the integer solutions $\mathbf{x}^{\mu^{(1)}}, \ldots, \mathbf{x}^{\mu^{(r)}}$ to the unique optimum $\mathbf{y}^I_\mu$, respectively. □

Notice that there is no integer lattice in $|\mathbf{y}^I_\mu - \mathbf{y}_\mu|$. Then $\mathbf{y}^I_\mu \in face_\mu(P^I_\mathbf{b})$, and the support of $\mathbf{y}^I_\mu$, $supp(\mathbf{y}^I_\mu)$, is a subset of $\mathcal{D}_\mu$. Finally, we can find the position of point with respect to non-generic cost function μ in $P^I_\mathbf{b}$.

Example 1 cont. *The unique point* $\boldsymbol{y}^I_{\mu_1}$ *can be obtained by reduced path. By the above, we obtain reduced path* $RP_{\mu_1} = \{x_3^2 - x_2x_4\}$. *And* $\boldsymbol{x}^{\mu^{(1)}} = (0,4,7,0)$, $\boldsymbol{x}^{\mu^{(4)}} = (0,7,1,3)$. *Then, we have* $x_2^4x_3^7 \xrightarrow[2\ times]{x_3^2 - x_2x_4} x_2^6x_3^3x_4^2$, $\;x_2^7x_3x_4^3 \xrightarrow[1\ time]{x_2x_4 - x_3^2} x_2^6x_3^3x_4^2$. *That is to say, the point* $\boldsymbol{y}^I_{\mu_1} = (0,6,3,2)$ *with respect to* $\mu_1 = (2,1,2,3)$ *lies in the* $face_{\mu_1}(P^I_{\boldsymbol{b}}) = [(0,4,7,0),(0,\ 7,1,3)]$. *Besides, there is no integer lattice in* $|\boldsymbol{y}^I_{\mu_1} - \boldsymbol{y}_{\mu_1}|$, *and the* $supp(\boldsymbol{y}^I_{\mu_1}) = \{2,3,4\}$ *is a subset of* D_{μ_1}, *The precise positions of the points* $\boldsymbol{y}^I_{\mu_1}$ *with respect to* μ_1.

We define the *universal Gröbner basis* $\mathcal{U}_\mathcal{A}$ to be the union of all reduced Gröbner basis $\mathcal{G}_\mu$ of the toric ideal $I_\mathcal{A}$ when μ runs over all term orders. And $\mathcal{U}_\mathcal{A}$ is a finite set that consists of binomials $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$, where $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in ker_\mathbb{Z}(\mathcal{A})$. Thus the line segment $[\mathbf{u}^+, \mathbf{u}^-]$ is an edge of the $\mathcal{A}\mathbf{u}^+$-fiber of $IP_\mathcal{A}$. We shall view polytope $P^I_{\mathcal{A}\mathbf{u}^+}$ as the fiber of $\mathbf{u}$. The definition of *Gröbner fiber* of $IP_{\mathcal{A}}$ is the fiber of an element $\mathbf{u} \in \mathcal{U}_{\mathcal{A}}$. The symbol $St(\mathcal{A})$ denotes the Minkowski sum of all Gröbner fibers, and this polytope is called the *state polytope* of $\mathcal{A}$ [3,8].

Definition 8. *If a reduced path RP_μ induces s different regular decompositions, then we call s the multiple number of RP_μ, denoted by $|RP_\mu| = s$. Especially, RP_μ is single when $s = 1$.*

Proposition 1. *The elements of reduced path are contained in the universal Gröbner basis $\mathcal{U}_{\mathcal{A}}$.*

Proof. According to the definition of reduced path, the result holds. □

Proposition 2. *The binomial $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-}$ is an element of reduced path if and only if an edge of $St(\mathcal{A})$ is given by this binomial.*

Proof. The elements of reduced path are the elements in $\mathcal{U}_{\mathcal{A}}$, and each edge of $St(\mathcal{A})$ is given by an element in $\mathcal{U}_{\mathcal{A}}$ according to reference [10]. So a binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in RP_\mu$ if and only if an edge of $St(\mathcal{A})$ is given by this binomial. □

