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Vanishing theorems for f-CC harmonic maps with potential H into sub-Riemannian manifolds

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Abstract. In this paper, we introduce the notion of f-CC harmonic maps with potential H from a Riemannian manifold into sub-Riemannian manifolds, and achieve some vanishing theorems for f-CC harmonic maps with potential H via the stress-energy tensor and the monotonicity formulas.

§1 Introduction

As a generalization of harmonic maps, f-harmonic maps between Riemannian manifolds was introduced by Lichnerowicz in [23] (see also [14]). Subsequently, many researchers have taken part in this field and obtained a lot of results on f-harmonic maps (cf. [10, 25, 29, 30]). On the other hand, another generalized harmonic map of a certain kind, harmonic maps with potential, was introduced by Fardoun, Ratto in [15], which had its own mathematical and physical background, for example, the classical Neumann motion and the static Landau-Lifshitz equation. Furthermore, they discovered some properties different from those of ordinary harmonic maps due to the presence of the potential. Since then, many researchers have focused on the work of harmonic maps with potential (cf. [8, 16, 24, 27, 35, 36]).

Sub-Riemannian geometry, as a generalization of Riemannian geometry, has important applications in physics (cf. [28]) and it has been paid much attention in recent years, especially the geometric analysis in sub-Riemannian geometry (cf. [3, 4]). For instance, many important geometric-functional inequalities were obtained in sub-Riemannian manifolds. The isoperimetric inequality was first proved by Pausu [31], for the Heisenberg group H^1. The Poincaré and Sobolev inequalities for Carnot groups or more generally for Hörmander's vector fields were well studied in [17, 21, 26]. Meanwhile, the subelliptic harmonic theory has been well developed. In [34], Wang investigated some regularity results for subelliptic harmonic maps

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from Carnot groups. In [7], Chang, Chang proved an existence result for pseudo-harmonic maps from closed pseudo-Hermitian manifolds into Riemannian manifolds. In [12], Dong discussed subelliptic harmonic maps from more general sub-Riemannian manifolds and obtained Eells-Sampson type results. On the other hand, Jost, Yang in [22] studied the heat flows of horizontal harmonic maps from Riemannian manifolds into a class of CC spaces. In [9], Chong, Dong, Ren introduced CC-harmonic maps associated with horizontal energy functional from Riemannian manifolds to pseudo-Hermitian manifolds and established Liouville-type theorems for CC-harmonic maps. Recently, in [20], He, Li, Zhao obtained Liouville-type theorems for CC-F-harmonic maps into Carnot groups.

Inspired by all the above-mentioned interesting literature on Riemannian geometry and sub-Riemannian geometry, we aim to study vanishing results for the generalized CC harmonic maps associated with sub-Riemannian geometry. To this end, in the present paper, we consider the f-CC harmonic maps with potential H (see Section 2) from a Riemannian manifold into sub-Riemannian manifolds. According to structural characteristics of sub-Riemannian manifolds, we obtain some vanishing theorems for f-CC harmonic maps with potential H under diverse proper conditions.

The paper is organized as follows. In Section 2, we give the notion of f-CC harmonic maps with potential H for the horizontal functional $E^f_{H,\mathcal{H}}$ from a Riemannian manifold into a sub-Riemannian manifold. In Section 3, we introduce the stress-energy tensor $S^f_{H,\mathcal{H}}$ which is naturally linked to conservation law. Afterwards, by using the stress-energy tensor $S^f_{H,\mathcal{H}}$, we obtain some vanishing results for f-CC harmonic maps with potential H under three different proper conditions: small energy conditions, slow-divergent energy conditions and the boundary vanishing conditions in Section 4. In the last section, we give some examples that are appropriate for the results of this paper.

§2 f-CC harmonic maps with potential H

In this section, we introduce a horizontal functional $E^f_{H,\mathcal{H}}$ and give the notion of f-CC harmonic maps with potential H associated with horizontal functional $E^f_{H,\mathcal{H}}$ from a Riemannian manifold into sub-Riemannian manifolds.

To this end, we first provide some basic knowledge of sub-Riemannian manifolds. Let N be a real $(\hat{n}+k)$-dimensional manifold of class C^∞. Suppose that there exists a distribution $\mathcal{H}(N)$ with rank $\hat{n}$ on N, and a Riemannian metric $g_{\mathcal{H}}$ on $\mathcal{H}(N)$. Then the triple $(N, \mathcal{H}(N), g_{\mathcal{H}})$, in the literature, is known as a sub-Riemannian manifold (cf. [6, 28, 32, 33]). Note that when $\mathcal{H}(N) = TN$, the sub-Riemannian manifold N is just a Riemannian manifold.

We also suppose that there exists a complementary distribution $\mathcal{V}(N)$ to $\mathcal{H}(N)$ in the tangent bundle TN of N. Note that $\mathcal{V}(N)$ exists on any paracompact sub-Riemannian manifold. And we usually call $\mathcal{H}(N)$ (resp. $\mathcal{V}(N)$) the horizontal distribution (resp. vertical distribution) on N. In this case, there exists a Riemannian metric $\tilde{g}$ on N and $\mathcal{V}(N)$ is taken as the complementary orthogonal distribution to $\mathcal{H}(N)$ in TN with respect to $\tilde{g}$. And $\tilde{g}$ is called a Riemannian extension of $g_{\mathcal{H}}$ if $\tilde{g}|_{\mathcal{H}} = g_{\mathcal{H}}$. Actually we may choose any Riemannian metric $\bar{g}$ on N and setting $g_{\mathcal{V}} = \bar{g}|_{\mathcal{V}}$, then we get a Riemannian extension $\tilde{g}$ of $g_{\mathcal{H}}$ and $\tilde{g} = g_{\mathcal{H}} + g_{\mathcal{V}}$, by requiring $\tilde{g}(\mathfrak{v}, \mathfrak{k}) = 0$ for any $\mathfrak{v} \in \mathcal{H}(N)$ and $\mathfrak{k} \in \mathcal{V}(N)$.

For our purpose, we consider a suitable linear connection compatible to the sub-Riemannian structure on $(N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$. The generalized Bott connection $\nabla^{\mathcal{B}}$ [5] is given by

$$\nabla^{\mathcal{B}}_X Y = \begin{cases} \Pi_{\mathcal{H}}(\nabla^R_X Y), & X, Y \in \Gamma(\mathcal{H}(N)), \\ \Pi_{\mathcal{H}}([X, Y]), & X \in \Gamma(\mathcal{V}(N)), Y \in \Gamma(\mathcal{H}(N)), \\ \Pi_{\mathcal{V}}([X, Y]), & X \in \Gamma(\mathcal{H}(N)), Y \in \Gamma(\mathcal{V}(N)), \\ \Pi_{\mathcal{V}}(\nabla^R_X Y), & X, Y \in \Gamma(\mathcal{V}(N)), \end{cases} \tag{2.1}$$

where ∇^R denotes the Riemannian connection of $\tilde{g}$, $\Pi_{\mathcal{H}} : TN \to \mathcal{H}(N)$ and $\Pi_{\mathcal{V}} : TN \to \mathcal{V}(N)$ are the horizontal and vertical projection, respectively. Clearly $\nabla^{\mathcal{B}}$ satisfies

$$\nabla^{\mathcal{B}}_X g_{\mathcal{H}} = 0 \qquad \text{and} \qquad \nabla^{\mathcal{B}}_Y g_{\mathcal{V}} = 0, \tag{2.2}$$

for any $X \in \mathcal{H}(N)$ and $Y \in \mathcal{V}(N)$.

Next, we will introduce the horizontal functional $E^f_{H,\mathcal{H}}$ and give the notion of f-CC harmonic maps with potential H.

Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map between Riemannian manifold M^m and sub-Riemannian manifold N. In this paper, we consider the horizontal functional $E^f_{H,\mathcal{H},\mathfrak{D}}(u)$

$$E^f_{H,\mathcal{H},\mathfrak{D}}(u) = \int_{\mathfrak{D}} (f(x)\frac{|du_{\mathcal{H}}|^2}{2} - H \circ u) dv_g, \tag{2.3}$$

where $du_{\mathcal{H}} = \Pi_{\mathcal{H}} \circ du$, $f : (M, g) \to (0, \infty)$ is a smooth function, H is a smooth function on N and $\mathfrak{D}$ is any bounded domain with smooth boundary and $\mathfrak{D} \subseteq M^m$. In particular, if M^m is compact, we may define the horizontal energy $E^f_{H,\mathcal{H}}(u)$ on M^m.

Definition 2.1. Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map between Riemannian manifold M^m and sub-Riemannian manifold N. The map u is called an f Carnot-Carathéodory harmonic map with potential H or f-CC harmonic map with potential H for simplicity for the horizontal functional $E^f_{H,\mathcal{H},\mathfrak{D}}(u)$ if it is a critical point of $E^f_{H,\mathcal{H},\mathfrak{D}}(u)$ for any horizontal vector field $Q \in \Gamma(u^{-1}\mathcal{H})$ whose support is contained in any bounded domain $\mathfrak{D}$ with smooth boundary.

Remark 2.2. If $H = 0$ and $f(x) = 1$, then we have $E^f_{H,\mathcal{H},\mathfrak{D}} = E_{\mathcal{H},\mathfrak{D}}$. If $H = 0$, then we have $E^f_{H,\mathcal{H},\mathfrak{D}} = E^f_{\mathcal{H},\mathfrak{D}}$. If $f(x) = 1$, then we have $E^f_{H,\mathcal{H},\mathfrak{D}} = E_{H,\mathcal{H},\mathfrak{D}}$. From Definition 2.1, we can give the notion of CC harmonic maps associated with $E_{\mathcal{H},\mathfrak{D}}$, f-CC harmonic maps associated with $E^f_{\mathcal{H},\mathfrak{D}}$ and CC harmonic maps with potential H associated with $E_{H,\mathcal{H},\mathfrak{D}}$, respectively.

In the following, we will introduce some preparatory work to get the first variation formula for $E^f_{H,\mathcal{H},\mathfrak{D}}$. In this paper, we assume that $\nabla^{\mathcal{B}}$ satisfies

$$\nabla^{\mathcal{B}}_Y g_{\mathcal{H}} = 0 \tag{2.4}$$

for any $Y \in \mathcal{V}(N)$. Then, it follows from (2.2) and (2.4) that $\nabla^{\mathcal{B}}$ is compatible to $g_{\mathcal{H}}$. In fact, this compatible connection indeed exists for some sub-Riemannian manifolds and we give two such examples in Section 5.

Then, for the smooth map $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ between Riemannian manifold M^m and sub-Riemannian manifold N, we may define the second fundamental form of u with respect to $(\nabla^M, \nabla^{\mathcal{B}})$ by

$$\mathfrak{B}(X, Y) = \nabla^{\mathcal{B}}_X du(Y) - du(\nabla^M_X Y),$$

where ∇^M denotes the Levi-Civita connection of M and $\nabla^{\mathcal{B}}$ also denotes the induced connection on $u^{-1}TN$.

We also need the following lemma (see also ([20])).

Lemma 2.3. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map between Riemannian manifold M^m and sub-Riemannian manifold N. Then, for any $X, Y \in \Gamma(TM)$, we have*

$$\nabla^{\mathcal{B}}_X du(Y) - \nabla^{\mathcal{B}}_Y du(X) = du([X, Y]) - \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(Y)]) - \Pi_{\mathcal{V}}([du_{\mathcal{H}}(X), du_{\mathcal{H}}(Y)]),$$

where $du_{\mathcal{V}} = \Pi_{\mathcal{V}} \circ du$.

For later use, we choose a local orthonormal frame field $\{e_i\}$ on M and define the f-CC-H tension field $\tau^f_{H,\mathcal{H}}(u)$ of u by

$$\tau^f_{H,\mathcal{H}}(u) = f\tau_{\mathcal{H}}(u) + du_{\mathcal{H}}(gradf) + grad_{\mathcal{H}}H \circ u,$$

where $\tau_{\mathcal{H}}(u) = \sum_{i=1}^m \{\nabla^{\mathcal{B}}_{e_i} du_{\mathcal{H}}(e_i) - du_{\mathcal{H}}\nabla^M_{e_i} e_i\}$ is the horizontal tensor field of u and $grad_{\mathcal{H}}H$ is the horizontal gradient of smooth function H defined by $\langle (grad_{\mathcal{H}}H)_q, X\rangle_{g_{\mathcal{H}}} = dH(X)$ for any $X \in \mathcal{H}_q$ and $q \in (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$.

Now, we give the first variation formula for $E^f_{H,\mathcal{H},\mathfrak{D}}$. For convenience, we write g, $g_{\mathcal{H}}$ and $\tilde{g}$ all as $\langle \cdot, \cdot \rangle$.

Lemma 2.4. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map between Riemannian manifold M^m and sub-Riemannian manifold N. Then*

$$\frac{d}{dt} E^f_{H,\mathcal{H},\mathfrak{D}}(u_t) = -\int_{\mathfrak{D}} \langle \tau^f_{H,\mathcal{H}}(u), Q\rangle, \tag{2.5}$$

where $\frac{d}{dt}u_t|_{t=0} = Q \in \Gamma(u^{-1}\mathcal{H})$ which support contained in $\mathfrak{D}$ and $\mathfrak{D}$ is any bounded domain with smooth boundary and $\mathfrak{D} \subseteq M^m$.

Proof. Let $\Psi : (-\epsilon, \epsilon) \times M$ be defined by $\Psi(t, x) = u_t(x)$ and ϵ is a positive constant. We also use ∇^M and $\nabla^{\mathcal{B}}$ for the Levi-Civita connection on $(-\epsilon, \epsilon) \times M$ and the induced connection on $\Psi^{-1}TN$. Then we get

$$\begin{aligned}
&\frac{d}{dt} E^f_{H,\mathcal{H},\mathfrak{D}}(u_t)|_{t=0} \\
&= \int_{\mathfrak{D}} \Big\{ f \sum_{i=1}^m (e_i \langle d\Psi(\frac{\partial}{\partial t}), d\Psi_{\mathcal{H}}(e_i)\rangle - \langle d\Psi(\frac{\partial}{\partial t}), \nabla^{\mathcal{B}}_{e_i} d\Psi_{\mathcal{H}}(e_i)\rangle) \\
&\quad - \langle grad_{\mathcal{H}}H \circ \Psi, d\Psi(\frac{\partial}{\partial t})\rangle \Big\}|_{t=0} dv_g \\
&= -\int_{\mathfrak{D}} \Big\{ \langle \sum_{i=1}^m f(\nabla^{\mathcal{B}}_{e_i} du_{\mathcal{H}})(e_i) + d\Psi_{\mathcal{H}}(gradf) + grad_{\mathcal{H}}H \circ \Psi, d\Psi(\frac{\partial}{\partial t})\rangle \Big\}|_{t=0} dv_g \\
&= -\int_{\mathfrak{D}} \langle f\tau_{\mathcal{H}}(u) + du_{\mathcal{H}}(gradf) + grad_{\mathcal{H}}H \circ u, Q\rangle dv_g,
\end{aligned}$$

where we use Lemma 2.3 and the divergent theorem. □

From Lemma 2.4, we can easily get that u is an f-CC harmonic map with potential H if and only if $\tau^f_{H,\mathcal{H}}(u) = 0$.

