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Weighted estimates for multilinear commutators of multilinear singular integral operators with generalized kernels

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Abstract. In this paper, the authors study the multilinear commutators generated by a class of multilinear singular integral operators with generalized kernels and Lipschitz functions. By establishing the sharp maximal estimates, the boundedness of this kind of multilinear commutators on product of weighted Lebesgue spaces can be obtained.

§1 Introduction

Coifman and Meyer first studied the multilinear Calderón-Zygmund theory in [1–3]. Then this theory was further investigated by many authors in the last few decades such as [7,8,10]. It is noticed that the commutator is more singular than the singular integral operator. In recent years there has been an explosion of interest in the study of the multilinear commutators generated by multilinear Calderón-Zygmund operators or multilinear fractional integrals. Details can be found in the references [4, 11, 16, 19, 21]. Pérez and Trujillo-González introduced the multilinear commutator as a generalization of the commutator in [17]. And the sharp weighted estimates for vector valued singular integral operators and commutators was obtained in [18]. There are a number of studies concerning multilinear singular integral operators whose kernel conditions are more weaker than the standard Calderón-Zygmund class. Indeed, we are motivated by the work of Lin and Xiao, who studied a class of multilinear singular integral operators with generalized kernels and their multilinear commutators with BMO functions in [12]. In this paper, we consider replacing the above BMO functions by Lipschitz functions and finding out the corresponding conclusions.

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The multilinear operators T that we study are initially defined on the m-fold product of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and take their values into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. We will assume that the distributional kernel on $(\mathbb{R}^n)^{m+1}$ of the operator coincides away from the diagonal $y_0 = y_1 = \cdots = y_m$ with a function K so that

$$T\vec{f}(x) = T(f_1, \cdots f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \cdots y_m) \prod_{j=1}^{m} f_j(y_j) dy_1 \cdots dy_m, \tag{1}$$

where $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and $f_j (j = 1, \cdots m)$ are smooth functions with compact support. Moreover, we will also assume that the function K satisfies the standard estimates

$$|K(y_0, y_1, \cdots y_m)| \leq \frac{C}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}, \tag{2}$$

and, for some $C > 0$,

$$|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \leq \frac{C|y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}, \tag{3}$$

provided that $0 \leq j \leq m$ and $|y_j - y_j'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$. Such kernels are called m-linear Calderón-Zygmund kernel.

For the multilinear singular integral operator T defined by (1) associated with a standard m-linear Calderón-Zygmund kernel K, T is called an m-linear Calderón-Zygmund operator if it satisfies either of the following two conditions

(C1) T map $L^{t_1,1} \times \cdots \times L^{t_m,1}$ into $L^{t,\infty}$ if $t > 1$,

(C2) T map $L^{t_1,1} \times \cdots \times L^{t_m,1}$ into L^1 if $t = 1$,

for $1 \leq t_1, \cdots t_m, t < \infty$ and $\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_m}$, where $L^{t_1,1}, \cdots L^{t_m,1}$ and $L^{t,\infty}$ are the Lorentz spaces.

First of all, we introduce the m-linear Calderón-Zygmund kernel of type φ. For details, one can see [13,14,20,22] and the references therein. Let $\varphi(t)$ be a non-negative and non-decreasing function on $\mathbb{R}^+$. A locally integrable function $K(y_0, y_1, \cdots, y_m)$ defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ is called an m-linear Calderón-Zygmund kernel of type φ if it satisfies the size condition (2) and

$$\begin{aligned} &|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \\ &\leq \frac{C}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{mn}} \varphi\Big(\frac{|y_j - y_j'|}{|y_0 - y_1| + \cdots + |y_0 - y_m|}\Big), \end{aligned} \tag{4}$$

whenever $0 \leq j \leq m$ and $|y_j - y_j'| \leq \frac{1}{2} \max_{1 \leq k \leq m} |y_0 - y_k|$. In addition, let $\varphi(t) = t^\varepsilon$ for some $\varepsilon > 0$ in condition (4), then the condition (4) becomes (3) and the m-linear Calderón-Zygmund kernel of this type φ is exactly the standard m-linear Calderón-Zygmund kernel.

