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Approximation by modified Durrmeyer type Jakimovski-Leviatan operators

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Abstract. In the present paper, the modified Durrmeyer type Jakimovski-Leviatan operators are presented and their approximation properties are examined. It has shown that the new operators are the Gamma transform of the Jakimovski-Leviatan operators. The degree of approximation is given by the modulus of continuity. It has been stressed that, there are other operators having the same error estimation with the operators, arising from the Szász-Durrmeyer operators. Then the degree of global approximation is obtained in a special Lipschitz type function space. Further, a Voronovskaja type asymptotic formula and Grüss-Voronovskaja type theorem are given. The approximation with these operators is visualized with the help of error tables and graphical examples.

§1 Introduction

In the approximation theory, one of the focus of research area is the improvement of the degree of approximation to functions with sequences of positive linear operators. Therefore, the new operator sequences are defined whose approximation properties are at least as good as those described in the literature [1–4]. There are several application areas of positive linear approximation processes such as computer aided geometric design [5] and 3D-wavelet filter banks [6]. On the other hand there are many usage areas, for instance, recently, sequences of linear auxiliary positive operators have been considered in the construction of approximating operators to approximate fractional calculus operators. Remarkable numerical results have been obtained in this direction of research [7–9]. In the last decades, special polynomials were used in the construction of positive linear operators especially for the approximation to a function defined on an unbounded interval. Jakimovski and Leviatan [10] defined as a generalized version

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of the Szász operators that include Appell polynomials

$$P_{\varrho}\left(\phi;\xi\right)=\frac{e^{-\varrho\xi}}{\vartheta(1)}\sum_{j=0}^{\infty}p_j(\varrho\xi)\phi\left(\frac{j}{\varrho}\right),\ \xi\geq 0, \varrho\in\mathbb{N},$$

where

$$\vartheta(u)=\sum_{\varrho=0}^{\infty}a_{\varrho}u^{\varrho}, a_0\neq 0,$$

is an analytic function in the disk $|u|<R$ $(R>1)$ with $\vartheta(1)\neq 0$ and

$$p_j(\xi)=\sum_{i=0}^{j}a_i\frac{\xi^{j-i}}{(j-i)!}$$

are the corresponding Appell polynomials generated by

$$\vartheta(u)e^{u\xi}=\sum_{j=0}^{\infty}p_j(\xi)u^j.$$

It has become inevitable to examine the properties of this sequence of operators and to define new operators with its help. Additionally, sequences of positive linear operators have been defined through the utilisation of well-known special functions and their approximation properties have been the focus of study by various mathematicians [11–25].

In the paper [26], Karaisa introduced Durrmeyer type modification of Jakimovski-Leviatan operators for all real valued bounded and continuous functions ϕ on $[0,\infty)$ as follows

$$L_{\varrho}\left(\phi;\xi\right)=\frac{e^{-\varrho\xi}}{\vartheta\left(1\right)}\sum_{j=1}^{\infty}\frac{p_j(\varrho\xi)}{\beta\left(\varrho+1,j\right)}\int_0^{\infty}\frac{t^{j-1}}{(1+t)^{\varrho+j+1}}\phi\left(t\right)dt+\frac{e^{-\varrho\xi}}{\vartheta\left(1\right)}a_0\phi\left(0\right),\xi\geq 0, \tag{1}$$

where $\beta\left(\varrho+1,j+1\right)=\int\limits_0^{\infty}\frac{u^{\varrho}}{(1+u)^{\varrho+j+2}}du$ is the Beta function. They examined the rate of convergence in a weighted space of continuous functions and also via continuity. Further, the authors gave Voronovskaja type theorem and a local approximation theorem for L_{ϱ} operators.

In this study, we introduce a generalization of the L_{ϱ} operators through the instrument of Gamma transform. We examine the approximation properties of these operators. First, we give the degree of approximation by the modulus of continuity. Then, we obtain the degree of global approximation in a certain Lipschitz type function space. Further, we consider a Voronovskaja type asymptotic formula and Grüss-Voronovskaja type theorem. Finally, we give error tables and graphical examples.

§2 Construction of the $L_{\varrho}^{(\alpha)}$ operators

Now, with the help of Durrmeyer type Jakimovski-Leviatan operators defined in (1), we present the following positive linear operators for $\alpha>0$ and $\xi>0$

$$L_{\varrho}^{(\alpha)}\left(\phi;\xi\right)=\frac{1}{(1+\varrho\alpha)^{\frac{\xi}{\alpha}}\,\vartheta(1)}\left[\sum_{j=1}^{\infty}\frac{1}{\beta(\varrho+1,j)}\left(\int_0^{\infty}\frac{u^{j-1}\phi\left(u\right)}{(1+u)^{\varrho+j+1}}du\right)\widetilde{p_j}^{(\alpha)}(\xi;\varrho)+a_0\phi\left(0\right)\right] \tag{2}$$

where

$$\widetilde{p_j}^{(\alpha)}(\xi;\varrho)=\sum_{i=0}^{j}\frac{a_i}{(j-i)!}\left(\frac{\varrho}{1+\varrho\alpha}\right)^{j-i}\xi^{(j-i,-\alpha)}$$

with $\xi^{(j-i,-\alpha)}=\xi(\xi+\gamma)(\xi+2\gamma)\cdots(\xi+(j-i-1)\,\alpha)$, $i=0,1,2,...,j$, $j\in\mathbb{N}$ and $\xi^{(0,-\alpha)}=1$. Here, $\phi\in C_\gamma[0,\infty)=\{\phi\in C[0,\infty):\phi(y)=O(y^\gamma)\text{ as }y\to\infty\}$ where $\gamma>\varrho$ and $C[0,\infty)$ denotes the space of all continuous functions defined on $[0,\infty)$.

