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Intuitionistic fuzzy projective modules and intuitionistic fuzzy homomorphisms

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Abstract. In this paper, we discuss the structure of intuitionistic fuzzy (IF) homomorphisms, exact sequences and some other concepts in category of IF modules. We study on IF exact sequences and IF Hom functors in $IFR - Mod$ and obtain some results about them. If R is a commutative ring and $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C$ is an exact sequence in $IFR - Mod$, where $\tilde{f}$ is IF split homomorphism, then we show that $Hom_{IF-R}(D, -)$ preserves the sequence for every $D \in \mathrm{IFR - Mod}$. Also IF projective modules will be introduced and investigated in this paper. Finally we define product and coproduct of IF modules and show that if M is an R-module, $A = (\mu_A,\ \nu_A) \leq_{IF} M$ and $e_i \in E(R)$ *for any* $i \in I$, then

$$Hom(\coprod_{i\in I} 0^{IF}_{Re_i},\ A) \cong \prod_{i\in I} Hom(0^{IF}_{Re_i},\ A).$$

§1 Introduction

After the definition of fuzzy sets by Zadeh [28], a number of applications of this fundamental concept have come up. Rosenfeld [23] was the first one who defined the concept of fuzzy subgroups of a group. Negoita and Ralescu [18] applied this concept to modules and defined fuzzy submodules of a module.

A category C is given by a collection C_0 of *objects* and a collection C_1 of *morphisms* which have the following structure:

(i) Each arrow has a *domain* and a *codomain* which are objects; one writes $f : X \longrightarrow Y$ if X is the domain of the morphism f, and Y its codomain.

(ii) Given two morphisms f and g such that $cod(f) = dom(g)$, the composition of f and g, written gf, is defined and has domain $dom(f)$ and codomain $cod(g)$: $X \xrightarrow{f} Y \xrightarrow{g} Z$

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such that $X \xrightarrow{gf} Z$.

(iii) Composition is *associative*, that is given $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow W$, $h(gf) = (hg)f$.

(iv) For every object X there is an identity morphism $id_X : X \longrightarrow X$, satisfying $id_X g = g$ for every $g : Y \longrightarrow X$ and $f id_X = f$ for every $f : X \longrightarrow Y$.

For example the category of (fuzzy) $R-$modules has objects $R-$ modules and morphisms (fuzzy) $R-$homomorphisms. We denote the categories of R-modules, fuzzy R-modules and intuitionistic fuzzy R-modules by $R-$Mod, $FR-$Mod and $IFR-$Mod, respectively.

As a generalization of fuzzy sets, the concept of fuzzy intuitionistic sets was introduced by K T Atanassov in [2]. Using this idea, B Davvaz [10] established the intuitionistic fuzzification of the concept of submodules of a module.

[3–5, 9, 22] are some other researches about intuitionistic fuzzy groups and modules and [6, 7, 11, 15, 17, 24–27] are some recent researches and applications of intuitionistic fuzzy sets, graphs and modules.

Pan [19, 21] made the $Hom(\mu_A, \nu_B)$ into a fuzzy module and investigated the properties of the functors $Hom(\mu_A, -) : FR - Mod \longrightarrow FR - Mod$ and $Hom(-, \nu_B) : FR - Mod \longrightarrow FR - Mod$. Isaac [12] gave an alternate definition for projective L-modules and investigated these fuzzy modules. Chen [7] studied the relation between projective S-acts and *Hom* functors in the category of S-acts and Liu [14] studied the *Hom* functors and tensor product functors in the category of fuzzy S-acts. In this paper, we study the properties of *Hom* functor in intuitionistic fuzzy modules category. We obtain some properties about IF modules category and *Hom* functors in this category. IF projective modules and their relationship with exact sequences in this category will be introduced and investigated.

Definition 1.1. [28] By a *fuzzy set (or fuzzy subset)* of a module M, we mean the map μ from M to $[0, 1]$. By $[0, 1]^M$ we will denote the set of all fuzzy subsets of M.

For each fuzzy subset μ of M and any $\alpha \in [0, 1]$, we define two sets $U(\mu, \alpha) = \{x \in M | \mu(x) \geq \alpha\}$, $L(\mu, \alpha) = \{x \in M | \mu(x) \leq \alpha\}$, which are called an *upper level cut* and a *lower level cut* of μ, respectively. The complement of μ, denoted by μ^c, is the fuzzy set on M defined by $\mu^c(x) = 1 - \mu(x)$.

Definition 1.2. [20] If $N \subseteq M$ and $\alpha \in [0, 1]$ then α_N is defined as,

$$\alpha_N(x) = \begin{cases} \alpha, & x \in N, \\ 0, & \text{otherwise}. \end{cases}$$

If $N = \{x\}$ then α_x is often called a *fuzzy point* and is denoted by x_α. When $\alpha = 1$ then 1_N is known as the characteristic function of N. We will denote the characteristic function of N as χ_N.

Now let $X \subseteq M$ and $\mu,\ \sigma \in [0, 1]^X$, then

(1) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$, for every $x \in X$;

(2) $(\mu \cup \sigma)(x) = max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$, for every $x \in X$;

(3) $(\mu \cap \sigma)(x) = min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$, for every $x \in X$;

(4) $\mu \times \sigma(x, y) = min\{\mu(x), \sigma(y)\} = \mu(x) \wedge \sigma(y)$, for every $x, y \in X$;

For any family $\{\mu_i | i \in I\}$ of fuzzy subsets of M, where I is any nonempty index set,

(5) $(\bigcup_{i\in I} \mu_i)(x) = sup_{i\in I}\mu_i(x) = \bigvee_{i\in I} \mu_i(x)$, for every $x \in M$;

(6) $(\bigcap_{i\in I} \mu_i)(x) = inf_{i\in I}\mu_i(x) = \bigwedge_{i\in I} \mu_i(x)$, for every $x \in M$.

Definition 1.3. [18] Let M be a left R-module. The $\mu \in [0, 1]^M$ is called a *fuzzy (left) R-module* (fuzzy R-submodule of M) if

1)$\mu(x + y) \geq \mu(x) \wedge \mu(y)$, for every $x, y \in M$;

2) $\mu(rx) \geq \mu(x)$ for every $x \in M$, $r \in R$;

3) $\mu(0) = 1$.

Similarly, we can define fuzzy right R-modules.

Definition 1.4. [18] Let A, B be two R-modules. For two fuzzy R-modules μ_A of A and ν_B of B, a function $\tilde{f} : \mu_A \longrightarrow \nu_B$ is called a *fuzzy R-homomorphism*, if $f : A \longrightarrow B$ is an R-homomorphism and satisfies $\nu(f(a)) \geq \mu(a)$ for every $a \in A$.

For simplicity, denote by $Hom(\mu_A, \nu_B)$ the set of all fuzzy R-homomorphisms from μ_A to ν_B.

