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Regularity criteria of weak solutions to the 3D axisymmetric Navier-Stokes equations

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Abstract. We investigate a sufficient condition, in terms of the azimuthal component ω^θ of $\omega = \operatorname{curl} u$ in cylindrical coordinates, for the regularity of axisymmetric weak solutions to the 3D incompressible Navier-Stokes equations. More precisely, we prove that if

$$\int_0^T \|\omega^\theta(\cdot,t)\|_{\dot{B}^0_{p,\frac{2p}{3}}}^q dt < \infty \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} < p \le \infty,$$

then the weak solution u is actually a regular solution. Similar regularity criterion still holds in the homogeneous Triebel-Lizorkin spaces.

§1 Introduction

In this paper, we are concerned with the regularity problem of axisymmetric weak solutions to the incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\pi = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $u : \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$ is the fluid velocity field, and $\pi : \mathbb{R}^3 \times (0,T) \to \mathbb{R}$ is a scalar pressure, and u_0 is a given initial velocity satisfying $\nabla\cdot u_0 = 0$ in the sense of distributions.

By the classical results of Leray [26] and Hopf [17], we know that for given initial data $u_0 \in L^2(\mathbb{R}^n)$ with $\nabla\cdot u_0 = 0$ in the sense of distributions, the n-dimensional version of system (1) admits at least one global weak solution $u \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n))$ satisfying the following energy inequality:

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \|u_0\|_{L^2}^2 \quad \text{for a.e. } t\in[0,T] \tag{2}$$

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and

$$\int_0^T \int_{\mathbb{R}^n} (u \cdot \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \cdot u + u \cdot \Delta \phi) dxdt + \int_{\mathbb{R}^n} u_0(x) \cdot \phi(x, 0) dx = 0,$$

for all $\phi \in C_0^\infty(\mathbb{R}^n \times [0, T))$ with $\nabla \cdot \phi = 0$. It is well-known that in two dimensions, this weak solution is unique and regular for all $t > 0$. However, in three dimensions, the regularity of weak solutions is a challenging open problem in mathematical fluid mechanics. In 1962, Serrin [34] proved that if u is a Leray-Hopf weak solution such that

$$u \in L^q(0, T; L^p(\mathbb{R}^3)) \ \text{ with } \ \frac{2}{q} + \frac{3}{p} \leq 1, \ \ 2 < q < \infty, \ \ 3 < p < \infty, \tag{3}$$

then u is a regular solution in $\mathbb{R}^3 \times (0, T)$. From then on, many mathematicians are interested in finding sufficient regularity conditions to ensure the regularity of weak solutions, the literatures listed here are far from being complete, and we refer the readers to see [1,3-7,9,12,13,15,16,19,20,31-33,36,42-44] and references therein.

In this paper, we are interested in finding sufficient conditions to ensure the regularity of weak solutions for the 3D axisymmetric Navier-Stokes equations. For a point in $\mathbb{R}^3$ by $x = (x_1, x_2, x_3)$, let us consider the cylindrical coordinates of $\mathbb{R}^3$

$$x_1 = r \cos\theta, \ \ x_2 = r \sin\theta, \ \ x_3 = x_3,$$

where $r > 0$, $0 \leq \theta < 2\pi$, $x_3 \in \mathbb{R}$, and

$$r = \sqrt{x_1^2 + x_2^2}, \ \ \theta = \arctan \frac{x_2}{x_1}, \ \ x_3 = x_3.$$

Let

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_3 = (0, 0, 1)$$

be the corresponding basis vectors. Then a solution u of the Navier-Stokes equations (1) is called an axisymmetric solution if the three components u^r, u^θ and u^3 are independent of the angular variable θ, i.e., the solution u has the following form

$$u(x, t) = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u^3(r, x_3, t)e_3,$$

and (u^r, u^θ, u^3) satisfies the following equations:

$$\begin{cases} \frac{D}{Dt}u^r - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u^r - \frac{(u^\theta)^2}{r} + \partial_r \pi = 0, \\ \frac{D}{Dt}u^\theta - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u^\theta + \frac{u^r u^\theta}{r} = 0, \\ \frac{D}{Dt}u^3 - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u^3 + \partial_3 \pi = 0, \\ \partial_r u^r + \partial_3 u^3 + \frac{1}{r}u^r = 0, \\ (u^r, u^\theta, u^3)|_{t=0} = (u_0^r, u_0^\theta, u_0^3), \end{cases} \tag{4}$$

where u^r, u^θ and u^3 are called the radial, swirl (or azimuthal) and axial components of the velocity field u, respectively, and $\frac{D}{Dt} := \partial_t + u^r \partial_r + u^3 \partial_3$ stands for the convection derivative.