Example 1 cont. *We can construct state polytope by using universal Gröbner basis, and the relationship is clear between the state polytope and all reduced paths. By the above, there are 5 reduced paths in total. Using software Macaulay 2, the universal Gröbner basis is $\mathcal{U}_{\mathcal{A}} = \{x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2, x_1x_4^2 - x_3^3, x_1^2x_4 - x_2^3\}$. Fig. 2 shows the relationship between the state polytope $St(\mathcal{A})$ and these five reduced paths, where the solid lines represent the reduced paths.*

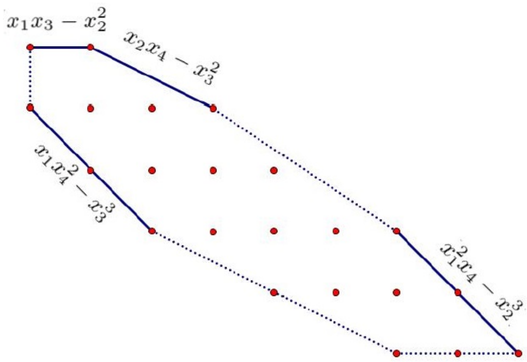


Figure 2. $St(\mathcal{A})$ and the reduced paths.

Theorem 1. *The number of regular triangulations plus the sum of multiple numbers for all reduced paths equals the number of regular decompositions of $\mathcal{A}$.*

Proof. By the above analysis, the regular triangulation is a special case of the regular decompositions. Besides, reduced paths are supported by all non-generic cost functions. Hence, the number of regular triangulations and the sum of multiple numbers for all reduced paths is equal to the number of regular decompositions of $\mathcal{A}$. □

Example 1 cont. *There are four regular triangulations and five reduced paths in total. So, the number of regular decompositions of $\mathcal{A}$ is $4 + 5 = 9$.*

§4 Regular control surfaces of toric patch

4.1 Regular control surfaces and regular decompositions

Definition 9. *([6]) A* toric patch *associated with a finite points set $\mathcal{A}$ is a patch parameterized by the rational map $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}} : U \to \mathbb{R}^3$*

$$\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(x) = \frac{\sum_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{\omega_a b_a} \eta_{\boldsymbol{a},\mathcal{A}}(x)}{\sum_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{\omega_a} \eta_{\boldsymbol{a},\mathcal{A}}(x)}, \quad x \in U, \tag{5}$$

where $U = conv(\mathcal{A}) \subset \mathbb{R}^d$, weights $\boldsymbol{\omega} = \{\boldsymbol{\omega_a} > 0 | \boldsymbol{a} \in \mathcal{A}\}$, control points $\mathcal{B} = \{\boldsymbol{b_a} | \boldsymbol{a} \in \mathcal{A}\} \subset \mathbb{R}^3$, toric-Bernstein basis functions $\eta_{\boldsymbol{a},\mathcal{A}}(x) = c_{\boldsymbol{a}} f_1(x)^{f_1(\boldsymbol{a})} \cdots f_r(x)^{f_r(\boldsymbol{a})}$ and positive coefficients $c_{\boldsymbol{a}} > 0$ indexed by lattice points of $\mathcal{A}$.

The image of $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(x)$ on U is called toric patch of shape of $\mathcal{A}$, denoted by $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$. Using cost function μ and weight $\boldsymbol{\omega} = \{\boldsymbol{\omega}_{\mathbf{a}} | \mathbf{a} \in \mathcal{A}\} \in \mathbb{R}^{\mathcal{A}}_{>}$, a set of weights is defined which depends on a parameter $\boldsymbol{\omega}_\mu(t) = \{t^{\mu(\mathbf{a})} \boldsymbol{\omega}_{\mathbf{a}} | \mathbf{a} \in \mathcal{A}\}$. So we can define toric patch parameterized by t as

$$\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B},\mu}(x;t) = \frac{\sum_{\mathbf{a}\in\mathcal{A}} t^{\mu(\mathbf{a})} \boldsymbol{\omega}_{\mathbf{a}} \mathbf{b}_{\mathbf{a}} \eta_{\mathbf{a},\mathcal{A}}(x)}{\sum_{\mathbf{a}\in\mathcal{A}} t^{\mu(\mathbf{a})} \boldsymbol{\omega}_{\mathbf{a}} \eta_{\mathbf{a},\mathcal{A}}(x)}, \quad x \in U. \tag{6}$$