Definition 1.5. [16] If μ and σ are two fuzzy submodules of a module M, then define $(\mu + \sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) \mid y, z \in M, y + z = x\}$, for every $x \in M$. It is not difficult to see that if μ and σ are two fuzzy submodules of M, then $\mu + \sigma$ is a fuzzy submodule of M.

§2 Intuitionistic fuzzy modules, basic notions and properties

Throughout this article R means an associative ring with unity and M denotes a unitary left R-module while R-Mod denotes the category of all left R-modules.

Definition 2.1. [2] An *intuitionistic fuzzy set* (briefly as IFS) A of a non-void set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \longrightarrow [0, 1]$ and $\nu_A : X \longrightarrow [0, 1]$, are fuzzy subsets of X, denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$.
For the sake of simplicity, we denote an IFS, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set X by $A = (\mu_A, \nu_A)$ or briefly A, and the set of all IFS of X by $IFS(X)$.

If X is a nonempty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ are two IFS of X, then
$A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for every $x \in X$;
$A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for every $x \in X$;

$A^c = (\nu_A, \mu_A)$;
$A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)); x \in X\}$;
$A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)); x \in X\}$.

Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i\in I}$ be a family of IFS of X. Then
$\bigcap_{i\in I} A_i = (\mu_{(\cap_{i\in I} A_i)}, \nu_{(\cap_{i\in I} A_i)}) = \{(x, \bigwedge_{i\in I} \mu_{A_i}(x), \bigvee_{i\in I} \nu_{A_i}(x)); x \in X\}$ and
$\bigcup_{i\in I} A_i = (\mu_{(\cup_{i\in I} A_i)}, \nu_{(\cup_{i\in I} A_i)}) = \{(x, \bigvee_{i\in I} \mu_{A_i}(x), \bigwedge_{i\in I} \nu_{A_i}(x)); x \in X\}$.

Definition 2.2. [5] Let G be a group. An IFS $A = (\mu_A, \nu_A)$ of G is called an *intuitionistic fuzzy subgroup of* G if the following conditions hold for every $x, y \in G$
(1) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$;
(2) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$;
(3) $\mu_A(x^{-1}) \geq \mu_A(x)$ (consequently $\mu_A(x^{-1}) = \mu_A(x)$);
(4) $\nu_A(x^{-1}) \geq \nu_A(x)$ (consequently $\nu_A(x^{-1}) = \nu_A(x)$).

Definition 2.3. [13] Let M be an $R-$module and $A = (\mu_A, \nu_A)$ an IFS of M. Then A is called an *intuitionistic fuzzy submodule of* M if A satisfies the following conditions:

(1) $\mu_A(0) = 1$, $\nu_A(0) = 0$,

(2) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, for every $x, y \in M$,
$\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$, for every $x, y \in M$.

(3) $\mu_A(rx) \geq \mu_A(x)$, for every $x \in M$ and $r \in R$,
$\nu_A(rx) \leq \nu_A(x)$, for every $x \in M$ and $r \in R$.

If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy submodule of an R-module M, we write A is an IFM of M and denote by $A \leq_{IF} M$. In this case we say A is an intuitionistic fuzzy module too.
We use by $IFS(M)$, the set of all IFM of M and $IFR-Mod$, the category of all IF R-modules.

Definition 2.4. [13] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFM's of M. Then the IFM, $A + B$ of M is $A + B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x); x \in M\}$ defined as

$$\mu_{A+B}(x) = \bigvee\{\mu_A(y) \wedge \mu_B(z) \mid x = y + z;\ y, z \in M\},$$
$$\nu_{A+B}(x) = \bigwedge\{\nu_A(y) \vee \nu_B(z) \mid x = y + z;\ y, z \in M\}.$$

For an IFM, $A = (\mu_A, \nu_A)$ of M and for any $r \in R$, define the IFS, $rA = (\mu_{rA}, \nu_{rA})$ such that for every $x \in M$

$$\mu_{rA}(x) = \bigvee\{\mu_A(y) \mid x = ry \ ; y \in M\},$$

and

$$\nu_{rA}(x) = \bigwedge\{\nu_A(y) \mid x = ry \ ; y \in M\}.$$

So the IFM, $-A = (\mu_{-A}, \nu_{-A})$ will be defined as $\mu_{-A}(x) = \mu_A(-x)$ and $\nu_{-A}(x) = \nu_A(-x)$ for every $x \in M$.

Proposition 2.5. *Let A, B be two IFM's of an R-module M. Then $A+B$ and rA for every $r \in R$, are IFM's of M.*

Proof. It is straightforward and follows from definitions. □

Definition 2.6. Let M be an R-module, $N \subseteq M$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. We define the IFS $\alpha_N = (\mu_{\alpha_N}, \nu_{\alpha_N})$ of M as follows

$$\mu_{\alpha_N}(x) = \begin{cases} \alpha, & x \in N, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \nu_{\alpha_N}(\mathrm{x}) = \begin{cases} \beta, & x \in N, \\ 1, & \text{otherwise}, \end{cases}$$

for every $x \in M$.

If $\alpha = 1$, then $\mu_{\alpha_N} = \chi_N$ and $\nu_{\alpha_N} = \chi_N^c$, where χ_N denotes the characteristic function of N. In this case we write $\alpha_N = \chi_N^{IF} = (\chi_N, \chi_N^c)$.

We denote χ_0^{IF} by $\bar{0}$ or 0_N^{IF} and χ_N^{IF} by 1_N^{IF}, too. If $A \leq_{IF} M$, then $\chi_0^{IF} \leq_{IF} A \leq_{IF} \chi_M^{IF}$.

Proposition 2.7. *Let M be an R-module and $N \subseteq M$. Then $N \leq M$ if and only if $\chi_N^{IF} \leq_{IF} M$.*

Proof. Suppose that N is a submodule of M. Then $0 \in N$ and hence
$\chi_N(0) = 1$ and $\chi_N^c(0) = 0$.
Now let $x,\ y \in M$. If $x, y \in N$, then $x + y \in N$, so $1 = \chi_N(x+y) \geq \chi_N(x) \wedge \chi_N(y)$ and $0 = \chi_N^c(x+y) \leq \chi_N^c(x) \vee \chi_N^v(y)$.
If $x \notin N$, then

$$\chi_N(x+y) \geq \chi_N(x) \wedge \chi_N(y) = 0 \quad \textit{and} \quad \chi_N^c(x+y) \leq \chi_N^c(x) \vee \chi_N^c(y) = 1.$$

Similar to this case we get if $y \notin N$.
Now let $x \in M$ and $r \in R$. If $x \in N$, then $rx \in N$ and so we have
$1 = \chi_N(rx) \geq \chi_N(x)$ and $0 = \chi_N^c(x) \leq \chi_N^c(rx)$.
If $x \notin N$, then $0 = \chi_N(x) \leq \chi_N(rx)$ and also $1 = \chi_N^c(x) \geq \chi_N^c(rx)$.
Therefore χ_N^{IF} is an IFM of M.
Conversely suppose that χ_N^{IF} is an IFM of M. So $\chi_N(0) = 1$ and hence $0 \in N$. Now let $x, y \in N$ and $r \in R$, then $\chi_N(rx+y) \geq \chi_N(rx) \wedge \chi_N(y) \geq \chi_N(x) \wedge \chi_N(y) = 1$. So $rx + y \in N$. That is N is a submodule of M. □

Example 2.8.

