Appl. Math. J. Chinese Univ.
2025, 40(3): 752-762

Stability analysis of conformable fractional order systems

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Abstract. In this paper, we study the stability of a class of conformable fractional-order systems using the Lyapunov function. We assume that the nonlinear part of the system satisfies the one-sided Lipschitz condition and the quadratic inner-bounded condition. We provide some sufficient conditions that ensure the asymptotic stability of the system. Furthermore, we present the construction of a feedback stabilizing controller for conformable fractional bilinear systems.

§1 Introduction

Fractional differential equations have proven to be powerful tools for modeling many physical phenomena. They have been widely applied in areas such as nonlinear oscillations in earthquakes, seepage flow in porous media, and fluid-dynamic traffic models. Significant progress has been made in the study of fractional ordinary and partial differential equations. For more details on fractional calculus theory, we refer the reader to the monographs of Kilbas et al. [9], Miller and Ross [10], Podlubny [11], Tarasov [12], and the papers of Agarwal et al. [3,4].

In recent years, several definitions of fractional derivatives have been introduced, such as the Riemann-Liouville, Grunwald-Letnikov, and Caputo definitions. Recently, Khalil et al. [8] proposed a new definition of the fractional derivative that is highly compatible with the classical derivative. Unlike other definitions, this new approach satisfies the product and quotient rules for derivatives of two functions and has a simpler chain rule. In addition to the conformable fractional derivative, the corresponding conformable fractional integral, Rolle's theorem, and the mean value theorem for conformable derivatives have also been established.

The concept of fractional differentiable functions was introduced, and another study [2] by Abdeljawad contributed to this new field. He presented left and right conformable fractional derivatives, as well as fractional integrals of higher-order concepts. Moreover, he provided the fractional chain rule, the fractional integration by parts formulas, the Gronwall inequality, the

Received: 2022-09-19. Revised: 2022-12-16.
MR Subject Classification: 93D05, 93D20.
Keywords: conformable fractional, exponential stability, asymptotical stability, one-sided Lipschitz.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-4851-3.

fractional power series expansion, and the fractional Laplace transform definition. In a short time, many studies related to this new fractional derivative definition were conducted [6,7,14].

For obvious reasons, the stability of differential systems is one of the fundamental topics in science. Recently, stability problems of nonlinear fractional systems have been extensively investigated by many authors [5], particularly regarding conformable fractional-order nonlinear systems. Souahi et al. presented some results on the stability and asymptotic stability of conformable fractional-order nonlinear systems using the Lyapunov function. These results can be taken as a starting point in this vast field. However, few contributions addressing the asymptotic stability of conformable fractional systems have been reported in the literature, which motivates us to carry out this work.

In this paper, motivated by previous works, we study the stability of conformable fractional-order derivative systems using Lyapunov functions. We assume that the nonlinear part of the system satisfies the one-sided Lipschitz condition and the quadratic inner-bounded condition, and we provide some sufficient conditions for the asymptotic stability of the system. The construction of a feedback-stabilizing controller for conformable fractional bilinear systems is also presented.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and the necessary lemmas. In Section 3, we present our main result. In Section 4, an example illustrates the validity of the proposed method. Finally, the conclusions are presented in Section 5.

§2 Preliminary

Notation 1. *Throughout the paper, A^T denotes the transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of a matrix A, respectively. $P > 0$ means that the matrix P is symmetric and positive definite. I denotes an identity matrix of appropriate dimension. $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$; that is, given $x, y \in \mathbb{R}^n$, we have $\langle x, y \rangle = x^T y$. The notation $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^n$, defined by $\|x\| = \sqrt{\langle x, x \rangle}$.*

We next recall some classical definitions and results that are essential for our study.

Definition 1. *([8]) Given a function g defined on $[a, \infty)$, the conformable fractional derivative of g of order α is defined by*

$$T^{\alpha}_{t_0} g(t) = \lim_{\varepsilon \to 0} \frac{g(t + \varepsilon t^{1-\alpha}) - g(t)}{\varepsilon}, \tag{1}$$

for all $t > a$ and $\alpha \in (0, 1]$.