By the uniqueness of local smooth solutions, it is easy to verify that if $u_0^\theta = 0$, then $u^\theta = 0$ for all later time. For the study of axisymmetric solutions of the Navier-Stokes equations without swirl, Ladyzhenskaya [23] and Ukhovskii-Yudovich [37] independently proved global existence, uniqueness and regularity of axisymmetric weak solutions. Later on, Leonardi et. al [25] gave a refined proof. However, for the axisymmetric Navier-Stokes equations with nonzero swirl component, it is not clear the effect of the vortex stretching in the vorticity equations and the regularity problem of weak solutions is still open. Many studies and interesting progresses have been made on the regularity issues of the axisymmetric weak solutions, see [10,15,18,21,22,24,28-30,38-41] and references therein. In 2002, Chae-Lee [8] proved that if the azimuthal component ω^θ of $\omega = \operatorname{curl} u$ satisfies

$$\int_0^T \|\omega^\theta(\cdot,t)\|_{L^p}^q dt < \infty \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \le 2, \quad \frac{3}{2} < p < \infty, \quad 1 < q \le \infty, \tag{5}$$

then the weak solution u is regular in $\mathbb{R}^3 \times (0,T)$. Subsequently, Chen-Zhang [11] proved that if

$$\int_0^T \|\omega^\theta(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty, \tag{6}$$

then u is a regular solution in $\mathbb{R}^3 \times (0,T)$.

In this paper, we aim at improving the regularity criteria (5) and (6) to the following Serrin's regularity criterion in the framework of Besov spaces and Triebel-Lizorkin spaces. The main results are as follows.

Theorem 1.1. *Let u be an axisymmetric weak solution of the Navier-Stokes equations* (1) *with $u_0 \in H^1(\mathbb{R}^3)$, $\nabla \cdot u_0 = 0$. If ω^θ satisfies the following condition:*

$$\int_0^T \|\omega^\theta(\cdot,t)\|_{\dot{B}^0_{p,\frac{2p}{3}}}^q dt < \infty \quad \textit{with} \quad \frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} < p \le \infty, \tag{7}$$

then the weak solution u is regular in $\mathbb{R}^3 \times (0,T)$.

Obviously, (7) becomes the condition (6) if we choose "$p = \infty$" in (7), thus (7) can be regarded as an extension of the regularity criteria (5) and (6). Moreover, by using similar argument in the proof of Theorem 1.1, we can prove that similar regularity criterion as (7) still holds in the homogeneous Triebel-Lizorkin spaces.

Theorem 1.2. *Let u be an axisymmetric weak solution of the Navier-Stokes equations* (1) *with $u_0 \in H^1(\mathbb{R}^3)$, $\nabla \cdot u_0 = 0$. If ω^θ satisfies the following condition:*

$$\int_0^T \|\omega^\theta(\cdot,t)\|_{\dot{F}^0_{p,\frac{2p}{3}}}^q dt < \infty \quad \textit{with} \quad \frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} < p \le \infty, \tag{8}$$

then the weak solution u is regular in $\mathbb{R}^3 \times (0,T)$, where $\dot{F}^s_{p,r}(\mathbb{R}^3)$ denotes the homogeneous Triebel-Lizorkin spaces.

The paper is organized in the following way. In Section 2, we shall introduce the homogeneous Besov spaces and review some known estimates. In Section 3, we shall complete the proofs of Theorems 1.1 and 1.2. Throughout the paper, C stands for some real positive constant which may be different in each occurrence.

§2 Preliminary

We start with the Fourier transform. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function, and $\mathcal{S}'(\mathbb{R}^3)$ of temperate distributions be the dual set of $\mathcal{S}(\mathbb{R}^3)$. Given $f \in \mathcal{S}(\mathbb{R}^3)$, the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi} f(x) dx.$$

More generally, the Fourier transform $\widehat{f}$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3)$ is defined by the dual argument in the standard way.

Let us now introduce a dyadic decomposition in $\mathbb{R}^3$. Let $\phi : \mathbb{R}^3 \to [0,1]$ be a smooth cut-off function which equals one on the ball $\mathcal{B}(0,\frac{5}{4}) := \{\xi \in \mathbb{R}^3 : \ |\xi| \le \frac{5}{4}\}$ and equals zero outside the ball $\mathcal{B}(0,\frac{3}{2})$. Let

$$\varphi(\xi) := \phi(\xi) - \phi(2\xi), \quad \phi_j(\xi) := \phi(2^{-j}\xi), \quad \varphi_j(\xi) := \varphi(2^{-j}\xi), \quad j \in \mathbb{Z}.$$

Then for all $j \in \mathbb{Z}$, the Littlewood-Paley projection operators Δ_j and S_j are respectively defined by

$$\Delta_j f := \varphi(2^{-j}D)f, and \ \ S_j f := \phi(2^{-j}D)f.$$

Let $\mathcal{S}'_h(\mathbb{R}^3)$ be the space of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\lim_{j\to-\infty} S_j f = 0.$$

By telescoping the series, we thus have the Littlewood-Paley decomposition

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f,$$

which holds for all $f \in \mathcal{S}'_h(\mathbb{R}^3)$. Now we recall the definition of the homogeneous Besov spaces. Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$ and $f \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\|f\|_{\dot{B}^s_{p,r}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j f\|^r_{L^p}\right)^{\frac{1}{r}} & \text{for } \ 1 \le r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^p} & \text{for } \ r = \infty. \end{cases}$$

Then the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined by

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define
$$\dot{B}^s_{p,r}(\mathbb{R}^3) := \Big\{f \in \mathcal{S}'_h(\mathbb{R}^3) : \ \ \|f\|_{\dot{B}^s_{p,r}} < \infty\Big\}.$$
- If $k \in \mathbb{N}$ and $\frac{3}{p} + k \le s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta f \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta| = k$.