(1) Since $n\mathbb{Z} \leq \mathbb{Z}$ so $\chi_{n\mathbb{Z}}^{IF} = (\chi_{n\mathbb{Z}},\ \chi_{n\mathbb{Z}}^c)$ is IFM of $\mathbb{Z}$ for every $n \in \mathbb{Z}$.

(2) $\mathbb{Z} \leq \mathbb{Q}$ and hence $\chi_{\mathbb{Z}}^{IF}$ is an IF submodule of $\mathbb{Q}$.

(3) $\mathbb{Z}_p^\infty \leq \frac{\mathbb{Q}}{\mathbb{Z}}$ and hence $\chi_{\mathbb{Z}_p^\infty}^{IF}$ is an IF submodule of $\frac{\mathbb{Q}}{\mathbb{Z}}$.

Definition 2.9. Let $M,\ N$ be two R-modules and $f : M \longrightarrow N$ an R-homomorphism. Let $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$. Then we define $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ and

$f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ by

$$(\mu_{f(A)})(y) = \begin{cases} \bigvee\{\mu_A(x) \mid \ y = f(x)\}, & y \in Im(f), \\ 0, & y \notin Im(f), \end{cases}$$

and

$$(\nu_{f(A)})(y) = \begin{cases} \bigwedge\{\nu_A(x) \mid \ y = f(x)\}, & y \in Im(f), \\ 1, & y \notin Im(f), \end{cases}$$

and for every $x \in M$

$$(\mu_{f^{-1}(B)})(x) = \mu_B(f(x)) \qquad \text{and} \qquad (\nu_{f^{-1}(B)})(x) = \nu_B(f(x)).$$

Proposition 2.10. *If $A \leq_{IF} M$, $B \leq_{IF} N$ and $f : M \longrightarrow N$ be an R-homomorphism. Then $f(A) \leq_{IF} N$ and $f^{-1}(B) \leq_{IF} M$.*

Proof. It is clear. □

§3 Intuitionistic fuzzy homomorphisms

In this section we introduce intuitionistic fuzzy R-homomorphisms and investigate various properties of them.

Definition 3.1. Let R be a ring and $M,\ N$ be R-modules such that $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$. We call the function $\tilde{f} : A \to B$ an *intuitionistic fuzzy R-homomorphism*, if $f : M \longrightarrow N$ is an R-homomorphism and $\mu_B(f(x)) \geq \mu_A(x)$ and $\nu_B(f(x)) \leq \nu_A(x)$ for every $x \in M$.

For simplicity, we denote by $Hom_{IF-R}(A, B)$ the set of all intuitionistic fuzzy R-homomorphisms from A to B .

Example 3.2.

(1) The identity map $id : M \longrightarrow M$ is an IF homomorphism ($\tilde{f} : A \longrightarrow A$) for every IF submodule A of M.

(2) Let A, B be two IF submodules of $\mathbb{Z}$ defined by

$$\mu_A(x) = \begin{cases} 1, & x \in 2\mathbb{Z} \\ \frac{1}{2}, & x \notin 2\mathbb{Z} \end{cases},$$

$$\nu_A(x) = \begin{cases} 0, & x \in 2\mathbb{Z} \\ \frac{1}{3}, & x \notin 2\mathbb{Z} \end{cases},$$

and

$$\mu_B(x) = \begin{cases} 1, & x \in 3\mathbb{Z} \\ \frac{1}{5}, & x \notin 3\mathbb{Z} \end{cases},$$

$$\nu_B(x) = \begin{cases} 0, & x \in 3\mathbb{Z} \\ \frac{1}{5}, & x \notin 3\mathbb{Z} \end{cases}.$$

Define $\tilde{f} : A \longrightarrow B$ such that ($f : \mathbb{Z} \longrightarrow \mathbb{Z}$) $f(x) = 2x$ for every $x \in \mathbb{Z}$. Then f is an R-homomorphism but $\tilde{f}$ is not an IF homomorphism (See that $\frac{1}{5} = \mu_B(f(2)) < \mu_A(2) = 1$).

Definition 3.3. We call an intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$, *fuzzy split*, if there is an intuitionistic fuzzy R-homomorphism $\bar{t} \in Hom_{IF-R}(B, A)$ such that the composition $\tilde{t}\tilde{f} = id$.

Definition 3.4. We call an intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$, *intuitionistic fuzzy quasi-isomorphism*, if f is an isomorphism.

If $\tilde{f} : A \to B$ is an IF R-homomorphism, define $\ker \tilde{f} = \left\{ a \in A \mid \begin{array}{l} \mu_B(\tilde{f}(a)) = 1, \\ \nu_B(\tilde{f}(a)) = 0, \end{array} \right\}$

and $Im\tilde{f} = \{\tilde{f}(a) | a \in A\}$.

Definition 3.5. We call an intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$, *intuitionistic fuzzy isomorphism*, if f is an isomorphism and $\mu_B(\bar{f}(a)) = \mu_A(a)$, $\nu_B(\bar{f}(a)) = \nu_A(a)$ for every $a \in M$.

Note that $\ker \tilde{f} = \ker f$ is not true in general, but $\ker f \subseteq \ker \tilde{f}$.

If $\ker \tilde{f} = \{0\}$ then $\tilde{f}$ is monomorphism as

$$f(x) = f(y) \Rightarrow f(x-y) = 0 \Rightarrow \left\{ \begin{array}{l} \mu_N(\tilde{f}(x-y)) = 1; \\ \nu_N(\tilde{f}(x-y)) = 0. \end{array} \right. \Rightarrow x - y \in \ker \tilde{f} = \{0\} \Rightarrow x = y.$$

But the reverse is not true, it means if $\tilde{f}$ is a monomorphism then it need not that $\ker \tilde{f} = \{0\}$.

Example 3.6. If $B = 1_M^{IF}$, then $\ker \tilde{f} = M$, for every $A \in IFR{-}Mod$ and $\tilde{f} \in Hom_{IF-R}(A, B)$. Especially let $M = N = \mathbb{Z}$, $A = B = 1_M^{IF}$ and $\tilde{f} : A \to B$ be the identity map. Then $\ker f = \{0\}$ but $\ker \tilde{f} = \mathbb{Z}$.