Definition 2.5. Let $u : (M, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}})$ be a smooth map between Riemannian manifold M and sub-Riemannian manifold N, u is called horizontal if $(d_x u)(T_x M) \subseteq \mathcal{H}_{u(x)}(N)$ for any point $x \in M$.

Remark 2.6. The notions of CC harmonic maps, f-CC harmonic maps, CC harmonic maps with potential H and f-CC harmonic maps with potential H defined above are all with respect to horizontal variational vector fields, but they are not required to be horizontal themselves. They are different from the notion of horizontal harmonic maps defined in [22], where horizontal harmonic maps is both horizontal and harmonic usually.

§3 Stress-energy tensor

In this section, we introduce the stress-energy tensor $S^f_{H,\mathcal{H}}(u)$ associated with the horizontal functional $E^f_{H,\mathcal{H}}$ and give the conditions that f-CC harmonic maps with potential H satisfy the conservation law.

The stress-energy tensor for maps between Riemannian manifolds was introduced by Baird, Eells [2] in 1980 and it unifies various results on harmonic maps. Since then, it has become a useful tool for studying the energy behaviour of related functional, the readers may refer to [1, 2, 11] and so on. Following [1, 16], we associated a symmetric 2-tensor $S^f_{H,\mathcal{H}}(u)$ to the horizontal functional $E^f_{H,\mathcal{H}}(u)$, which is called the stress-energy tensor

$$S^f_{H,\mathcal{H}}(u) = (f\frac{|du_{\mathcal{H}}|^2}{2} - H \circ u)g - f(u^*\tilde{g})_{\mathcal{H}},$$

where $(u^*\tilde{g})_{\mathcal{H}} = g_{\mathcal{H}}(du_{\mathcal{H}}(\cdot), du_{\mathcal{H}}(\cdot))$. Analogously, we may define the stress-energy tensor $S^f_{\mathcal{H}}(u)$ associated with the horizontal functional $E^f_{\mathcal{H}}(u)$, i.e., $S^f_{\mathcal{H}}(u) = f(\frac{|du_{\mathcal{H}}|^2}{2}g - (u^*\tilde{g})_{\mathcal{H}})$.

Proposition 3.1. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map. Then for any vector field $X \in \Gamma(TM)$,*

$$(divS^f_{H,\mathcal{H}}(u))(X) = \frac{|du_{\mathcal{H}}|^2}{2}df(X) - \langle \tau^f_{H,\mathcal{H}}(u), du_{\mathcal{H}}(X)\rangle - \sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle,$$

where $\tau^f_{H,\mathcal{H}}(u) = f\tau_{\mathcal{H}}(u) + du_{\mathcal{H}}(gradf) + grad_{\mathcal{H}}H \circ u$.

Proof. We choose a local orthonormal basis $\{e_i\}$ of TM such that it is an normal frame at a point $x \in M$. Let X be any vector field on M. Then, at the point x, one has

$$(divS^f_{H,\mathcal{H}}(u))(X) = \frac{|du_{\mathcal{H}}|^2}{2}df(X) - \sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle - \langle grad_{\mathcal{H}}H \circ u, du(X)\rangle$$

$$
\begin{aligned}
&-\sum_{i=1}^{m} e_i(f)\langle du_{\mathcal{H}}(e_i), du_{\mathcal{H}}(X)\rangle-\sum_{i=1}^{m} f\langle\nabla^{\mathcal{B}}_{e_i} du_{\mathcal{H}}(e_i), du_{\mathcal{H}}(X)\rangle\\
=&\frac{|du_{\mathcal{H}}|^2}{2}df(X)-\langle\tau^f_{H,\mathcal{H}}(u), du_{\mathcal{H}}(X)\rangle-\sum_{i=1}^{m} f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle, \quad (3.1)
\end{aligned}
$$

where we used Lemma 2.3. □

Let X be any smooth vector field on M. Recall that for any 2-tensor field $\mathcal{T}\in\Gamma(T^*M\otimes T^*M)$, its divergence div$\mathcal{T}\in\Gamma(T^*M)$ is defined by div$\mathcal{T}(X)=\sum_{i=1}^m(\nabla^M_{e_i}\mathcal{T})(e_i,X)$. Let θ_X be the dual one form for any vector field $X\in\Gamma(TM)$, i.e., $\theta_X(Y)=\langle X,Y\rangle$, where $Y\in\Gamma(TM)$. The covariant derivative of θ_X is given by $(\nabla^M\theta_X)(Y,Z)=(\nabla^M_Y\theta_X)(Z)=\langle\nabla^M_Y X,Z\rangle$ for any $X,Y,Z\in\Gamma(TM)$. If $X=\nabla^M\varphi$ is the gradient field of some C^2 function φ on M, then $\theta_X=d\varphi$ and $\nabla^M\theta_X=Hess\varphi$. Let $\mathcal{T}$ be a symmetric (0,2)-type tensor field. It follows from [1,13] that $div(i_X\mathcal{T})=div(\mathcal{T})(X)+\langle\mathcal{T},\nabla^M\theta_X\rangle=div(\mathcal{T})(X)+\frac{1}{2}\langle\mathcal{T},L_Xg\rangle$ for any $X\in\Gamma(TM)$. Let $\mathfrak{D}$ be any bounded domain of M with C^1 boundary. By using the Stokes' theorem, we get

$$
\int_{\partial\mathfrak{D}}\mathcal{T}(X,\tilde{v})ds_g=\int_{\mathfrak{D}}(\langle\mathcal{T},\frac{1}{2}L_Xg\rangle+div(\mathcal{T})(X))dv_g, \quad (3.2)
$$

where $\tilde{v}$ is the unit outward normal vector field along $\partial\mathfrak{D}$.

Therefore, we can infer from Proposition 3.1 that

$$
(divS^f_{H,\mathcal{H}}(u))(X)=\frac{|du_{\mathcal{H}}|^2}{2}df(X)-\sum_{i=1}^{m} f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle. \quad (3.3)
$$

Then, by using (3.2) and (3.3), we get

$$
\begin{aligned}
\int_{\partial\mathfrak{D}}S^f_{\mathcal{H}}(X,\bar{v})ds_g=&\int_{\mathfrak{D}}\Big\{\langle S^f_{\mathcal{H}},\frac{1}{2}L_Xg\rangle+\langle grad_{\mathcal{H}}H\circ u, du(X)\rangle+\frac{|du_{\mathcal{H}}|^2}{2}df(X)\\
&-\sum_{i=1}^{m} f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle\Big\}dv_g, \quad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\partial\mathfrak{D}}S^f_{H,\mathcal{H}}(X,\bar{v})ds_g=&\int_{\mathfrak{D}}\Big\{\langle S^f_{H,\mathcal{H}},\frac{1}{2}L_Xg\rangle+\frac{|du_{\mathcal{H}}|^2}{2}df(X)\\
&-\sum_{i=1}^{m} f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle\Big\}dv_g. \quad (3.5)
\end{aligned}
$$

Definition 3.2. Let $\mathfrak{u}:(M,g)\to(N,\mathcal{H}(N),g_{\mathcal{H}})$ be a smooth map. We call $\mathfrak{u}$ satisfies the conservation law if $divS_{\mathcal{H}}(\mathfrak{u})=0$, where $S_{\mathcal{H}}(\mathfrak{u})$ is the stress-energy tensor of $\mathfrak{u}$.