In the sequel we shall frequently use the Bernstein's inequalities (see for example [2]).

Lemma 2.1. *Let $\mathcal{B}$ be a ball, and $\mathcal{C}$ a ring in $\mathbb{R}^3$. There exists a constant C such that for any positive real number λ, any nonnegative integer k and any couple of real numbers (p,q) with $1 \le p \le q \le \infty$, we have*

$$\operatorname{supp} \widehat{f} \subset \lambda\mathcal{B} \quad \Rightarrow \quad \sup_{|\alpha|=k} \|\Lambda^\alpha f\|_{L^q} \le C^{k+1}\lambda^{k+3(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}, \tag{9}$$

$$\operatorname{supp} \widehat{f} \subset \lambda\mathcal{C} \quad \Rightarrow \quad C^{-1-k}\lambda^k\|f\|_{L^p} \le \sup_{|\alpha|=k} \|\Lambda^\alpha f\|_{L^p} \le C^{1+k}\lambda^k\|f\|_{L^p}. \tag{10}$$

Let us recall the well-known Biot-Savart law, which reveals the relation between the divergence free velocity field u and the vorticity $\omega = \operatorname{curl} u$ (see for example [27]).

Lemma 2.2. *Let u be a smooth vector field with $\nabla \cdot u = 0$. Then we have*

$$\nabla u(x) = M\omega(x) + K * \omega(x), \tag{11}$$

where M is a constant matrix, and K is a matrix valued function with homogeneous of degree of -3. Moreover, for any $1 < p < \infty$, we have

$$\|\nabla u\|_{L^p} \le C\|\omega\|_{L^p}. \tag{12}$$

Finally, in the following, we shall use two notations for the axisymmetric vector field u

$$\widetilde{u} := u^r e_r + u^3 e_3,$$

and

$$\widetilde{\nabla} := (\partial_r, \partial_3).$$

We can easily compute the vorticity $\omega = \operatorname{curl} u$ as follows:

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,$$

where

$$\omega^r = -\partial_3 u^\theta,\ \ \omega^3 = \partial_r u^\theta + \frac{u^\theta}{r},\ \ \omega^\theta = -\partial_r u^3 + \partial_3 u^r.$$

Moreover, if we denote

$$\tilde{\omega} := \omega^r e_r + \omega^3 e_3,$$

then we can infer from [8] and [11] to get the following equalities and inequalities.

Lemma 2.3. *Let u be an axisymmetric vector field. Then we have*

$$|\nabla\widetilde{u}|^2 = \left|\frac{u^r}{r}\right|^2 + |\widetilde{\nabla} u^r|^2 + |\widetilde{\nabla} u^3|^2, \tag{13}$$

$$|\nabla(u^\theta e_\theta)|^2 = \left|\frac{u^\theta}{r}\right|^2 + |\widetilde{\nabla} u^\theta|^2. \tag{14}$$

Lemma 2.4. *Let u be an axisymmetric vector field with $\nabla\cdot u = 0$, and let $\omega = \operatorname{curl} u$ vanish sufficiently fast near infinity in $\mathbb{R}^3$. Then $\nabla\widetilde{u}$ and $\nabla(u^\theta e_\theta)$ can be represented as the singular integral form*

$$\nabla\widetilde{u}(x) = M\omega^\theta e_\theta(x) + [K * (\omega^\theta e_\theta)](x), \tag{15}$$

$$\nabla(u^\theta e_\theta(x)) = M'\widetilde{\omega}(x) + [H * \widetilde{\omega}](x), \tag{16}$$

where the kernels $K(x)$ and $H(x)$ are matrix valued functions homogeneous of degree -3, defining a singular integral operator by convolution, and $f * g(x) = \int_{\mathbb{R}^3} f(x-y)g(y)dy$ denotes the standard convolution operator. The matrices M and M' are the constant matrices.

Lemma 2.5. *Let $1 < p < \infty$. Then we have*

$$\|\nabla\widetilde{u}\|_{L^p} \le C\|\omega^\theta\|_{L^p}, \qquad \|\nabla(u^\theta e_\theta)\|_{L^p} \le C\|\widetilde{\omega}\|_{L^p}, \tag{17}$$

where C is a constant depending only on p.