Above all, we will introduce a class of more general multilinear singular integral operators T defined by (1) with the kernel K satisfying the size condition (2) and a weaker condition. For any $k_1, \cdots, k_m \in \mathbb{N}_+$, there are positive constant $C_{k_i}, i = 1, \cdots, m$, such that

$$\begin{aligned} &\Big(\int_{2^{k_m}|y_0 - y_0'| \leq |y_m - y_0| < 2^{k_m+1}|y_0 - y_0'|} \cdots \int_{2^{k_1}|y_0 - y_0'| \leq |y_1 - y_0| < 2^{k_1+1}|y_0 - y_0'|} \\ &|K(y_0, y_1, \cdots, y_m) - K(y_0', y_1, \cdots, y_m)|^q dy_1 \cdots dy_m\Big)^{\frac{1}{q}} \end{aligned}$$

$$\leq C|y_0 - y_0'|^{-\frac{mn}{q'}} \prod_{i=1}^{m} C_{k_i} 2^{-\frac{nk_i}{q'}}, \tag{5}$$

where (q, q') is a fixed pair of positive numbers with $\frac{1}{q} + \frac{1}{q'} = 1$ and$1 < q < \infty$. Obviously, when $C_{k_i} = \varphi(2^{-k_i})^{\frac{1}{m}}$, $i = 1, \cdots, m$, the condition (5) includes the condition (4) with any $1 < q < \infty$. Thus the multilinear singular integral with the kernel of type φ can be regarded as a special case of the multilinear singular integral operator we will focus on.

Now we define the multilinear commutator generated by the multilinear integral operator and the Lipschitz function. First we recall the following definition of the Lipschitz function.

Definition 1.1 Let $0 < \beta < 1$. The Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined by

$$\|f\|_{Lip_\beta} = \sup_{x,h \in \mathbb{R}^n, h \neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

The notation $\vec{b} = (b_1, \cdots, b_m) \in Lip_\beta^m$ stands for $b_j \in Lip_\beta(\mathbb{R}^n)$ for $j = 1, \cdots, m$. We denote $\|\vec{b}\|_{Lip_\beta^m} = \max_{1 \leq j \leq m} \|b_j\|_{Lip_\beta}$.

Let T be an m-linear operator defined by (1). Given a collection of locally integrable functions $\vec{b} = (b_1, \cdots, b_m)$, then the m-linear commutator of T with $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \cdots, f_m) = \sum_{j=1}^{m} T_{\vec{b}}^j(\vec{f}),$$

where

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \cdots, f_m) - T(f_1, \cdots, f_{j-1}, b_j f_j, f_{j+1}, \cdots, f_m).$$

Before stating our main results, firstly we introduce some necessary notations and definitions.

In this paper, for $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. $E^c = \mathbb{R}^n \setminus E$ is the complementary set of E. $B(x, R)$ is the ball centered at x with radius $R > 0$, $CB(X, R) = B(x, CB)$ for $C > 0$, the Lebesgue measure of $B(x, R)$ is $|B(x, R)|$, and $f_{B(x,R)} = \frac{1}{|B(x,R)|} \int_{B(x,R)} f(y) d(y)$ denotes the average of f over B.

Definition 1.2 We say that a non-negative measurable function ω on $\mathbb{R}^n$ is in the Muckenhoupt class A_p with $1 < p < \infty$, if there exists a constant $C > 0$ such that for any cube Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes,

$$\left(\frac{1}{|Q|} \int_Q \omega(x) dx\right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'} dx\right)^{p-1} \leq C.$$

And for the situation $p = 1$, we say that a non-negative measurable function ω on $\mathbb{R}^n$ belongs to A_1, if there exists a constant $C > 0$ such that for any cube Q in $\mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q \omega(y) dy \leq C\omega(x), \qquad a.e. \quad x \in Q.$$

Denote by $A_\infty = \bigcup_{p \geq 1} A_p$. It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_q$ for some $1 < q < p$.

Definition 1.3 A locally integrable non-negative function ω on $\mathbb{R}^n$ is said to belong to the weight class $A(p, q)$, $1 < p, q < \infty$, if there exists a constant $C > 0$ such that for any cube Q,

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx\right)^{\frac{1}{p'}} \leq C.$$

Definition 1.4 The Hardy-Littlewood maximal operator M is defined by
$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy.$$
We set $M_s(f) = M(|f|^s)^{\frac{1}{s}}$, where $0 < s < \infty$. For a function $f \in L_{loc}(\mathbb{R}^n)$, the sharp maximal operator $M^\sharp$ is defined by
$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B| dy \sim \sup_{B \ni x} \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(y) - a| dy,$$
where the supremum is taken over all balls B containing x.