Similarly, let $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B},\mu}(t)$ be the image of $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B},\mu}(x;t)$ on U with some $t \in \mathbb{R}_>$. Fix $\mathcal{A}$, weights $\boldsymbol{\omega}$ and control points $\mathcal{B}$ for a toric patch $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ of shape $\mathcal{A}$. Given a regular decomposition $\mathcal{D}_\mu$ induced by cost function μ, we consider the weights $\boldsymbol{\omega}$ and control points $\mathcal{B}$ indexed by elements of a facet $\mathbf{I}$ of $\mathcal{D}_\mu$ as weights and control points for a toric patch of shape $\mathbf{I}$, denoted by $\mathbf{P}_{\mathbf{I},\boldsymbol{\omega}|_{\mathbf{I}},\mathcal{B}|_{\mathbf{I}}}$. The union

$$\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(\mathcal{D}_\mu) = \bigcup_{\mathbf{I}\in\mathcal{D}_\mu} \mathbf{P}_{\mathbf{I},\boldsymbol{\omega}|_{\mathbf{I}},\mathcal{B}|_{\mathbf{I}}} \tag{7}$$

of these patches is called the *regular control surface* of $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ induced by the regular decomposition $\mathcal{D}_\mu$ [3]. This procedure is called *toric degeneration* of toric patch.

Theorem 2. *([3])*

$$\lim_{t\to+\infty} \boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B},\mu}(t) = \boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(\mathcal{D}_\mu).$$

This result explains the geometric meaning of the limiting form of toric patch is the regular control surface by a regular decomposition $\mathcal{D}_\mu$ of $\mathcal{A}$ when all weights tend to infinity. The specific degeneration methods of toric patch can be found in [3].

Theorem 3. *The number of regular control surfaces of a toric patch $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ is equal to the number of regular decompositions of $\mathcal{A}$.*

Proof. Each regular control surface of the patch is associated to one of regular decompositions of $\mathcal{A}$ by a rational map. In other words, for arbitrary given weights $\boldsymbol{\omega}$ and control points $\mathcal{B}$, the number of regular control surfaces of a toric patch $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ is equal to the number of the regular decompositions of $\mathcal{A}$. □

By the above analysis and results, we present an algorithm to calculate the number of regular control surfaces of a toric patch. And we use N to be the number of regular decompositions of

$\mathcal{A}$ in total, T to be the number of regular triangulations of $\mathcal{A}$ and S to be the sum of multiple numbers for all reduced paths.

Algorithm 1.
Input: $\mathcal{A}, \boldsymbol{\omega}, \mathcal{B}, N := T, M := 0, S := 0.$
Output: *The number of regular control surfaces of toric patch* $R_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$.

1 Calculate the toric ideal $I_{\mathcal{A}}$ *and the universal Gröbner basis* $\mathcal{U}_{\mathcal{A}}$ *of* $\mathcal{A}$.

2 Calculate the number of regular decompositions of $\mathcal{A}$ *by* $\mathcal{U}_{\mathcal{A}}$ *and integer programming method.*

2.1 If $\mathcal{U}_{\mathcal{A}} = \emptyset$ *then go to Step 3. Else select* $\dot{\boldsymbol{u}} \in \mathcal{U}_{\mathcal{A}}$ *and set* $\mathcal{U}_{\mathcal{A}} := \mathcal{U}_{\mathcal{A}} \backslash \{\dot{\boldsymbol{u}}\}$.