Proposition 3.7. *Let R be a ring. If $\tilde{f} \in Hom_{IF-R}(A, B)$, where A and B are two IF R-modules, such that $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$, then*
(1) $\ker \tilde{f}$ is a submodule of M.
(2) Define $\mu' \mid_{\ker \tilde{f}}: \ker \tilde{f} \to [0, 1]$, $\nu' \mid_{\ker \tilde{f}}: \ker \tilde{f} \to [0, 1]$ by $\left\{ \begin{array}{l} \mu'(k) = \mu(k); \\ \nu'(k) = \nu(k). \end{array} \right.$ for every $k \in \ker \tilde{f}$.

Then $A' = (\mu'|_{\ker \tilde{f}}\ ,\ \nu'|_{\ker \tilde{f}})$ is an IF submodule of A.

Proof. (1) Let 0 be the zero element of M. Obviously, we have $0 \in \ker \tilde{f}$. Given $a \in \ker \tilde{f}$ and $r \in R$, then

$$\mu_B(\tilde{f}(ra)) = \mu_B(r\tilde{f}(a)) \geq \mu_B(\tilde{f}(a)) = 1,$$
$$\nu_B(\tilde{f}(ra)) = \nu_B(r\tilde{f}(a)) \leq \nu_B(\tilde{f}(a)) = 0.$$

So we get $ra \in \ker \tilde{f}$. Particularly we have $-a \in \ker \tilde{f}$. If $a, b \in \ker \tilde{f}$, we can easily get $a + b \in \ker \tilde{f}$. This proves that $\ker \tilde{f}$ is a submodule of A.
(2) The proof is obvious. □

§4 IF exact sequences and IF Hom functors in IFR-Mod

In the following, we will define a function from $Hom_{IF-R}(A, B)$ to $[0, 1]$ and make $Hom_{IF-R}(A, B)$ into an intuitionistic fuzzy R-module.

In R-Mod, the sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0$ is called a *short exact sequence* if f is a monomorphism, g is an epimorphism and $Imf = \ker g$. We know that f is monomorphism iff $\ker f = \{0\}$. We denote by $\bar{0}$ the IF zero module.

Definition 4.1. Let A, B and C be IF R-modules of M, N and K respectively. A short exact sequence is a sequence of the form $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$, where $\tilde{f}$ is a monomorphism, $\tilde{g}$ is an epimorphism and $Im\tilde{f} = \ker \tilde{g}$.
Note that $\ker \tilde{f}$ is usually larger than $\{0\}$ by Definition 3.5. Hence, the crisp case of the definition is different from the well-known notion of short exact sequence in the category IFR-mod.

If $C = 1_K$, we get $Im\tilde{f} = \ker \tilde{g} = N$. As $\tilde{f}$ is monic, so we conclude that $\tilde{f}$ is quasi-isomorphism.

Definition 4.2. An IF R-module $P = (\mu_P, \nu_P)$ is called *projective* if for every surjective IF R-homomorphism $\tilde{f} : A \longrightarrow B$ and for every IF R-homomorphism $\tilde{g} : P \longrightarrow B$, there exists an IFR-homomorphism $\tilde{h} : P \longrightarrow A$ such that $\tilde{f}\tilde{h} = \tilde{g}$.

Remark 4.3. *If P is a projective IF R-module, then $P = \bar{0}$.*

Proof. Let $P = (\mu_P, \nu_P) \leq_{IF} M$ be a projective IF R-module and
$A = \bar{0} \leq_{IF} M$, $B = P \leq_{IF} M$. Let $\tilde{f} = \tilde{g} = id_M$. Then there exists $\tilde{h} = id : P \longrightarrow A$, since P is projective. We must have $\mu_A(h(x)) \geq \mu_P(x)$ and $\nu_A(h(x)) \leq \nu_P(x)$ for every $x \in P$. This implies $P \subseteq A = \bar{0}$. □

Theorem 4.4. *Let R be a commutative ring and $A = (\mu_A, \nu_A) \leq_{IF} M$,
$B = (\mu_B, \nu_B) \leq_{IF} N$ be two intuitionistic fuzzy R- modules. Then
$Hom_{IF-R}(A, B) = (\alpha, \beta)$ is an IF R-module with membership function $\alpha : Hom_{IF-R}(A, B) \longrightarrow [0, 1]$ and non-membership function $\beta : Hom_{IF-R}(A, B) \longrightarrow [0, 1]$ defined by*

$$\alpha(\tilde{f}) = \bigwedge\{\mu_B(\tilde{f}(x)) \mid x \in M\} \text{ and } \beta(\tilde{f}) = \bigvee\{\nu_B(\tilde{f}(x)) \mid x \in M\}.$$

Proof. Assume $r \in R$ and $\tilde{f} \in Hom_{IF-R}(A, B)$. Define a function
$r.\tilde{f} : A \longrightarrow B$ by $r.\tilde{f}(x) = r\tilde{f}(x)$ for every $x \in M$.
Then we have

$$\mu_B(r.\tilde{f}(x)) = \mu_B(r\tilde{f}(x)) \geq \mu_B(\tilde{f}(x)) \geq \mu_A(x)$$
$$\nu_B(r.\tilde{f}(x)) = \nu_B(r\tilde{f}(x)) \leq \nu_B(\tilde{f}(x)) \leq \nu_A(x)$$

This concludes that $r.\tilde{f} \in Hom_{IF-R}(A, B)$. Hence we show that $Hom_{IF-R}(A, B)$ is an R-module.
We now have to prove that $Hom_{IF-R}(A, B)$ is an IF R-module. Suppose that
$r \in R, \tilde{f} \in Hom_{IF-R}(A, B)$ and $x \in M$. By

$$\mu_B(r.\tilde{f}(x)) = \mu_B(r\tilde{f}(x)) \geq \mu_B(\tilde{f}(x)),$$

$$\nu_B(r.\tilde{f}(x)) = \nu_B(r\tilde{f}(x)) \leq \nu_B(\tilde{f}(x)),$$

we have, $\bigwedge\{\mu_B(r.\tilde{f}(x)) \mid x \in M\} \geq \bigwedge\{\mu_B(\tilde{f}(x)) \mid x \in M\}$ and $\bigvee\{\nu_B(r.\tilde{f}(x)) \mid x \in M\} \leq \bigvee\{\nu_B(\tilde{f}(x)) \mid x \in M\}$. This implies that $\alpha(r.\tilde{f}) \geq \alpha(\tilde{f})$ and $\beta(r.\tilde{f}) \leq \beta(\tilde{f})$.