It is easy to check that the above definition is equivalent to the one by taking the supremum over all balls B centered at x. For $0 < \delta < \infty$, we denote by $M_\delta^\sharp$ the operator
$$M_\delta^\sharp(f) = [M^\sharp(|f|^\delta)]^{\frac{1}{\delta}}.$$

Definition 1.5 For $0 < \alpha, l < \infty$, the fractional maximal operator $M_{\alpha,l}$ is defined by
$$M_{\alpha,l}(f)(x) = \sup_{r>0} \left(\frac{1}{|B|^{1-\frac{\alpha l}{n}}} \int_B |f(y)|^l dy \right)^{\frac{1}{l}}.$$
Obviously, $M_{\alpha,l}(f)(x) = [M_{\alpha l,1}(|f|^l)(x)]^{\frac{1}{l}}$.

§2 Main results

Firstly, we will give the pointwise estimates for the sharp maximal functions of multilinear commutators generated by the multilinear singular operators T with generalized kernels and Lipschitz functions.

Theorem 2.1 *Let $m \geq 2$, T be an m-linear operator defined by (1) whose kernel satisfies the conditions (2) and (5) with $\sum_{k_i=1}^{\infty} C_{k_i} < \infty, i = 1, \cdots, m$. Suppose for fixed $1 \leq r_1, \cdots, r_m \leq q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in Lip_\beta^m$, $0 < \beta < 1, 0 < \delta < 1/m$, then*
$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \leq C \|\vec{b}\|_{Lip_\beta^m} \sum_{j=1}^m \Big(M_{\beta,\delta}(T(\vec{f}))(x) + M_{\beta,q'}(f_j)(x) \prod_{i \neq j, i=1}^m M_{q'}(f_i)(x) \Big),$$
for all m-tuples $\vec{f} = (f_1, \cdots, f_m)$ of bounded measurable functions with compact support.

Then, as applications of the maximal function estimates, we can establish the boundedness of multilinear commutators generated by multilinear singular operators T with generalized kernels and Lipschitz functions on product of weighted Lebesgue spaces.

Theorem 2.2 *Let $m \geq 2$, T be an m-linear operator defined by (1) whose kernel satisfies the conditions (2) and (5) with $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \leq r_1, \cdots, r_m \leq q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in Lip_\beta^m$, $0 < \beta < \min\{1, n/q'\}$, then for any $q' < p_1, \cdots, p_m < n/\beta$, $1/q_j = 1/p_j - \beta/n$, $j = 1, \cdots, m$, $p > 1$, $1/p = \sum_{i=1}^m 1/p_i$, $1/q_0 = 1/p - \beta/n$, $T_{\vec{b}}$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ into $L^{q_0}(\omega)$, where $\omega_j \in A_{p_j/q'}$, $\omega_j^{q'/p_j} \in A(p_j/q', q_j/q')$, $j = 1, \cdots, m$, $\omega = \prod_{j=1}^m \omega_j^{q_0/p_j}$ and $\omega^{1/q_o} \in A(p, q_0)$.*

§3 Lemmas

Before giving the proof of our main results, we need some lemmas as follows.

Lemma 3.1 ([6,9]) *Let $0 < p < q < \infty$, then there is a positive constant $C = C_{p,q}$ such that for any measurable function f there has*

$$|Q|^{-1/p}\|f\|_{L^p(Q)} \leq C|Q|^{-1/q}\|f\|_{L^{q,\infty}(Q)}.$$

Lemma 3.2 ([5]) *Let $0 < p, \delta < \infty$ and $\omega \in A_\infty$. Then there exists a constant $C > 0$ depending only on the A_∞ constant of ω such that*

$$\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p \omega(x)dx \leq C \int_{\mathbb{R}^n} [M^\sharp_\delta(f)(x)]^p \omega(x)dx,$$

for every function f such that the left-hand side is finite.

Lemma 3.3 ([12]) *Let $m \geq 2$, T be an m-linear operator defined by (1) whose kernel satisfies the conditions (2) and (5) with $\sum_{k_i=1}^\infty C_{k_i} < \infty$, $i = 1, \cdots, m$. Suppose for fixed $1 \leq r_1, \cdots, r_m \leq q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. Then for any $q' < p_1, \cdots, p_m < \infty$, with $1/p = 1/p_1 + \cdots + 1/p_m$, T is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ into $L^p(\omega)$, where $(\omega_1, \cdots, \omega_m) \in (A_{p_1/q'}, \cdots, A_{p_m/q'})$ and $\omega = \prod_{j=1}^m \omega_j^{p/p_j}$.*

Lemma 3.4 *For $1 < p, q < \infty$, $\omega \in A(p, q)$ if and only if $\omega^q \in A_{q/p'+1}$.*

The result of Lemma 3.4 directly comes from the definitions of the two kinds of weights.