We will consider $\widetilde{p_j}^{(\alpha)}$ polynomials that satisfy the following conditions for the positivity of linear $L_\varrho^{(\alpha)}$ operators:

(1) $\widetilde{p_j}^{(\alpha)}(\xi;\varrho)\geq 0$, $j=0,1,2...$,

(2) $\vartheta(\xi)=\sum\limits_{\varrho=0}^{\infty}a_\varrho\xi^\varrho, a_0\neq 0$, is an analytic function in the disk $|\xi|<R$ $(R>1)$ with $\vartheta(1)\neq 0$.

Remark 1. *(1) For $\vartheta(u)=1$ the operators L_ϱ return to the Szász-beta-Durrmeyer operators given in paper [27]. Thus, the operators $L_\varrho^{(\alpha)}$ reduce to the Gamma variant of these operators.*

(2) For $\alpha\to 0^+$, the operators $L_\varrho^{(\alpha)}$ reduce to the operators L_ϱ and furthermore for the choice $\vartheta(u)=1$, we get the Szász-Durrmeyer operators [28].

Remark 2. *The family of operators given by (2) can be written explicitly as follows*

$$\begin{aligned}
L_\varrho^{(\alpha)}(\phi;\xi)=&\frac{1}{\Gamma\left(\frac{\xi}{\alpha}\right)}\int_0^\infty e^{-t}t^{\frac{\xi}{\alpha}-1}L_\varrho(\phi;\alpha t)\,dt\\
=&\frac{1}{\Gamma\left(\frac{\xi}{\alpha}\right)}\int_0^\infty e^{-t}t^{\frac{\xi}{\alpha}-1}\left(\frac{e^{-\varrho\alpha t}}{\vartheta(1)}\sum_{j=1}^{\infty}\frac{p_j(\varrho\alpha t)}{\beta(\varrho+1,j)}\int_0^\infty\frac{u^{j-1}\phi(u)}{(1+u)^{\varrho+j+1}}du+\frac{e^{-\varrho\alpha t}}{\vartheta(1)}a_0\phi(0)\right)dt\\
=&\frac{1}{\vartheta(1)\Gamma\left(\frac{\xi}{\alpha}\right)}\sum_{j=1}^{\infty}\frac{1}{\beta(\varrho+1,j)}\left(\int_0^\infty\frac{u^{j-1}}{(1+u)^{\varrho+j+1}}\phi(u)\,du\right)\sum_{i=0}^{j}\frac{a_i(\varrho\alpha)^{j-i}}{(j-i)!}\\
&\times\int_0^\infty e^{-(1+\varrho\alpha)t}t^{\frac{\xi}{\alpha}+j-i-1}dt+\frac{1}{\vartheta(1)\Gamma\left(\frac{\xi}{\alpha}\right)}\int_0^\infty e^{-(1+\varrho\alpha)t}t^{\frac{\xi}{\alpha}-1}a_0\phi(0)\,dt\\
=&\frac{1}{\vartheta(1)\Gamma\left(\frac{\xi}{\alpha}\right)}\left(\sum_{j=1}^{\infty}\frac{1}{\beta(\varrho+1,j)}\left(\int_0^\infty\frac{u^{j-1}\phi(u)}{(1+u)^{\varrho+j+1}}du\right)\sum_{i=0}^{j}\frac{a_i(\varrho\alpha)^{j-i}}{(j-i)!}\frac{\Gamma\left(\frac{\xi}{\alpha}+j-i\right)}{(1+\varrho\alpha)^{\frac{\xi}{\alpha}+j-i}}\right.\\
&\left.+\frac{\Gamma\left(\frac{\xi}{\alpha}\right)a_0\phi(0)}{(1+\varrho\alpha)^{\frac{\xi}{\alpha}}}\right).
\end{aligned}$$

In addition, since

$$\frac{\Gamma\left(\frac{\xi}{\alpha}+j-i\right)}{\Gamma\left(\frac{\xi}{\alpha}\right)}=\left(\frac{\xi}{\alpha}\right)\left(\frac{\xi}{\alpha}+1\right)\cdots\left(\frac{\xi}{\alpha}+j-i-1\right)$$

$$= \frac{\xi(\xi+\alpha)(\xi+2\alpha)\cdots(\xi+(j-i-1)\,\alpha)}{\alpha^{j-i}} = \frac{\xi^{(j-i,-\alpha)}}{\alpha^{j-i}},$$

one can get the desired result. Here Γ *and* β *are the well-known Gamma and Beta functions, respectively.*

Now we calculate the values of the first five moments and the central moments of the $L_{\varrho}^{(\alpha)}$ operators.

