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Monotonicity of limit wave speed for the perturbed gKdV equation with general even m

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Abstract. This paper concerns the monotonicity of limit wave speed $c_0(h)$ for the perturbed gKdV equation with general even m. We show that $c_0(h)$ is decreasing. Our results give partial answer to the open problem presented by Yan et al. (Math. Model. Anal., 19, 537-555, 2014).

§1 Introduction

Shallow water wave models play an important role in describing natural phenomena, and the research of their solutions is a very active subject in the field of differential equations and has significant application. Many approaches have been developed to study them [1–11]. In recent years, the perturbed shallow water wave equations and their traveling waves have attracted considerable interest since the perturbed models maybe more realistic [12–21]. Among these models, the KdV-type equations are of particular interest and have attracted continuous attention due to the fact that they possess a type of traveling wave solutions, the so-called solitary wave solutions and are believed to be able to describe the soliton phenomenon first discovered by Scott Russell in 1834. The KdV equation is also well-known as the prototypical example of an exactly solvable equation. Some modified or generalized forms of the KdV equation have been introduced and studied more recently for both physical and mathematical interest. When modelling the move of wave, small perturbations due to the existence of uncertainty in real world and the unavoidable error in modeling may be neglected. However, the perturbations should be included to obtain a more realistic model and to better understand the dynamical behavior of the equation. For example, Owaga [23] studied the perturbed KdV equation

$$v_t + vv_x + v_{xxx} + \varepsilon(v_{xx} + v_{xxxx}) = 0, \tag{1}$$

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and established the persistence of its solitary waves and periodic waves for sufficiently small parameter $\varepsilon > 0$, which include the perturbation terms: the backward diffusion u_{xx} and dissipation u_{xxxx}. As a generalization, Yan et al. [22] further proved the existence of these two kinds of waves to the perturbed generalized KdV (gKdV) equation

$$v_t + v^m v_x + v_{xxx} + \varepsilon(v_{xx} + v_{xxxx}) = 0. \tag{2}$$

However, the authors also proposed an open problem about the monotonicity of limit wave speed, which will further help understand the water wave. About this open problem, there have been partial results for some particular (small) m. For $m = 1$, Owaga [23] showed that $c_0(h)$ is decreasing. For $m = 2, 3, 4$, Chen et al. [24,25] established that $c_0(h)$ is also decreasing. However, as suggested in [25], for $m \geq 5$, the monotonicity of $c_0(h)$ is difficult to prove analytically and remains an open problem. We made a tentative study to address this problem for $m \geq 5$ from the numerical perspective [26]. In this paper, inspired by [26], we further study the monotonicity of $c_0(h)$ to Eq.(2) with general even m. From now on, we suppose that m is even.

Obviously, Eq.(2) can be transformed into the following ODE

$$-cv'(\zeta) + v^m(\zeta)v'(\zeta) + v'''(\zeta) + \varepsilon(v''(\zeta) + v''''(\zeta)) = 0, \tag{3}$$

by letting $\zeta = x - ct$ with the wave speed $c > 0$, from which we easily obtain

$$-cv(\zeta) + \frac{1}{m+1}v^{m+1}(\zeta) + v''(\zeta) + \varepsilon(v'(\zeta) + v'''(\zeta)) = 0. \tag{4}$$

Introducing the scale transformation $\rho = \sqrt{c}\zeta$, $v = \sqrt[m]{c}X$, we arrive at

$$-X(\rho) + \frac{1}{m+1}X^{m+1}(\rho) + X''(\rho) + \varepsilon\left(\frac{1}{\sqrt{c}}X'(\rho) + \sqrt{c}X'''(\rho)\right) = 0. \tag{5}$$

