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Maximum likelihood estimation of the parameters of weighted exponential distribution in simple random sampling and ranked set sampling

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Abstract. Weighted exponential distribution $WED(\alpha,\ \lambda)$ with shape parameter α and scale parameter λ possesses some good properties and can be used as a good fit to survival time data compared to other distributions such as gamma, Weibull, or generalized exponential distribution. In this article, we proved the existence and uniqueness of the maximum likelihood estimator (MLE) of the parameters of $WED(\alpha,\ \lambda)$ in simple random sampling (SRS) and provided explicit expressions for the Fisher information number in SRS. Moreover, we also proved the existence and uniqueness of the MLE of the parameters of $WED(\alpha,\ \lambda)$ in ranked set sampling (RSS) and provided explicit expressions for the Fisher information number in RSS. Simulation studies show that these MLEs in RSS can be real competitors for those in SRS.

§1 Introduction

The distribution function of weighted exponential distribution $WED(\alpha,\ \lambda)$ with shape parameter α and scale parameter λ is

$$F(x;\alpha,\lambda)=\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{\alpha+1}\left(1-e^{-(\alpha+1)\lambda x}\right)\right]I(x), \tag{1}$$

where $\alpha>0$, $\lambda>0$ and $I(x)=1$ if $x>0$, otherwise $I(x)=0$. The probability density function (pdf) corresponding to the distribution function in (1) is then given by

$$f(x;\alpha,\lambda)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right).$$

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$WED(\alpha,\ \lambda)$ was first proposed by Gupta et al. (2009). They showed that $WED(\alpha,\ \lambda)$ possesses some good properties and can be used as a good fit to survival time data compared to other distributions such as gamma, Weibull, or generalized exponential distribution. Al-Mutairi et al. (2011) pointed out that $WED(\alpha,\ \lambda)$ is very flexible and can be used quite effectively to analyze skewed data. For further details on the importance and applications of $WED(\alpha,\ \lambda)$, one may refer to Shakhatreh (2012) and Mahdavi (2015).

The estimation problem for $WED(\alpha,\ \lambda)$ has not been discussed extensively in the literature yet. There are a few works regarding the inference problem for $WED(\alpha,\ \lambda)$. For example, Gupta et al. (2009) considered the maximum likelihood estimators (MLEs) of the parameters of $WED(\alpha,\ \lambda)$ in simple random sampling (SRS). However, they did not discuss the existence and uniqueness of the MLE in SRS and did not provide explicit expressions for the Fisher information number in SRS. In this paper, we will discuss the existence and uniqueness of the MLE in SRS and provide explicit expressions for the Fisher information number in SRS. Moreover, we focus on the MLEs in ranked set sampling (RSS).

The remainder of the paper is organized as follows. In Section 2, RSS is introduced. In Section 3, we will prove the existence and uniqueness of the MLE of λ in SRS and obtain the Fisher information number of λ in SRS. In Section 4, we will obtain the Fisher information matrix of α and λ in SRS. In Sections 5 and 6, we will focus on MLEs of the parameters of $WED(\alpha,\ \lambda)$ in RSS. In Section 7, we will compare the asymptotic efficiencies of the MLEs. Conclusions are given in Section 8.

§2 Introduction to RSS

RSS was introduced by McIntyre (1952, 2005) for estimating the pasture yields. It is appropriate for situations where quantification of sampling units is either costly or difficult, but ranking the units in a small set is easy and inexpensive. The procedure of RSS is described as follows:

(1)The experimenter draws m^2 units from the population and then randomly partitioning them into m sets of size m with equal probability and without replacement from the population.

(2)The units are ranked within each set. Here ranking could be judgement, visual perception, covariates, or any other method that does not require actual measurement of the units.

(3)Units within the $ith\,(i = 1, 2, \cdots, m)$ sample are subjected to judgement ordering, with negligible cost, and the unit possessing ith lowest rank is identified.

(4)The identified units are measured. Proceeding in this way, a ranked set sample of size m can be attained. If needed, this process can be replicated r times (cycles).

Yang et al. (2020) used RSS to obtain the MLEs of the parameters of the log-extended exponential-geometric distribution. They showed that the MLEs under RSS always performed better than that under SRS numerically. Wang et al. (2023) used RSS to obtain the MLEs of the parameters of the inverse Gaussian distribution. Simulation studies show that the MLEs in RSS can be real competitors for those in SRS. Shen et al. (2022) used RSS to obtain the

MLEs of the parameters of the generalized Rayleigh distribution. They showed that the MLEs under RSS always performed better than that under SRS numerically. For more applications of RSS, refer to Chen et al. (2003), Dong and Zhang (2020), He et al. (2021), Mahdizadeh and Zamanzade (2018), Qiu and Eftekharian (2021) and Zamanzade (2019).

§3 MLE of λ in SRS

In this section, we prove the existence and uniqueness of the MLE of λ in SRS and obtain the Fisher information number of λ in SRS.

Let $\{x_1, x_2, x_3, \cdots, x_m\}$ be a simple random sample of size m from (1) in which α is known. The log-likelihood function based on this sample is

$$lnL_{SRS}(\lambda) = d + mln\lambda - \lambda \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} ln\left(1 - e^{-\alpha\lambda x_i}\right),$$

where d is a value independent of λ. The first derivative of $lnL_{SRS}(\lambda)$ is

$$\frac{\partial lnL_{SRS}(\lambda)}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \frac{\alpha x_i e^{-\alpha\lambda x_i}}{1 - e^{-\alpha\lambda x_i}}. \tag{2}$$

Since

$$\lim_{\lambda\to 0} \frac{\partial lnL_{SRS}(\lambda)}{\partial\lambda} = \infty \tag{3}$$

and

$$\lim_{\lambda\to\infty} \frac{\partial lnL_{SRS}(\lambda)}{\partial\lambda} = -\sum_{i=1}^{m} x_i, \tag{4}$$

a solution of (2) exists. The second derivative of $lnL_{SRS}(\lambda)$ is

$$\frac{\partial^2 lnL_{SRS}(\lambda)}{\partial\lambda^2} = -\frac{m}{\lambda^2} - \sum_{i=1}^{m} \frac{\alpha^2 x_i^2 e^{-\alpha\lambda x_i}}{1 - e^{-\alpha\lambda x_i}} - \sum_{i=1}^{m} \frac{\alpha^2 x_i^2 e^{-2\alpha\lambda x_i}}{\left(1 - e^{-\alpha\lambda x_i}\right)^2} < 0. \tag{5}$$