§3 The proofs of Theorems 1.1 and 1.2

Let u be an axisymmetric solution of the Navier-Stokes equations (1). Taking the curl on both sides of (1), we obtain

$$\partial_t\omega + (u\cdot\nabla)\omega - \Delta\omega = (\omega\cdot\nabla)u. \tag{18}$$

Multiplying both sides of (18) by w and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 = \int_{\mathbb{R}^3}(\omega\cdot\nabla u)\omega dx, \tag{19}$$

where we used the fact

$$\int_{\mathbb{R}^3}(u\cdot\nabla\omega)\omega dx=\frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla)\omega^2dx=-\frac{1}{2}\int_{\mathbb{R}^3}(\nabla\cdot u)\omega^2dx=0,$$

since $\nabla\cdot u=0$. Noting that

$$\omega=\omega^re_r+\omega^\theta e_\theta+\omega^3e_3,\qquad \nabla=e_r\partial_r-\frac{1}{r}e_\theta\partial_\theta+e_3\partial_3,$$

we can rewrite the right-hand side of (19) as

$$\begin{aligned}\int_{\mathbb{R}^3}(\omega\cdot\nabla u)\omega dx=&\int_{\mathbb{R}^3}\Big(\omega^r\partial_ru^r\omega^r+\omega^r\partial_ru^\theta\omega^\theta+\omega^r\partial_ru^3\omega^3+\frac{1}{r}\omega^\theta u^r\omega^\theta\\&-\frac{1}{r}\omega^\theta u^\theta\omega^r+\omega^3\partial_3u^r\omega^r+\omega^3\partial_3u^\theta\omega^\theta+\omega^3\partial_3u^3\omega^3\Big)dx\\:=&I_1(t)+I_2(t)+\cdots+I_8(t).\end{aligned}\tag{20}$$

We estimate $I_i(t)$ $(i=1,2,\cdots,8)$ as follows. For $I_1(t)$, it is easily seen from the fact $|\partial_ru^r|\le|\nabla\tilde{u}|$ and using Lemma 2.4, one has

$$|I_1(t)|\le\int_{\mathbb{R}^3}|\omega^r|^2|\partial_ru^r|dx\le C\int_{\mathbb{R}^3}|\omega^r|^2(|\omega^\theta|+|K*(\omega^\theta e_\theta)|)dx.\tag{21}$$

By means of the Littlewood-Paley dyadic decomposition, we divide ω^θ into the following three parts:

$$\omega^\theta=\sum_{j=-\infty}^{+\infty}\Delta_j\omega^\theta=\sum_{j<-N}\Delta_j\omega^\theta+\sum_{j=-N}^{N}\Delta_j\omega^\theta+\sum_{j>N}\Delta_j\omega^\theta,$$

where N is a positive integer to be determined later. Thus we have

$$\begin{aligned}|I_1(t)|\le&\sum_{j<-N}\int_{\mathbb{R}^3}|\omega^r|^2(|\Delta_j\omega^\theta|+|K*(\Delta_j\omega^\theta e_\theta)|)dx\\&+\sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\omega^r|^2(|\Delta_j\omega^\theta|+|K*(\Delta_j\omega^\theta e_\theta)|)dx\\&+\sum_{j>N}\int_{\mathbb{R}^3}|\omega^r|^2(|\Delta_j\omega^\theta|+|K*(\Delta_j\omega^\theta e_\theta)|)dx\\:=&I_{11}(t)+I_{12}(t)+I_{13}(t).\end{aligned}\tag{22}$$

For $I_{11}(t)$, applying the Hölder's inequality and (9), we can bound it as

$$\begin{aligned}|I_{11}(t)|=&\sum_{j<-N}\int_{\mathbb{R}^3}|\omega^r|^2(|\Delta_j\omega^\theta|+|K*(\Delta_j\omega^\theta e_\theta)|)dx\\\le&C\|\omega^r\|_{L^2}^2\sum_{j<-N}(\|\Delta_j\omega^\theta\|_{L^\infty}+\|K*(\Delta_j\omega^\theta e_\theta)\|_{L^\infty})\\\le&C\|\omega^r\|_{L^2}^2\sum_{j<-N}2^{\frac{3}{2}j}\|\Delta_j\omega^\theta\|_{L^2}\\\le&C\|\omega^r\|_{L^2}^2\Big(\sum_{j<-N}2^{3j}\Big)^{\frac{1}{2}}\Big(\sum_{j<-N}\|\Delta_j\omega^\theta\|_{L^2}^2\Big)^{\frac{1}{2}}\\\le&C2^{-\frac{3}{2}N}\|\omega\|_{L^2}^3.\end{aligned}\tag{23}$$