2.2 If $\dot{\boldsymbol{u}} \in \mathcal{U}_{\mathcal{A}}$ *is a reduced path, then we can conclude cost function* μ *and the regular decompositions* $\mathcal{D}_\mu$ *of* $\mathcal{A}$ *by Equation (4).*

2.2.1 Gather the counts of multiple numbers s *of the reduced path by Definition 8, denoted by* $S := S + s$.

2.2.2 According to the Theorem 1, compute $N := N + S$ *and return to Step 2.1.*

2.3 Else $\dot{\boldsymbol{u}} \in \mathcal{U}_{\mathcal{A}}$ *is a non-reduced path, then return to Step 2.1.*

3 Output N *according to Theorem 3.*

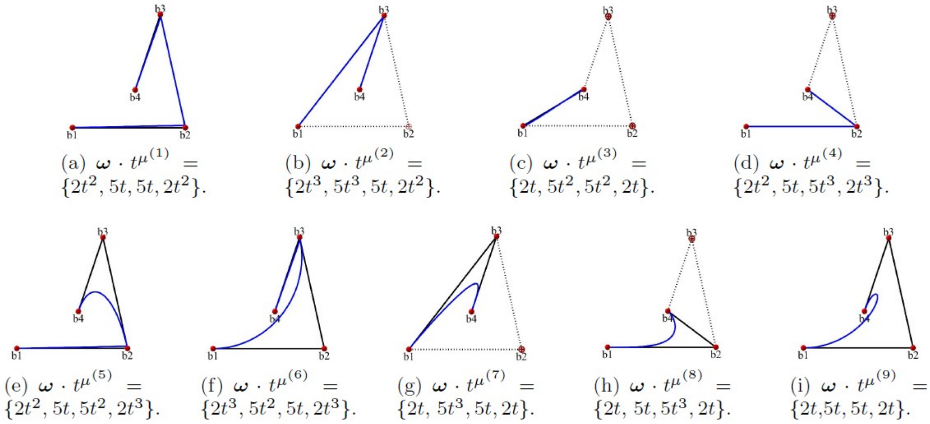
Remark 2. *This algorithm not only can calculate the number of regular control surfaces of a toric patch, but also provides an method to construct all the regular control surfaces. In Step 2.2, we get all regular decompositions of* $\mathcal{A}$*, and then we can construct every regular control surface* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(\mathcal{D}_\mu)$ *of the toric patch* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *induced by* $\mathcal{D}_\mu$ *according to Equation (7).*

4.2 Examples

Example 1 cont. *According the theories and results raised above, we obtain the number and all the forms of regular decompositions of* $\mathcal{A}$*. So, the number and all the form of regular control surfaces of a toric patch can be obtained by Theorem 3 and Algorithm 1. Since there are* 9 *different regular decompositions of* $\mathcal{A}$*, the number of the regular control curves of the cubic Bézier curve* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *is* 9*. We can construct all regular control curves of this curve* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *(see Fig. 3) by these regular decompositions of* $\mathcal{A}$ *and Equation (7) as below.*

In Definition 9, we don't need to fix the coefficients $c_{\mathbf{a}}$ of the basis function $\eta_{\mathbf{a},\mathcal{A}}$, as they can vary from case to case.

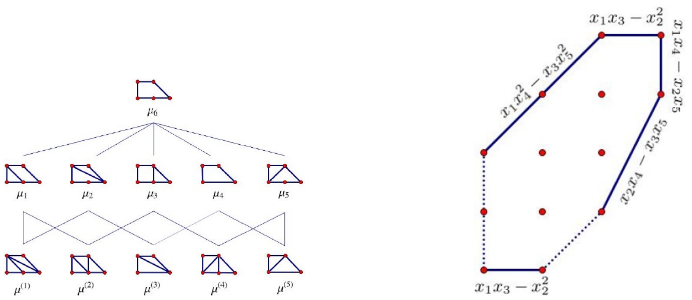
Example 2. *We set a* 5 *point configuration* $\mathcal{A}$ *with index set* $\{1, 2, 3, 4, 5\}$*, which is shown in Fig. 5(a). Fig. 5(b) also shows the toric patch* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *associated by* $\mathcal{A}$ *for given control points* $\mathcal{B}$ *and weights* $\boldsymbol{\omega}$*. There are five regular triangulations of* $\mathcal{A}$*. Using Algorithm 1, we have universal Gröbner basis* $\mathcal{U}_{\mathcal{A}} = \{x_1x_3 - x_2^2,\ x_1x_4 - x_2x_5, x_1x_4^2 - x_3x_5^2, x_2x_4 - x_3x_5\}$*. Exactly, the elements of* $\mathcal{U}_{\mathcal{A}}$ *are the reduced paths of* $\mathcal{A}$*, where* $|x_1x_3 - x_2^2| = 2$ *and the rest is single. Fig 4(a) shows the relationship between these* 11 *regular decompositions of* $\mathcal{A}$*. Using Theorem 1, the*