Therefore, we get that if f is a constant function and either u is horizontal or the vertical distribution $\mathcal{V}$ on N is integrable, then the f-CC harmonic map with potential H satisfies the conservation law.

§4 Vanishing results

In this section, we use the stress-energy tensor to obtain vanishing results for f-CC harmonic maps with potential H under small energy conditions, under slowly divergent energy conditions and under boundary vanishing conditions in Subsections 4.1, 4.2 and 4.3, respectively.

4.1 Vanishing theorems under small energy conditions

In this subsection, we will prove some vanishing theorems for f-CC harmonic maps with potential H under small energy conditions. Furthermore, by using these vanishing theorems, we get another class of vanishing results on pinched Riemannian manifolds.

Before we present vanishing results, we first suppose that (M^m, g_1) is a complete Riemannian manifold with a pole x_1 and $r(x) = dist_{g_1}(x, x_1)$ is g_1-distance function relative to the pole x_1. Set $B(r) = \{x \in M : r(x) \leq r\}$. Denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$) the maximum (resp. minimal) eigenvalues of $Hess_{g_1}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_1\}$. Next, we suppose that $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H, where $g = \omega^2 g_1$ and $0 < \omega \in C^\infty(M)$. We assume that ω satisfies (I) $\frac{\partial \log \omega}{\partial r} \geq 0$ and (II) there is a constant $\mathcal{K}_1$ such that $(m-2)r\frac{\partial \log \omega}{\partial r} + \frac{m-1}{2}\lambda_{\min} + 1 - \max\{2, \lambda_{\max}\} \geq \mathcal{K}_1$. We also assume that $\vartheta = \sup_M r|\frac{\partial \log f}{\partial r}| < +\infty$.

Definition 4.1. A map $u : (M, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}})$ between a complete Riemannian manifold M with a pole x_1 and a sub-Riemannian manifold N is called radial horizontal, if $du(\frac{\partial}{\partial r})$ is horizontal.

Theorem 4.2. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H. If $\mathcal{K}_1 - \vartheta > 0$, ω satisfies (I) (II), $\frac{\partial H \circ u}{\partial r} \geq 0$ and u is radial horizontal, then*

$$\frac{\int_{B(\varrho_1)} f\frac{|du_{\mathcal{H}}|^2}{2} dv_g}{\varrho_1^{\mathcal{K}_1 - \vartheta}} \leq \frac{\int_{B(\varrho_2)} f\frac{|du_{\mathcal{H}}|^2}{2} dv_g}{\varrho_2^{\mathcal{K}_1 - \vartheta}},$$

for any $0 < \varrho_1 \leq \varrho_2$. In particular, if $\int_{B(R)} f\frac{|du_{\mathcal{H}}|^2}{2} dv_g = o(R^{\mathcal{K}_1 - \vartheta})$, then $du_{\mathcal{H}} = 0$.

Proof. According to the assumption $\frac{\partial H \circ u}{\partial r} \geq 0$ and taking $\mathfrak{D} = B(R)$ and $X = r\frac{\partial}{\partial r} = \frac{1}{2}\nabla^1 r^2$ into (3.4), one has

$$\int_{\partial B(R)} S^f_{\mathcal{H}}(X, \tilde{v}) ds_g \geq \int_{B(R)} \Big\{\langle S^f_{\mathcal{H}}, \frac{1}{2}L_X g\rangle + \frac{|du_{\mathcal{H}}|^2}{2} df(X)\Big\} dv_g, \tag{4.1}$$

where ∇^1 denotes the covariant derivative determined by g_1. Then direct computation yields

$$\langle S^f_{\mathcal{H}}, \frac{1}{2}L_X g\rangle = r\frac{\partial \log \omega}{\partial r}\langle S^f_{\mathcal{H}}, g\rangle + \frac{1}{2}\omega^2\langle S^f_{\mathcal{H}}, Hess_{g_1}(r^2)\rangle. \tag{4.2}$$

Now, we choose an orthonormal basis $\{e_i\}_{i=1}^m$ with respect to g_1 and $e_m = \frac{\partial}{\partial r}$ and suppose that $Hess_{g_1}(r^2)$ becomes a diagonal matrix w.r.t. $\{e_i\}$. Then we have

$$r\frac{\partial \log \omega}{\partial r}\langle S^f_{\mathcal{H}}, g\rangle = r\frac{\partial \log \omega}{\partial r}(m-2)f\frac{|du_{\mathcal{H}}|^2}{2}, \tag{4.3}$$

and

$$\frac{1}{2}\omega^2\langle S^f_{\mathcal{H}}, Hess_{g_1}(r^2)\rangle \geq \frac{1}{2}f\Big\{(m-1)\lambda_{\min} + 2 - 2\max\{2, \lambda_{\max}\}\Big\}\frac{|du_{\mathcal{H}}|^2}{2}. \tag{4.4}$$

Combining (I), (II) and (4.2)-(4.4), one has

$$\langle S^f_{\mathcal{H}}, \frac{1}{2}L_X g\rangle + \frac{|du_{\mathcal{H}}|^2}{2} df(X) \geq (\mathcal{K}_1 - \vartheta)f\frac{|du_{\mathcal{H}}|^2}{2}. \tag{4.5}$$

By using the coarea formula and $|\nabla r| = \omega^{-1}$, one has

$$\int_{\partial B(R)} S^f_{\mathcal{H}}(X,\tilde{v})ds_g = R\frac{d}{dR}\int_{B(R)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g - \int_{\partial B(R)} fR\omega^{-1}\langle du_{\mathcal{H}}(\frac{\partial}{\partial r}), du_{\mathcal{H}}(\frac{\partial}{\partial r})\rangle ds_g. \quad (4.6)$$

Then, it follows form (4.1), (4.5) and (4.6) that

$$\frac{\int_{B(\varrho_1)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g}{\varrho_1^{\mathcal{K}_1-\vartheta}} \le \frac{\int_{B(\varrho_2)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g}{\varrho_2^{\mathcal{K}_1-\vartheta}}, \quad (4.7)$$

for any $0 < \varrho_1 \le \varrho_2$. In particular, if $\int_{B(R)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g = o(R^{\mathcal{K}_1-\vartheta})$, we obtain that $du_{\mathcal{H}} = 0$ by (4.7) and yields the claim. □

Theorem 4.3. *Let $u : (M,g) \to (N,\mathcal{H}(N),g_{\mathcal{H}},\tilde{g})$ be an f-CC harmonic map with potential H. If $\mathcal{K}_1 - \vartheta > 0$, $H \le 0$ (or $H|_{u(M)} \le 0$), ω satisfies (I) (II) and u is radial horizontal, then*

$$\frac{\int_{B(\varrho_1)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g}{\varrho_1^{\mathcal{K}_1-\vartheta}} \le \frac{\int_{B(\varrho_2)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g}{\varrho_2^{\mathcal{K}_1-\vartheta}},$$

for any $0 < \varrho_1 \le \varrho_2$. In particular, if $\int_{B(R)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g = o(R^{\mathcal{K}_1-\vartheta})$, then $du_{\mathcal{H}} = 0$.