For $I_{12}(t)$, it follows from the Hölder's inequality that

$$\begin{aligned}|I_{12}(t)| &= \sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\omega^r|^2(|\Delta_j\omega^\theta|+|K*(\Delta_j\omega^\theta e_\theta)|)dx\\
&\le C\|\omega^r\|^2_{L^{\frac{2p}{p-1}}}\sum_{j=-N}^{N}(\|\Delta_j\omega^\theta\|_{L^p}+\|K*(\Delta_j\omega^\theta e_\theta)\|_{L^p})\\
&\le C\|\omega^r\|^2_{L^{\frac{2p}{p-1}}}\Big(\sum_{j=-N}^{N}1^{\frac{2p}{2p-3}}\Big)^{\frac{2p-3}{2p}}\Big(\sum_{j=-N}^{N}\|\Delta_j\omega^\theta\|^{\frac{2p}{3}}_{L^p}\Big)^{\frac{3}{2p}}\\
&\le CN^{\frac{2p-3}{2p}}\|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}\|\omega\|^2_{L^{\frac{2p}{p-1}}}\\
&\le CN^{\frac{2p-3}{2p}}\|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}\|\omega\|^{2(1-\frac{3}{2p})}_{L^2}\|\nabla\omega\|^{\frac{3}{p}}_{L^2}\\
&\le \frac{1}{16}\|\nabla\omega\|^2_{L^2}+CN\|\omega^\theta\|^{\frac{2p}{2p-3}}_{\dot{B}^0_{p,\frac{2p}{3}}}\|\omega\|^2_{L^2},\end{aligned}\tag{24}$$

where we used the following Gagliardo-Nirenberg inequality:

$$\|w\|_{L^{\frac{2p}{p-1}}}\le\|w\|^{1-\frac{3}{2p}}_{L^2}\|\nabla w\|^{\frac{3}{2p}}_{L^2}\ \text{ for all }\ p\ge\frac{3}{2}.$$

For $I_{13}(t)$, applying the Hölder's inequality, (9) and (10) yield that

$$\begin{aligned}|I_{13}(t)| &= \sum_{j>N}\int_{\mathbb{R}^3}|\omega^r|^2(|\Delta_j\omega^\theta|+|K*(\Delta_j\omega^\theta e_\theta)|)dx\\
&\le C\|\omega^r\|_{L^2}\|w^r\|_{L^6}\sum_{j>N}(\|\Delta_j\omega^\theta\|_{L^3}+\|K*(\Delta_j\omega^\theta e_\theta)\|_{L^3})\\
&\le C\|\omega^r\|_{L^2}\|w^r\|_{L^6}\sum_{j>N}2^{\frac{j}{2}}\|\omega^\theta\|_{L^2}\\
&\le C\|\omega^r\|_{L^2}\|\nabla\omega^r\|_{L^2}\Big(\sum_{j>N}2^{-j}\Big)^{\frac{1}{2}}\Big(\sum_{j>N}2^{2j}\|\omega^\theta\|^2_{L^2}\Big)^{\frac{1}{2}}\\
&\le C2^{-\frac{N}{2}}\|\omega^r\|_{L^2}\|\nabla\omega^r\|_{L^2}\|\nabla\omega^\theta\|_{L^2}\\
&\le C2^{-\frac{N}{2}}\|\omega\|_{L^2}\|\nabla\omega\|^2_{L^2}.\end{aligned}\tag{25}$$

Plugging estimates (23)–(25) into (22), we see that

$$|I_1(t)|\le\frac{1}{16}\|\nabla\omega\|^2_{L^2}+C\Big(2^{-\frac{3}{2}N}\|\omega\|^3_{L^2}+N\|\omega^\theta\|^{\frac{2p}{2p-3}}_{\dot{B}^0_{p,\frac{2p}{3}}}\|\omega\|^2_{L^2}+2^{-\frac{N}{2}}\|\omega\|_{L^2}\|\nabla\omega\|^2_{L^2}\Big).\tag{26}$$

Similarly, by means of the fact

$$|\partial_r u^3|,\Big|\frac{u^r}{r}\Big|,|\partial_3 u^r|,|\partial_3 u^3|\le|\nabla\tilde{u}|\le C(|M\omega^\theta|+|K*(\omega^\theta e_\theta)|),$$

the terms $I_3(t)$, $I_4(t)$, $I_6(t)$ and $I_8(t)$ can be estimated as $I_1(t)$, thus for $i=3,4,6,8$, we have

$$|I_i(t)|\le\frac{1}{16}\|\nabla\omega\|^2_{L^2}+C\Big(2^{-\frac{3}{2}N}\|\omega\|^3_{L^2}+N\|\omega^\theta\|^{\frac{2p}{2p-3}}_{\dot{B}^0_{p,\frac{2p}{3}}}\|\omega\|^2_{L^2}+2^{-\frac{N}{2}}\|\omega\|_{L^2}\|\nabla\omega\|^2_{L^2}\Big).\tag{27}$$

For $I_2(t)$, similar to the derivation of (22), it can be rewritten as the following three terms

$$|I_2(t)| = \sum_{j<-N}\int_{\mathbb{R}^3}\omega^r\partial_r u^\theta \Delta_j\omega^\theta dx + \sum_{j=-N}^{N}\int_{\mathbb{R}^3}\omega^r\partial_r u^\theta \Delta_j\omega^\theta dx + \sum_{j>N}\int_{\mathbb{R}^3}\omega^r\partial_r u^\theta \Delta_j\omega^\theta dx$$
$$:= I_{21}(t) + I_{22}(t) + I_{23}(t). \tag{28}$$