If $\tilde{f},\ \tilde{g} \in Hom_{IF-R}(A, B)$, then

$$\alpha(\tilde{f}+\tilde{g}) = \bigwedge\{\mu_B(\tilde{f}(x)+\tilde{g}(x)) \mid x \in M\} \geq \bigwedge\{\bigwedge(\mu_B(\tilde{f}(x)), \mu_B(\tilde{g}(x))) \mid x \in M\}$$
$$\geq \bigwedge\{\{\bigwedge(\mu_B(\tilde{f}(x)) \mid x \in M\}, \{\bigwedge(\mu_B(\tilde{g}(x)) \mid x \in M\}\} = \bigwedge\{\alpha(\tilde{f}), \alpha(\tilde{g})\},$$
$$\beta(\tilde{f}+\tilde{g}) = \bigvee\{\nu_B(\tilde{f}(x)+\tilde{g}(x)) \mid x \in M\} \leq \bigvee\{\bigvee(\nu_B(\tilde{f}(x)), \nu_B(\tilde{g}(x))) \mid x \in M\}$$
$$\leq \bigvee\{\{\bigvee(\nu_B(\tilde{f}(x)) \mid x \in M\}, \{\bigvee(\nu_B(\tilde{g}(x)) \mid x \in M\}\} = \bigvee\{\beta(\tilde{f}), \beta(\tilde{g})\}.$$

We obviously have that $\begin{cases} \alpha(0) = 1; \\ \beta(0) = 0. \end{cases}$. Therefore $Hom_{IF-R}(A, B)$ is an IF R-module. □

Let M be an R-module and A IF submodules of M. Then we define the functor Hom_{IF-R}(A,-) from IFR-Mod to IFR-Mod such that for every two IF submodules B and C of M and IF homomorphism $\tilde{f} : B \longrightarrow C$, $Hom_{IF-R}(A, -)\left(B\right) = Hom_{IF-R}(A, B)$ and

$$Hom_{IF-R}(A, -)\left(\tilde{f}\right) = Hom_{IF-R}(A, \tilde{f}) : Hom_{IF-R}(A, B) \longrightarrow Hom_{IF-R}(A, C)$$

such that $Hom_{IF-R}(A, \tilde{f})\left(f\right) = \tilde{f}of$. Similarly the functor Hom_{IF-R}(-,A) can be defined. We call these two functors, the Hom functors from $IFR-$Mod to $IFR-$Mod.

Example 4.5. Let $\mathbb{Z}$ be the set of integers and $M = 4\mathbb{Z}$. Define $B = (\mu_B, \nu_B) \leq_{IF} \mathbb{Z}$ such that $\mu : \mathbb{Z} \longrightarrow [0, 1]$, $\nu : \mathbb{Z} \longrightarrow [0, 1]$ by

$$\mu_B(n) = \begin{cases} \frac{1}{10}, & if\ n \neq 0; \\ 1, & if\ x = 0. \end{cases}$$
$$\nu_B(n) = \begin{cases} \frac{4}{10}, & if\ n \neq 0; \\ 0, & if\ x = 0. \end{cases}$$

$C = (\mu_C, \nu_C) \leq_{IF} \mathbb{Z}$

$$\mu_C(\bar{k}) = \begin{cases} \frac{1}{2}, & if\ \bar{k} \neq \bar{0}; \\ 1, & if\ \bar{k} = \bar{0}. \end{cases}$$
$$\nu_C(\bar{k}) = \begin{cases} \frac{1}{3}, & if\ \bar{k} \neq \tilde{0}; \\ 1, & if\ \bar{k} = \bar{0}. \end{cases}$$

Then both B and C are IF $\mathbb{Z} - modules.$ Also, we have a short exact sequence

$$\bar{0} \longrightarrow 0_M \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$$

where $\tilde{f}$ is the inclusion homomorphism and $\tilde{g}$ is the natural epimorphism. Let $F = Hom_{IF-R}(B, -)$. We claim that the sequence

$$\bar{0} \longrightarrow Hom_{IF-R}(B, 0_M) \xrightarrow{F\tilde{f}} Hom_{IF-R}(B, B) \xrightarrow{F\tilde{g}} Hom_{IF-R}(B, C)$$

is not exact. Define $\tilde{h_1} : \mathbb{Z} \longrightarrow \mathbb{Z}$ by putting $\tilde{h_1}(n) = 6n$ and define $\tilde{h_2} : \mathbb{Z} \longrightarrow \mathbb{Z}$ by putting $\tilde{h_2}(n) = 12n$. We can check that both $\tilde{h_1}$ and $\tilde{h_2}$ are in $\ker f\tilde{g}$. Hence $\mid \ker f\tilde{g} \mid \geq 2$. Since

$Hom(B, 0_M)$ contains only zero morphism, we have $ImF\tilde{f} \neq \ker f\tilde{g}$. So $Hom_{IF-R}(B,-)$ is not exact.

Theorem 4.6. *Let R be a commutative ring and let $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C$ be an exact sequence in IFR-Mod, where $\tilde{f}$ is IF split homomorphism. Then Hom_{IF-R}(D,-) preserves the sequence, for every $D \in$ IFR − Mod.*

Proof. Let $F = Hom_{IF-R}(D,-)$ such that $A = (\mu_A, \nu_A) \leq_{IF} M$, $B = (\mu_B, \nu_B) \leq_{IF} N$, $C = (\mu_c, \nu_c) \leq_{IF} K$ and $D = (\mu_D, \nu_D) \leq_{IF} H$. We will show that the sequence

$$\bar{0} \longrightarrow Hom_{IF-R}(D,A) \xrightarrow{F\tilde{f}} Hom_{IF-R}(D,B) \xrightarrow{F\tilde{g}} Hom_{IF-R}(D,C)$$

is exact. Put $F\tilde{f} = f_*$ *and* $F\tilde{g} = g_*$. $Hom_{IF-R}(D,-)$ is left exact, so it is clear that f_* is monic.

Let $Hom_{IF-R}(D,A) = (\alpha_1, \alpha_2)$ that $\alpha_1 : Hom_{IF-R}(D,A) \longrightarrow [0,1]$, $\alpha_2 : Hom_{IF-R}(D,A) \longrightarrow [0,1]$ are membership and non-membership functions, respectively, such that $\alpha_1(\tilde{\psi}) = \bigwedge\{\mu_A(\psi(x)) \mid x \in H\}$, $\alpha_2(\tilde{\psi}) = \bigvee\{\nu_A(\psi(x)) \mid x \in H\}$. Also let $Hom_{IF-R}(D,C) = (\delta_1, \delta_2)$.

Now is $\tilde{\lambda} \in Imf_*$ then $\delta_2(\tilde{g}o\tilde{f}o\tilde{\psi}) = \bigvee\{\nu_C((\tilde{g}o\tilde{f}o\tilde{\psi})(x)) \mid x \in H\} = \bigvee\{\nu_C((\tilde{g}o\tilde{f}(\tilde{\psi}))(x)) \mid x \in H\} = \bigvee\{0\} = 0$. So $ImF\tilde{f} \subseteq \ker F\tilde{g}$.

Now we will show that $ImF\tilde{f} \supseteq \ker F\tilde{g}$.

Assume $\tilde{\lambda} \in \ker g_*$. Hence we have

(1) $1 = \delta_1(g_*(\tilde{\lambda})) = \bigwedge\{\mu_C(go\lambda(x)) \mid x \in H\}$, hence $\mu_C(go\lambda(x)) = 1$, for every $x \in H$.