Lemma 3.5 ([15]) *If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $\omega \in A(p, q)$, then there is a constant $C > 0$, independent of f, such that*

$$\Big(\int_{\mathbb{R}^n} [M_{\alpha,1}(f)(x)\omega(x)]^q dx\Big)^{1/q} \leq C\Big(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\Big)^{1/p}.$$

§4 Proof of main results

Now we are able to prove our main results. First, we will give the proof of Theorem 2.1.

Proof. Without loss of generality, we will only consider the situation $m = 2$ and omit other situations since their similarities.

Let f_1, f_2 be bounded measurable functions with compact support. Then for any ball $B = B(x_0, r_B)$ with center at x_0 and radius $r_B > 0$, we decompose f_1 and f_2 as follows

$$f_1 = f_1\chi_{16B} + f_1\chi_{(16B)^c} := f_1^1 + f_1^2,$$
$$f_2 = f_2\chi_{16B} + f_2\chi_{(16B)^c} := f_2^1 + f_2^2.$$

Choose a $z_0 \in 3B \setminus 2B$, then

$$\begin{aligned} T^1_{\vec{b}}(\vec{f})(z) &= (b_1(z) - b^1_B)T(f_1, f_2)(z) - T((b_1 - b^1_B)f_1, f_2)(z) \\ &= (b_1(z) - b^1_B)T(f_1, f_2)(z) - T((b_1 - b^1_B)f_1^1, f_2^1)(z) \\ &\quad - T((b_1 - b^1_B)f_1^1, f_2^2)(z) - T((b_1 - b^1_B)f_1^2, f_2^1)(z) \\ &\quad - T((b_1 - b^1_B)f_1^2, f_2^2)(z), \end{aligned}$$

where $b_B^1 = \frac{1}{|B|}\int_B b_1(z)dz$. Thus,

$$\begin{aligned}
&\bigg(\frac{1}{|B|}\int_B |T_{\vec{b}}^1(\vec{f})(z) + T((b_1 - b_B^1)f_1^2, f_2^2)(z_0)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\le C\bigg(\frac{1}{|B|}\int_B |(b_1(z) - b_B^1)T(f_1, f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)f_1^1, f_2^1)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)f_1^1, f_2^2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)f_1^2, f_2^1)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1 - b_B^1)f_1^2, f_2^2)(z) - T((b_1 - b_B^1)f_1^2, f_2^2)(z_0)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&:= \sum_{j=1}^{5} I_j.
\end{aligned}$$

We can estimate the five terms as follows. By the definition of the Lipschitz function, we have

$$\begin{aligned}
I_1 &\le C\bigg(\frac{1}{|B|}\int_B |b_1(z) - b_B^1|^\delta |T(f_1, f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\le C\|b_1\|_{Lip_\beta} r_B^\beta \bigg(\frac{1}{|B|}\int_B |T(f_1, f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\le C\|b_1\|_{Lip_\beta} \bigg(\frac{1}{|B|^{1-\frac{\beta\delta}{n}}}\int_B |T(f_1, f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\le C\|b_1\|_{Lip_\beta} M_{\beta,\delta}(T(\vec{f}))(x_0).
\end{aligned}$$

It follows from $0 < \delta < r < \infty$ and Lemma 3.1 that

$$\begin{aligned}
I_2 &= C|B|^{-1/\delta}\|T((b_1 - b_B^1)f_1^1, f_2^1)\|_{L^\delta(B)} \\
&\le C|B|^{-1/r}\|T((b_1 - b_B^1)f_1^1, f_2^1)\|_{L^{r,\infty}(B)} \\
&\le C\bigg(\frac{1}{|16B|}\int_{16B} |b_1(y_1) - b_B^1|^{r_1}|f_1(y_1)|^{r_1} dy_1\bigg)^{\frac{1}{r_1}} \\
&\quad \times \bigg(\frac{1}{|16B|}\int_{16B} |f_2(y_2)|^{r_2} dy_2\bigg)^{\frac{1}{r_2}} \\
&\le C\|b_1\|_{Lip_\beta} r_B^\beta \bigg(\frac{1}{|16B|}\int_{16B} |f_1(y_1)|^{r_1} dy_1\bigg)^{\frac{1}{r_1}} M_{r_2}(f_2)(x_0) \\
&\le C\|b_1\|_{Lip_\beta} \bigg(\frac{1}{|16B|^{1-\frac{\beta q'}{n}}}\int_{16B} |f_1(y_1)|^{q'} dy_1\bigg)^{\frac{1}{q'}} M_{r_2}(f_2)(x_0) \\
&\le C\|b_1\|_{Lip_\beta} M_{\beta,q'}(f_1)(x_0) M_{q'}(f_2)(x_0).
\end{aligned}$$