It follows from Lemmas 2.3 and 2.4 that

$$|\partial_r u^\theta| \le |\nabla(u^\theta e_\theta)| \le |\tilde{\omega}| + |H * \tilde{\omega}|,$$

and since the singular operator is bounded on L^p for all $1 < p < \infty$, we can use Lemma 2.5 and perform the same procedure as (26) to obtain

$$|I_2(t)| \le \frac{1}{16}\|\nabla\omega\|_{L^2}^2 + C\Big(2^{-\frac{3}{2}N}\|\omega\|_{L^2}^3 + N\|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}}\|\omega\|_{L^2}^2 + 2^{-\frac{N}{2}}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}^2\Big). \tag{29}$$

For the last two terms $I_5(t)$ and $I_7(t)$, it follows Lemmas 2.3 and 2.4 again that

$$\Big|\frac{u^\theta}{r}\Big|, |\partial_3 u^\theta| \le |\nabla(u^\theta e_\theta)| \le |\tilde{\omega}| + |H * \tilde{\omega}|,$$

thus these two terms can be bounded analogously as the term $I_2(t)$. Combining all above estimates (26)–(29) altogether, we obtain

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 \le C\Big(2^{-\frac{3}{2}N}\|\omega\|_{L^2}^3 + N\|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}}\|\omega\|_{L^2}^2 + 2^{-\frac{N}{2}}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}^2\Big). \tag{30}$$

Now we can choose N large enough such that

$$C2^{-\frac{N}{2}}\|\omega\|_{L^2} \le \frac{1}{2},$$

i.e.,

$$N \ge \frac{2\ln^+(C\|\omega\|_{L^2}^2)}{\ln 2} + 2,$$

where $\ln^+ t = \ln t$ for $t \ge 1$ and $\ln^+ t = 0$ for $0 < t < 1$, then we know from (30) that

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \frac{1}{2}\|\nabla\omega\|_{L^2}^2 \le C(1 + \|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}})(\|\omega\|_{L^2}^2 + e)\ln(\|\omega\|_{L^2}^2 + e). \tag{31}$$

Denoting $Y(t) := \|\omega(t)\|_{L^2}^2 + e$, the inequality (31) implies that

$$\frac{d}{dt}\ln Y(t) \le C(1 + \|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}})\ln Y(t).$$

By solving this ordinary differential inequality, we have

$$Y(t) \le Y(0)\exp\Big\{\exp\{Ct + C\int_0^t \|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}} d\tau\}\Big\}$$

for all $0 < t \le T$, which implies further that

$$\sup_{0\le t\le T}\|\omega\|_{L^2}^2 \le (\|\nabla u_0\|_{L^2}^2 + e)\exp\left\{\exp\left\{CT + C\int_0^T \|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}} dt\right\}\right\}. \tag{32}$$

Combining the above estimate (32) with the energy inequality (2), we finally get

$$\sup_{0\le t\le T}\|u(t)\|_{H^1}^2 \le (\|u_0\|_{H^1}^2 + e)\exp\left\{\exp\left\{CT + C\int_0^T \|\omega^\theta\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}} dt\right\}\right\}. \tag{33}$$

Proof of Theorem 1.1. In [14], Fujita and Kato proved the local existence of strong solution for $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and it follows that there exists $T_* > 0$ and an axisymmetric solution v of the Navier-Stokes equations (1) satisfying

$$v(t) \in C([0,T_*),H^1) \cap C^1((0,T_*),H^1) \cap C((0,T_*),H^3), \quad v(0) = u_0.$$

Since the weak solution u satisfies the energy inequality (2), we can apply Serrin's uniqueness criterion in [35] to conclude that

$$u \equiv v \quad \text{on} \quad [0,T_*).$$

Thus it is sufficient to show that $T_* = T$. Suppose $T_* < T$. Without loss of generality, we may assume that T_* is the maximal existence time for $v(t)$. Notice that $u(t) = v(t)$ on $[0,T_*)$, by the assumption (7), we have

$$\int_0^{T_*} \|(\operatorname{curl} v)^\theta(t)\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}} \, dt < \infty.$$

Then it follows from (33) that the existence time of $v(t)$ can be extended after $t = T_*$, which contradicts the maximality of T_*. The proof of Theorem 1.1 is achieved.