(2) $0 = \delta_2(g_*(\tilde{\lambda})) = \bigvee\{\nu_C(go\lambda(x)) \mid x \in H\}$, hence $\nu_C(go\lambda(x)) = 0$, for every $x \in H$.

From (1), (2) we conclude that $Im\tilde{\lambda} \in \ker\tilde{g} = Im\tilde{f}$.

Now define $\tilde{\rho} : D \longrightarrow A$ by $\tilde{\rho}(x) = m$ for every $x \in H$, where $\tilde{\lambda}(x) = \tilde{f}(m)$.

It is not difficult to see that $\tilde{f}o\tilde{\rho} = \tilde{\lambda}$(i.e, $\tilde{\lambda} \in Imf_*$), which completes the proof. ☐

Theorem 4.7. *Let R be a commutative ring and $A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ be an exact sequence of IF R-modules, where $\tilde{f}$ is IF split.*

Let $G = Hom_{IF-R}(-,D)$, then the following sequence is exact

$$\bar{0} \longrightarrow Hom_{IF-R}(C,D) \xrightarrow{G\tilde{g}} Hom_{IF-R}(B,D) \xrightarrow{G\tilde{f}} Hom_{IF-R}(A,D).$$

Proof. It is similar to the proof of Theorem 4.6. ☐

Let $E(R)$ be the set of all idempotent members of the ring R.

Now, we study the functor $Hom_{IF-R}(M,-)$, where $M = 0^{IF}_{Re}$ for $e \in E(R)$. First suppose that $A \leq_{IF-R} M$ and $e \in E(R)$. Then the IF R-module $eA \leq_{IF} eM$ is defined by $eA = (\mu_{eA}, \nu_{eA})$ such that $\begin{cases} \mu_{eA}(em) = \mu_A(em) \\ \nu_{eA}(em) = \nu_A(em) \end{cases}$.

Lemma 4.8. *Let R be a commutative ring and $A \in IFR-Mod$. Then $\Gamma_A : Hom(0^{IF}_{Re}, A) \longrightarrow eA$ defined by $\tilde{f} \longmapsto \tilde{f}(e)$, is an IF R-module isomorphism.*

Proof. Assume $em \in eM$ where $A = (\mu_A, \nu_A) \leq_{IF} M$. Define a map $\tilde{f} : 0^{IF}_{Re} \longrightarrow eA$ by putting $\tilde{f}(re) = rem$. We can easily check that $\tilde{f} \in Hom_{IF-R}(0^{IF}_{Re}, A)$ and $\Gamma_A(\tilde{f}) = em$. It follows that Γ_A is a surjective map. If $\tilde{f} \in Hom_{IF-R}(0^{IF}_{Re}, A)$, we can see that $\tilde{f}$ is determined by $\tilde{f}(e)$. This shows that Γ_A is an injective map. Let $\tilde{f} \in Hom_{IF-R}(0^{IF}_{Re}, A)$. We have $\alpha_1(\tilde{f}) = \mu_A(\tilde{f}(e))$, $\alpha_2(\tilde{f}) = \nu_A(\tilde{f}(e))$ where $Hom_{IF-R}(0^{IF}_{Re}, A) = (\alpha_1, \alpha_2)$. This shows Γ_A is an intuitionistic fuzzy isomorphism. □

Proposition 4.9. *Let R be a ring and the following diagram of IF R-modules is commutative:*

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & A & \xrightarrow{\tilde{f}} & B & \xrightarrow{\tilde{g}} & C & \longrightarrow & \bar{0} \\ & & \downarrow\tilde{\alpha} & & \Big\downarrow\tilde{\beta} & & \downarrow\tilde{\gamma} & & \\ \bar{0} & \longrightarrow & D & \xrightarrow{\tilde{h}} & E & \xrightarrow{\tilde{p}} & F & \longrightarrow & \bar{0} \end{array}$$

where $\tilde{\alpha}$, $\tilde{\gamma}$ are IF isomorphisms and $\tilde{\beta}$ is an IF quasi-isomorphism. Then the bottom row is a short exact sequence if and only if so is the top row.

Proof. Let $A \leq_{IF} M$, $B \leq_{IF} N$, $C \leq_{IF} K$, $D \leq_{IF} H$, $E \leq_{IF} X$ and $F \leq_{IF} Y$.

Suppose that the bottom row is exact. First we prove that $\tilde{f}$ is monomorphism. Assume $f(m_1) = f(m_2)$ $(m_1, m_2 \in M)$. We have $\tilde{h}\tilde{\alpha}(m_1) = \tilde{\beta}\tilde{f}(m_1) = \tilde{\beta}\tilde{f}(m_2) = \tilde{h}\tilde{\alpha}(m_2)$. So $m_1 = m_2$, as $\tilde{h}\tilde{\alpha}$ is monomorphism. So $\tilde{f}$ is monic.

Let $k \in K$ and $\tilde{\gamma}(k) = y$. Since $\tilde{p}$ is epimorphism, so there exists $x \in X$ such that $\tilde{p}(x) = y$. Suppose that $x = \tilde{\beta}(n)$ for some $n \in N$. Now $y = \tilde{p}(x) = \tilde{p}\tilde{\beta}(n) = \tilde{\gamma}\tilde{g}(n)$ *i.e.*, $\tilde{\gamma}(x) = \tilde{\gamma}\tilde{g}(x)$ and hence $x = \tilde{g}(x)$ as $\tilde{\gamma}$ is monic. This implies $\tilde{g}$ is an epimorphism.

Now let $m \in M$. We will show that $\tilde{f}(m) \in \ker \tilde{g}$. For this first we have $\tilde{h}\tilde{\alpha}(m) \in Im\tilde{h} = \ker \tilde{p}$. Therefore $\begin{cases} \mu_F(\tilde{p}(\tilde{h}\tilde{\alpha}(m))) = 1; \\ \nu_F(\tilde{p}(\tilde{h}\tilde{\alpha}(m))) = 0; \end{cases}$ and so

$\begin{cases} \mu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 1; \\ \nu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 0 \end{cases}$. Then since $\tilde{\gamma}$ is isomorphism, hence

$\begin{cases} \mu_C(\tilde{g}(\tilde{f}(m))) = \mu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 1; \\ \nu_C(\tilde{g}(\tilde{f}(m))) = \nu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 0; \end{cases}$ and this implies $\tilde{f}(m) \in \ker \tilde{g}$. So $Im\tilde{f} \subseteq \ker \tilde{g}$.