By the size condition (2) of the kernel, we have

$$
\begin{aligned}
I_3 &= C\Big(\frac{1}{|B|}\int_B\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(z,y_1,y_2)(b_1(y_1)-b_B^1)f_1^1(y_1)f_2^2(y_2)dy_1dy_2\Big|^\delta dz\Big)^{\frac{1}{\delta}}\\
&\le C\Big(\frac{1}{|B|}\int_B\Big(\int_{(16B)^c}\int_{16B}|K(z,y_1,y_2)||b_1(y_1)-b_B^1|\\
&\quad\times|f_1(y_1)||f_2(y_2)|dy_1dy_2\Big)^\delta dz\Big)^{\frac{1}{\delta}}\\
&\le C\left(\frac{1}{|B|}\int_B\left(\int_{(16B)^c}\Big(\int_{16B}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1\Big)\frac{|f_2(y_2)|}{|z-y_2|^{2n}}dy_2\right)^\delta dz\right)^{\frac{1}{\delta}}\\
&\le C\Big(\int_{16B}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1\Big)\sum_{k=4}^\infty\int_{2^{k+1}B\setminus2^kB}\frac{|f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2\\
&\le C\|b_1\|_{Lip_\beta}r_B^\beta\left(\frac{1}{|16B|}\int_{16B}|f_1(y_1)|^{q'}dy_1\right)^{1/q'}\sum_{k=4}^\infty2^{-kn}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_2(y_2)|dy_2\\
&\le C\|b_1\|_{Lip_\beta}\left(\frac{1}{|16B|^{1-\frac{\beta q'}{n}}}\int_{16B}|f_1(y_1)|^{q'}dy_1\right)^{1/q'}M(f_2)(x_0)\sum_{k=4}^\infty2^{-kn}\\
&\le C\|b_1\|_{Lip_\beta}M_{\beta,q'}(f_1)(x_0)M_{q'}(f_2)(x_0).
\end{aligned}
$$

Similarly, we can estimate I_4

$$
\begin{aligned}
I_4 &\le C\left(\frac{1}{|B|}\int_B\left(\int_{(16B)^c}\Big(\int_{16B}|b_1(y_1)-b_B^1||f_2(y_2)|dy_2\Big)\frac{|f_1(y_1)|}{|z-y_1|^{2n}}dy_1\right)^\delta dz\right)^{\frac{1}{\delta}}\\
&\le C\|b_1\|_{Lip_\beta}\Big(\int_{16B}|f_2(y_2)|dy_2\Big)\sum_{k=4}^\infty\int_{2^{k+1}B\setminus2^KB}\frac{|f_1(y_1)||x_0-y_1|^\beta}{|x_0-y_1|^{2n}}dy_1\\
&\le C\|b_1\|_{Lip_\beta}|B|M(f_2)(x_0)\sum_{k=4}^\infty(2^kr_B)^{-n}\frac{1}{|2^{k+1}B|^{1-\frac{\beta}{n}}}\int_{2^{k+1}B}|f_1(y_1)|dy_1\\
&\le C\|b_1\|_{Lip_\beta}M(f_2)(x_0)\sum_{k=4}^\infty2^{-kn}\left(\frac{1}{|2^{k+1}B|^{1-\frac{\beta q'}{n}}}\int_{2^{k+1}B}|f_1(y_1)|^{q'}dy_1\right)^{\frac{1}{q'}}\\
&\le C\|b_1\|_{Lip_\beta}M_{\beta,q'}(f_1)(x_0)M_{q'}(f_2)(x_0).
\end{aligned}
$$