Definition 2. *([8]) The conformable fractional integral of order $0 < \alpha \leq 1$ starting from a for a function g is defined by*

$$I^{\alpha}_{a} g(t) = \int_a^t (x - a)^{\alpha - 1} g(x)\, dx. \tag{2}$$

Considering the following conformable fractional nonlinear system

$$T_{t_0}^{\alpha} x(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{3}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying $f(t, 0) = 0$ for every $t \geq 0$.

Definition 3. *(Fractional exponential stability [2]): The solution of the system (3) is called fractional exponentially stable if*

$$\|x(t)\| \leq M\|x_0\| E_{\alpha}(-\lambda, t - t_0), \quad t \geq t_0, \tag{4}$$

where $0 < \alpha \leq 1$, $\lambda > 0$, $M > 0$, *and* $E_{\alpha}(\lambda, t) = \exp\left(\lambda \frac{t^{\alpha}}{\alpha}\right)$.

Definition 4. *The origin of the system (3) is said to be stable if, for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\|x(t)\| < \varepsilon$ *for all* $t \geq t_0$ *whenever* $\|x_0\| < \delta$.

The origin is said to be asymptotically stable if it is stable and additionally satisfies

$$\lim_{t \to +\infty} x(t) = 0.$$

Theorem 1. *([13]) Let* $x = 0$ *be the equilibrium point of the fractional-order system (3). Assume that there exists a fractional Lyapunov function* $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ *and positive constants* λ_i, $i = 1, 2, 3$, *satisfying*

(i) $\lambda_1 \|x\|^2 \leq V(t, x(t)) \leq \lambda_2 \|x\|^2$;

(ii) $T_{t_0}^{\alpha} V(t, x(t)) \leq -\lambda_3 \|x\|^2$.

Then, the origin of system (3) is fractionally exponentially stable.

Remark 1. *Fractional exponential stability implies asymptotic stability.*

Lemma 1. *[13] Let* $0 < \alpha < 1$ *and let* $g : [t_0, \infty) \to \mathbb{R}_+$ *be a continuous function and* α-*differentiable on* (t_0, ∞), *such that*

$$T_{t_0}^{\alpha} g(t) \leq -\lambda g(t),$$

where λ *is a positive constant. Then,*

$$g(t) \leq E_{\alpha}(-\lambda, t - t_0) g(t_0).$$

Lemma 2. *([13]) Let* $x : [a, \infty) \to \mathbb{R}^n$ *such that* $T_a^{\alpha} x(t)$ *exists on* (a, ∞) *and* P *a symmetric positive definite matrix. Then* $T_a^{\alpha} x(t)^T P x(t)$ *exists on* (a, ∞) *and*

$$T_a^{\alpha} x(t)^T P x(t) = 2x(t)^T P T_a^{\alpha} x(t), \forall t > a.$$

Lemma 3. *([2]) Let* $A \in \mathbb{R}^{n \times n}$ *be a constant matrix. Then, the solution of the following conformable fractional differential system*

$$T_{t_0}^{\alpha} x(t) = Ax(t) + f(t, x(t)), \; x(t_0) = x_0, \tag{5}$$

is given by

$$x(t) = x_0 \exp(A\frac{(t - t_0)^{\alpha}}{\alpha}) + \int_{t_0}^{t} \exp(A\frac{(t - t_0)^{\alpha}}{\alpha}) \times \exp(-A\frac{(s - t_0)^{\alpha}}{\alpha}) f(s, x(s))(s - t_0)^{\alpha} ds. \tag{6}$$

Definition 5. *([1]) The nonlinear function $f(t,x)$ is said to be one-sided Lipschitz if there exists $\rho \in \mathbb{R}$ such that*

$$\langle f(t,x_1) - f(t,x_2), x_1 - x_2 \rangle \leq \rho \|x_1 - x_2\|^2, \tag{7}$$

where ρ is called the one-sided Lipschitz constant.