Proof. Taking $\mathfrak{D} = B(R)$ and $X = r\frac{\partial}{\partial r} = \frac{1}{2}\nabla^1 r^2$ into (3.5), one has

$$\langle S^f_{H,\mathcal{H}}, \frac{1}{2}L_X g\rangle = \langle S^f_{H,\mathcal{H}}, r\frac{\partial \log\omega}{\partial r}g\rangle + \langle S^f_{H,\mathcal{H}}, \frac{1}{2}\omega^2 Hess_{g_1}(r^2)\rangle, \quad (4.8)$$

where ∇^1 denotes the covariant derivative determined by g_1. Analogously, using (I), we have

$$\langle S^f_{H,\mathcal{H}}, r\frac{\partial \log\omega}{\partial r}g\rangle \ge (m-2)r\frac{\partial \log\omega}{\partial r}(f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u), \quad (4.9)$$

and

$$\langle S^f_{H,\mathcal{H}}, \frac{1}{2}\omega^2 Hess_{g_1}(r^2)\rangle \ge \frac{1}{2}(f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)\Big\{2 + (m-1)\lambda_{\min} - 2\max\{2,\lambda_{\max}\}\Big\}. \quad (4.10)$$

From (4.8)-(4.10) and (II), one has

$$\langle S^f_{H,\mathcal{H}}, \frac{1}{2}L_X g\rangle \ge \mathcal{K}_1(f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u). \quad (4.11)$$

By using the coarea formula and $|\nabla r| = \omega^{-1}$, we have

$$\int_{\partial B(R)} S^f_{H,\mathcal{H}}(X,\tilde{v})ds_g \le R\frac{d}{dR}\int_{B(R)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g. \quad (4.12)$$

Then, it follows form (3.5), (4.11) and (4.12) that

$$\frac{\int_{B(\varrho_1)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g}{\varrho_1^{\mathcal{K}_1-\vartheta}} \le \frac{\int_{B(\varrho_2)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g}{\varrho_2^{\mathcal{K}_1-\vartheta}}, \quad (4.13)$$

for any $0 < \varrho_1 \le \varrho_2$. In particular, if $\int_{B(R)} (f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)dv_g = o(R^{\mathcal{K}_1-\vartheta})$, we obtain that $du_{\mathcal{H}} = 0$ by (4.13) and yields the claim. □

The rest of this subsection is devoted to obtain some vanishing results for f-CC harmonic maps with potential H on pinched Riemannian manifolds. To this end we will make use of the following lemmas.

Lemma 4.4. *(cf. [11, 13, 18, 19]) Let (M^m, g) be a complete Riemannian manifold with a pole x_1. Let K_r be the radial curvature of M^m.*

(i) if $-\alpha^2 \le K_r \le -\beta^2$ *with* $\alpha \ge \beta > 0$*, then*

$$\beta\coth(\beta r)[g_1 - dr\otimes dr] \le Hess(r) \le \alpha\coth(\alpha r)[g_1 - dr\otimes dr];$$

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$*,* $A \ge 0$ *and* $0 \le B < 2\varepsilon$ *then*

$$\frac{1-\frac{B}{2\varepsilon}}{r}[g_1 - dr\otimes dr] \le Hess(r) \le \frac{e^{\frac{A}{2\varepsilon}}}{r}[g_1 - dr\otimes dr];$$

(iii) if $-\frac{a^2}{1+r^2} \le K_r \le \frac{b^2}{1+r^2}$ *with* $a \ge 0$ *and* $0 \le b^2 \le \frac{1}{4}$*, then*

$$\frac{1+\sqrt{1-4b^2}}{2r}[g_1 - dr\otimes dr] \le Hess(r) \le \frac{1+\sqrt{1+4a^2}}{2r}[g_1 - dr\otimes dr].$$

In [16], Feng, Han obtained the following lemma by Lemma 4.4.

Lemma 4.5. *Let* (M^m, g) *be a complete Riemannian manifold with a pole* x_1*. Let* K_r *be the radial curvature of* M^m*.*

(i) if $-\alpha^2 \le K_r \le -\beta^2$ *with* $\alpha \ge \beta > 0$ *and* $m - 1 - \frac{2\alpha}{\beta} \ge 0$*, then*

$$\frac{m-1}{2}\lambda_{\min} + 1 - \max\{2, \lambda_{\max}\} \ge m - \frac{2\alpha}{\beta};$$

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$*,* $A \ge 0$ *and* $0 \le B < 2\varepsilon$*, then*

$$\frac{m-1}{2}\lambda_{\min} + 1 - \max\{2, \lambda_{\max}\} \ge 1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 2e^{\frac{A}{2\varepsilon}};$$

(iii) if $-\frac{a^2}{1+r^2} \le K_r \le \frac{b^2}{1+r^2}$ *with* $a \ge 0$ *and* $0 \le b^2 \le \frac{1}{4}$*, then*

$$\frac{m-1}{2}\lambda_{\min} + 1 - \max\{2, \lambda_{\max}\} \ge 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - (1 + \sqrt{1+4a^2}).$$

Using Lemma 4.5 and the proof of Theorem 4.2-Theorem 4.3, we easily get the following corollaries on pinched Riemannian manifolds.

Corollary 4.6. *Let* (M^m, g) *be a complete Riemannian manifold with a pole* x_1*. Assume that the radial curvature* K_r *of* M *satisfies one of the following three conditions:*

(i) if $-\alpha^2 \le K_r \le -\beta^2$ *with* $\alpha \ge \beta > 0$ *and* $m - 1 - \frac{2\alpha}{\beta} \ge 0$*;*

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$*,* $A \ge 0$*,* $0 \le B < 2\varepsilon$ *and* $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 2e^{\frac{A}{2\varepsilon}} > 0$*;*

(iii) if $-\frac{a^2}{1+r^2} \le K_r \le \frac{b^2}{1+r^2}$ *with* $a \ge 0$*,* $0 \le b^2 \le \frac{1}{4}$ *and* $1 + \frac{m-1}{2}(1+\sqrt{1-4b^2}) - (1+\sqrt{1+4a^2}) > 0$*.*

Let $u : (M, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ *be an* f*-CC harmonic map with potential* H*. If* $\mathcal{A} - \vartheta > 0$*,* $\frac{\partial H\circ u}{\partial r} \ge 0$ *and* u *is radial horizontal, then*

$$\frac{\int_{B(\varrho_1)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g}{\varrho_1^{\mathcal{A}-\vartheta}} \le \frac{\int_{B(\varrho_2)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g}{\varrho_2^{\mathcal{A}-\vartheta}},$$

for any $0 < \varrho_1 \le \varrho_2$ *and*

$$\mathcal{A} = \begin{cases} m - \frac{2\alpha}{\beta}, & \text{when } K_r \text{ satisfies } (i), \\ 1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 2e^{\frac{A}{2\varepsilon}}, & \text{when } K_r \text{ satisfies } (ii), \\ 1 + \frac{m-1}{2}(1+\sqrt{1-4b^2}) - (1+\sqrt{1+4a^2}), & \text{when } K_r \text{ satisfies } (iii). \end{cases}$$

In particular, if $\int_{B(R)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g = o(R^{\mathcal{A}-\vartheta})$*, then* $du_{\mathcal{H}} = 0$*.*

Corollary 4.7. *Let $u : (M, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H. Let (M^m, g), K_r and $\mathcal{A}$ be as in Corollary 4.6. If $\mathcal{A} - \vartheta > 0$, $H \le 0$ (or $H|_{u(M)} \le 0$) and u is radial horizontal, then*

$$\frac{\int_{B(\varrho_1)}(f\frac{|du_{\mathcal{H}}|^2}{2} - H \circ u)dv_g}{\varrho_1^{\mathcal{A}-\vartheta}} \le \frac{\int_{B(\varrho_2)}(f\frac{|du_{\mathcal{H}}|^2}{2} - H \circ u)dv_g}{\varrho_2^{\mathcal{A}-\vartheta}},$$

for any $0 < \varrho_1 \le \varrho_2$. In particular, if $\int_{B(R)}(f\frac{|du_{\mathcal{H}}|^2}{2} - H \circ u)dv_g = o(R^{\mathcal{A}-\vartheta})$, then $du_{\mathcal{H}} = 0$.