Now suppose that $n \in \ker \tilde{g}$, so $\begin{cases} \mu_C(\tilde{g}(n)) = 1; \\ \nu_C(\tilde{g}(n)) = 0; \end{cases}$ and hence

$\begin{cases} \mu_F(\tilde{\gamma}\tilde{g}(n)) = 1; \\ \nu_F(\tilde{\gamma}\tilde{g}(n)) = 0 \end{cases}$. Now by commutativity we can obtain $\begin{cases} \mu_F(\tilde{p}\tilde{\beta}(n)) = 1; \\ \nu_F(\tilde{p}\tilde{\beta}(n)) = 0 \end{cases}$. This implies $\tilde{\beta}(n) \in \ker \tilde{p} = Im\tilde{h}$ and so there exists $t \in H$ such that $\tilde{h}(t) = \tilde{\beta}(n)$. Also there exists $m \in M$ such that $\tilde{\alpha}(m) = t$.

Now $\tilde{\beta}\tilde{f}(m) = \tilde{h}\tilde{\alpha}(m) = \tilde{h}(t) = \tilde{\beta}(n)$ and hence $n = \tilde{f}(m)$, as $\tilde{\beta}$ is isomorphism. Thus $\ker \tilde{g} \subseteq Im\tilde{f}$.

Similarly the converse can be shown. □

Definition 4.10. Let M, N be two $R-$modules and $A \leq_{IF} M$, $B \leq_{IF} N$. If $\tilde{f} : A \longrightarrow B$ is an IF homomorphism and $e \in E(R)$, we define $\tilde{ef} : eA \longrightarrow eB$ by $\tilde{ef}(em) = \tilde{f}(em) = e\tilde{f}(m)$, for every $m \in M$.

Proposition 4.11. *Let R be a commutative ring and $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ be a short exact sequence of IF R-module. Let $e \in E(R)$, $\tilde{ef} = \tilde{f}|_{eA}$ and $\tilde{eg} = \tilde{g}|_{eB}$. Then the sequence $\bar{0} \longrightarrow eA \xrightarrow{\tilde{ef}} eB \xrightarrow{\tilde{eg}} eC \longrightarrow \bar{0}$ is exact.*

Proof. Suppose that $A \leq_{IF} M$, $B \leq_{IF} N$, $C \leq_{IF} K$. Let $ek \in eK$. Since $\tilde{g}$ is an epimorphism, there exists $n \in N$ satisfying $\tilde{g}(n) = ek$. We have $en \in eN$ and $\tilde{eg}(en) = e\tilde{g}(n) = ek$, since e is an idempotent.

This proves that $\tilde{eg}$ is an epimorphism. Since $Im\tilde{f} \subseteq \ker\tilde{g}$, it is clear that $Im(\tilde{ef}) \subseteq \ker(\tilde{eg})$. Suppose that $en \in \ker(\tilde{eg})$. We have an element $m \in M$ satisfying $\tilde{f}(m) = en$, because $\ker(\tilde{eg}) \subseteq \ker\tilde{g}$. Hence $em \in eM$ and $\tilde{ef}(em) = \tilde{f}(em) = e\tilde{f}(m) = en$, i.e., $\ker(\tilde{eg}) \subseteq Im(\tilde{ef})$, as desired. □

Proposition 4.12. *Let R be a commutative ring and Let $e \in E(R)$. The functor $Hom(0^{IF}_{Re}, -)$ preserves the sequence $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ of IF R-modules.*

Proof. The sequence $\bar{0} \longrightarrow eA \xrightarrow{\tilde{ef}} eB \xrightarrow{\tilde{eg}} eC \longrightarrow \bar{0}$ is a short exact sequence by Proposition 4.11. Consider the following commutative diagram of IF R-modules

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & Hom(0^{IF}_{Re}, A) & \xrightarrow{\tilde{f}} & Hom(0^{IF}_{Re}, B) & \xrightarrow{\tilde{g}} & Hom(0^{IF}_{Re}, C) & \longrightarrow & \bar{0} \\ & & \downarrow\eta_A & & \downarrow\eta_B & & \downarrow\eta_C & & \\ \bar{0} & \longrightarrow & eA & \xrightarrow{\tilde{h}} & eB & \xrightarrow{\tilde{p}} & eC & \longrightarrow & \bar{0} \end{array}.$$

Where η_A, η_B, η_C are IF isomorphisms by Lemma 4.8. Now by Proposition 4.9, the top row is exact. □

§5 Product and coproduct in IFR-Mod

In this section we will introduce the operations product and coproduct in category IFR-Mod and present some equivalence for them. We investigate the functor Hom for these operations in IF case and crisp case and present some isomorphisms between them.

Definition 5.1. Let $\{A_i = (\mu_{A_i}, \ \nu_{A_i}) \mid i \in I\}$ be a family of IF R-modules. Then we define $\coprod_{i\in I} A_i = (\mu, \nu)$ is the *coproduct* of $\{A_i = (\mu_{A_i}, \ \nu_{A_i}) \mid i \in I\}$ such that the maps $(\mu, \nu) : \coprod_{i\in I} A_i \to [0, 1]$ are defined by putting

$$\mu((a_i)_{i\in I}) = \wedge\{\mu_i(a_i) \mid i \in I\} \text{ and } \nu((a_i)_{i\in I}) = \vee\{\nu_i(a_i) \mid i \in I\}.$$

Similarly, the *product* of $\{A_i = (\mu_{A_i}, \ \nu_{A_i}) \mid i \in I\}$ denoted by $\prod_{i\in I} A_i = (\mu, \nu)$, where $(\mu, \nu) : \prod_{i\in I} A_i \to [0, 1]$ are defined by
$\mu((a_i)_{i\in I}) = \wedge \ \{\mu_i(a_i) \mid i \in I\}$, $\nu((a_i)_{i\in I}) = \vee \ \{\nu_i(a_i) \mid i \in I\}$.

If $\{A_i \mid i \in I\}$ is a family of IF R-modules such that $A_i \leq_{IF} M_i$ for every $i \in I$, then it is not difficult to see that $\prod_{i\in I} A_i$ and $\coprod_{i\in I} A_i$ are IF submodules of $\prod_{i\in I} M_i$ and $\coprod_{i\in I} M_i$, respectively.

Proposition 5.2. *Let M be an R-module, $A = (\mu_A,\ \nu_A) \leq_{IF} M$ and $e_i \in E(R)$ for any $i \in I$. Then*

$$Hom(\coprod_{i\in I} 0^{IF}_{Re_i},\ A) \cong \prod_{i\in I} Hom(0^{IF}_{Re_i},\ A).$$

Proof. Let

$$\kappa : Hom(\coprod_{i\in I} Re_i,\ M) \longrightarrow \prod_{i\in I} Hom(Re_i,\ M)$$

given by

$$f \longmapsto (f\lambda_i = f_i)_{i\in I}$$

be the isomorphism, where λ_j is the injection $Re_j \longrightarrow \coprod_{i\in I} Re_i$, for every $j \in I$.