Setting $\varepsilon = 0$, we obtain the unperturbed system of Eq.(5)

$$X'' + \frac{1}{m+1}X^{m+1} - X = 0, \tag{6}$$

or in the form of planar dynamical system

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}\rho} = Y, \\ \frac{\mathrm{d}Y}{\mathrm{d}\rho} = X - \frac{1}{m+1}X^{m+1}, \end{cases} \tag{7}$$

with first integral

$$H(X,Y) = -Y^2 + X^2 - \frac{2}{(m+1)(m+2)}X^{m+2} = h. \tag{8}$$

Obviously, system (7) has one saddle $(0,0)$, two centers $(\pm\sqrt[m]{m+1}, 0)$, and the phase portrait is given in Figure 1. In addition, we have $H(0,0) = 0$ and $H(\pm\sqrt[m]{m+1}, 0) = \frac{m\sqrt[m]{(m+1)^2}}{m+2} \triangleq H^*$. Then we can parameter the traveling waves of system (7) through h, and display the monotonicity of $c_0(h)$ in the following theorem.

Theorem 1. *There exists $\varepsilon_0 > 0$, such that for $\varepsilon \in [0, \varepsilon_0)$ and $h \in [0, H^*)$, Eq.(2) with general even m has solutions $v = \pm\sqrt[m]{c}X(\varepsilon, h, c, \rho)$, where $X(\varepsilon, h, c, \rho)$ is a solution of Eq.(5). $c = c(\varepsilon, h)$ is a smooth function of ε and h, with the limit $c_0(h)$ as $\varepsilon \to 0$. Furthermore, $c_0(h)$ is a smooth decreasing function for $h \in [0, H^*)$ and satisfies*

$$\frac{1}{m} \leq c_0(h) < \frac{3m+4}{m(2m+3)}, \quad c_0(0) = \frac{3m+4}{m(2m+3)}, \quad \text{and} \quad \lim_{h \to H^*-0} c_0(h) = \frac{1}{m}. \tag{9}$$

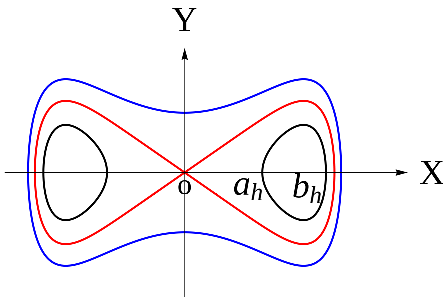

Figure 1. The phase portrait of system (7).

§2 The theoretic derivations of the monotonicity of $c_0(h)$

In this section, we exploit the Abelian integral theory and numerical technique to deal with the monotonicity of $c_0(h)$ only, since other parts of Theorem 1 were proved in [22].

Assume that $X(\rho)$ is the solution of system (7), and Q and R are defined by

$$Q = \frac{1}{2}\int (X'')^2 \mathrm{d}\rho, \ R = \frac{1}{2}\int (X')^2 \mathrm{d}\rho,$$

where the integrals are performed along the orbits of system (7). Then from [22], it is known that $c_0(h)$ can be expressed as

$$c_0(h) = \frac{R}{Q}.$$

Now it is time to analyze Q and R in detail. Suppose that $a(h)$ and $b(h)$ are two roots of $X^2 - \frac{2}{(m+1)(m+2)}X^{m+2} = h$, where $0 \le h < H^*$, satisfying $0 \le a(h) < b(h)$. Therefore, we can express Q and R as

$$Q = \int_{a(h)}^{b(h)} \frac{\left(X - \frac{1}{m+1}X^{m+1}\right)^2}{E(X)}\mathrm{d}X, \ R = \int_{a(h)}^{b(h)} E(X)\mathrm{d}X, \tag{10}$$

where $E(X) = \sqrt{X^2 - \frac{2}{(m+1)(m+2)}X^{m+2} - h}$, through system (7).

For convenience, we introduce the following integrals:

$$J_m(h) = \int_{a(h)}^{b(h)} X^m E(X)\mathrm{d}X, \ m = 0, 1, 2, \cdots, \tag{11}$$

which satisfy

$$\int_{a(h)}^{b(h)} \frac{X^m}{E(X)}\mathrm{d}X = -2J'_m(h). \tag{12}$$

To study the monotonicity of $c_0(h)$, we turn to its reciprocal $Z(h) = \frac{Q}{R}$, and present its properties in Proposition 1.