Thus (2) has a unique solution and this solution is a unique MLE of λ, denoted by $\hat{\lambda}_{SRS}$. Under the usual regularity assumptions (see Azzalini, 1996, p.71), the Fisher information number for λ in SRS is

$$\begin{aligned}
I_{11,\ SRS} &= -E\left(\frac{\partial^2 lnL_{SRS}(\lambda)}{\partial\lambda^2}\right) \\
&= \frac{m}{\lambda^2} + E\left(\sum_{i=1}^{m} \frac{\alpha^2 x_i^2 e^{-\alpha\lambda x_i}}{1 - e^{-\alpha\lambda x_i}}\right) + E\left(\sum_{i=1}^{m} \frac{\alpha^2 x_i^2 e^{-2\alpha\lambda x_i}}{\left(1 - e^{-\alpha\lambda x_i}\right)^2}\right) \\
&= \frac{m}{\lambda^2} + m\alpha^2 E\left(\frac{x^2 e^{-\alpha\lambda x}}{1 - e^{-\alpha\lambda x}}\right) + m\alpha^2 E\left(\frac{x^2 e^{-2\alpha\lambda x}}{\left(1 - e^{-\alpha\lambda x}\right)^2}\right) \\
&= \frac{m}{\lambda^2} + \frac{2m\alpha}{(\alpha+1)^2 \lambda^2} + \frac{m(\alpha+1)}{\alpha^2\lambda^2} \int_0^1 \frac{t^{\frac{1}{\alpha}+1} \ln^2 t}{1-t} dt \\
&= \frac{m\left(\alpha^2 + 4\alpha + 1\right)}{(\alpha+1)^2 \lambda^2} + \frac{2m(\alpha+1)}{\alpha^2\lambda^2} \Phi\left(1, 3, \frac{1}{\alpha} + 2\right),
\end{aligned} \tag{6}$$

where $\Phi(z, \lambda, v) = \frac{1}{\Gamma(\lambda)} \int_0^1 \frac{u^{v-1} \log^{\lambda-1}(1/u)}{1-uz} du$ for all complex λ, $z \in C - [1, \infty)$ and $v > 0$ (see Pedro and Jimenez-Gamero (2014)).

§4 MLEs of λ and α in SRS

In this section, we consider the MLEs of α and λ in SRS and obtain the Fisher information matrix of α and λ in SRS. Let $\{x_1, x_2, x_3, \cdots, x_m\}$ be a simple random sample of size m from (1) in which λ and α are both unknown. The log-likelihood function based on this sample is

$$lnL_{SRS}(\lambda,\alpha) = mln(\alpha+1) - mln\alpha + mln\lambda - \lambda\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} ln\left(1-e^{-\alpha\lambda x_i}\right).$$

This function implies that

$$\frac{\partial lnL_{SRS}(\lambda,\alpha)}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \frac{\alpha x_i e^{-\alpha\lambda x_i}}{1-e^{-\alpha\lambda x_i}}, \tag{7}$$

and

$$\frac{\partial lnL_{SRS}(\lambda,\alpha)}{\partial\alpha} = \frac{m}{\alpha+1} - \frac{m}{\alpha} + \sum_{i=1}^{m} \frac{\lambda x_i e^{-\alpha\lambda x_i}}{1-e^{-\alpha\lambda x_i}}. \tag{8}$$

Solutions of (7) and (8) are the MLEs of λ and α, denoted as $\left(\hat{\lambda}_{SRS}, \hat{\alpha}_{SRS}\right)$. Then under the usual regularity assumptions (see Azzalini, 1996, p.71), the Fisher information matrix for λ and α in SRS is given by

$$I_{SRS}(\lambda,\alpha) = \begin{pmatrix} I_{11,\ SRS} & I_{12,\ SRS} \\ I_{12,\ SRS} & I_{22,\ SRS} \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
I_{12,\ SRS} &= -E\left(\frac{\partial^2 lnL_{SRS}(\lambda,\alpha)}{\partial\lambda\partial\alpha}\right) \\
&= -E\left(\sum_{i=1}^{m}\frac{x_i e^{-\alpha\lambda x_i}(1-\alpha\lambda x_i)}{1-e^{-\alpha\lambda x_i}}\right) + E\left(\sum_{i=1}^{m}\frac{\alpha\lambda x_i^2 e^{-2\alpha\lambda x_i}}{(1-e^{-\alpha\lambda x_i})^2}\right) \\
&= -mE\left(\frac{xe^{-\alpha\lambda x}(1-\alpha\lambda x)}{1-e^{-\alpha\lambda x}}\right) + m\alpha\lambda E\left(\frac{x^2 e^{-2\alpha\lambda x}}{(1-e^{-\alpha\lambda x})^2}\right) \\
&= -\frac{m(1-\alpha)}{\alpha\lambda(\alpha+1)^2} + \frac{m(\alpha+1)}{\alpha^3\lambda}\int_0^1 \frac{t^{\frac{1}{\alpha}+1}\ln^2 t}{1-t}dt \\
&= -\frac{m(1-\alpha)}{\alpha\lambda(\alpha+1)^2} + \frac{2m(\alpha+1)}{\alpha^3\lambda}\Phi\left(1,3,\frac{1}{\alpha}+2\right)
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
I_{22,\ SRS} &= -E\left(\frac{\partial^2 lnL_{SRS}(\lambda,\alpha)}{\partial\alpha^2}\right) \\
&= \frac{m}{(\alpha+1)^2} - \frac{m}{\alpha^2} + E\left(\sum_{i=1}^{m}\frac{\lambda^2 x_i^2 e^{-\alpha\lambda x_i}}{1-e^{-\alpha\lambda x_i}}\right) + E\left(\sum_{i=1}^{m}\frac{\lambda^2 x_i^2 e^{-2\alpha\lambda x_i}}{(1-e^{-\alpha\lambda x_i})^2}\right) \\
&= \frac{m}{(\alpha+1)^2} - \frac{m}{\alpha^2} + m\lambda^2 E\left(\frac{x^2 e^{-\alpha\lambda x}}{1-e^{-\alpha\lambda x}}\right) + m\lambda^2 E\left(\frac{x^2 e^{-2\alpha\lambda x}}{(1-e^{-\alpha\lambda x})^2}\right) \\
&= \frac{m}{(\alpha+1)^2} - \frac{m}{\alpha^2} + \frac{2m}{\alpha(\alpha+1)^2} + \frac{m(\alpha+1)}{\alpha^4}\int_0^1 \frac{t^{\frac{1}{\alpha}+1}\ln^2 t}{1-t}dt \\
&= -\frac{m}{\alpha^2(\alpha+1)^2} + \frac{m(\alpha+1)}{\alpha^4}\int_0^1 \frac{t^{\frac{1}{\alpha}+1}\ln^2 t}{1-t}dt \\
&= -\frac{m}{\alpha^2(\alpha+1)^2} + \frac{2m(\alpha+1)}{\alpha^4}\Phi\left(1,3,\frac{1}{\alpha}+2\right).
\end{aligned} \tag{11}$$