Lemma 1. *For each* $\xi > 0$*, we have*

$$\begin{aligned}
L_{\varrho}^{(\alpha)}(1;\xi) &= 1,\\
L_{\varrho}^{(\alpha)}(y;\xi) &= \xi + \frac{\vartheta'(1)}{\varrho\vartheta(1)},\\
L_{\varrho}^{(\alpha)}\left(y^2;\xi\right) &= \frac{\varrho}{\varrho-1}\xi^2 + \left(\frac{\varrho\alpha}{\varrho-1} + \frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right]\right)\xi + \frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}, \quad \varrho > 1,\\
L_{\varrho}^{(\alpha)}\left(y^3;\xi\right) &= \frac{\varrho^2}{(\varrho-1)(\varrho-2)}\xi^3 + \left[\frac{3\alpha\varrho^2}{(\varrho-1)(\varrho-2)} + \frac{\varrho\,(3\vartheta'(1)+7\vartheta(1))}{(\varrho-1)(\varrho-2)\,\vartheta(1)}\right]\xi^2\\
&+ \left[\frac{2\varrho^2\alpha^2}{(\varrho-1)(\varrho-2)} + \frac{\varrho\alpha\,(3\vartheta'(1)+7\vartheta(1))}{(\varrho-1)(\varrho-2)\,\vartheta(1)} + \frac{3\vartheta''(1)+14\vartheta'(1)+5\vartheta(1)}{(\varrho-1)(\varrho-2)\,\vartheta(1)}\right]\xi\\
&+ \frac{\vartheta'''(1)+7\vartheta''(1)+5\vartheta'(1)}{\varrho\,(\varrho-1)(\varrho-2)\vartheta(1)}, \quad \varrho > 2,\\
L_{\varrho}^{(\alpha)}\left(y^4;\xi\right) &= \frac{\varrho^3}{(\varrho-1)(\varrho-2)(\varrho-3)}\xi^4\\
&+ \left[\frac{6\alpha\varrho^3}{(\varrho-1)(\varrho-2)(\varrho-3)} + \frac{\varrho^2\,(4\vartheta'(1)+16\vartheta(1))}{(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}\right]\xi^3\\
&+ \left[\frac{11\alpha^2\varrho^3}{(\varrho-1)(\varrho-2)(\varrho-3)} + \frac{3\alpha\varrho^2\,(4\vartheta'(1)+16\vartheta(1))}{(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}\right.\\
&\left.+ \frac{\varrho\,(6\vartheta''(1)+48\vartheta'(1)+45\vartheta(1))}{(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}\right]\xi^2\\
&+ \left[\frac{6\alpha^3\varrho^3}{(\varrho-1)(\varrho-2)(\varrho-3)} + \frac{2\alpha^2\varrho^2\,(4\vartheta'(1)+16\vartheta(1))}{(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}\right.\\
&+ \frac{\alpha\varrho\,(6\vartheta''(1)+48\vartheta'(1)+45\vartheta(1))}{(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}\\
&\left.+ \frac{4\vartheta'''(1)+48\vartheta''(1)+90\vartheta'(1)+17\vartheta(1)}{(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}\right]\xi\\
&+ \frac{\vartheta^{(4)}(1)+16\vartheta'''(1)+45\vartheta''(1)+17\vartheta'(1)}{\varrho\,(\varrho-1)(\varrho-2)(\varrho-3)\,\vartheta(1)}, \quad \varrho > 3,\\
L_{\varrho}^{(\alpha)}(y-\xi;\xi) &= \frac{\vartheta'(1)}{\varrho\vartheta(1)},\\
L_{\varrho}^{(\alpha)}\left((y-\xi)^2;\xi\right) &= \frac{1}{\varrho-1}\xi^2 + \left(\frac{\varrho\alpha}{\varrho-1} + \frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right] - \frac{2\vartheta'(1)}{\varrho\vartheta(1)}\right)\xi\\
&+ \frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}, \quad \varrho > 1.
\end{aligned}$$

Proof. Using the facts that from [26]

$$
\begin{aligned}
&L_\varrho(1;\xi) = 1,\\
&L_\varrho(y;\xi) = \xi + \frac{\vartheta'(1)}{\varrho\vartheta(1)},\\
&L_\varrho(y^2;\xi) = \frac{1}{\varrho(\varrho-1)}\left[\varrho^2\xi^2 + \varrho\xi\left(\frac{2\vartheta'(1)+2\vartheta(1)}{\vartheta(1)}\right) + \frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)}\right], \quad \varrho > 1,\\
&L_\varrho(y^3;\xi) = \frac{1}{\varrho(\varrho-1)(\varrho-2)}\left[\varrho^3\xi^3 + \varrho^2\xi^2\left(\frac{3\vartheta'(1)+7\vartheta(1)}{\vartheta(1)}\right)\right.\\
&\qquad\left. + \varrho\xi\left(\frac{3\vartheta''(1)+14\vartheta'(1)+5\vartheta(1)}{\vartheta(1)}\right) + \frac{\vartheta'''(1)+7\vartheta''(1)+5\vartheta'(1)}{\vartheta(1)}\right], \quad \varrho > 2,\\
&L_\varrho(y^4;\xi) = \frac{1}{\varrho(\varrho-1)(\varrho-2)(\varrho-3)}\left[\varrho^4\xi^4 + \varrho^3\xi^3\left(\frac{4\vartheta'(1)+16\vartheta(1)}{\vartheta(1)}\right)\right.\\
&\qquad + \varrho^2\xi^2\left(\frac{6\vartheta''(1)+48\vartheta'(1)+45\vartheta(1)}{\vartheta(1)}\right)\\
&\qquad + \varrho\xi\left(\frac{4\vartheta'''(1)+48\vartheta''(1)+90\vartheta'(1)+17\vartheta(1)}{\vartheta(1)}\right)\\
&\qquad \left. + \frac{\vartheta^{(4)}(1)+16\vartheta'''(1)+45\vartheta''(1)+17\vartheta'(1)}{\vartheta(1)}\right], \quad \varrho > 3,
\end{aligned}
$$

we get the desired results.