Proof of Theorem 1.2. Back to (20), we need to estimate $I_i(t)$ $(i = 1,2,\cdots,8)$ one by one. For $I_1(t)$, it suffices to tackle with $I_{12}(t)$ as follows:

$$\begin{aligned}
|I_{12}(t)| &= \int_{\mathbb{R}^3} |\omega^r|^2 \sum_{j=-N}^{N} (|\Delta_j\omega^\theta| + |K*(\Delta_j\omega^\theta e_\theta)|)dx \\
&\le C\int_{\mathbb{R}^3} |\omega^r|^2 \Big(\sum_{j=-N}^{N} 1^{\frac{2p}{2p-3}}\Big)^{\frac{2p-3}{2p}} \Big(\sum_{j=-N}^{N} (|\Delta_j\omega^\theta|^{\frac{2p}{3}} + |K*(\Delta_j\omega^\theta e_\theta)|^{\frac{2p}{3}})\Big)^{\frac{3}{2p}} dx \\
&\le CN^{\frac{2p-3}{2p}} \|\omega^r\|^2_{L^{\frac{2p}{p-1}}} \Big\| \Big(\sum_{j=-N}^{N} \big(|\Delta_j\omega^\theta|^{\frac{2p}{3}} + |K*(\Delta_j\omega^\theta e_\theta)|^{\frac{2p}{3}}\big)\Big)^{\frac{3}{2p}} \Big\|_{L^p} \\
&\le CN^{\frac{2p-3}{2p}} \Big(\|\omega^\theta\|_{\dot{F}^0_{p,\frac{2p}{3}}} + \|K*\omega^\theta\|_{\dot{F}^0_{p,\frac{2p}{3}}}\Big) \|\omega\|^2_{L^{\frac{2p}{p-1}}} \\
&\le CN^{\frac{2p-3}{2p}} \|\omega^\theta\|_{\dot{F}^0_{p,\frac{2p}{3}}} \|\omega\|_{L^2}^{2(1-\frac{3}{2p})} \|\nabla\omega\|_{L^2}^{\frac{3}{p}} \\
&\le \frac{1}{16}\|\nabla\omega\|^2_{L^2} + CN\|\omega^\theta\|_{\dot{F}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}} \|\omega\|^2_{L^2},
\end{aligned}$$

which together with (23) and (25) yield that

$$|I_1(t)| \le \frac{1}{16}\|\nabla\omega\|^2_{L^2} + C\Big(2^{-\frac{3}{2}N}\|\omega\|^3_{L^2} + N\|\omega^\theta\|_{\dot{F}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}} \|\omega\|^2_{L^2} + 2^{-\frac{N}{2}}\|\omega\|_{L^2}\|\nabla\omega\|^2_{L^2}\Big).$$

Based on the above argument, the remaining terms $I_i(t)(i = 2,\cdots,8)$ can be similarly estimated by only using "the homogeneous Triebel-Lizorkin norm" instead of "the homogeneous Besov norm", and we can exactly proceed the same lines as the proof of Theorem 1.1 to complete the proof of Theorem 1.2.

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] R P Agarwal, S Gala, M A Ragusa. *A regularity criterion in weak spaces to Boussinesq equations*, Mathematics, 2020, 8(6): 920, DOI: 10.3390/math8060920.

[2] H Bahouri, J Y Chemin, R Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, Berlin: Springer, 2011.

[3] J T Beale, T Kato, A Majda. *Remarks on breakdown of smooth solutions for the 3D Euler equations*, Comm Math Phys, 1984, 94: 61-66.

[4] H Beirão da Veiga. *A new regularity class for the Navier-Stokes equations in* $\mathbb{R}^n$, Chinese Ann Math Ser B, 1995, 16: 407-412.

[5] S Benbernou, S Gala, M A Ragusa. *On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO space*, Math Method Appl Sci, 2014, 37(15): 2320-2325.

[6] C S Cao, E S Titi. *Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor*, Arch Ration Mech Anal, 2011, 202: 919-932.

[7] D Chae, H Choe. *Regularity of solutions to the Navier-Stokes equations*, Electron J Differ Equ, 1999, 1999: 1-7.

[8] D Chae, J Lee. *On the regularity of axisymmetric solutions of the Navier-Stokes equations*, Math Z, 2002, 239: 645-671.

[9] J Y Chemin, P Zhang. *On the critical one component regularity for 3-D Navier-Stokes systems*, Ann Sci Éc Norm Supér, 2016, 49(1): 131-167.

[10] H Chen, D Fang, T Zhang. *Regularity of 3D axisymmetric Navier-Stokes equations*, Discrete Contin Dyn Syst, 2017, 37(4): 1923-1939.

[11] Q Chen, Z Zhang. *Regularity criterion of axisymmetric weak solutions to the 3D Navier-Stokes equations*, J Math Anal Appl, 2007, 331(2): 1384-1395.

[12] L Eskauriaza, G A Serëgin, V Šverák. $L_{3,\infty}$*-solutions of Navier-Stokes equations and backward uniqueness*, Russ Math Surv, 2003, 58: 211-250.

[13] D Fang, C Qian. *The regularity criterion for the 3D Navier-Stokes equations involving one velocity gradient component*, Nonlinear Anal, 2013, 78: 86-103.

[14] H Fujita, T Kato. *On the Navier-Stokes initial value problem I*, Arch Ration Mech Anal, 1964, 16: 269-315.