Definition 6. *([1]) The nonlinear function $f(t,x)$ is called quadratically inner-bounded if there exist constants $\beta, \gamma \in \mathbb{R}$ such that*

$$[f(t,x_1) - f(t,x_2)]^T [f(t,x_1) - f(t,x_2)] \leq \beta \|x_1 - x_2\|^2 + \gamma \langle x_1 - x_2, f(t,x_1) - f(t,x_2) \rangle. \tag{8}$$

Remark 2. *If a function $f(t,x)$ satisfies the Lipschitz condition, then it also satisfies the one-sided Lipschitz condition and the quadratic inner-bounded condition. However, the converse is not necessarily true. In many cases, the one-sided Lipschitz constant can be much smaller than the standard Lipschitz constant. Moreover, the constants ρ, β, and $\gamma \in \mathbb{R}$ can take arbitrary real values, whereas the Lipschitz constant must be positive. Therefore, the class of nonlinearities considered in this paper is quite general.*

§3 Main results

3.1 Stability

In this section, we focus on the following conformable fractional differential system. The main objective is to analyze the asymptotic stability of the system

$$T_{t_0}^{\alpha} x(t) = Ax(t) + f(t, x(t)), \quad x(t_0) = x_0, \tag{9}$$

where $A \in \mathbb{R}^{n \times n}$ is a constant matrix and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying $f(t, 0) = 0$ for all $t \geq 0$.

Let S be the symmetric positive definite solution of the Lyapunov equation

$$A^T S + SA = -Q, \tag{10}$$

for a given symmetric positive definite matrix Q. Then, the following stability result holds

Theorem 2. *Let $\alpha \in (0,1]$. Suppose that the function $f(t,x)$ satisfies the conditions* (7) *and* (8) *with constants ρ, β, and γ. Suppose also that there exist a scalar $r \neq 0$ and two matrices S and Q satisfying the Lyapunov equation* (10)*, such that*

$$\left\{ (r^2\beta + 1) + \rho(\gamma + 2r) \right\} \lambda_{\max}(S) < \lambda_{\min}(S) + \lambda_{\min}(Q). \tag{11}$$

Then, the origin of system (9) *is fractionally exponentially stable.*

Proof Let us consider the following Lyapunov function candidate:

$$V(t, x(t)) = x^T(t) S x(t). \tag{12}$$

From Lemma 2, we obtain

$$\begin{aligned} T_{t_0}^{\alpha} V(t, x(t)) &= x^T(t)(A^T S + SA)x(t) + 2x^T(t) S f(t, x(t)) \\ &\leq -x^T(t) Q x(t) + 2x^T(t) S f(t, x(t)). \end{aligned}$$

For brevity, denote $f(t,x(t))$ by f_t. Note that

$$2x^T S f_t = [x+f_t]^T S [x+f_t] - x^T S x - f_t^T S f_t.$$

Since S is symmetric positive definite, it satisfies

$$\lambda_{\min}(S)\|f_t\|^2 \le f_t^T S f_t \le \lambda_{\max}(S)\|f_t\|^2. \tag{13}$$

Moreover

$$[x+f_t]^T S [x+f_t] \le \lambda_{\max}(S)\|x+f_t\|^2. \tag{14}$$

Using the quadratic inner-boundedness condition (8) and the fact that $f(t,0)=0$, we have

$$f_t^T f_t \le \beta x^T x + \gamma\langle x, f_t\rangle.$$

Thus,

$$\begin{aligned}\frac{1}{r}\|x+rf_t\|^2 &= x^T x + 2rx^T f_t + r^2 f_t^T f_t\\ &\le x^T x + 2rx^T f_t + r^2\beta x^T x + r^2\gamma\langle x, f_t\rangle\\ &\le (r^2\beta+1)x^T x + (\gamma+2r)x^T f_t.\end{aligned}$$

Now, combining inequalities (14) and (7), we get

$$\begin{aligned}2x^T S f_t &\le \lambda_{\max}(S)\left[(r^2\beta+1)x^T x + (\gamma+2r)x^T f_t\right] - x^T S x - \lambda_{\min}(S)\|f_t\|^2\\ &\le \left\{\lambda_{\max}(S)\left[(r^2\beta+1)+\rho(\gamma+2r)\right] - \lambda_{\min}(S)\right\}x^T x,\end{aligned}$$

where we used $x^T f_t \le \rho\|x\|^2$ from condition (7).