Further, from the linearity of the $L_\varrho^{(\alpha)}$ operators, we get

$$L_\varrho^{(\alpha)}(y-\xi;\xi) = L_\varrho^{(\alpha)}(y;\xi) - \xi L_\varrho^{(\alpha)}(1;\xi) = \xi + \frac{\vartheta'(1)}{\varrho\vartheta(1)} - \xi = \frac{\vartheta'(1)}{\varrho\vartheta(1)},$$

and

$$
\begin{aligned}
L_\varrho^{(\alpha)}\left((y-\xi)^2;\xi\right) &= L_\varrho^{(\alpha)}(y^2,\xi) - 2\xi L_\varrho^{(\alpha)}(y,\xi) + \xi^2 L_\varrho^{(\alpha)}(1,\xi)\\
&= \frac{1}{\varrho-1}\xi^2 + \left(\frac{\varrho\alpha}{\varrho-1} + \frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right] - \frac{2\vartheta'(1)}{\varrho\vartheta(1)}\right)\xi\\
&\quad + \frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}.
\end{aligned}
$$

□

Remark 3. *If we choose the determining function ϑ such that $\vartheta'(1) = 0$ (for example $\vartheta(u) = e^{-(u-1)^2}$), then the resulting sequence of operators preserves linear functions. In addition, if we choose the determining function ϑ such that $\vartheta'(1) = \vartheta''(1) = 0$ (for example $\vartheta(u) = e^{-(u-1)^3}$), then the expression $L_\varrho^{(\alpha)}\left((y-\xi)^2;\xi\right)$ determines the error as the same.*

Lemma 2. *Let $\alpha = \alpha_\varrho \to 0$ as $\varrho \to \infty$ and $\lim\limits_{\varrho\to\infty} \varrho\alpha_\varrho = l \in \mathbb{R}$. Then, we have*

$$\lim_{\varrho\to\infty} \varrho L_\varrho^{(\alpha_\varrho)}(y-\xi;\xi) = \frac{\vartheta'(1)}{\vartheta(1)},$$

$$\lim_{\varrho\to\infty} \varrho L_\varrho^{(\alpha_\varrho)}\left((y-\xi)^2;\xi\right) = \xi^2 + (2+l)\,\xi,$$

$$\lim_{\varrho\to\infty}\varrho^2 L_\varrho^{(\alpha_\varrho)}\left((y-\xi)^4;\xi\right) = 3\xi^4+(6l+24)\xi^3 + \left(3l^2+20l+\frac{24\vartheta''(1)+116\vartheta'(1)+65\vartheta(1)}{\vartheta(1)}\right)\xi^2.$$

Proof. From Lemma 1 and the linearity of the $L_\varrho^{(\alpha)}$ operators, one can obtain above limits. □

Here, we recall some function spaces that will be used throughout the paper. Let $B_{1+\xi^2}[0,\infty)$ denote the space of all functions ϕ satisfying $|\phi(\xi)|\le M_\phi(1+\xi^2)$ and M_ϕ is a nonnegative real number depending only on ϕ. By $C_{1+\xi^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_{1+\xi^2}[0,\infty)$ and finally, $E_{1+\xi^2}[0,\infty)$ is the space of all functions $\phi\in C_{1+\xi^2}[0,\infty)$ such that $\lim_{\xi\to\infty}\frac{|\phi(\xi)|}{1+\xi^2}<\infty$ which is endowed with the norm

$$\|\phi\|_{1+\xi^2} := \sup_{\xi\in[0,\infty)}\frac{|\phi(\xi)|}{1+\xi^2}.$$

In order to give the approximation properties, in the rest of the paper we let $\alpha_\varrho:=\alpha_\varrho(\xi)$ such that $\alpha_\varrho(\xi)$ satisfies the condition $0\le\dfrac{\xi\alpha_\varrho(\xi)}{1+\xi^2}\le a_\varrho\le C$ where (α_ϱ) is a positive numerical sequence and C is a constant.

Lemma 3. *For $\psi(\xi)=\dfrac{1}{1+\xi^2}$, the inequality*

$$\psi(\xi)L_\varrho^{(\alpha_\varrho(\xi))}\left(\frac{1}{\psi};\xi\right)\le A$$

holds for all $\xi\in(0,\infty)$ and $\varrho\in\mathbb{N}$, where $A\in\mathbb{R}$. Moreover, for all $\phi\in E_{1+\xi^2}$ and $\xi\in(0,\infty)$ we have

$$\left\|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;.)\right\|_{1+\xi^2}\le A\|\phi\|_{1+\xi^2}$$

for a positive numerical sequence $(\alpha_\varrho)=(\alpha_\varrho(\xi))$ satisfying the inequality $0\le\dfrac{\xi\alpha_\varrho(\xi)}{1+\xi^2}\le a_\varrho\le C$.