For $z\in B$ and $y_1,y_2\in(16B)^c$, there are $|y_1-z_0|\ge2|z-z_0|$, $|y_2-z_0|\ge2|z-z_0|$, and $r_B\le|z-z_0|\le4r_B$. It follows from Hölder's inequality and the condition (5) that

$$
\begin{aligned}
I_5 &\le C\Big(\frac{1}{|B|}\int_B\Big(\int_{(16B)^c}\int_{(16B)^c}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|\\
&\quad\times|b_1(y_1)-b_B^1||f_1(y_1)||f_2(y_2)|dy_1dy_2\Big)^\delta dz\Big)^{\frac{1}{\delta}}\\
&\le C\|b_1\|_{Lip_\beta}\Big(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^\infty\sum_{k_2=1}^\infty\int_{2^{k_2}|z-z_0|\le|y_2-z_0|<2^{k_2+1}|z-z_0|}\\
&\quad\times\int_{2^{k_1}|z-z_0|\le|y_1-z_0|<2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)||y_1-x_0|^\beta
\end{aligned}
$$

$$
\begin{aligned}
&\times |f_1(y_1)||f_2(y_2)|dy_1dy_2\Big)^{\delta}dz\Bigg)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{Lip_\beta}\Bigg(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|<2^{k_2+1}|z-z_0|}|f_2(y_2)|\\
&\times\Big(\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|<2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|^q dy_1\Big)^{\frac{1}{q}}\\
&\times\Big(\int_{2^{k_1+4}B}|y_1-x_0|^{\beta q'}|f_1(y_1)|^{q'}dy_1\Big)^{\frac{1}{q'}}dy_2\Big)^{\delta}dz\Bigg)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{Lip_\beta}\Bigg(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1+4}B}|y_1-x_0|^{\beta q'}|f_1(y_1)|^{q'}dy_1\Big)^{\frac{1}{q'}}\\
&\times\Big(\int_{2^{k_2+4}B}|f_2(y_2)|^{q'}dy_2\Big)^{\frac{1}{q'}}\Big(\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|<2^{k_2+1}|z-z_0|}\\
&\times\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|<2^{k_1+1}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|^q dy_1dy_2\Big)^{\frac{1}{q}}\Big)^{\delta}dz\Bigg)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{Lip_\beta}\Bigg(\frac{1}{|B|}\int_B\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\frac{1}{|2^{k_1+4}B|^{1-\frac{\beta q'}{n}}}\int_{2^{k_1+4}B}|f_1(y_1)|^{q'}dy_1\Big)^{\frac{1}{q'}}\\
&\times\Big(\frac{1}{|2^{k_2+4}B|}\int_{2^{k_2+4}B}|f_2(y_2)|^{q'}dy_2\Big)^{\frac{1}{q'}}|2^{k_1+4}B|^{\frac{1}{q'}}|2^{k_2+4}B|^{\frac{1}{q'}}\\
&\times|z-z_0|^{\frac{-2n}{q'}}C_{k_1}2^{\frac{-k_1n}{q'}}C_{k_2}2^{\frac{-k_2n}{q'}}\Big)^{\delta}dz\Bigg)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{Lip_\beta}M_{\beta,q'}(f_1)(x_0)M_{q'}(f_2)(x_0)\Big(\sum_{k_1=1}^{\infty}C_{k_1}\Big)\Big(\sum_{k_2=1}^{\infty}C_{k_2}\Big)\\
&\times|B|^{\frac{2}{q'}}\Big(\frac{1}{|B|}\int_B|z-z_0|^{-\frac{2n}{q'}\delta}dz\Big)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{Lip_\beta}M_{\beta,q'}(f_1)(x_0)M_{q'}(f_2)(x_0).
\end{aligned}
$$

Combining the estimates of I_j, $j=1,\cdots,5$, we get

$$
\begin{aligned}
&\Big(\frac{1}{|B|}\int_B\Big|T_{\vec{b}}^1(\vec{f})(z)+T((b_1-b_B^1)f_1^2,f_2^2)(z_0)\Big|^{\delta}dz\Big)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{Lip_\beta}(M_{\beta,\delta}(T(\vec{f}))(x_0)+M_{\beta,q'}(f_1)(x_0)M_{q'}(f_2)(x_0)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Big(\frac{1}{|B|}\int_B\Big|T_{\vec{b}}^2(\vec{f})(z)+T(f_1^2,(b_2-b_B^2)f_2^2)(z_0)\Big|^{\delta}dz\Big)^{\frac{1}{\delta}}\\
&\leq C\|b_2\|_{Lip_\beta}(M_{\beta,\delta}(T(\vec{f}))(x_0)+M_{q'}(f_1)(x_0)M_{\beta,q'}(f_2)(x_0)).
\end{aligned}
$$