Therefore,

$$\begin{aligned}T_{t_0}^{\alpha} V(t,x(t)) &\le -x^T Q x + \left\{\lambda_{\max}(S)\left[(r^2\beta+1)+\rho(\gamma+2r)\right] - \lambda_{\min}(S)\right\}x^T x\\ &\le -\left\{\lambda_{\min}(Q) + \lambda_{\min}(S) - \lambda_{\max}(S)\left[(r^2\beta+1)+\rho(\gamma+2r)\right]\right\}\|x\|^2.\end{aligned}$$

By condition (11), the coefficient of $\|x\|^2$ is positive. Hence, according to Theorem 1, the origin of system (9) is fractionally exponentially stable. The proof is complete. □

3.2 Stabilization

Let us now consider the following conformable fractional differential system:

$$T_{t_0}^{\alpha} x(t) = Ax(t) + Bu + f(t,x(t)), \quad x(t_0) = x_0, \tag{15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are constant matrices of dimensions $n \times n$ and $n \times m$, respectively, and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying $f(t,0)=0$ for all $t \ge 0$. Then, the following theorem holds.

Theorem 3. *For $0 < \alpha \le 1$, suppose that the function $f(t,x(t))$ satisfies the conditions* (7) *and* (8) *with constants ρ, β, and γ. Suppose also that there exist a symmetric positive definite matrix P, a constant matrix $K \in \mathbb{R}^{m\times n}$, a scalar $r \ne 0$, and a positive scalar $\epsilon > 0$ such that the following inequalities hold:*

$$(A+BK)^T P + P(A+BK) = -\epsilon I, \tag{16}$$

$$\left\{(r^2\beta+1)+\rho(\gamma+2r)\right\}\lambda_{\max}(P) < \lambda_{\min}(P) + \epsilon. \tag{17}$$

Then, the control law $u(x) = Kx$ renders system (15) *fractionally exponentially stable.*

Proof Consider the following Lyapunov function candidate:

$$V(x) = x^T(t)Px(t). \tag{18}$$

From Lemma 2, we have

$$\begin{aligned} T_{t_0}^{\alpha} V(t) &= x^T(t)\left[(A+BK)^T P + P(A+BK)\right]x(t) + 2x^T(t)Pf(t,x(t)) \\ &= -\epsilon\, x^T(t)x(t) + 2x^T(t)Pf(t,x(t)) \\ &\le -\epsilon\,\|x(t)\|^2 + 2x^T(t)Pf(t,x(t)). \end{aligned}$$

Using the same argument as in the proof of Theorem 2, and under conditions (7) and (8), we obtain the inequality

$$2x^T(t)Pf(t,x(t)) \le \left\{\lambda_{\max}(P)\left[(r^2\beta+1)+\rho(\gamma+2r)\right] - \lambda_{\min}(P)\right\}\|x(t)\|^2.$$

Therefore,

$$\begin{aligned} T_{t_0}^{\alpha} V(t) &\le -\epsilon\,\|x(t)\|^2 + \left\{\lambda_{\max}(P)\left[(r^2\beta+1)+\rho(\gamma+2r)\right] - \lambda_{\min}(P)\right\}\|x(t)\|^2 \\ &= -\left\{\epsilon + \lambda_{\min}(P) - \lambda_{\max}(P)\left[(r^2\beta+1)+\rho(\gamma+2r)\right]\right\}\|x(t)\|^2. \end{aligned}$$