Proof. From Lemma 1, we have

$$\begin{aligned}\psi(\xi)L_\varrho^{(\alpha_\varrho(\xi))}\left(\frac{1}{\psi};\xi\right) &= \frac{1}{1+\xi^2}\left[L_\varrho^{(\alpha_\varrho(\xi))}(1;\xi)+L_\varrho^{(\alpha_\varrho(\xi))}(y^2;\xi)\right]\\ &\le \frac{1}{1+\xi^2}\left[1+\frac{\varrho}{\varrho-1}\xi^2+\left(\frac{\varrho\alpha_\varrho(\xi)}{\varrho-1}+\frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right]\right)\xi\right.\\ &\quad\left.+\frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}\right]\\ &\le A.\end{aligned}$$

Since

$$\begin{aligned}\psi(\xi)\left|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)\right| &= \psi(\xi)\left|L_\varrho^{(\alpha_\varrho(\xi))}\left(\psi\frac{\phi}{\psi};\xi\right)\right|\\ &\le \|\phi\|_{1+\xi^2}\,\psi(\xi)L_\varrho^{(\alpha_\varrho(\xi))}\left(\frac{1}{\psi};\xi\right)\le A\|\phi\|_{1+\xi^2},\end{aligned}$$

we can obtain $\left\|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;.)\right\|_{1+\xi^2}\le A\|\phi\|_{1+\xi^2}$ by taking supremum over $\xi\in(0,\infty)$ in the above expression. □

§3 Main results

In this section, the degree of approximation of the operators $L_\varrho^{(\alpha)}$ defined by (2) is computed.

Theorem 1. *Let $\phi \in E_{1+\xi^2}$ and $\omega_{\gamma+1}(\phi,\delta)$ be the modulus of continuity on the interval $[b,\gamma+1] \subset [0,\infty)$, $b>0$. Then*

$$\left\| L_\varrho^{(\alpha_\varrho(\xi))}(\phi;.) - \phi \right\|_{C[b,\gamma]} \leq 6M_\phi \left(1+\gamma^2\right)\delta_\varrho + 2\omega_{\gamma+1}\left(\phi, \delta_\varrho^{\frac{1}{2}}\right),$$

where

$$\delta = \delta_\varrho^{\frac{1}{2}} = \left[\frac{1}{\varrho-1}\gamma^2 + \frac{\varrho a_\varrho}{\varrho-1}\left(1+\gamma^2\right) + \frac{2\gamma}{\varrho-1}\left(\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right) - \frac{2\gamma}{\varrho}\frac{\vartheta'(1)}{\vartheta(1)} + \frac{1}{\varrho(\varrho-1)}\left(\frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)}\right)\right]^{\frac{1}{2}},$$

M_ϕ is a number depending on ϕ, $1<\varrho\in\mathbb{N}$ and $\omega_\gamma(\phi,\delta)$ is the usual modulus of continuity defined by the formula $\omega_\gamma(\phi,\delta) = \sup\limits_{\substack{|y-\xi|\leq\delta \\ \xi,y\in[0,\gamma]}} |\phi(y)-\phi(\xi)|$.

Proof. Let $\xi\in[b,\gamma]$ and $y\leq\gamma+1$. We have

$$|\phi(y)-\phi(\xi)| \leq \omega_{\gamma+1}(\phi,|y-\xi|) \leq \left(1+\frac{|y-\xi|}{\delta}\right)\omega_{\gamma+1}(\phi,\delta). \tag{3}$$

Now, let $\xi\in[b,\gamma]$ and $y>\gamma+1$. In this case, since $y-\xi>1$, we have

$$\begin{aligned} |\phi(y)-\phi(\xi)| &\leq M_\phi(2+\xi^2+y^2) \leq M_\phi(2+3\xi^2+2(y-\xi)^2) \\ &\leq 6M_\phi\left(1+\gamma^2\right)(y-\xi)^2, \end{aligned} \tag{4}$$

where M_ϕ is a positive number depending on ϕ. According to (3) and (4), we have

$$|\phi(y)-\phi(\xi)| \leq 6M_\phi\left(1+\gamma^2\right)(y-\xi)^2 + \left(1+\frac{|y-\xi|}{\delta}\right)\omega_{\gamma+1}(\phi,\delta)$$

for all $\xi\in[b,\gamma]$ and $y\geq 0$. Thus,

$$\begin{aligned} \left| L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)-\phi(\xi)\right| &\leq 6M_\phi\left(1+\gamma^2\right)L_\varrho^{(\alpha_\varrho(\xi))}((y-\xi)^2;\xi) \\ &\quad + \left(1+\frac{L_\varrho^{(\alpha_\varrho(\xi))}(|y-\xi|;\xi)}{\delta}\right)\omega_{\gamma+1}(\phi,\delta). \end{aligned}$$