Thus

$$M_{\delta}^{\sharp}(T_{\vec{b}}(\vec{f}))(x_0)$$

$$
\begin{aligned}
&\sim \sup_{r_B>0}\inf_{a\in\mathbb{C}}\left(\frac{1}{|B(x_0,r_B)|}\int_{B(x_0,r_B)}\Big||T_{\vec{b}}(\vec{f})(z)|^{\delta}-a\Big|dz\right)^{\frac{1}{\delta}}\\
&\le \sup_{r_B>0}\left(\frac{1}{|B(x_0,r_B)|}\int_{B(x_0,r_B)}\Big||T_{\vec{b}}(\vec{f})(z)|^{\delta}\right.\\
&\quad\left.-|T((b_1-b_B^1)f_1^2,f_2^2)(z_0)+T(f_1^2,(b_2-b_B^2)f_2^2)(z_0)|^{\delta}\Big|dz\right)^{\frac{1}{\delta}}\\
&\le C\sup_{r_B>0}\left\{\left(\frac{1}{|B(x_0,r_B)|}\int_{B(x_0,r_B)}\Big|T_{\vec{b}}^1(\vec{f})(z)+T((b_1-b_B^1)f_1^2,f_2^2)(z_0)\Big|^{\delta}dz\right)^{\frac{1}{\delta}}\right.\\
&\quad\left.+\left(\frac{1}{|B(x_0,r_B)|}\int_{B(x_0,r_B)}\Big|T_{\vec{b}}^2(\vec{f})(z)+T(f_1^2,(b_2-b_B^2)f_2^2)(z_0)\Big|^{\delta}dz\right)^{\frac{1}{\delta}}\right\}\\
&\le C\|b_1\|_{Lip_\beta^2}\sum_{j=1}^{2}\Big(M_{\beta,\delta}(T(\vec{f}))(x_0)+M_{\beta,q'}(f_j)(x_0)\prod_{i=1,i\ne j}^{2}M_{q'}(f_i)(x_0)\Big).
\end{aligned}
$$

This completes the proof of Theorem 2.1.

Then, we will give the proof of Theorem 2.2.

Proof. It follows from $\omega^{1/q_0}\in A(p,q_0)$ and Lemma 3.4 that $\omega\in A_{q_0/p'+1}$. Take a δ such that $0<\delta<1/m$. Then by Lemma 3.2 and Theorem 2.1, we have

$$
\begin{aligned}
&\|T_{\vec{b}}(\vec{f})\|_{L^{q_0}(\omega)}\le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^{q_0}(\omega)}\le C\|M_\delta^{\sharp}(T_{\vec{b}}(\vec{f}))\|_{L^{q_0}(\omega)}\\
&\le C\|\vec{b}\|_{Lip_\beta^m}\sum_{j=1}^{m}\Big(\|M_{\beta,\delta}(T(\vec{f}))\|_{L^{q_0}(\omega)}+\Big\|M_{\beta,q'}(f_j)\prod_{i\ne j,i=1}^{m}M_{q'}(f_i)\Big\|_{L^{q_0}(\omega)}\Big).
\end{aligned}
$$

Since $\omega\in A_{q_0/p'+1}$, there exists an s such that $1<s<q_0/p'+1$ and $\omega\in A_s$. Denote by $t=\frac{pq_0}{p(s-1)+q_0}$, then $s=\frac{q_0/t}{(p/t)'}+1$, $1<t<p<n/\beta$, and $\omega\in A_{\frac{q_0/t}{(p/t)'}+1}$. Let $\widetilde{\beta}=\beta t$, $\widetilde{p}=p/t$ and $\widetilde{q}=q_0/t$, then $0<\widetilde{\beta}<n$, $1<\widetilde{p}<n/\widetilde{\beta}$ and $1/\widetilde{q}=1/\widetilde{p}-\widetilde{\beta}/n$.