4.2 Vanishing theorems under slowly divergent energy conditions

In this subsection, we will prove some vanishing theorems for f-CC harmonic maps with potential H under slowly divergent energy conditions. Furthermore, by using these vanishing theorems, we get another class of vanishing results on pinched Riemannian manifolds.

We call the functional $E^f_{\mathcal{H}}(u)$ (or $E^f_{H,\mathcal{H}}(u)$) is slowly divergent if there exists a positive function $\Psi(r)$ with $\int_{R_1}^{\infty} \frac{dr}{r\Psi(r)} = +\infty$ ($R_1 > 0$), such that

$$\lim_{R\to\infty} \int_{B(R)} \frac{f\frac{|du_{\mathcal{H}}|^2}{2}}{\Psi(r(x))} dv_g < \infty \ \Big(\text{or } \lim_{R\to\infty} \int_{B(R)} \frac{(f\frac{|du_{\mathcal{H}}|^2}{2} - H \circ u)}{\Psi(r(x))} dv_g < \infty\Big). \tag{4.14}$$

Theorem 4.8. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H. If ω satisfies (I) (II), $\mathcal{K}_1 - \vartheta > 0$, $\frac{\partial H \circ u}{\partial r} \ge 0$, u is radial horizontal and $E^f_{\mathcal{H}}(u)$ is slowly divergent, then $du_{\mathcal{H}} = 0$.*

Proof. It follows from the proof of Theorem 4.2 that

$$R \int_{\partial B(R)} \omega f \frac{|du_{\mathcal{H}}|^2}{2} ds_g \ge (\mathcal{K}_1 - \vartheta) \int_{B(R)} f \frac{|du_{\mathcal{H}}|^2}{2} dv_g. \tag{4.15}$$

Assume that $du_{\mathcal{H}} \ne 0$, then there is a $R_1 > 0$ such that $\int_{B(R)} f \frac{|du_{\mathcal{H}}|^2}{2} dv_g \ge C_1$, for $R \ge R_1$ and for a positive constant C_1. Combining this inequality with (4.15), for $R \ge R_1$, one has

$$\int_{\partial B(R)} \omega f \frac{|du_{\mathcal{H}}|^2}{2} ds_g \ge \frac{C_1(\mathcal{K}_1 - \vartheta)}{R}.$$

This allows us to infer

$$\lim_{R\to\infty} \int_{B(R)} \frac{f\frac{|du_{\mathcal{H}}|^2}{2}}{\Psi(r(x))} dv_g \ge C_1(\mathcal{K}_1 - \vartheta) \int_{R_1}^{\infty} \frac{dR}{R\Psi(R)} = \infty,$$

which contradicts (4.14) and then completes the proof of Theorem 4.8. □

Theorem 4.9. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H. If $\mathcal{K}_1 - \vartheta > 0$, $H \le 0$ (or $H|_{u(M)} \le 0$), ω satisfies (I) (II), u is radial horizontal and $E^f_{H,\mathcal{H}}(u)$ is slowly divergent, then $du_{\mathcal{H}} = 0$.*

Proof. Analogously, using Theorem 4.3, we can finish the proof of Theorem 4.9. □

Theorem 4.10. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map which satisfies*

$$\int_M (divS^f_{\mathcal{H}})(X) dv_g = \int_M \langle grad_{\mathcal{H}} H \circ u, du(X)\rangle dv_g + \int_M \frac{|du_{\mathcal{H}}|^2}{2} df(X) dv_g$$

$$-\int_M \sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle dv_g, \tag{4.16}$$

for any $X \in \Gamma(TM)$. If $\mathcal{K}_1 - \vartheta > 0$, ω satisfies (I) (II), $\frac{\partial H\circ u}{\partial r} \geq 0$, u is radial horizontal and $E^f_{\mathcal{H}}(u)$ is slowly divergent, then $du_{\mathcal{H}} = 0$.

Proof. Taking $\mathfrak{D} = B(r)$, $\mathcal{T} = S^f_{\mathcal{H}}$ and $X = r\frac{\partial}{\partial r}$ in (3.2), we have

$$\int_{B(r)} \langle S^f_{\mathcal{H}}, \frac{1}{2}L_X g\rangle dv_g + \int_{B(r)} (divS^f_{\mathcal{H}})(X)dv_g \leq r\int_{\partial B(r)} \omega f\frac{|du_{\mathcal{H}}|^2}{2}ds_g. \tag{4.17}$$

From the proof of Theorem 4.2, we have

$$\langle S^f_{\mathcal{H}}, \frac{1}{2}L_X g\rangle + \frac{|du_{\mathcal{H}}|^2}{2}df(X) \geq (\mathcal{K}_1 - \vartheta)f\frac{|du_{\mathcal{H}}|^2}{2}. \tag{4.18}$$

It follows from (4.16) that

$$\lim_{R\to\infty}\int_{B(R)} (divS^f_{\mathcal{H}})(X)dv_g = \lim_{R\to\infty}\int_{B(R)} \Big\{\langle grad_{\mathcal{H}} H\circ u, du(X)\rangle + \frac{|du_{\mathcal{H}}|^2}{2}df(X)dv_g - \sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle\Big\}dv_g. \tag{4.19}$$

Assume that $du_{\mathcal{H}} \neq 0$, then there is a positive constant R_3 such that $\int_{B(R)} f\frac{|du_{\mathcal{H}}|^2}{2}dv_g \geq C_3$ for $R \geq R_3$ and a positive constant C_3. Then, there exists a positive constant $R_4 > R_3$ such that

$$|\int_{B(R)} (divS^f_{\mathcal{H}})(X)dv_g - \int_{B(R)} \Big\{\langle grad_{\mathcal{H}} H\circ u, du(X)\rangle + \frac{|du_{\mathcal{H}}|^2}{2}df(X)dv_g - \sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle\Big\}dv_g| \leq \frac{(\mathcal{K}_1 - \vartheta)C_3}{2}, \tag{4.20}$$

for $R \geq R_4$. From (4.17), (4.18) and (4.20) for $R > R_4$, one has $\int_{\partial B(R)} \omega f\frac{|du_{\mathcal{H}}|^2}{2}ds_g \geq \frac{(\mathcal{K}_1-\vartheta)C_3}{2R}$. Combining this with $|\nabla r| = \omega^{-1}$ leads to

$$\lim_{R\to\infty}\int_{B(R)} \frac{f\frac{|du_{\mathcal{H}}|^2}{2}}{\Psi(r(x))}dv_g \geq C_3(\mathcal{K}_1 - \vartheta)\int_{R_4}^\infty \frac{dR}{2R\Psi(R)} = \infty,$$

which contradicts (4.14), and then we complete the proof of Theorem 4.10. □

Theorem 4.11. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be a smooth map which satisfies*

$$\int_M (divS^f_{H.\mathcal{H}})(X)dv_g = \int_M \frac{|du_{\mathcal{H}}|^2}{2}df(X)dv_g - \int_M \sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i), \Pi_{\mathcal{H}}([du_{\mathcal{V}}(X), du_{\mathcal{V}}(e_i)])\rangle dv_g, \tag{4.21}$$

for any $X \in \Gamma(TM)$. If $\mathcal{K}_1 - \vartheta > 0$, $H \leq 0$ (or $H_{u(M)} \leq 0$), ω satisfies (I) (II), u is radial horizontal and $E^f_{H,\mathcal{H}}(u)$ is slowly divergent, then $du_{\mathcal{H}} = 0$.