§5 MLE of λ in RSS

RSS was introduced by McIntyre (1952, 2005) for estimating the pasture yields. It is appropriate for situations where quantification of sampling units is either costly or difficult, but ranking the units in a small set is easy and inexpensive. An important advantage of RSS approach is that it improves the efficiency of estimators of the population parameters by providing more representative sample from the target population.

In this section, we prove the existence and uniqueness of the MLE of λ in RSS and obtain the Fisher information number of λ in RSS. Let $\left\{x_{1(1)}, x_{2(2)}, x_{3(3)}, \cdots, x_{m(m)}\right\}$ be a ranked set sample of size m from (1) in which α is known. Then the pdf of $X_{i(i)}$ is

$$\begin{aligned} f_i(x;\alpha,\lambda) = c(i,m)&\left\{1+\frac{\alpha+1}{\alpha}e^{-\lambda x}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x}-1\right)\right\}^{i-1} \\ &\times\left\{\frac{\alpha+1}{\alpha}e^{-\lambda x}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x}\right)\right\}^{m-i} \\ &\times\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right), \end{aligned}$$

where $c(i,m) = \frac{m!}{(m-i)!(i-1)!}$. Then the log-likelihood function based on the observed sample is

$$\begin{aligned} lnL_{RSS}(\lambda) = d + mln\lambda &- \lambda\sum_{i=1}^{m} x_{i(i)} + \sum_{i=1}^{m} ln\left(1-e^{-\alpha\lambda x_{i(i)}}\right) \\ &+\sum_{i=1}^{m}(i-1)ln\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right) \\ &+\sum_{i=1}^{m}(m-i)ln\left(\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)\right), \end{aligned}$$

where d is a value independent of λ. The first derivative of $lnL_{RSS}(\lambda)$ is

$$\begin{aligned} \frac{\partial lnL_{RSS}(\lambda)}{\partial\lambda} = \frac{m}{\lambda} &- \sum_{i=1}^{m} x_{i(i)} + \sum_{i=1}^{m}\frac{\alpha x_{i(i)}e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}} \\ &+\sum_{i=1}^{m}(m-i)\frac{x_{i(i)}\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}} \\ &+\sum_{i=1}^{m}(i-1)\frac{\frac{\alpha+1}{\alpha}x_{i(i)}e^{-\lambda x_{i(i)}}\left(1-e^{-\alpha\lambda x_{i(i)}}\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}. \end{aligned} \tag{12}$$

Based on the above preparation, we have the following result.

Theorem 1. The MLE of λ under RSS exists and is unique, denoted by $\hat{\lambda}_{RSS}$.

Proof. Since

$$\lim_{\lambda\to 0}\frac{\partial lnL_{RSS}(\lambda)}{\partial\lambda} = \infty, \tag{13}$$

and

$$\lim_{\lambda\to\infty}\frac{\partial lnL_{RSS}(\lambda)}{\partial\lambda} = -\sum_{i=1}^{m}(m-i+1)\,x_{i(i)} < 0, \tag{14}$$

a solution of (13) exists. Considering that

$$\begin{aligned}
&\frac{\partial^2 lnL_{RSS}(\lambda)}{\partial\lambda^2}\\
&=\sum_{i=1}^{m}(i-1)\frac{\frac{\alpha+1}{\alpha}x_{i(i)}^2e^{-\lambda x_{i(i)}}\left((\alpha+1)e^{-\alpha\lambda x_{i(i)}}-1\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\\
&\quad-\sum_{i=1}^{m}(i-1)\frac{\left(\frac{\alpha+1}{\alpha}\right)^2x_{i(i)}^2e^{-2\lambda x_{i(i)}}\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^2}\\
&\quad-\sum_{i=1}^{m}(m-i)\frac{\alpha x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}}{1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}}-\sum_{i=1}^{m}(m-i)\frac{\frac{\alpha}{\alpha+1}x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)^2}\\
&\quad-\sum_{i=1}^{m}\frac{\alpha^2x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}-\sum_{i=1}^{m}\frac{\alpha^2x_{i(i)}^2e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}-\frac{m}{\lambda^2}\\
&=-\sum_{i=1}^{m}(i-1)\frac{\frac{\alpha+1}{\alpha}x_{i(i)}^2e^{-\lambda x_{i(i)}}\left(1-(\alpha+1)e^{-\alpha\lambda x_{i(i)}}\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\\
&\quad-\sum_{i=1}^{m}(i-1)\frac{\left(\frac{\alpha+1}{\alpha}\right)^2x_{i(i)}^2e^{-2\lambda x_{i(i)}}\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^2}\\
&\quad-\left(\sum_{i=1}^{m}(m-i)\frac{\alpha x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)^2}+\sum_{i=1}^{m}(m-i)\frac{\frac{\alpha}{\alpha+1}x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)^2}\right)\\
&\quad-\sum_{i=1}^{m}\frac{\alpha^2x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}-\sum_{i=1}^{m}\frac{\alpha^2x_{i(i)}^2e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}-\frac{m}{\lambda^2}\\
&=-\sum_{i=1}^{m}(i-1)\frac{\frac{\alpha+1}{\alpha}x_{i(i)}^2e^{-\lambda x_{i(i)}}\left(1-(\alpha+1)e^{-\alpha\lambda x_{i(i)}}\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\\
&\quad-\sum_{i=1}^{m}(i-1)\frac{\left(\frac{\alpha+1}{\alpha}\right)^2x_{i(i)}^2e^{-2\lambda x_{i(i)}}\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^2}\\
&\quad-\sum_{i=1}^{m}(m-i)\frac{\frac{\alpha^2}{\alpha+1}x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)^2}-\frac{m}{\lambda^2}-\sum_{i=1}^{m}\frac{\alpha^2x_{i(i)}^2e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}-\sum_{i=1}^{m}\frac{\alpha^2x_{i(i)}^2e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}<0.
\end{aligned} \tag{15}$$

Thus (13) has a unique solution and this solution is a unique MLE of λ. This completes the proof of Theorem 1. In Theorem 2, we give the Fisher information number of λ in RSS.