By condition (17) and according to Theorem 1, the origin of the closed-loop system

$$T_{t_0}^{\alpha} x(t) = (A+BK)x(t) + f(t,x(t))$$

is fractionally exponentially stable. The proof is complete. □

3.3 Stabilization of conformable fractional perturbed systems

Let us now consider the following system:

$$T_{t_0}^{\alpha} x(t) = Ax(t) + Bu + f(t,x(t)) + h(t,x(t)), \quad x(t_0) = x_0, \tag{19}$$

where $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear perturbation satisfying the following condition. There exist constants $\sigma > 0$ and $\omega > 0$ such that

$$\|h(t,x)\| < \omega\|x\| \quad \text{whenever} \quad \|x\| < \sigma. \tag{20}$$

Then, the following theorem holds:

Theorem 4. *For $0 < \alpha \le 1$, suppose that the function $f(t,x(t))$ satisfies the conditions (7) and (8) with constants ρ, β, and γ, and that the perturbation $h(t,x(t))$ satisfies condition (20) with constants $\sigma > 0$ and $\omega > 0$. Suppose also that there exist a symmetric positive definite matrix P, a constant matrix $K \in \mathbb{R}^{m\times n}$, and a positive scalar $\epsilon > 0$ such that the following inequalities hold.*

$$(A+BK)^T P + P(A+BK) = -\epsilon I, \tag{21}$$

$$\{(\beta+1)+\rho(\gamma+2)\}\,\lambda_{\max}(P) < \lambda_{\min}(P) - \omega + \epsilon. \tag{22}$$

Then, the control law $u(x) = Kx$ renders system (19) locally asymptotically stable.

Proof Consider the Lyapunov function candidate:

$$V(x) = x^T(t)Px(t). \tag{23}$$

From Lemma 2, we can conclude

$$
\begin{aligned}
lllT_{t_0}^{\alpha}V(t) &= x^T(t)((A+BK)^TP + P(A+BK))x(t) + 2x^T(t)Pf(t,x(t)) + 2x^T(t)Ph(t,x(t)) \\
&\leq -\epsilon x^T(t)x(t) + 2x^T(t)Pf(t,x(t)) + 2x^T(t)Ph(t,x(t)) \\
&\leq -\epsilon x^T(t)x(t) + \{\lambda_{\max}(P)\left[(\beta+1)+\rho(\gamma+2)\right] - \lambda_{\min}(P)\}\, x^T(t)x(t) + \omega||x(t)||^2 \\
&\leq -\{\epsilon + \lambda_{\min}(P) - \omega - \lambda_{\max}(P)\left[(\beta+1)+\rho(\gamma+2)\right]\}\, \|x(t)\|^2.
\end{aligned}
$$

By Theorem 1, it is easy to verify that the origin of the closed-loop system $T_{t_0}^{\alpha}x(t) = Ax(t) + BKx(t) + f(t,x(t)) + h(t,x(t))$ is locally asymptotically stable.

§4 Stabilization of conformable fractional bilinear systems

In this section, we study the stabilization of the following conformable fractional bilinear system with multiple inputs, in a constructive manner

$$T_{t_0}^{\alpha}x(t) = Ax(t) + \sum_{i=1}^{p} u_iB_ix(t), \quad t \in \mathbb{R}, \tag{24}$$

where $x(t) \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ for all $i \in \{1,\ldots,p\}$, and $A, B_i \in \mathbb{R}^{n\times n}$ are constant matrices.

Definition 7. *The control system* (24) *is said to be fractionally exponentially stabilizable via a feedback control* $u = u(x)$ *if the resulting closed-loop system*

$$T_{t_0}^{\alpha}x(t) = Ax(t) + \sum_{i=1}^{p} u_i(x)B_ix(t)$$

is fractionally exponentially stable.