It follows from Lemma 3.4 that $\omega^{1/\widetilde{q}}\in A(\widetilde{p},\widetilde{q})$. By Lemma 3.3 and Lemma 3.5, we have

$$
\begin{aligned}
\|M_{\beta,\delta}(T(\vec{f}))\|_{L^{q_0}(\omega)}&\le \|M_{\beta,t}(T(\vec{f}))\|_{L^{q_0}(\omega)}=\|M_{\widetilde{\beta},1}(|T(\vec{f})|^t)^{1/t}\|_{L^{\widetilde{q}t}(\omega)}\\
&=\Big(\int_{\mathbb{R}^n}[M_{\widetilde{\beta},1}(|T(\vec{f})|^t)(x)\omega(x)^{1/\widetilde{q}}]^{\widetilde{q}}dx\Big)^{\frac{1}{\widetilde{q}}\frac{1}{t}}\\
&\le C\Big(\int_{\mathbb{R}^n}[|T(\vec{f})(x)|^t\omega(x)^{\frac{1}{\widetilde{q}}}]^{\widetilde{p}}dx\Big)^{\frac{1}{\widetilde{p}}\frac{1}{t}}\\
&=C\|T(\vec{f})\|_{L^p(\omega^{p/q_0})}\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i)}.
\end{aligned}
$$

For every $j=1,\cdots,m$, $1/q_0=1/q_j+\sum_{i\ne j,i=1}^{m}1/p_i$ since $1/q_j=1/p_j-\beta/n$. By Hölder's inequality, we have

$$
\Big\|M_{\beta,q'}(f_j)\prod_{i\ne j,i=1}^{m}M_{q'}(f_i)\Big\|_{L^{q_0}(\omega)}\le\|M_{\beta,q'}(f_j)\|_{L^{q_j}(\omega_j^{q_j/p_j})}\prod_{i\ne j,i=1}^{m}\|M_{q'}(f_i)\|_{L^{p_i}(\omega_i)}.
$$

Denote by $\widetilde{\beta_j}=\beta q'$, $\widetilde{p_j}=p_j/q'$ and $\widetilde{q_j}=q_j/q'$, then $0<\widetilde{\beta_j}<n$, $1<\widetilde{p_j}<n/\widetilde{\beta_j}$ and $1/\widetilde{q_j}=1/\widetilde{p_j}-\widetilde{\beta_j}/n$. The fact $\omega_j^{q'/p_j}\in A(p_j/q',q_j/q')$ means that $\omega^{1/\widetilde{p_j}}\in A(\widetilde{p_j},\widetilde{q_j})$.

By Lemma 3.5, we have

$$\begin{aligned}\|M_{\beta,q'}(f_j)\|_{L^{q_j}(\omega_j^{q_j/p_j})} &= \|M_{\widetilde{\beta_j},1}(|f_j|^{q'})^{1/q'}\|_{L^{q_j}(\omega_j^{\widetilde{q_j}/\widetilde{p_j}})} \\ &= \Big(\int_{\mathbb{R}^n}[M_{\widetilde{\beta_j},1}(|f_j|^{q'})(x)\omega_j(x)^{1/\widetilde{p_j}}]^{\widetilde{q_j}}dx\Big)^{\frac{1}{\widetilde{q_j}}\frac{1}{q'}} \\ &\le C\Big(\int_{\mathbb{R}^n}[|f_j(x)|^{q'}\omega_j(x)^{1/\widetilde{p_j}}]^{\widetilde{p_j}}dx\Big)^{\frac{1}{\widetilde{p_j}}\frac{1}{q'}} \\ &= C\|f_j\|_{L^{p_j}(\omega_j)}.\end{aligned}$$

For every $i = 1, \cdots, m$, and $i \neq j$, since $\omega_i \in A_{p_i/q'}$ and $p_i/q' > 1$, M is bounded on $L^{p_i/q'}(\omega_i)$. Thus

$$\|M_{q'}(f_i)\|_{L^{p_i}(\omega_i)} = \|M(|f_i|^{q'})\|^{1/q'}_{L^{p_i/q'}(\omega_i)} \le C\||f_i|^{q'}\|^{1/q'}_{L^{p_i/q'}(\omega_i)} = C\|f_i\|_{L^{p_i}(\omega_i)}.$$

Therefore for every $j = 1, \cdots, m$,

$$\Big\|M_{\beta,q'}(f_j)\prod_{i\neq j,i=1}^{m} M_{q'}(f_i)\Big\|_{L^{q_0}(\omega)} \le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i)}.$$

In conclusion,

$$\|T_{\vec{b}}(\vec{f})\|_{L^{q_0}(\omega)} \le C\|\vec{b}\|_{Lip^m_\beta}\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i)},$$

which completes the proof of Theorem 2.2.

Declarations

Conflict of interest The authors declare no conflict of interest.

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