Theorem 2. Under the usual regularity assumptions (see Azzalini, 1996, p.71), the Fisher information number for λ in RSS is

$$I_{11,RSS}=I_{11,\ SRS}+\frac{m(m-1)\left(\alpha^2+6\alpha+6\right)}{4\lambda^2(\alpha+2)^2}+\frac{m(m-1)\left(7\alpha^2+18\alpha+12\right)}{4\lambda^2(\alpha+1)(\alpha+2)^3}$$

$$
\begin{aligned}
&+\frac{m\left(m-1\right)\left(\alpha+1\right)^{3}}{\alpha^{3}\lambda^{2}}\int_{0}^{1}\frac{t^{2}\ln^{2}t(1-t^{\alpha})^{3}}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^{\alpha}-1\right)}dt \\
&-\frac{m\left(m-1\right)\left(\alpha+1\right)}{\alpha\lambda^{2}}\int_{0}^{1}\frac{t^{2}\ln^{2}t(t^{\alpha}-1)^{2}}{1-\frac{1}{\alpha+1}t^{\alpha}}dt.
\end{aligned}
\tag{16}
$$

Proof. Using the basic identities

$$\sum_{i=1}^{m}E\left[g_{i}\left(x;\alpha,\lambda\right)\right]=mE\left[g\left(x;\alpha,\lambda\right)\right],\tag{17}$$

$$\sum_{i=1}^{m}\left(i-1\right)E\left[g_{i}\left(x;\alpha,\lambda\right)\right]=m\left(m-1\right)E\left[g\left(x;\alpha,\lambda\right)F\left(x;\alpha,\lambda\right)\right],\tag{18}$$

and

$$\sum_{i=1}^{m}\left(m-i\right)E\left[g_{i}\left(x;\alpha,\lambda\right)\right]=m\left(m-1\right)E\left[g\left(x;\alpha,\lambda\right)\left(1-F\left(x;\alpha,\lambda\right)\right)\right],\tag{19}$$

of order statistics in RSS (see Stokes (1995)), we have

$$
\begin{aligned}
I_{11,\ RSS}&=-E\left(\frac{\partial^{2}lnL_{RSS}\left(\lambda\right)}{\partial\lambda^{2}}\right)\\
&=-E\left(\sum_{i=1}^{m}\left(i-1\right)\frac{\frac{\alpha+1}{\alpha}x_{i(i)}^{2}e^{-\lambda x_{i(i)}}\left(\left(\alpha+1\right)e^{-\alpha\lambda x_{i(i)}}-1\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\right)\\
&\quad+E\left(\sum_{i=1}^{m}\left(i-1\right)\frac{\left(\frac{\alpha+1}{\alpha}\right)^{2}x_{i(i)}^{2}e^{-2\lambda x_{i(i)}}\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^{2}}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^{2}}\right)\\
&\quad+E\left(\sum_{i=1}^{m}\left(m-i\right)\frac{\alpha x_{i(i)}^{2}e^{-\alpha\lambda x_{i(i)}}}{1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}}\right)+E\left(\sum_{i=1}^{m}\left(m-i\right)\frac{\frac{\alpha}{\alpha+1}x_{i(i)}^{2}e^{-\alpha\lambda x_{i(i)}}\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)^{2}}\right)\\
&\quad+E\left(\sum_{i=1}^{m}\frac{\alpha^{2}x_{i(i)}^{2}e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}\right)+E\left(\sum_{i=1}^{m}\frac{\alpha^{2}x_{i(i)}^{2}e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^{2}}\right)+\frac{m}{\lambda^{2}}\\
&=-\frac{m\left(m-1\right)\left(\alpha+1\right)}{\alpha}E\left(x^{2}e^{-\lambda x}\left(\left(\alpha+1\right)e^{-\alpha\lambda x}-1\right)\right)\\
&\quad+\frac{m\left(m-1\right)\left(\alpha+1\right)^{2}}{\alpha^{2}}E\left(\frac{x^{2}e^{-2\lambda x}\left(1-e^{-\alpha\lambda x}\right)^{2}}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x}-1\right)}\right)\\
&\quad+m\left(m-1\right)\left(\alpha+1\right)E\left(x^{2}e^{-(\alpha+1)\lambda x}\right)+m\left(m-1\right)E\left(\frac{x^{2}e^{-(\alpha+1)\lambda x}\left(e^{-\alpha\lambda x}-1\right)}{1-\frac{1}{\alpha+1}e^{-\alpha\lambda x}}\right)\\
&\quad+m\alpha^{2}E\left(\frac{x^{2}e^{-\alpha\lambda x}}{1-e^{-\alpha\lambda x}}\right)+m\alpha^{2}E\left(\frac{x^{2}e^{-2\alpha\lambda x}}{\left(1-e^{-\alpha\lambda x}\right)^{2}}\right)+\frac{m}{\lambda^{2}}\\
&=-\frac{m\left(m-1\right)\left(\alpha+1\right)^{2}}{\alpha^{2}\lambda^{2}}\int_{0}^{1}t\ln^{2}t\left(\left(\alpha+1\right)t^{\alpha}-1\right)\left(1-t^{\alpha}\right)dt\\
&\quad+\frac{m\left(m-1\right)\left(\alpha+1\right)^{2}}{\alpha\lambda^{2}}\int_{0}^{1}t^{\alpha+1}\ln^{2}t\left(1-t^{\alpha}\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{m(m-1)(\alpha+1)^3}{\alpha^3\lambda^2}\int_0^1\frac{t^2\ln^2 t(1-t^\alpha)^3}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt\\
&-\frac{m(m-1)(\alpha+1)}{\alpha\lambda^2}\int_0^1\frac{t^2\ln^2 t(t^\alpha-1)^2}{1-\frac{1}{\alpha+1}t^\alpha}dt\\
&+\frac{m\alpha(\alpha+1)}{\lambda^2}\int_0^1 t^\alpha\ln^2 t dt+\frac{m(\alpha+1)}{\alpha^2\lambda^2}\int_0^1\frac{t^{\frac{1}{\alpha}+1}\ln^2 t}{1-t}dt+\frac{m}{\lambda^2}\\
=&\frac{m(m-1)\left(\alpha^2+6\alpha+6\right)}{4\lambda^2(\alpha+2)^2}+\frac{m(m-1)\left(7\alpha^2+18\alpha+12\right)}{4\lambda^2(\alpha+1)(\alpha+2)^3}\\
&+\frac{m(m-1)(\alpha+1)^3}{\alpha^3\lambda^2}\int_0^1\frac{t^2\ln^2 t(1-t^\alpha)^3}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt\\
&-\frac{m(m-1)(\alpha+1)}{\alpha\lambda^2}\int_0^1\frac{t^2\ln^2 t(t^\alpha-1)^2}{1-\frac{1}{\alpha+1}t^\alpha}dt\\
&+\frac{m\left(\alpha^2+4\alpha+1\right)}{(\alpha+1)^2\lambda^2}+\frac{2m(\alpha+1)}{\alpha^2\lambda^2}\Phi\left(1,3,\frac{1}{\alpha}+2\right)\\
=&I_{11,\ SRS}+\frac{m(m-1)\left(\alpha^2+6\alpha+6\right)}{4\lambda^2(\alpha+2)^2}+\frac{m(m-1)\left(7\alpha^2+18\alpha+12\right)}{4\lambda^2(\alpha+1)(\alpha+2)^3}\\
&+\frac{m(m-1)(\alpha+1)^3}{\alpha^3\lambda^2}\int_0^1\frac{t^2\ln^2 t(1-t^\alpha)^3}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt\\
&-\frac{m(m-1)(\alpha+1)}{\alpha\lambda^2}\int_0^1\frac{t^2\ln^2 t(t^\alpha-1)^2}{1-\frac{1}{\alpha+1}t^\alpha}dt,
\end{aligned}
$$