By the proof of Lemma 4.8, for every $i \in I$, when $Hom(0^{IF}_{Re_i}, A) = (\alpha_i, \beta_i)$, $Hom(\coprod_{i\in I} 0^{IF}_{Re_i},\ A) = (\alpha, \beta)$, we have $\alpha_i(f_i) = \mu_A(f_i(e_i))$, $\beta_i(f_i) = \nu_A(f_i(e_i))$. Note that $\mu_A(\sum_{i\in I} f_i(r_ie_i)) = \bigwedge_{i\in I} \mu_A(f_i(r_ie_i)$,
$\nu_A(\sum_{i\in I} f_i(r_ie_i)) = \bigvee_{i\in I} \nu_A(f_i(r_ie_i))$ and $\mu_A(f_i(r_ie_i)) = \mu_A(r_if_i(e_i)) \geq \mu_A(f_i(e_i))$, $\nu_A(f_i(r_ie_i))$ $= \nu_A(r_if_i(e_i)) \leq \nu_A(f_i(e_i))$, where $r_i \in R$ forevery $i \in I$. For every $f \in Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, A)$, we have

$$\begin{aligned}\alpha(f) &= \bigwedge\{\mu_A(f((r_ie_i)_{i\in I})) \mid (r_ie_i)_{i\in I} \in \coprod_{i\in I} Re_i\}\\ &= \bigwedge\{\mu_A(\sum_{i\in I} f_i(r_ie_i)) \mid r_ie_i \in Re_i\}\\ &= \bigwedge\{\mu_A(f_i(e_i))_{i\in I} \mid i \in I\} = \bigwedge\{\alpha_i(f_i) \mid i \in I\}\\ &= (\prod_{i\in I} \alpha_i)((f_i)_{i\in I}) = (\prod_{i\in I} \alpha_i)_{(x(f))}\end{aligned}$$

and

$$\begin{aligned}\beta(f) &= \bigvee\{\mu_B(f((r_ie_i)_{i\in I})) \mid (r_ie_i)_{i\in I} \in \coprod_{i\in I} Re_i\}\\ &= \bigvee\{\mu_B(\sum_{i\in I} f_i(r_ie_i)) \mid r_ie_i \in Re_i\}\\ &= \bigvee\{\mu_B(f_i(e_i))_{i\in I} \mid i \in I\} = \bigvee\{\beta_i(f_i) \mid i \in I\}\\ &= (\prod_{i\in I} \beta_i)((f_i)_{i\in I}) = (\prod_{i\in I} \beta_i)_{(x(f))}.\end{aligned}$$

Similarly, note that $\{f_i(e_i) \mid e_i \in Re_i\} \subseteq \{\sum_{i\in I} f_i(r_ie_i) \mid r_ie_i \in Re_i\}$. So κ is an IF isomorphism, that is

$$Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, A) \cong \prod_{i\in I} Hom(0^{IF}_{Re_i}, A).$$

□

Proposition 5.3. *If $e_iA = (\mu_{e_iA}, \nu_{e_iA}) \leq_{IF} e_iM$, where $e_i \in E(R)$ for $i \in I$, then*

$$Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, A) \cong \prod_{i\in I} e_iA.$$

Proof. By Lemma 4.8 and Proposition 5.2. □

Let R be a ring and I an ideal of R. Then we say *idempotents lift modulo I* if for every idempotent $e+I$ in R/I, there exists an idempotent e' of R such that $e+I=e'+I$.
A ring R is called *semiperfect* if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$, where $J(R)$ is the Jacobson radical of R.

Proposition 5.4. *Let R be a semiperfect ring and P a projective R-module. Then $P \cong \amalg_{i\in I} Re_i$, for some $e_i \in E(R)$.*

Proof. See [1, Theorem 27.11]. □

By Proposition 5.4, for any semiperfect ring R and projective IF R-module P, we have $P \cong \coprod_{i\in I} 0^{IF}_{Re_i}$, for some $e_i \in E(R)$ (Since every projective IF R-module is zero IF R-module).

Proposition 5.5. *Let R be a commutative semiperfect ring. Then $Hom(P,-)$ preserves the short exact sequence $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ of IF R-modules if and only if P is an IF projective R-module.*

Proof. Suppose that Hom(P,-) reserves that exact sequence. Let $B \leq_{IF} M$ and $C \leq_{IF} N$ and $\tilde{g}: B \longrightarrow C$ be an If epimorphism. Let

$$K = \ker \tilde{g} = \left\{ x \in M \mid \begin{array}{c} \mu_c(g(x)) = 1; \\ \nu_c(g(x)) = 0 \end{array} \right\}$$

and $B' = (\mu_B,\ \nu_B)|_K$. Then B' is an IF submodule of B. So we obtain the short exact sequence $\bar{0} \longrightarrow K \xrightarrow{\tilde{i}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ where $\tilde{i}$ is the inclusion map. Since $Hom(P,-)$ preserves the sequence, $Hom(P,-)$ preserves the epimorphism $\tilde{g}$. By this we can conclude that P is an IF projective R-module.

Conversely since P is an IF projective R-module, we have $P \cong \coprod_{i\in I} 0^{IF}_{Re_i}$ where $e_i \in E(R)$. Let $\bar{0} \longrightarrow A \xrightarrow{\tilde{h}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ be a short exact sequence of IF R-modules. Then the sequence

$$\bar{0} \longrightarrow \prod_{i\in I} e_i A \longrightarrow \prod_{i\in I} e_i B \longrightarrow \prod_{i\in I} e_i C \longrightarrow \bar{0}$$

is also a short exact sequences by Proposition 4.11. Using Proposition 5.3 we have the following commutative diagram

$$\begin{array}{ccccccc} \bar{0} & \longrightarrow & Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, A) & \longrightarrow & Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, B) & \longrightarrow & Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ \bar{0} & \longrightarrow & \prod_{i\in I} e_i A & \longrightarrow & \prod_{i\in I} e_i B & \longrightarrow & \prod_{i\in I} e_i C \end{array}$$

$\longrightarrow \bar{0}$
$\longrightarrow \bar{0}$

Now since the bottom row is a short exact sequence, so the top row is also a short exact sequence, by Proposition 4.9. This completes the proof.

□

Example 5.6. Let M be a $\mathbb{Z}_n$-module for some natural number $n > 1$. Let $P = \bar{0}$ be an IF submodule of M. Then $Hom(\bar{0}, -)$ preserves the short exact sequence $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ of IF $\mathbb{Z}_n$-modules, by Proposition 5.5.

§6 Conclusion

The conclusion of this paper is that under the concepts of homomorphisms, exact sequences, Hom functors, product and coproduct in modules theory, we use these concepts on the intuitionistic fuzzy case of modules theory and get some interesting results about them. Reader can use this paper to get some other results for intuitionistic fuzzy modules such as tensor product.

Also we would like to notice several open questions on the topic of this paper as below

(1) Are the functors $Hom(D, -)$ and $Hom(-, D)$ right exact?

(2) Are the Theorems 4.4, 4.6, 4.7, Lemma 4.8, Proposition 4.11 and Proposition 4.12 true, when R is not commutative?

(3) Is Theorem 5.5 true, when R is not semiperfect?

Declarations

Conflict of interest The authors declare no conflict of interest.

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