By the Cauchy-Schwarz inequality and Lemma 1, we get

$$\begin{aligned} &\left| L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)-\phi(\xi)\right| \\ &\leq 6M_\phi\left(1+\gamma^2\right)L_\varrho^{(\alpha_\varrho(\xi))}\left((y-\xi)^2;\xi\right) + \left(1+\frac{\left[L_\varrho^{(\alpha_\varrho(\xi))}\left((y-\xi)^2;\xi\right)\right]^{1/2}}{\delta}\right)\omega_{\gamma+1}(\phi,\delta) \\ &\leq 6M_\phi\left(1+\gamma^2\right)\left[\frac{1}{\varrho-1}\gamma^2+\frac{\varrho a_\varrho}{\varrho-1}\left(1+\gamma^2\right)+\frac{2\gamma}{\varrho-1}\left(\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right)\right. \\ &\left.-\frac{2\gamma}{\varrho}\frac{\vartheta'(1)}{\vartheta(1)}+\frac{1}{\varrho(\varrho-1)}\left(\frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)}\right)\right] \end{aligned}$$

$$+\left(1+\frac{\left[\frac{1}{\varrho-1}\gamma^2+\frac{\varrho a_\varrho}{\varrho-1}\left(1+\gamma^2\right)+\frac{2\gamma}{\varrho-1}\left(\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right)-\frac{2\gamma}{\varrho}\frac{\vartheta'(1)}{\vartheta(1)}+\frac{1}{\varrho(\varrho-1)}\left(\frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)}\right)\right]^{1/2}}{\delta}\right)$$

$$\times\,\omega_{\gamma+1}(\phi,\delta)$$

$$\leq 6M_\phi\left(1+\gamma^2\right)\delta_\varrho+2\omega_{\gamma+1}\left(\phi,\delta_\varrho^{1/2}\right),$$

where

$$\delta=\delta_\varrho^{\frac{1}{2}}=\left[\frac{1}{\varrho-1}\gamma^2+\frac{\varrho a_\varrho}{\varrho-1}\left(1+\gamma^2\right)+\frac{2\gamma}{\varrho-1}\left(\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right)-\frac{2\gamma}{\varrho}\frac{\vartheta'(1)}{\vartheta(1)}\right.$$
$$\left.+\frac{1}{\varrho(\varrho-1)}\left(\frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)}\right)\right]^{\frac{1}{2}},$$

which completes the proof. □

Corollary 1. *If the sequence (a_ϱ) converges to 0, then for all $\phi\in E_{1+\xi^2}[0,\infty)$,*

$$\lim_{\varrho\to\infty}L_\varrho^{(\alpha_\varrho)}(\phi;\xi)=\phi(\xi)$$

uniformly on each compact subset of $(0,\infty)$.

Now, we give the Voronovskaja type theorem for $L_\varrho^{(\alpha)}$ operators.

Theorem 2. *Let $\phi\in C_{1+\xi^2}[0,\infty)$ such that $\phi',\phi''\in C_{1+\xi^2}[0,\infty)$ and $\lim_{\varrho\to\infty}\varrho\alpha_\varrho=l\in\mathbb{R}$. Then we have the following equality*

$$\lim_{\varrho\to\infty}\varrho\left(L_\varrho^{(\alpha_\varrho)}(\phi;\xi)-\phi(\xi)\right)=\phi'(\xi)\frac{\vartheta'(1)}{\vartheta(1)}+\frac{1}{2}\phi''(\xi)\left[\xi^2+(l+2)\xi\right],$$

uniformly for each compact subset of $(0,\infty)$.

Proof. From the Taylor expansion of ϕ, we may write

$$\phi(y)=\phi(\xi)+\phi'(\xi)(y-\xi)+\frac{1}{2}f''(\xi)(y-\xi)^2+\epsilon(y,\xi)(y-\xi)^2,$$

where $\epsilon(y,\xi)\to 0$ as $y\to\xi$. By applying operators $L_\varrho^{(\alpha_\varrho)}$ to both sides on the above equation, we get

$$L_\varrho^{(\alpha_\varrho)}(\phi;\xi)-\phi(\xi)=\phi'(\xi)L_\varrho^{(\alpha_\varrho)}((y-\xi);\xi)+\frac{1}{2}\phi''(\xi)L_\varrho^{(\alpha_\varrho)}\left((y-\xi)^2;\xi\right)$$
$$+L_\varrho^{(\alpha_\varrho)}\left(\epsilon(y,\xi)(y-\xi)^2;\xi\right).$$

From Lemma 1, we have

$$L_\varrho^{(\alpha_\varrho)}(\phi;\xi)-\phi(\xi)=\phi'(\xi)\frac{\vartheta'(1)}{\varrho\vartheta(1)}$$
$$+\frac{1}{2}\phi''(\xi)\left[\frac{1}{\varrho-1}\xi^2+\left(\frac{\varrho\alpha_\varrho}{\varrho-1}+\frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right]-\frac{2\vartheta'(1)}{\varrho\vartheta(1)}\right)\xi\right.$$
$$\left.+\frac{\vartheta''(1)+2\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}\right]+L_\varrho^{(\alpha_\varrho)}\left(\epsilon(y,\xi)(y-\xi)^2;\xi\right)$$

and via Cauchy-Schwarz inequality, we write

$$\lim_{\varrho\to\infty}\varrho L_\varrho^{(\alpha_\varrho)}\left(\epsilon(y,\xi)(y-\xi)^2;\xi\right)\leq\sqrt{\lim_{\varrho\to\infty}L_\varrho^{(\alpha_\varrho)}\left(\epsilon^2(y,\xi);\xi\right)}\sqrt{\lim_{\varrho\to\infty}\varrho^2L_\varrho^{(\alpha_\varrho)}\left((y-\xi)^4;\xi\right)}.$$