[15] S Gala. *On the regularity criterion of axisymmetric weak solutions to the 3D Navier-Stokes equations*, Nonlinear Anal, 2011, 74: 775-782.

[16] S Gala, E Galakhov, M A Ragusa, et al. *Beale-Kato-Majda regularity criterion of smooth solutions for the Hall-MHD equations with zero viscosity*, Bull Braz Math Soc, New Series, 2022, 53: 229-241.

[17] E Hopf. *Über die Anfangwertaufgaben für die hydromischen Grundgleichungen*, Math Nachr, 1951, 4: 213-321.

[18] O Kreml, M Pokorný. *A regularity criterion for the angular velocity component in axisymmetric Navier-Stokes equations*, Electron J Differ Equ, 2007, 8: 1-10.

[19] H Kozono, T Ogawa, Y Taniuchi. *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations*, Math Z, 2002, 242: 251-278.

[20] H Kozono, Y Taniuchi. *Bilinear estimates in BMO and the Navier-Stokes equations*, Math Z, 2000, 235: 173-194.

[21] A Kubica, M Pokorný, W Zajaczkowski. *Remarks on regularity criteria for axially symmetric weak solutions to the Navier-Stokes equations*, Math Method Appl Sci, 2012, 35: 360-371.

[22] I Kukavica, M Ziane. *One component regularity for the Navier-Stokes equations*, Nonlinearity, 2006, 19: 453-469.

[23] O A Ladyzhenskaya. *Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry*, Zap Nauchn Sem LOMI, 1968, 7: 155-177.

[24] Z Lei, Q S Zhang. *A Liouville theorem for the axially symmetric Navier-Stokes equations*, J Funct Anal, 2011, 261(8): 2323-2345.

[25] S Leonardi, J Málek, J Nečas, et al. *On axially symmetric flows in* $\mathbb{R}^3$, Z Anal Anwendungen, 1999, 18: 639-649.

[26] J Leray. *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math, 1934, 63: 193-248.

[27] A J Majda, A L Bertozzi. *Vorticity and Incompressible Flow*, Cambridge: Cambridge University Press, 2002.

[28] J Neustupa, M Pokorný. *An interior regularity criterion for an axially symmetric suitable weak solution to the Navier-Stokes equations*, J Math Fluid Mech, 2000, 2: 381-399.

[29] J Neustupa, M Pokorný. *Axisymmetric flow of Navier-Stokes fluid in the whole space with non-zero angular velocity component*, Math Bohem, 2001, 126: 469-481.

[30] X Pan. *A regularity condition of 3d axisymmetric Navier-Stokes equations*, Acta Appl Math, 2017, 150: 103-109.

[31] P Penel, M Pokorný. *Some new regularity criteria for the Navier-Stokes equations containing gradient of the velocity*, Appl Math, 2004, 49(5): 483-493.

[32] P Penel, M Pokorný. *On anisotropic regularity criteria for the solutions to 3D Navier-Stokes equations*, J Math Fluid Mech, 2011, 13: 341-353.

[33] C Qian. *A generalized regularity criterion for 3D Navier-Stokes equations in terms of one velocity component*, J Differ Equ, 2016, 260(4): 3477-3494.

[34] J Serrin. *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch Ration Mech Anal, 1962, 9: 187-195.

[35] J Serrin. *The initial value problem for the Navier-Stokes equations*, in: R E Langer (Ed.), *Nonlinear Problems*, Madison: University of Wisconsin Press, 1963, 69-98.

[36] H Sohr. *Zur Regularitätstheorie der instationären Gleichungen von Navier-Stokes*, Math Z, 1983, 184: 359-375.

[37] M R Ukhovskii, V I Yudovich. *Axially symmetric flows of ideal and viscous fluids filling the whole space*, J Appl Math Mech, 1968, 32: 52-61.

[38] D Wei. *Regularity criterion to the axially symmetric Navier-Stokes equations*, J Math Anal Appl, 2016, 435(1): 402-413.

[39] P Zhang, T Zhang. *Global axisymmetric solutions to the three-dimensional Navier-Stokes equations system*, Int Math Res Notices, 2014, 2014: 610-642.

[40] Z Zhang. *Remarks on regularity criteria for the Navier-Stokes equations with axisymmetric data*, Ann Polon Math, 2016, 117(2): 181-196.

[41] Z Zhang. *A pointwise regularity criterion for axisymmetric Navier-Stokes system*, J Math Anal Appl, 2018, 461(1): 1-6.

[42] X Zheng. *A regularity criterion for the tridimensional Navier-Stokes equations in terms of one velocity component*, J Differ Equ, 2014, 256(1): 283-309.

[43] Y Zhou. *A new regularity criterion for weak solutions to the Navier-Stokes equations*, J Math Pures Appl, 2005, 84(11): 1496-1514.

[44] Y Zhou, M Pokorný. *On the regularity of the solutions of the Navier-Stokes equations via one velocity component*, Nonlinearity, 2010, 23: 1097-1107.

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