under the usual regularity assumptions of Thoerem 2. This completes the proof of Theorem 2.

§6 MLEs of λ and α in RSS

In this section, we consider the MLEs of α and λ in RSS and obtain the Fisher information matrix of α and λ in RSS.

Let $\left\{x_{1(1)}, x_{2(2)}, x_{3(3)}, \cdots, x_{m(m)}\right\}$ be a ranked set sample of size m from (1) in which λ and α are both unknown. The log-likelihood function based on this sample is

$$
\begin{aligned}
lnL_{RSS}(\lambda,\alpha)=d+mln(\alpha+1)-mln\alpha+mln\lambda-\lambda\sum_{i=1}^m x_{i(i)}+\sum_{i=1}^m ln\left(1-e^{-\alpha\lambda x_{i(i)}}\right)\\
+\sum_{i=1}^m(i-1)ln\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)\\
+\sum_{i=1}^m(m-i)ln\left(\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{(i)}}\right)\right),
\end{aligned}
$$

where d is a value independent of λ and α. This function implies that

$$\begin{aligned}\frac{\partial lnL_{RSS}\left(\lambda,\alpha\right)}{\partial\lambda}=&\frac{m}{\lambda}-\sum_{i=1}^{m}x_{i(i)}+\sum_{i=1}^{m}\frac{\alpha x_{i(i)}e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}+\sum_{i=1}^{m}\left(m-i\right)\frac{x_{i(i)}\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}}\\&+\sum_{i=1}^{m}\left(i-1\right)\frac{\frac{\alpha+1}{\alpha}x_{i(i)}e^{-\lambda x_{(i)}}\left(1-e^{-\alpha\lambda x_{i(i)}}\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)},\end{aligned}\tag{20}$$

and

$$\begin{aligned}\frac{\partial lnL_{RSS}\left(\lambda,\alpha\right)}{\partial\alpha}=&\frac{m}{\alpha+1}-\frac{m}{\alpha}+\sum_{i=1}^{m}\frac{\lambda x_{i(i)}e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}\\&+\sum_{i=1}^{m}\left(i-1\right)\frac{\frac{1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha}-\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)e^{-\alpha\lambda x_{i(i)}}\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\\&-\sum_{i=1}^{m}\left(m-i\right)\frac{\frac{1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha}-\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)e^{-\alpha\lambda x_{i(i)}}\right)}{\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)}.\end{aligned}\tag{21}$$

The solutions of (20) and (21) are the MLEs of λ and α, denoted as $\left(\hat{\lambda}_{RSS},\hat{\alpha}_{RSS}\right)$. The Fisher information matrix for λ and α under RSS is given as follows.

Theorem 3. Under the usual regularity assumptions of Theorem 2, the Fisher information matrix for λ and α in RSS is given by

$$I_{RSS}\left(\lambda,\alpha\right)=\begin{pmatrix}I_{11,\ RSS} & I_{12,\ RSS}\\ I_{12,\ RSS} & I_{22,\ RSS}\end{pmatrix},\tag{22}$$

where

$$\begin{aligned}I_{12,\ RSS}=&I_{12,\ SRS}+\frac{m\left(m-1\right)\alpha}{4\lambda\left(\alpha+2\right)^{3}}+\frac{m\left(m-1\right)\left(7\alpha^{2}+18\alpha+12\right)}{4\alpha\lambda\left(\alpha+1\right)\left(\alpha+2\right)^{3}}\\&+\frac{m\left(m-1\right)\left(\alpha+1\right)^{2}}{\alpha^{3}\lambda}\int_{0}^{1}\frac{t^{2}\ln t\left(t^{\alpha}\left(\frac{1}{\alpha}-\ln t\right)-\frac{1}{\alpha}\right)\left(1-t^{\alpha}\right)^{2}}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^{\alpha}-1\right)}dt\\&-\frac{m\left(m-1\right)\left(\alpha+1\right)^{2}}{\alpha^{2}\lambda}\int_{0}^{1}\frac{t^{\alpha+1}\ln t(t^{\alpha}-1)^{2}\left(\frac{1}{\alpha+1}\ln t-\frac{1}{(\alpha+1)^{2}}\right)}{1-\frac{1}{\alpha+1}t^{\alpha}}dt,\end{aligned}\tag{23}$$

and

$$\begin{aligned}I_{22,RSS}=&I_{22,SRS}+\frac{m\left(m-1\right)\left(\alpha+1\right)}{\alpha^{3}}\int_{0}^{1}\frac{t^{2}\left(\frac{1}{\alpha}-t^{\alpha}\left(\frac{1}{\alpha}-\ln t\right)\right)^{2}\left(1-t^{\alpha}\right)}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^{\alpha}-1\right)}dt\\&+\frac{m\left(m-1\right)}{\alpha^{2}}\int_{0}^{1}\frac{t\left(\frac{1}{\alpha}-t^{\alpha}\left(\frac{1}{\alpha}-\ln t\right)\right)^{2}\left(1-t^{\alpha}\right)}{1-\frac{1}{\alpha+1}t^{\alpha}}dt.\end{aligned}\tag{24}$$