Proposition 1. *For $0 < h < H^*$, $Z'(h) > 0$ and $\frac{m(2m+3)}{3m+5} < Z(h) < m$. Additionally, $Z(0) = \frac{m(2m+3)}{3m+5}$ and $\lim\limits_{h \to H^*-0} Z(h) = m$.*

To prove the above proposition, we cite Lemmas 1, 2, 3 and 4 from [22].

Lemma 1. *[22]. We have* $\frac{J_m(0)}{J_0(0)} = \frac{2(m+1)(m+2)}{3m+4}$ *and* $\lim\limits_{h\to H^*-0} \frac{J_m(h)}{J_0(h)} = m+1$.

Lemma 2. *[22]. We have* $J = A(h)J'$*, where* $J = (J_0, J_2, \cdots, J_m)^T$

$$\text{and } A(h) = \begin{pmatrix} \frac{M}{m+4} & \frac{-2m}{m+4} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{M}{m+8} & \frac{-2m}{m+8} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{M}{m+12} & \frac{-2m}{m+12} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-2m}{3m-4} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{M}{3m} & \frac{-2m}{3m} \\ F & G & 0 & 0 & \cdots & 0 & \frac{M}{3m+4} \end{pmatrix}_{\frac{m+2}{2}\times\frac{m+2}{2}},$$

with $F = \frac{2m(m+1)(m+2)h}{(m+4)(3m+4)}, G = -\frac{4m(m+1)(m+2)}{(m+4)(3m+4)}$*, and* $M = 2(m+2)h$.

Lemma 3. *[22]. Let* $V = M^{\frac{m+2}{2}} + (-1)^{\frac{m+2}{2}}(m+1)M(-2m)^{\frac{m}{2}}$*,* $W = \prod_{j=1}^{\frac{m+2}{2}}(m+4j)$*, and we have* $|A| = \frac{V}{W}, A^{-1} = \frac{\Omega}{V}, \Omega = (\omega_{ij})_{\frac{m+2}{2}\times\frac{m+2}{2}}$*, where*

$$\omega_{11} = (m+4)M^{\frac{m}{2}} + (-1)^{\frac{m+2}{2}}2(m+1)(m+2)(-2m)^{\frac{m}{2}},$$

$$\omega_{1j} = (-1)^{1+j}(m+4j)M^{\frac{m}{2}-(j-1)}(-2m)^{j-1},\ 2 \le j \le \frac{m+2}{2},$$

$$\omega_{ij} = (-1)^{i+j}(m+4j)M^{\frac{m}{2}-(j-i)}(-2m)^{j-i},\ 2 \le i \le j \le \frac{m+2}{2},$$

$$\omega_{i1} = (-1)^{\frac{m}{2}+i}m(m+1)M^{i-1}(-2m)^{\frac{m}{2}-(i-1)},\ 2 \le i \le \frac{m+2}{2},$$

$$\omega_{ij} = (-1)^{\frac{m}{2}+i+j}(m+1)(m+4j)M^{i-j}(-2m)^{\frac{m}{2}-(i-j)},\ 2 \le j < i \le \frac{m+2}{2}.$$

Lemma 4. *[22]. We have* $Q = J_m - J_0, R = J_0$*, and* $Z = \frac{Q}{R} = \frac{J_m}{J_0} - 1$.

For Lemma 2, we have $J'' = A^{-1}(I - A')J'$, where I is the $\frac{m+2}{2} \times \frac{m+2}{2}$ identity matrix, which yields Lemma 5.

Lemma 5. *We have*

$$J_0'' = \frac{1}{V}\left(-\frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{1\frac{m+2}{2}}J_0' + \sum_{j=1}^{\frac{m+2}{2}}\frac{-m+4j-4}{m+4j}\omega_{1j}J'_{2(j-1)}\right),$$

$$J_m'' = \frac{1}{V}\left(-\frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{\frac{m+2}{2}\frac{m+2}{2}}J_0' + \sum_{j=1}^{\frac{m+2}{2}}\frac{-m+4j-4}{m+4j}\omega_{\frac{m+2}{2}j}J'_{2(j-1)}\right).$$

Let

$$\bar{Z} = \frac{J_m}{J_0},\ \text{and}\ \tilde{Z} = \frac{J_m'}{J_0'}, \tag{13}$$

then we list their properties in Lemmas 6, 7, 8, and 9.