(a) $\boldsymbol{\omega} \cdot t^{\mu^{(1)}} = \{2t^2, 5t, 5t, 2t^2\}$. (b) $\boldsymbol{\omega} \cdot t^{\mu^{(2)}} = \{2t^3, 5t^3, 5t, 2t^2\}$. (c) $\boldsymbol{\omega} \cdot t^{\mu^{(3)}} = \{2t, 5t^2, 5t^2, 2t\}$. (d) $\boldsymbol{\omega} \cdot t^{\mu^{(4)}} = \{2t^2, 5t, 5t^3, 2t^3\}$.

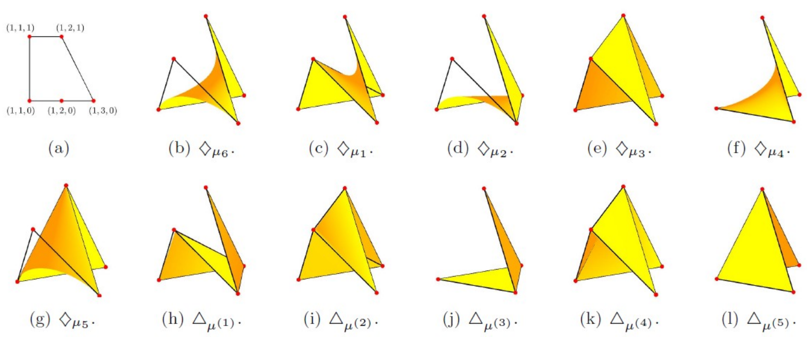
(e) $\boldsymbol{\omega} \cdot t^{\mu^{(5)}} = \{2t^2, 5t, 5t^2, 2t^3\}$. (f) $\boldsymbol{\omega} \cdot t^{\mu^{(6)}} = \{2t^3, 5t^2, 5t, 2t^3\}$. (g) $\boldsymbol{\omega} \cdot t^{\mu^{(7)}} = \{2t, 5t^3, 5t, 2t\}$. (h) $\boldsymbol{\omega} \cdot t^{\mu^{(8)}} = \{2t, 5t, 5t^3, 2t\}$. (i) $\boldsymbol{\omega} \cdot t^{\mu^{(9)}} = \{2t, 5t, 5t, 2t\}$.

Figure 3. Regular control curves of cubic rational Bézier curve.



(a) Regular decompositions. (b) $St(\mathcal{A})$ and RP_μ.

Figure 4. Regular decompositions and state polytope $St(\mathcal{A})$.



(a) (b) $\diamondsuit_{\mu_6}$. (c) $\diamondsuit_{\mu_1}$. (d) $\diamondsuit_{\mu_2}$. (e) $\diamondsuit_{\mu_3}$. (f) $\diamondsuit_{\mu_4}$.

(g) $\diamondsuit_{\mu_5}$. (h) $\triangle_{\mu^{(1)}}$. (i) $\triangle_{\mu^{(2)}}$. (j) $\triangle_{\mu^{(3)}}$. (k) $\triangle_{\mu^{(4)}}$. (l) $\triangle_{\mu^{(5)}}$.

Figure 5. Regular control surfaces of toric patch associated with a 5 points configuration.

number of regular decompositions of $\mathcal{A}$ is 11*. The relationship between reduced paths and state polytope $St(\mathcal{A})$ is shown in Fig 4(b), where the solid lines represent the reduced paths. According to Theorem 3 and Remark 2, the number of the regular control surfaces of toric patch* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$

associated by 5 *point configuration is* 11 *too. In Fig. 5, there are* 11 *regular decompositions of* $\mathcal{A}$ *can induce* 11 *regular control surfaces of the toric patch* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$.