In the sequel, we introduce the following assumption

$$\mathbf{(H)} : \bigcap_{i=1}^{p} \mathbf{S}_i = \{0\}, \tag{25}$$

where

$$\mathbf{S}_i = \{x \in \mathbb{R}^n : \langle B_ix, x\rangle = 0\}, \quad i \in \{1,2,\ldots,p\}.$$

Theorem 5. *If condition* ***(H)*** *holds, then there exist bounded feedback laws*

$$u_i(x) = -c\frac{\langle B_ix, x\rangle}{\|x\|^2}, \quad \forall i \in \{1,2,\ldots,p\}, \tag{26}$$

where $c > 0$ *is a constant to be chosen appropriately, such that the closed-loop system* (24) *is fractionally exponentially stable.*

Proof Let us consider the quadratic function

$$V(x) = \frac{1}{2}\|x\|^2.$$

which is positive definite. If condition **(H)** holds, then the fractional derivative of V along the solutions of the closed-loop system (24) under the feedback (26) becomes

$$\begin{aligned} T_{t_0}^{\alpha} V(x(t)) &= \langle Ax, x\rangle + \sum_{i=1}^{p} u_i(x)\langle B_i x, x\rangle \\ &= \langle Ax, x\rangle - \frac{c}{\|x\|^2}\sum_{i=1}^{p}\langle B_i x(t), x(t)\rangle^2. \end{aligned}$$

We have

$$\langle Ax, x\rangle = \langle A_s x, x\rangle,$$

where $A_s = \frac{A+A^T}{2}$, and let

$$\lambda_{\max} = \max_{x\neq 0}\frac{\langle A_s x, x\rangle}{\|x\|^2} = \max_{y\in \mathbf{S}^{n-1}}\langle A_s y, y\rangle.$$

Let

$$f(y) = \sum_{i=1}^{p}\langle B_i y, y\rangle^2, \quad \forall y \in \mathbf{S}^{n-1}.$$

It is easy to verify that the function f is continuous and differentiable on the compact set $\mathbf{S}^{n-1}$, so $f(y)$ attains both a maximum and a minimum on $\mathbf{S}^{n-1}$. Therefore, there exist two real numbers $\mathfrak{m}$ and $\mathfrak{M}$ such that

$$0 < \mathfrak{m} \leq f(y) \leq \mathfrak{M} \quad \text{for all } y \in \mathbf{S}^{n-1}.$$

Hence,

$$\begin{aligned} T_{t_0}^{\alpha} V(x) &\leq \lambda_{\max}\|x\|^2 - cf(y) \\ &\leq \lambda_{\max}\|x\|^2 - c\mathfrak{m}\|x\|^2 \\ &\leq (\lambda_{\max} - c\mathfrak{m})\|x\|^2. \end{aligned}$$

If we choose the positive constant c such that

$$c > \frac{\lambda_{\max}}{\mathfrak{m}},$$

then $\lambda_{\max} - c\mathfrak{m} < 0$, and it follows that

$$\begin{aligned} T_{t_0}^{\alpha} V(x(t)) &\leq (\lambda_{\max} - c\mathfrak{m})\|x(t)\|^2 \\ &\leq -2(c\mathfrak{m} - \lambda_{\max})V(x(t)). \end{aligned}$$

Using Lemma 1, we obtain

$$\begin{aligned} V(x(t)) &\leq E_\alpha\left(-2(c\mathfrak{m} - \lambda_{\max}), t - t_0\right) V(x(t_0)), \quad \forall t \geq t_0 \\ &\leq \exp\left(-2(c\mathfrak{m} - \lambda_{\max})\frac{(t-t_0)^\alpha}{\alpha}\right) V(x(t_0)). \end{aligned}$$

Therefore,

$$\|x(t)\| \leq \sqrt{2}\, E_\alpha\left(-(c\mathfrak{m} - \lambda_{\max}), t - t_0\right)\|x(t_0)\|,$$