Proof. Since

$$\begin{aligned}&\frac{\partial^{2}lnL_{RSS}\left(\lambda,\alpha\right)}{\partial\lambda\partial\alpha}\\&=-\sum_{i=1}^{m}\left(i-1\right)\frac{\frac{1}{\alpha}x_{i(i)}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\left(\alpha+1\right)\lambda x_{i(i)}\right)\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\\&\quad+\sum_{i=1}^{m}\left(i-1\right)\frac{\frac{\alpha+1}{\alpha^{2}}x_{i(i)}e^{-2\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)\right)\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^{2}}\end{aligned}$$

$$-\sum_{i=1}^{m}(m-i)\frac{x_{(i)}\left(e^{-\alpha\lambda x_{(i)}}-1\right)\left(\left(\frac{1}{(\alpha+1)^2}+\lambda x_{(i)}\frac{1}{\alpha+1}\right)e^{-\alpha\lambda x_{(i)}}\right)}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{(i)}}\right)^2}$$

$$+\sum_{i=1}^{m}(m-i)\frac{\lambda x_{i(i)}^2 e^{-\alpha\lambda x_{i(i)}}}{\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1}+\sum_{i=1}^{m}\frac{\left(x_{i(i)}-\alpha\lambda x_{i(i)}^2\right)e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}-\sum_{i=1}^{m}\frac{\alpha\lambda x_{i(i)}^2 e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2},$$

and

$$\begin{aligned}\frac{\partial^2 lnL_{RSS}(\lambda,\alpha)}{\partial\alpha^2}=&-\sum_{i=1}^{m}(i-1)\frac{\frac{1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}+\frac{2}{\alpha}\lambda x_{i(i)}+\lambda^2 x_{i(i)}^2\right)\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\\
&-\sum_{i=1}^{m}(i-1)\frac{\frac{1}{\alpha^2}e^{-2\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)\right)^2}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^2}\\
&+\sum_{i=1}^{m}(m-i)\frac{\frac{1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}+\frac{2}{\alpha}\lambda x_{i(i)}+\lambda^2 x_{i(i)}^2\right)\right)}{\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)}\\
&-\sum_{i=1}^{m}(m-i)\frac{\frac{1}{\alpha^2}e^{-2\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)\right)^2}{\left(\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)\right)^2}\\
&-\frac{m}{(\alpha+1)^2}+\frac{m}{\alpha^2}-\sum_{i=1}^{m}\frac{\lambda^2 x_{i(i)}^2 e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}-\sum_{i=1}^{m}\frac{\lambda^2 x_{i(i)}^2 e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2},\end{aligned}$$

under the usual regularity assumptions of Theorem 2, we can obtain

$$\begin{aligned}&I_{12,\ RSS}=-E\left(\frac{\partial^2 lnL_{RSS}(\lambda,\alpha)}{\partial\lambda\partial\alpha}\right)\\
&=E\left(\sum_{i=1}^{m}\frac{\alpha\lambda x_{i(i)}^2 e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}\right)-E\left(\sum_{i=1}^{m}\frac{\left(x_{i(i)}-\alpha\lambda x_{i(i)}^2\right)e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}\right)\\
&\quad-E\left(\sum_{i=1}^{m}(m-i)\frac{\lambda x_{i(i)}^2 e^{-\alpha\lambda x_{i(i)}}}{\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1}\right)\\
&\quad+E\left(\sum_{i=1}^{m}(i-1)\frac{\frac{1}{\alpha}x_{i(i)}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+(\alpha+1)\lambda x_{i(i)}\right)\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\right)\\
&\quad-E\left(\sum_{i=1}^{m}(i-1)\frac{\frac{\alpha+1}{\alpha^2}x_{i(i)}e^{-2\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)\right)\left(e^{-\alpha\lambda x_{i(i)}}-1\right)}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^2}\right)\\
&\quad+E\left(\sum_{i=1}^{m}(m-i)\frac{x_{i(i)}\left(e^{-\alpha\lambda x_{i(i)}}-1\right)\left(\left(\frac{1}{(\alpha+1)^2}+\lambda x_{i(i)}\frac{1}{\alpha+1}\right)e^{-\alpha\lambda x_{i(i)}}\right)}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)^2}\right)\\
&=\frac{m(m-1)}{\alpha}E\left(xe^{-\lambda x}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x}\left(\frac{1}{\alpha}+(\alpha+1)\lambda x\right)\right)\right)\end{aligned}$$