Lemma 6. *[22]. If* $\bar{Z}'(h_0) = 0$ *for some* $0 < h_0 < H^*$*, then* $\frac{2(m+1)(m+2)}{3m+4} < \bar{Z}(h_0) < m+1$.

Lemma 7. *If* $0 < h < H^*$*, then* $\tilde{Z}' > 0$.

Proof . *Applying Lemma 5, we derive the following Ricatti equation*

$$
\begin{aligned}
\tilde{Z}'(h) &= \left(\frac{J'_m}{J'_0}\right)' = \frac{J''_m}{J'_0} - \frac{J''_0 J'_m}{(J'_0)^2} \\
&= \frac{1}{VJ'_0}\left(-\frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{\frac{m+2}{2}\frac{m+2}{2}} J'_0 + \sum_{j=1}^{\frac{m+2}{2}} \frac{-m+4j-4}{m+4j}\omega_{\frac{m+2}{2}j} J'_{2(j-1)}\right) \\
&\quad - \frac{J'_m}{V(J'_0)^2}\left(-\frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{1\frac{m+2}{2}} J'_0 + \sum_{j=1}^{\frac{m+2}{2}} \frac{-m+4j-4}{m+4j}\omega_{1j} J'_{2(j-1)}\right) \\
&= \frac{1}{V}\left(-\frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{\frac{m+2}{2}\frac{m+2}{2}} - \frac{m}{m+4}\omega_{\frac{m+2}{2}1} + \sum_{j=2}^{\frac{m}{2}} \frac{-m+4j-4}{m+4j}\omega_{\frac{m+2}{2}j}\frac{J'_{2(j-1)}}{J'_0}\right. \\
&\quad \left. + \frac{2m}{3m+4}\omega_{\frac{m+2}{2}\frac{m+2}{2}}\frac{J'_m}{J'_0} - \sum_{j=2}^{\frac{m}{2}} \frac{-m+4j-4}{m+4j}\omega_{1j}\frac{J'_{2(j-1)}}{J'_0}\frac{J'_m}{J'_0} - \frac{m}{3m+4}\omega_{1\frac{m+2}{2}}\left(\frac{J'_m}{J'_0}\right)^2\right) \\
&= \frac{1}{V}\left(-\frac{m}{3m+4}\omega_{1\frac{m+2}{2}}\left(\frac{J'_m}{J'_0} - \frac{\omega_{\frac{m+2}{2}\frac{m+2}{2}}}{\omega_{1\frac{m+2}{2}}}\right)^2 + F(h)\right),
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
F_m(h) &= \frac{m}{3m+4}\frac{\left(\omega_{\frac{m+2}{2}\frac{m+2}{2}}\right)^2}{\omega_{1\frac{m+2}{2}}} - \frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{\frac{m+2}{2}\frac{m+2}{2}} - \frac{m}{m+4}\omega_{\frac{m+2}{2}1} \\
&\quad + \frac{1}{J'_0}\left(m+1-\frac{J'_m}{J'_0}\right)\sum_{j=2}^{\frac{m}{2}} \frac{-m+4j-4}{m+4j}\omega_{1j}J'_{2(j-1)}.
\end{aligned} \tag{15}
$$