Since $\lim\limits_{\varrho\to\infty} L_\varrho^{(\alpha_\varrho)}\left(\epsilon^2(y,\xi);\xi\right)=0$ and by Lemma 2, $\lim\limits_{\varrho\to\infty}\varrho^2 L_\varrho^{(\alpha_\varrho)}\left((y-\xi)^4;\xi\right)$ is finite, so $\lim\limits_{\varrho\to\infty}\varrho L_\varrho^{(\alpha_\varrho)}\left(\epsilon(y,\xi)(y-\xi)^2;\xi\right)=0$. Finally, by taking the limit as $\varrho\to\infty$, we obtain $\lim\limits_{\varrho\to\infty}\varrho\left(L_\varrho^{(\alpha_\varrho)}(\phi;\xi)-\phi(\xi)\right)=\phi'(\xi)\frac{\vartheta'(1)}{\vartheta(1)}+\frac{1}{2}\phi''(\xi)\left[\xi^2+(l+2)\xi\right]$.

□

In this part, we examine a global approximation result. In the paper [30], the following Lipschitz type space has been considered

$$\widetilde{Lip}_M(\lambda):=\left\{\phi\in C_B[0,\infty):|\phi(y)-\phi(\xi)|\le M\frac{|y-\xi|^\lambda}{(y+\xi^2+1)^{\lambda/2}};\ \xi,y\in(0,\infty)\right\},$$

where M is any positive number and $0<\lambda\le 1$. Here, $C_B[0,\infty)$ denotes the space of all bounded continuous functions defined on $[0,\infty)$.

Theorem 3. *For any $\phi\in\widetilde{Lip}_M(\lambda)$, $1<\varrho\in\mathbb{N}$ and $\lambda\in(0,1]$, we have*

$$\begin{aligned}\left|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)-\phi(\xi)\right|\le&M\left\{\frac{1}{\varrho-1}+\frac{\varrho a_\varrho}{\varrho-1}+\frac{2\gamma}{\varrho-1}\left|\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right|+\frac{2}{\varrho}\left|\frac{\vartheta'(1)}{\vartheta(1)}\right|\right.\\&\left.+\frac{1}{\varrho(\varrho-1)}\left|\frac{\vartheta''(1)+\vartheta'(1)}{\vartheta(1)}\right|\right\}^{\lambda/2}\end{aligned}$$

for $\xi\in(0,\infty)$.

Proof. Let $\lambda=1$. Then, for $\phi\in\widetilde{Lip}_M(1)$ and $\xi\in(0,\infty)$, we have

$$\begin{aligned}\left|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)-\phi(\xi)\right|&\le L_\varrho^{(\alpha_\varrho(\xi))}(|\phi(y)-\phi(\xi)|;\xi)\le L_\varrho^{(\alpha_\varrho(\xi))}\left(\frac{M|y-\xi|}{(y+\xi^2+1)^{\frac12}};\xi\right)\\&\le\frac{M}{(\xi^2+1)^{\frac12}}L_\varrho^{(\alpha_\varrho(\xi))}(|y-\xi|;\xi).\end{aligned}$$

Applying Cauchy-Schwarz inequality and using Lemma 1, we get

$$\begin{aligned}\left|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)-\phi(\xi)\right|&\le\frac{M}{(\xi^2+1)^{1/2}}\left(L_\varrho^{(\alpha_\varrho(\xi))}\left((y-\xi)^2;\xi\right)\right)^{1/2}\\&=\frac{M}{(\xi^2+1)^{1/2}}\left\{\frac{1}{\varrho-1}\xi^2+\xi\left(\frac{\varrho\alpha_\varrho(\xi)}{\varrho-1}+\frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right]-\frac{2\vartheta'(1)}{\varrho\vartheta(1)}\right)\right.\\&\left.+\frac{\vartheta''(1)+\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}\right\}^{1/2}\\&\le M\left\{\frac{1}{\varrho-1}+\frac{\varrho}{\varrho-1}a_\varrho+\frac{2\gamma}{\varrho-1}\left|\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right|\right.\\&\left.+\frac{2}{\varrho}\left|\frac{\vartheta'(1)}{\vartheta(1)}\right|+\frac{1}{\varrho(\varrho-1)}\left|\frac{\vartheta''(1)+\vartheta'(1)}{\vartheta(1)}\right|\right\}^{1/2}.\end{aligned}$$

Now assume that $\lambda\in(0,1)$. Similarly, for $\phi\in\widetilde{Lip}_M(\lambda)$ and $\xi\in(0,\infty)$, we have

$$\begin{aligned}\left|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi)-\phi(\xi)\right|&\le L_\varrho^{(\alpha_\varrho(\xi))}(|\phi(y)-\phi(\xi)|;\xi)\le L_\varrho^{(\alpha_\varrho(\xi))}\left(\frac{M|y-\xi|^\lambda}{(y+\xi^2+1)^{\frac{\lambda}{2}}};\xi\right)\\&\le\frac{M}{(\xi^2+1)^{\frac{\lambda}{2}}}L_\varrho^{(\alpha_\varrho(\xi))}\left(|y-\xi|^\lambda;\xi\right).\end{aligned}$$