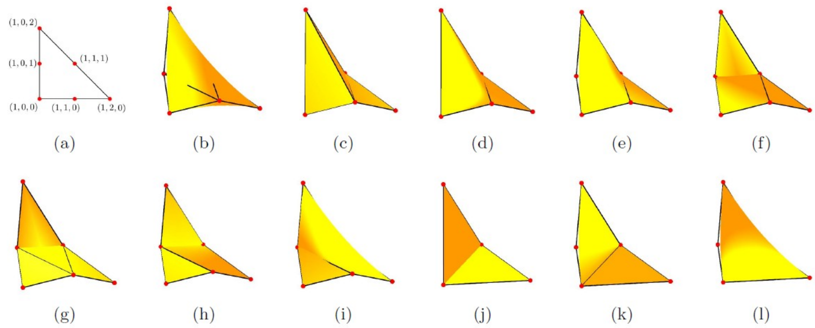


Figure 6. Regular control surfaces of a quadratic rational Bézier triangle.

Example 3. *Let* $\mathcal{A}$ *be an affine points set, where index set is* $\{1, 2, 3, 4, 5, 6\}$. *Fig. 6(a) indicates the structure of* $\mathcal{A}$ *and the convex hull* U. *A triangular Bézier patch* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *defined by* $\mathcal{A}$ *for given control points* $\mathcal{B}$ *and weights* $\boldsymbol{\omega}$ *is shown in Fig. 6(b). There are* 14 *regular triangulations of* $\mathcal{A}$. *Using Algorithm 1, we have universal Gröbner basis* $\mathcal{U}_{\mathcal{A}} = \{x_1x_3 - x_2^2, x_1x_5 - x_6^2, x_3x_5 - x_4^2, x_2x_5 - x_4x_6, x_2x_6 - x_1x_4, x_3x_6 - x_2x_4, x_2^2x_5 - x_1x_4^2, x_3x_6^2 - x_1x_4^2, x_2^2x_5 - x_3x_6^2\}$. *Exactly, the elements of* $\mathcal{U}_{\mathcal{A}}$ *are the reduced paths of* $\mathcal{A}$ *, where* $|x_1x_3 - x_2^2| = |x_1x_5 - x_6^2| = |x_3x_5 - x_4^2| = 5$ *and the rest is single. Besides, they are* 10 *combined reduced paths. So, the number of regular decompositions of* $\mathcal{A}$ *is* 45. *It means that the patch associated with* $\mathcal{A}$ *have* 45 *regular control surfaces by Remark 2. Fig. 6 shows some of these regular control surfaces of* $\boldsymbol{P}_{\mathcal{A},\omega,\mathcal{B}}$. *Likewise, we provide* 10 *of those* 45 *regular control surfaces from Fig. 6(c) to Fig. 6(l).*

Example 4. *In this example, we introduce the method to calculate the number of regular control surfaces of a biquadratic rational Bézier patch. In general, the patches of bidegree* (m, n) *can be dealt with in the same way. Let* $\mathcal{A}$ *be affine points set, where* $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *is index set. There are* 9 *points in configuration* $\mathcal{A}$ *as Fig. 1(a) shown, which defines a biquadratic rational Bézier patch* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *for given control points* $\mathcal{B}$ *and weights* $\boldsymbol{\omega}$ *(see Fig. 1(b)). There exist* 429 *regular triangulations of* $\mathcal{A}$. *Surprisingly, the number of regular decompositions is* 4279. *So there are* 4279 *regular control surfaces of biquadratic rational Bézier patch. In Fig. 1(d) – Fig. 1(f) show three of those regular control surfaces of* $\boldsymbol{P}_{\mathcal{A},\omega,\mathcal{B}}$.