$$+\frac{m\lambda(m-1)(\alpha+1)}{\alpha}E\left(x^2e^{-(\alpha+1)\lambda x}\right)$$
$$-\frac{m(m-1)(\alpha+1)}{\alpha^2}E\left(\frac{xe^{-2\lambda x}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x}\left(\frac{1}{\alpha}+\lambda x\right)\right)\left(e^{-\alpha\lambda x}-1\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x}-1\right)}\right)$$
$$+\frac{m(m-1)(\alpha+1)}{\alpha}E\left(\frac{xe^{-\lambda x}\left(e^{-\alpha\lambda x}-1\right)\left(\left(\frac{1}{(\alpha+1)^2}+\lambda x\frac{1}{\alpha+1}\right)e^{-\alpha\lambda x}\right)}{1-\frac{1}{\alpha+1}e^{-\alpha\lambda x}}\right)$$
$$-mE\left(\frac{\left(x-\alpha\lambda x^2\right)e^{-\alpha\lambda x}}{1-e^{-\alpha\lambda x}}\right)+m\alpha\lambda E\left(\frac{x^2e^{-2\alpha\lambda x}}{\left(1-e^{-\alpha\lambda x}\right)^2}\right)$$
$$=-\frac{m(m-1)(\alpha+1)}{\alpha^2\lambda}\int_0^1 t\ln t\left(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-(\alpha+1)\ln t\right)\right)(1-t^\alpha)\,dt$$
$$+\frac{m(m-1)(\alpha+1)^2}{\alpha^2\lambda}\int_0^1 t^{\alpha+1}\ln^2 t\,(1-t^\alpha)\,dt$$
$$+\frac{m(m-1)(\alpha+1)^2}{\alpha^3\lambda}\int_0^1\frac{t^2\ln t\left(t^\alpha\left(\frac{1}{\alpha}-\ln t\right)-\frac{1}{\alpha}\right)(1-t^\alpha)^2}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt$$
$$-\frac{m(m-1)(\alpha+1)^2}{\alpha^2\lambda}\int_0^1\frac{t^{\alpha+1}\ln t(t^\alpha-1)^2\left(\frac{1}{\alpha+1}\ln t-\frac{1}{(\alpha+1)^2}\right)}{1-\frac{1}{\alpha+1}t^\alpha}dt$$
$$+\frac{m(1-\alpha)}{\alpha\lambda(\alpha+1)^2}+\frac{m(\alpha+1)}{\alpha^3\lambda}\int_0^1\frac{t^{\frac{1}{\alpha}+1}\ln^2 t}{1-t}dt$$
$$=\frac{m(m-1)\alpha}{4\lambda(\alpha+2)^3}+\frac{m(m-1)\left(7\alpha^2+18\alpha+12\right)}{4\alpha\lambda(\alpha+1)(\alpha+2)^3}-\frac{m(\alpha-1)}{\alpha\lambda(\alpha+1)^2}+\frac{2m(\alpha+1)}{\alpha^3\lambda}\Phi\left(1,3,\frac{1}{\alpha}+2\right)$$
$$+\frac{m(m-1)(\alpha+1)^2}{\alpha^3\lambda}\int_0^1\frac{t^2\ln t\left(t^\alpha\left(\frac{1}{\alpha}-\ln t\right)-\frac{1}{\alpha}\right)(1-t^\alpha)^2}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt$$
$$-\frac{m(m-1)(\alpha+1)^2}{\alpha^2\lambda}\int_0^1\frac{t^{\alpha+1}\ln t(t^\alpha-1)^2\left(\frac{1}{\alpha+1}\ln t-\frac{1}{(\alpha+1)^2}\right)}{1-\frac{1}{\alpha+1}t^\alpha}dt$$
$$=I_{12,\ SRS}+\frac{m(m-1)\alpha}{4\lambda(\alpha+2)^3}+\frac{m(m-1)\left(7\alpha^2+18\alpha+12\right)}{4\alpha\lambda(\alpha+1)(\alpha+2)^3}$$
$$+\frac{m(m-1)(\alpha+1)^2}{\alpha^3\lambda}\int_0^1\frac{t^2\ln t\left(t^\alpha\left(\frac{1}{\alpha}-\ln t\right)-\frac{1}{\alpha}\right)(1-t^\alpha)^2}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt$$
$$-\frac{m(m-1)(\alpha+1)^2}{\alpha^2\lambda}\int_0^1\frac{t^{\alpha+1}\ln t(t^\alpha-1)^2\left(\frac{1}{\alpha+1}\ln t-\frac{1}{(\alpha+1)^2}\right)}{1-\frac{1}{\alpha+1}t^\alpha}dt,$$

and

$$I_{22,\ RSS}=-E\left(\frac{\partial^2 lnL_{RSS}(\lambda,\alpha)}{\partial\alpha^2}\right)$$

$$
\begin{aligned}
&= E\left(\sum_{i=1}^{m}(i-1)\frac{\frac{1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}+\frac{2}{\alpha}\lambda x_{i(i)}+\lambda^2 x_{i(i)}^2\right)\right)}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)}\right)\\
&\quad+E\left(\sum_{i=1}^{m}(i-1)\frac{\frac{1}{\alpha^2}e^{-2\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)\right)^2}{\left(1+\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}-1\right)\right)^2}\right)\\
&\quad-E\left(\sum_{i=1}^{m}(m-i)\frac{\frac{1}{\alpha}e^{-\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{2}{\alpha^2}+\frac{2}{\alpha}\lambda x_{i(i)}+\lambda^2 x_{i(i)}^2\right)\right)}{\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)}\right)\\
&\quad+E\left(\sum_{i=1}^{m}(m-i)\frac{\frac{1}{\alpha^2}e^{-2\lambda x_{i(i)}}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x_{i(i)}}\left(\frac{1}{\alpha}+\lambda x_{i(i)}\right)\right)^2}{\left(\frac{\alpha+1}{\alpha}e^{-\lambda x_{i(i)}}\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x_{i(i)}}\right)\right)^2}\right)\\
&\quad+E\left(\sum_{i=1}^{m}\frac{\lambda^2 x_{i(i)}^2 e^{-\alpha\lambda x_{i(i)}}}{1-e^{-\alpha\lambda x_{i(i)}}}\right)+E\left(\sum_{i=1}^{m}\frac{\lambda^2 x_{i(i)}^2 e^{-2\alpha\lambda x_{i(i)}}}{\left(1-e^{-\alpha\lambda x_{i(i)}}\right)^2}\right)+\frac{m}{(\alpha+1)^2}-\frac{m}{\alpha^2}\\
&= \frac{m(m-1)}{\alpha}E\left(e^{-\lambda x}\left(\frac{2}{\alpha^2}-e^{-\alpha\lambda x}\left(\frac{2}{\alpha^2}+\frac{2}{\alpha}\lambda x+\lambda^2x^2\right)\right)\right)\\
&\quad+\frac{m(m-1)}{\alpha^2}E\left(\frac{e^{-2\lambda x}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x}\left(\frac{1}{\alpha}+\lambda x\right)\right)^2}{1+\frac{\alpha+1}{\alpha}e^{-\lambda x}\left(\frac{1}{\alpha+1}e^{-\alpha\lambda x}-1\right)}\right)\\
&\quad-\frac{m(m-1)}{\alpha}E\left(e^{-\lambda x}\left(\frac{2}{\alpha^2}-e^{-\alpha\lambda x}\left(\frac{2}{\alpha^2}+\frac{2}{\alpha}\lambda x+\lambda^2x^2\right)\right)\right)\\
&\quad+\frac{m(m-1)}{\alpha(\alpha+1)}E\left(\frac{e^{-\lambda x}\left(\frac{1}{\alpha}-e^{-\alpha\lambda x}\left(\frac{1}{\alpha}+\lambda x\right)\right)^2}{\left(1-\frac{1}{\alpha+1}e^{-\alpha\lambda x}\right)}\right)\\
&\quad+m\lambda^2E\left(\frac{x^2e^{-\alpha\lambda x}}{1-e^{-\alpha\lambda x}}\right)+m\lambda^2E\left(\frac{x^2e^{-2\alpha\lambda x}}{(1-e^{-\alpha\lambda x})^2}\right)+\frac{m}{(\alpha+1)^2}-\frac{m}{\alpha^2}\\
&= \frac{m(m-1)(\alpha+1)}{\alpha^3}\int_0^1\frac{t^2\left(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-\ln t\right)\right)^2(1-t^\alpha)}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt\\
&\quad+\frac{m(m-1)}{\alpha^2}\int_0^1\frac{t\left(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-\ln t\right)\right)^2(1-t^\alpha)}{1-\frac{1}{\alpha+1}t^\alpha}dt-\frac{m}{\alpha^2(\alpha+1)^2}\\
&\quad+\frac{m(\alpha+1)}{\alpha^4}\int_0^1\frac{t^{\frac{1}{\alpha}+1}\ln^2 t}{1-t}dt\\
&= \frac{m(m-1)(\alpha+1)}{\alpha^3}\int_0^1\frac{t^2\left(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-\ln t\right)\right)^2(1-t^\alpha)}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt\\
&\quad+\frac{m(m-1)}{\alpha^2}\int_0^1\frac{t\left(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-\ln t\right)\right)^2(1-t^\alpha)}{1-\frac{1}{\alpha+1}t^\alpha}dt\\
&\quad-\frac{m}{\alpha^2(\alpha+1)^2}+\frac{2m(\alpha+1)}{\alpha^4}\Phi\left(1,3,\frac{1}{\alpha}+2\right)
\end{aligned}
$$