So, the closed-loop system (24) is fractionally exponentially stable. □

Remark 3. *For the single-input system $T_{t_0}^{\alpha} x = Ax + uBx$, our method remains effective. Indeed, by replacing condition **(H)** with*

$$\{x \in \mathbb{R}^n : \langle Bx, x\rangle = 0\} \subseteq \{x \in \mathbb{R}^n : \langle Ax, x\rangle < 0\},$$

we can construct the feedback control law

$$u_c(x) = -c\frac{\langle Bx, x\rangle}{\|x\|^2}.$$

Remark 4. *If there exist real scalars $k_1, k_2, \ldots, k_p$, not all zero, such that the linear combination $\sum_{i=1}^{p} k_i B_{i_s}$ is symmetric and either positive definite or negative definite, then condition **(H)** is satisfied, i.e.,*

$$\bigcap_{i=1}^{p} \mathbf{S}_i = \{0\}.$$

This means that the quadratic forms associated with the B_i's are jointly non-degenerate.

§5 Numerical example

Let us consider the following system:

$$\begin{aligned} T_{t_0}^{\alpha} x_1(t) &= -x_1(t) + x_2(t) - x_1(t)(x_1^2(t) + x_2^2(t)), \\ T_{t_0}^{\alpha} x_2(t) &= -2x_1(t) + x_2(t) - x_2(x_1^2(t) + x_2^2(t)) + u(t). \end{aligned} \tag{27}$$

System (27) can be rewritten as the form of (15) with

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$f(t, x(t)) = -(x_1^2(t) + x_2^2(t)) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The function f satisfies the one-sided Lipschitz condition and the quadratic inner-boundedness inequality with parameters $\rho = 0$, $\beta = -100$, and $\gamma = -99$, (see [1]). Now, select

$$K = \begin{bmatrix} -2 & 5 \end{bmatrix}$$

. Then $A_K = A + BK$ is Hurwitz. We also choose $\epsilon = 1$. The matrix P is given by

$$P = \begin{bmatrix} 2.2500 & -0.9167 \\ -0.9167 & 0.5833 \end{bmatrix},$$

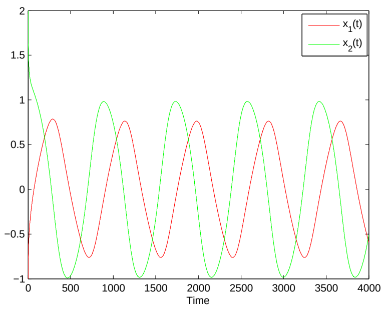
and

$$\begin{aligned} lllT_{t_0}^{\alpha} V(t) &= x^T(t)((A + BK)^T P + P(A + BK))x(t) + 2x^T(t)Pf(t, x(t)) \\ &\leq -\{\epsilon + \lambda_{\min}(P) - \lambda_{\max}(P)\left[(\beta + 1) + \rho(\gamma + 2)\right]\} \|x(t)\|^2 \\ &\leq -264.0723||x(t)||^2. \end{aligned}$$

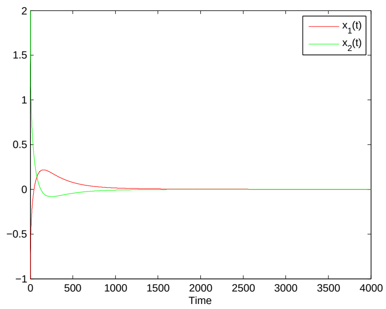
.

Hence, the system (27) is fractionally exponentially stable.

The numerical solution of system (27) is shown in Figure 1b for a fractional order $\alpha = 0.9$. It indicates that the zero solution is asymptotically stable.



(a) Open-loop response.



(b) Closed-loop response.

Figure 1. State trajectories $x_1(t)$ and $x_2(t)$ for Example 27 with initial conditions $x_1(0) = -1$, $x_2(0) = 2$, and $\alpha = 0.9$.

§6 Conclusion

In this paper, we start by the stability of a class of conformable fractional order systems using the Lyapunov function. We suppose that the nonlinear part of the system satisfies the one-sides Lipschitz and quadratic inner-bounded condition, and we give some sufficient conditions which imply the asymptotical stability of the system. The stabilization of conformable fractional bilinear systems is studied. A numerical example is given to illustrate the efficiency of the obtained results.

Declarations

Conflict of interest The authors declare no conflict of interest.

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