4.3 Application to shape deformation

In this section, a toric patch $\mathbf{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ can be deformed into a target surface based on the theories we raised. According to Equation (7), the regular control surface is the union of some patches which is induced by the subsets of regular decomposition. So, the shape of the regular control surface is predictable, if the regular decomposition is known. In this way, the regular control surface of the toric patch can be generated by selecting an appropriate regular decomposition and cost function. And this regular control surface is the desired target surface.

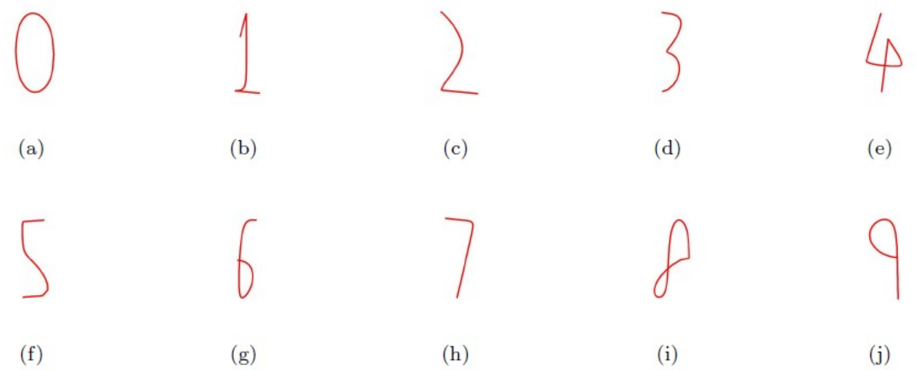


Figure 7. Handwritten Arabic numerals.

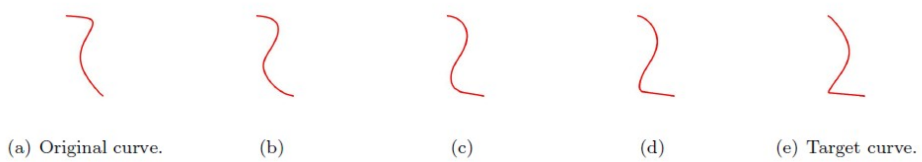
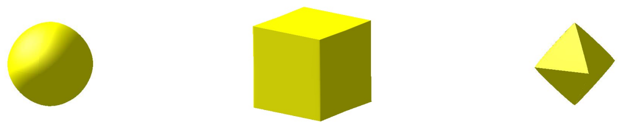


Figure 8. The procedure of shape deformation when $t \to +\infty$.

Example 5. *In this example, we want to construct ten target curves which are ten handwritten Arabic numerals. Chosen ten toric Bézier curves, also called original curves, we want to deform them into handwritten Arabic numerals by using the theories we raised above(see Fig. 7). Take the handwritten Arabic numeral two as an example, and other numbers can be dealt with in the same way. According to the characteristics of handwritten Arabic numeral two, a toric Bézier curve of degree* 5 $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}$ *(see Fig. 8(a)) is chosen, which is associated with a finite integer lattice points set* $\mathcal{A} = \{1, ..., 6\}$. *It can be deformed into handwritten Arabic numeral two (see Fig. 8(e)). According to Algorithm 1, the regular decomposition is* $\mathcal{D}_\mu = \{\{1,3,5\},\{5,6\}\}$ *and corresponding cost function is* $\mu = (2,3,2,3,2,5)$. *So, the regular control curve* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(\mathcal{D}_\mu)$ *is the union of a toric Bézier curve of degree* 2 *and a line segment, where the toric Bézier curve of degree* 2 *is associated with the subset* $\{1,3,5\}$ *of* $\mathcal{D}_\mu$ *and the line segment is associated with the subset* $\{5,6\}$ *of* $\mathcal{D}_\mu$. *Finally, the regular control curve* $\boldsymbol{P}_{\mathcal{A},\boldsymbol{\omega},\mathcal{B}}(\mathcal{D}_\mu)$ *is the target curve. And the final representations validate by the algebra expressions when* $t \to +\infty$. *Fig. 8(a) - Fig. 8(e) show the procedure of shape deformation.*

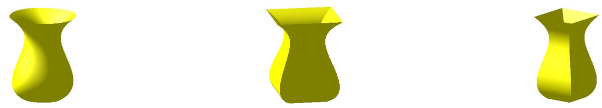
Example 6. *Given a sphere surface, we obtain the regular control surface by proposed result. The sphere surface is combined by eight patches. Using the result we raised, the sphere surface* *in Fig. 9(a) can be deformed into a cube shown in Fig. 9(b) and an octahedron shown in Fig. 9(c). What's more, all final target surfaces are one-step forming.*



(a) (b) (c)

Figure 9. Shape deformation of a sphere surface.