$$= I_{22,\ SRS} + \frac{m\,(m-1)\,(\alpha+1)}{\alpha^3}\int_0^1 \frac{t^2\big(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-\ln t\right)\big)^2(1-t^\alpha)}{1+\frac{\alpha+1}{\alpha}t\left(\frac{1}{\alpha+1}t^\alpha-1\right)}dt$$
$$+\frac{m\,(m-1)}{\alpha^2}\int_0^1 \frac{t\big(\frac{1}{\alpha}-t^\alpha\left(\frac{1}{\alpha}-\ln t\right)\big)^2(1-t^\alpha)}{1-\frac{1}{\alpha+1}t^\alpha}dt,$$

from (17), (18) and (19). This completes the proof of Theorem 3. □

§7 Numerical comparison

In this section, we will compare the MLEs in RSS with respect to (w.r.t.) those MLEs in SRS in terms of the asymptotic efficiency.

Table 1. Asymptotic efficiencies of MLEs of λ and α

α	m	AE_1	AE_2	AE_3
0.5	2	1.44	1.47	1.64
	3	1.89	1.94	2.37
	4	2.33	2.42	3.19
	5	2.77	2.88	4.11
	6	3.22	3.37	5.13
	7	3.66	3.84	6.23
	8	4.10	4.30	7.43
	9	4.54	4.79	8.72
1	2	1.45	1.50	1.66
	3	1.89	2.00	2.42
	4	2.34	2.50	3.29
	5	2.79	2.99	4.26
	6	3.24	3.48	5.41
	7	3.68	3.96	6.53
	8	4.13	4.49	7.83
	9	4.58	4.98	9.23
1.5	2	1.45	1.49	1.68
	3	1.90	1.98	2.49
	4	2.35	2.48	3.41
	5	2.77	2.97	3.98
	6	3.25	3.46	5.63
	7	3.70	3.95	6.42
	8	4.15	4.44	8.33
	9	4.60	4.93	9.87

Since under some regularity conditions, the asymptotic efficiencies of the MLEs can be obtained from the inverse of the Fisher information number (Barabesi and El-Sharaawi (2001)), the asymptotic efficiency of $\hat{\lambda}_{RSS}$ w.r.t $\hat{\lambda}_{SRS}$ may be defined as $AE_1 = \frac{I_{11,\ RSS}}{I_{11,\ SRS}}$. The asymptotic efficiencies of $\hat{\alpha}_{RSS}$ w.r.t $\hat{\alpha}_{SRS}$ and $\left(\hat{\lambda}_{RSS}, \hat{\alpha}_{RSS}\right)$ w.r.t $\left(\hat{\lambda}_{SRS}, \hat{\alpha}_{SRS}\right)$ may be respectively defined as $AE_2 = \frac{I_{22,\ RSS}}{I_{22,\ SRS}}$ and $AE_3 = \frac{\det\{I_{RSS}(\lambda,\alpha)\}}{\det\{I_{SRS}(\lambda,\alpha)\}}$. Since AE_i $(i = 1, 2, 3)$ are free of λ, without loss of generality, the simulation is given for $\lambda = 1$. The simulation results are given in Table 1.

From Table 1, we conclude the following:

(1) $AE_1 > 1$, which means $\hat{\lambda}_{RSS}$ is more efficient than $\hat{\lambda}_{SRS}$;

(2) $AE_2 > 1$, which means $\hat{\alpha}_{RSS}$ is more efficient than $\hat{\alpha}_{SRS}$;

(3) $AE_3 > 1$, which means $(\hat{\lambda}_{RSS}, \hat{\alpha}_{RSS})$ is more efficient than $(\hat{\lambda}_{SRS}, \hat{\alpha}_{SRS})$;

(4) In conclusion, the MLEs of λ and α in RSS are more efficient than that in SRS.

§8 Conclusion

In this article, we proved the existence and uniqueness of the MLE of the parameters of $WED\,(\alpha,\ \lambda)$ in SRS and provided explicit expressions for the Fisher information number in SRS. Moreover, we also proved the existence and uniqueness of the MLE of the parameters of $WED\,(\alpha,\ \lambda)$ in RSS and provided explicit expressions for the Fisher information number in RSS. Simulation studies show that these MLEs in RSS can be real competitors for those in SRS. A further stage would be to extend the use of minimum ranked set sampling with unequal samples to $WED\,(\alpha,\ \lambda)$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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