Proof. Taking $\mathfrak{D} = B(r)$, $\mathcal{T} = S^f_{H,\mathcal{H}}$ and $X = r\frac{\partial}{\partial r}$ in (3.2), we have

$$\int_{B(r)} \Big\{\langle S^f_{H,\mathcal{H}}, \frac{1}{2}L_X g\rangle + (divS^f_{H,\mathcal{H}})(X)\Big\}dv_g \leq r\int_{\partial B(r)} \omega(f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u)ds_g. \tag{4.22}$$

From the proof of Theorem 4.3, we have

$$\langle S^f_{H,\mathcal{H}}, \frac{1}{2}L_X g\rangle + \frac{|du_{\mathcal{H}}|^2}{2}df(X) \geq (\mathcal{K}_1 - \vartheta)(f\frac{|du_{\mathcal{H}}|^2}{2} - H\circ u). \tag{4.23}$$

It follows from (4.21) that

$$\lim_{R\to\infty}\int_{B(R)}(divS^f_{H,\mathcal{H}})(X)dv_g \tag{4.24}$$
$$=\lim_{R\to\infty}\int_{B(R)}\frac{|du_{\mathcal{H}}|^2}{2}df(X)dv_g-\lim_{R\to\infty}\int_{B(R)}\sum_{i=1}^m f\langle du_{\mathcal{H}}(e_i),\Pi_{\mathcal{H}}([du_{\mathcal{V}}(X),du_{\mathcal{V}}(e_i)])\rangle dv_g.$$

Assume that $du_{\mathcal{H}}\neq 0$, then there is a positive constant R_5 such that $\int_{B(R)}(f\frac{|du_{\mathcal{H}}|^2}{2}-H\circ u)dv_g\geq C_4$, for $R\geq R_5$ and a positive constant C_4. Then, there exists a positive constant $R_6>R_5$ such that

$$|\int_{B(R)}(divS^f_{H,\mathcal{H}})(X)dv_g-\int_{B(R)}\frac{|du_{\mathcal{H}}|^2}{2}df(X)dv_g \tag{4.25}$$
$$+\sum_{i=1}^m\int_{B(R)}f\langle du_{\mathcal{H}}(e_i),\Pi_{\mathcal{H}}([du_{\mathcal{V}}(X),du_{\mathcal{V}}(e_i)])\rangle dv_g|\leq\frac{(\mathcal{K}_1-\vartheta)C_4}{2},$$

for $R\geq R_6$. From (4.22), (4.23) and (4.25), for $R>R_6$, one has $\int_{\partial B(R)}\omega(f\frac{|du_{\mathcal{H}}|^2}{2}-H\circ u)ds_g\geq\frac{(\mathcal{K}_1-\vartheta)C_4}{2R}$. Combining this with $|\nabla r|=\omega^{-1}$ leads to

$$\lim_{R\to\infty}\int_{B(R)}\frac{(f\frac{|du_{\mathcal{H}}|^2}{2}-H\circ u)}{\Psi(r(x))}dv_g\geq C_4(\mathcal{K}_1-\vartheta)\int_{R_6}^{\infty}\frac{dR}{2R\Psi(R)}=\infty,$$

which contradicts (4.14) and then complete the proof of Theorem 4.11. □

On the other hand, using Theorem 4.8-Theorem 4.11, we obtain the following corollaries on pinched Riemannian manifolds.

Corollary 4.12. *Let $u:(M^m,g)\to(N,\mathcal{H}(N),g_{\mathcal{H}},\tilde{g})$ be an f-CC harmonic map with potential H. Let (M^m,g_1), K_r and $\mathcal{A}$ be as in Corollary 4.6. If $\mathcal{A}-\vartheta>0$, u is radial horizontal, and one of the following two conditions is satisfied*
(1) $\frac{\partial H\circ u}{\partial r}\geq 0$ and $E^f_{\mathcal{H}}(u)$ is slowly divergent,
(2) $H\leq 0$ (or $H|_{u(M)}\leq 0$) and $E^f_{H,\mathcal{H}}(u)$ is slowly divergent,
then $du_{\mathcal{H}}=0$.

Corollary 4.13. *Let $u:(M^m,g)\to(N,\mathcal{H}(N),g_{\mathcal{H}},\tilde{g})$ be a smooth map. Let (M^m,g_1), K_r and $\mathcal{A}$ be as in Corollary 4.6. If $\mathcal{A}-\vartheta>0$, u is radial horizontal, and one of the following two conditions is satisfied*
(1) the equality (4.16) holds, $\frac{\partial H\circ u}{\partial r}\geq 0$ and $E^f_{\mathcal{H}}(u)$ is slowly divergent,
(2) the equality (4.21) holds, $H\leq 0$ (or $H|_{u(M)}\leq 0$) and $E^f_{H,\mathcal{H}}(u)$ is slowly divergent,
then $du_{\mathcal{H}}=0$.

4.3 Vanishing theorems under boundary vanishing conditions

In this subsection, we will prove some vanishing results for f-CC harmonic maps with potential H under boundary vanishing conditions on Riemannian manifolds and pinched Riemannian manifolds. This kind of vanishing results under boundary vanishing conditions here may be regarded as a natural generalization of constant Dirichlet boundary value problems for the maps between two Riemannian manifolds. To this end, we recall starlike domains with C^1-boundaries which generalize C^1-convex domains as below.

Definition 4.14. (cf. [13]) A bounded domain $\mathfrak{D} \subset M$ with C^1 boundary $\partial\mathfrak{D}$ is called starlike if there exists an interior point $x_0 \in \mathfrak{D}$ such that $\langle \frac{\partial}{\partial r_{x_0}}, \hat{v}\rangle|_{\partial\mathfrak{D}} \geq 0$, where $\hat{v}$ is the unit outer normal to $\partial\mathfrak{D}$, and the vector field $\frac{\partial}{\partial r_{x_0}}$ is the unit vector field such that for any $x \in (\mathfrak{D} \setminus \{x_0\}) \cup \partial\mathfrak{D}$, $\frac{\partial}{\partial r_{x_0}}$ is the unit vector tangent to the unique geodesic joining x_0 and pointing away from x_0.

Theorem 4.15. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H and $\mathfrak{D} \subset M$ be a bounded statlike domain with C^1 boundary with the pole $x_\text{o} \in \mathfrak{D}$. If $\mathcal{K}_1 - \vartheta > 0$ and $\frac{\partial H\circ u}{\partial r} \geq 0$ on $\mathfrak{D}$, ω satisfies (I) (II), u is radial horizontal and $du_{\mathcal{H}}|_{\partial\mathfrak{D}} = 0$, then $du_{\mathcal{H}}|_{\mathfrak{D}} = 0$.*

Proof. According to $du_{\mathcal{H}}|_{\partial\mathfrak{D}} = 0$ and choosing $X = r_{x\text{o}}\frac{\partial}{\partial r_{x\text{o}}}$, one has

$$\int_{\partial\mathfrak{D}} S^f_{\mathcal{H}}(r_{x\text{o}}\frac{\partial}{\partial r_{x\text{o}}}, \hat{v})ds_g \leq 0. \tag{4.26}$$

It follows from the proof of Theorem 4.2 that

$$\int_{\mathfrak{D}} \langle S^f_{\mathcal{H}}, \frac{1}{2}L_{r_{x\text{o}}\frac{\partial}{\partial r_{x\text{o}}}} g\rangle dv_g + \int_{\mathfrak{D}} \frac{|du_{\mathcal{H}}|^2}{2} df(r_{x\text{o}}\frac{\partial}{\partial r_{x\text{o}}})dv_g \geq 0. \tag{4.27}$$

Then, from (4.1), (4.26) and (4.27), we obtain $du_{\mathcal{H}}|_{\mathfrak{D}} = 0$ and finish the proof of Theorem 4.15. □

Corollary 4.16. *Let $u : (M^m, g) \to (N, \mathcal{H}(N), g_{\mathcal{H}}, \tilde{g})$ be an f-CC harmonic map with potential H. Let (M^m, g_1), K_r and $\mathcal{A}$ be as in Corollary 4.6, and $\mathfrak{D} \subset M$ be a bounded statlike domain with C^1 boundary with the pole $x_\text{o} \in \mathfrak{D}$. If $\mathcal{A} - \vartheta > 0$ and $\frac{\partial H\circ u}{\partial r} \geq 0$ on $\mathfrak{D}$, u is radial horizontal and $du_{\mathcal{H}}|_{\partial\mathfrak{D}} = 0$, then $du_{\mathcal{H}}|_{\mathfrak{D}} = 0$.*

§5 Examples

In this last section, we give two sub-Riemannian manifolds that are appropriate for the vanishing results in this paper.