Example 7. *Similar to Example 8, given a vase surface, we obtain its regular control surface. The vase surface is combined by four patches which bidegree is* $(2, 5)$*. Using the result we raised, vase surface in Fig. 10(a) can be deformed into target surfaces shown in Fig. 10(b) and Fig. 10(c). And, all final target surfaces are one-step forming.*



(a) (b) (c)

Figure 10. Shape deformation of a vase surface.

§5 Conclusions

In this paper, we get the conclusion that each regular control surface of toric patch defined by $\mathcal{A}$ is associated to a regular decomposition by a rational map, which means the number of regular control surfaces of a toric patch is equal to the number of regular decompositions of $\mathcal{A}$. By the theories of integer programming and universal Gröbner bases, we present a method to calculate the number of regular decompositions of $\mathcal{A}$. And all regular decompositions of $\mathcal{A}$ can be constructed at the same time. An algorithm is provided to calculate the number of regular control surfaces of a toric patch and all of these regular control surfaces can be constructed too. At the end, the paper show three examples of shape deformation by the proposed result. The final target curve/surface can be viewed as a regular control curve/surface of original curve/surface. And the final target curve/surface is one-step forming.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] D Ahmad, S Naeem. *Quasi-Harmonic constraints for toric bézier surfaces*, 2018, DOI: 10.48550/-arXiv.1803.09328.

[2] D A Cox, J Little, D O'Shea. *Using algebraic geometry*, New York: Springer, 2006.

[3] L D García-Puente, F Sottile, C Zhu. *Toric degenerations of bézier patches*, ACM Transactions on Graphics (TOG), 2011, 30(5): 110.

[4] I M Gelfand, M M Kapranov, A V Zelevinsky. *Discriminants, resultants, and multidimensional determinants*, Boston, MA: Birkhäuser, 1994.

[5] S Hoşten, R R Thomas. *Gomory integer programs*, Mathematical Programming, 2003, 96(2): 271-292.

[6] R Krasauskas. *Toric surface patches*, Advances in Computational Mathematics, 2002, 17(1): 89-113.

[7] L Piegl, W Tiller. *The NURBS book*, New York: Springer, 1997.

[8] B Sturmfels. *Gröbner bases of toric varieties*, Tohoku Mathematical Journal, Second Series, 1991, 43 (2): 249-261.

[9] B Sturmfels. *Gröbner bases and convex polytopes*, Providence, RI: American Mathematical Society, 1996.

[10] B Sturmfels, R R Thomas. *Variation of cost functions in integer programming*, Mathematical Programming, 1997, 77(2): 357-387.

[11] R R Thomas. *Gröbner Bases in Integer Programming*, New York: Springer, 1997.

[12] D W Walkup, R J B Wets. *Lifting projections of convex polyhedra*, Pacific Journal of Mathematics, 1969, 28 (2): 465-475.

[13] J Warren. *Creating multisided rational Bézier surfaces using base points*, ACM Transactions on Graphics, 1992, 11(2): 127-139.

[14] H Wang, C G Zhu, X Y Zhao. *The number of regular control surfaces of toric patch*, Journal of Computational and Applied Mathematics, 2018, 329: 280-293.

[15] Y Y Yu, Y Ji, C G Zhu. *An improved algorithm for checking the injectivity of 2D toric surface patches*, Computers & Mathematics with Applications, 2020, 79(10): 2973-2986.

[16] X F Zhu, Y Ji, C G Zhu, et al. *Isogeometric analysis for trimmed CAD surfaces using multi-sided toric surface patches*, Computer Aided Geometric Design, 2020, 79: 101847.

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