Note that $J'_{2(j-1)} = \frac{1}{V}\sum_{k=1}^{\frac{m+2}{2}} \omega_{jk}J_{2(k-1)}$*, then we further have*

$$
\begin{aligned}
F_m(h) &= \frac{m}{3m+4}\frac{\left(\omega_{\frac{m+2}{2}\frac{m+2}{2}}\right)^2}{\omega_{1\frac{m+2}{2}}} - \frac{2m(m+1)(m+2)}{(m+4)(3m+4)}\omega_{\frac{m+2}{2}\frac{m+2}{2}} - \frac{m}{m+4}\omega_{\frac{m+2}{2}1} \\
&\quad + \frac{1}{VJ'_0}\left(m+1-\frac{J'_m}{J'_0}\right)\sum_{k=1}^{\frac{m+2}{2}}\sum_{j=2}^{\frac{m}{2}} \frac{-m+4j-4}{m+4j}\omega_{1j}\omega_{jk}J_{2(k-1)}.
\end{aligned} \tag{16}
$$

Since V *and the first term in the brackets in (14) are negative, if we can prove* $F_m(h) < 0$*, then the conclusion holds. Here we apply numerical technique to prove* $F_m(h) < 0$*. Specifically, for each even* m*, let* h *changes from* 0.01 *to* H^* *with step* 0.01*, then we can numerically evaluate* $a(h), b(h)$ *and* $J_0, J_2, \cdots, J_m, J'_0, J'_m$ *through (11) and (12), and the values of* $F_m(h)$ *follow. Further, for each even* m*, we find the maximum of* $F_m(h)$ *with respect to* h*, denoted by* $max(F_m(h))$*. The profile of* $max(F_m(h))$ *with respect to* m *from* 2 *to* 80 *are illustrated in Figure 2, which implies that* $F_m(h) < 0$*, for* $0 < h < H^*$*.*

Remark 1. *Although we choose a specific step* 0.01 *in the numerical technique, the step can be any small number.*

Remark 2. *As indicated in [25], it is difficult to prove* $\tilde{Z}'(h) > 0$ *for* $m \geq 5$ *analytically. Here,*

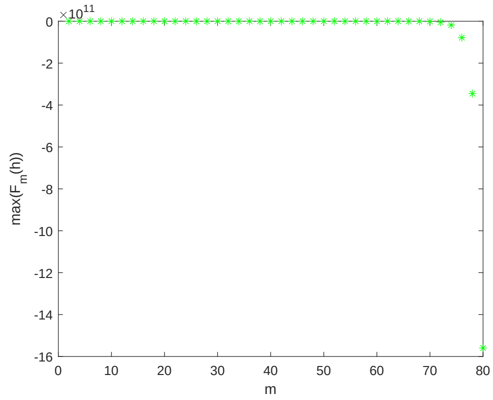

Figure 2. The profile of $max(F_m(h))$ with respect to m from 2 to 80.

we adopt the numerical technique to show that $\tilde{Z}'(h) > 0$ *for general even* m*, which partially solve the open question given in [22].*

Exploiting Lemma 7 and performing the similar analysis as that in [23–25], we then obtain Lemma 8.

Lemma 8. *If* $\bar{Z}'(h_0) = 0$ *for some* $0 < h_0 < H^*$*, then* $\bar{Z}''(h_0) < 0$*.*

Combining the results in Lemmas 1, 6 and 8, we derive the monotonicity of $\bar{Z}$ through the proof by contradiction.

Lemma 9. *If* $0 < h < H^*$*, then* $\bar{Z}'(h) > 0$*.*

Now we also complete the proof about the monotonicity of Z in Proposition 1 since $Z = \bar{Z} - 1$. Furthermore, the monotonicity of $c_0(h)$ and its upper and lower bounds in Theorem 1 follow from $c_0(h) = Z^{-1}$.

Remark 3. *Although we show that* $\tilde{Z}'(h) > 0$ *for general even* m*, unfortunately,* $\tilde{Z}'(h)$ *are not always positive for general odd* m *when we perform the same procedure.*

§3 Conclusions

In this paper, we develop the numerical technique to derive the monotonicity of $c_0(h)$ for the perturbed gKdV equation (2) with general even m, which partially solve the open problem presented by [22]. However, the present framework cannot be applied to general odd m, which is still under consideration.

Declarations

Conflict of interest The authors declare no conflict of interest.

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