Example 5.1. Let $\mathcal{X}_1 = \frac{1}{2}\partial x - \frac{1}{4}y\partial z$, $\mathcal{X}_2 = \partial y + \frac{3}{2}x\partial z$ and $\mathcal{X}_3 = \partial z$ be three vector fields on $\mathbb{R}^3$. Consider the distribution given by $\mathcal{D}_1 = span\{\mathcal{X}_1, \mathcal{X}_2\}$. Then, we get the matrix $\{\tilde{g}_{ij}\}$ has the following form

$$\tilde{g}_{ij} = \begin{pmatrix} 4 + \frac{1}{4}y^2 & -\frac{3}{4}xy & \frac{1}{2}y \\ -\frac{3}{4}xy & 1 + \frac{9}{4}x^2 & -\frac{3}{2}x \\ \frac{1}{2}y & -\frac{3}{2}x & 1 \end{pmatrix}.$$

Choosing $\mathcal{H} = \mathcal{D}_1$ and setting $g_{\mathcal{H}} = \tilde{g}|_{\mathcal{H}}$, then $(\mathbb{R}^3, \mathcal{H}, g_{\mathcal{H}}, \tilde{g})$ defines a sub-Riemannian manifold. Now, we prove that $\nabla^{\mathcal{B}}$ is compatible to $g_{\mathcal{H}}$.

Take any vector fields $\mathcal{U} = \mathcal{U}^a\mathcal{X}_a \in \Gamma(\mathcal{H})$, $\mathcal{W} = \mathcal{W}^b\mathcal{X}_b \in \Gamma(\mathcal{H})$ for $a, b = 1, 2$, $\Upsilon = \Upsilon^3\mathcal{X}_3 \in \Gamma(\mathcal{V})$ and $\Lambda = \Lambda^3\mathcal{X}_3 \in \Gamma(\mathcal{V})$. By (2.1), we easily get $(\nabla^{\mathcal{B}}_{\Upsilon} g_{\mathcal{H}})(\mathcal{U}, \mathcal{W}) = 0$ and $(\nabla^{\mathcal{B}}_{\mathcal{U}} g_{\mathcal{V}})(\Upsilon, \Lambda) = 0$. Combining this two equalities with (2.2), we conclude that $\nabla^{\mathcal{B}}$ is compatible to $g_{\mathcal{H}}$ and (2.4) holds. Furthermore, we know $\nabla^{\mathcal{B}}$ is compatible to $\tilde{g}$.

Example 5.2. Let $\mathcal{X}_1 = \partial x$, $\mathcal{X}_2 = x\partial y + \partial z + \frac{1}{2}x^2\partial w$, $\mathcal{X}_3 = \partial y + x\partial w$ and $\mathcal{X}_4 = \partial w$ be four vector fields on $\mathbb{R}^4$. Consider the distribution given by $\mathcal{D}_2 = span\{\mathcal{X}_1, \mathcal{X}_2\}$. Then we get the matrix $\{\tilde{g}_{ij}\}$ as below

$$\tilde{g}_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+x^2 & -x-\frac{1}{2}x^3 & -x \\ 0 & -x-\frac{1}{2}x^3 & 1+x^2+\frac{1}{4}x^4 & \frac{1}{2}x^2 \\ 0 & -x & \frac{1}{2}x^2 & 1 \end{pmatrix}.$$

Choosing $\mathcal{H} = \mathcal{D}_2$ and setting $g_{\mathcal{H}} = \tilde{g}|_{\mathcal{H}}$. Then we have a sub-Riemannian manifold $(\mathbb{R}^4, \mathcal{H}, g_{\mathcal{H}}, \tilde{g})$. Now, we prove that $\nabla^{\mathcal{B}}$ is compatible to $g_{\mathcal{H}}$.

Take any vector fields $\mathcal{U} = \mathcal{U}^a\mathcal{X}_a \in \Gamma(\mathcal{H})$, $\mathcal{W} = \mathcal{W}^b\mathcal{X}_b \in \Gamma(\mathcal{H})$ for $a, b = 1, 2$, $\Upsilon = \Upsilon^\alpha\mathcal{X}_\alpha \in \Gamma(\mathcal{V})$ and $\Lambda = \Lambda^\beta\mathcal{X}_\beta \in \Gamma(\mathcal{V})$ for $\alpha, \beta = 3, 4$. From (2.1), we get

$$\begin{aligned}(\nabla^{\mathcal{B}}_{\Upsilon}g_{\mathcal{H}})(\mathcal{U}, \mathcal{W}) = {} & \Upsilon^\alpha\mathcal{X}_\alpha\langle\mathcal{U}^a\mathcal{X}_a, \mathcal{W}^b\mathcal{X}_b\rangle_{g_{\mathcal{H}}} - \langle\Pi_{\mathcal{H}}([\Upsilon^\alpha\mathcal{X}_\alpha, \mathcal{U}^a\mathcal{X}_a]), \mathcal{W}^b\mathcal{X}_b\rangle_{g_{\mathcal{H}}} \\ & - \langle\mathcal{U}^a\mathcal{X}_a, \Pi_{\mathcal{H}}([\Upsilon^\alpha\mathcal{X}_\alpha, \mathcal{W}^b\mathcal{X}_b])\rangle_{g_{\mathcal{H}}} = 0,\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}(\nabla^{\mathcal{B}}_{\mathcal{U}}g_{\mathcal{V}})(\Upsilon, \Lambda) = {} & \mathcal{U}^a\mathcal{X}_a(\Upsilon^\alpha\Lambda^\alpha) - \langle\mathcal{U}^a\mathcal{X}_a(\Upsilon^\alpha)\mathcal{X}_\alpha, \Lambda^\alpha\mathcal{X}_\alpha\rangle_{g_{\mathcal{V}}} - \langle\mathcal{U}^a\Upsilon^\alpha[\mathcal{X}_a, \mathcal{X}_\alpha], \Lambda^\beta\mathcal{X}_\beta\rangle_{g_{\mathcal{V}}} \\ & - \langle\Upsilon^\alpha\mathcal{X}_\alpha, \mathcal{U}^a\mathcal{X}_a(\Lambda^\alpha)\mathcal{X}_\alpha\rangle_{g_{\mathcal{V}}} - \langle\Upsilon^\alpha\mathcal{X}_\alpha, \mathcal{U}^a\Lambda^\beta[\mathcal{X}_a, \mathcal{X}_\beta]\rangle_{g_{\mathcal{V}}} \neq 0.\end{aligned} \tag{5.2}$$

Combining (5.1), (5.2) with (2.2), we conclude that $\nabla^{\mathcal{B}}$ is compatible to $g_{\mathcal{H}}$ and (2.4) holds. Furthermore, we know $\nabla^{\mathcal{B}}$ is not compatible to $\tilde{g}$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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