Taking $p=\frac{1}{\lambda}$ and $q=\frac{1}{1-\lambda}$ and with the help of the Hölder's inequality, we get

$$\left|L_{\varrho}^{(\alpha_{\varrho}(\xi))}(\phi;\xi)-\phi(\xi)\right| \leq \frac{M}{(\xi^2+1)^{\lambda/2}}\left(L_{\varrho}^{(\alpha_{\varrho}(\xi))}\left(|y-\xi|\,;\xi\right)\right)^{\lambda}.$$

Finally, by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|L_{\varrho}^{(\alpha_{\varrho}(\xi))}(\phi;\xi)-\phi(\xi)\right| &\leq \frac{M}{(\xi^2+1)^{\frac{\lambda}{2}}}\left(L_{\varrho}^{(\alpha_{\varrho}(\xi))}\left((y-\xi)^2;\xi\right)\right)^{\frac{\lambda}{2}}\\ &= \frac{M}{(\xi^2+1)^{\frac{\lambda}{2}}}\left\{\frac{1}{\varrho-1}\xi^2+\xi\left(\frac{\varrho\alpha_{\varrho}(\xi)}{\varrho-1}+\frac{2}{\varrho-1}\left[\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right]-\frac{2\vartheta'(1)}{\varrho\vartheta(1)}\right)\right.\\ &\left.+\frac{\vartheta''(1)+\vartheta'(1)}{\vartheta(1)\varrho(\varrho-1)}\right\}^{\frac{\lambda}{2}}\\ &\leq M\left\{\frac{1}{\varrho-1}+\frac{\varrho}{\varrho-1}a_{\varrho}+\frac{2\gamma}{\varrho-1}\left|\frac{\vartheta'(1)+\vartheta(1)}{\vartheta(1)}\right|\right.\\ &\left.+\frac{2}{\varrho}\left|\frac{\vartheta'(1)}{\vartheta(1)}\right|+\frac{1}{\varrho(\varrho-1)}\left|\frac{\vartheta''(1)+\vartheta'(1)}{\vartheta(1)}\right|\right\}^{\frac{\lambda}{2}},\end{aligned}$$

which gives the desired result. □

Remark 4. *In Theorems 1 and 3, if the sequence (a_{ϱ}) converges to 0, then the sequence $\left(L_{\varrho}^{(\alpha_{\varrho})}(\phi;\xi)\right)$ converges uniformly to ϕ on each compact subset of $(0,\infty)$ for all $\phi\in E_{1+\xi^2}$ and $\phi\in\widetilde{Lip}_M(\lambda)$, respectively.*

In 1935, Grüss [31] gave an inequality for integrable functions on a closed bounded interval. After that, many mathematicians used this inequality as an auxiliary tool in their studies [32–35].

In the following theorem, we gave the Grüss-Voronovskaja type theorem for the operators $L_{\varrho}^{(\alpha)}$.

Theorem 4. *Let $f'(\xi), g'(\xi), f''(\xi), g''(\xi), (fg)'(\xi)$ and $(fg)''(\xi)\in C_{1+\xi^2}[0,\infty)$, then we have*

$$\lim_{\varrho\to\infty}\varrho\left(L_{\varrho}^{(\alpha_{\varrho})}(fg;\xi)-L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)\,L_{\varrho}^{(\alpha_{\varrho})}(g;\xi)\right)=\left(\xi^2+(2+l)\,\xi\right)f'(\xi)g'(\xi).$$

Proof. Since we have the following equality,

$$(fg)''(\xi)=f''(\xi)g(\xi)+2f'(\xi)g'(\xi)+g''(\xi)f(\xi)$$

by using Taylor expansion, we can write

$$\begin{aligned}&\varrho\left\{L_{\varrho}^{(\alpha_{\varrho})}(fg;\xi)-L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)\,L_{\varrho}^{(\alpha_{\varrho})}(g;\xi)\right\}\\ &=\varrho\left\{L_{\varrho}^{(\alpha_{\varrho})}(fg;\xi)-f(\xi)\,g(\xi)-(fg)'(\xi)\,L_{\varrho}^{(\alpha_{\varrho})}(y-\xi;\xi)-\frac{L_{\varrho}^{(\alpha_{\varrho})}\left((y-\xi)^2;\xi\right)}{2}(fg)''(\xi)\right.\end{aligned}$$

$$-g(\xi)\left(L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)-f(\xi)-f'(\xi)L_{\varrho}^{(\alpha_{\varrho})}(y-\xi;\xi)-\frac{L_{\varrho}^{(\alpha_{\varrho})}\left((y-\xi)^{2};\xi\right)}{2}f''(\xi)\right)$$
$$-L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)\left(L_{\varrho}^{(\alpha_{\varrho})}(g;\xi)-g(\xi)-g'(\xi)L_{\varrho}^{(\alpha_{\varrho})}(y-\xi;\xi)-\frac{L_{\varrho}^{(\alpha_{\varrho})}\left((y-\xi)^{2};\xi\right)}{2}g''(\xi)\right)$$
$$+L_{\varrho}^{(\alpha_{\varrho})}\left((y-\xi)^{2};\xi\right)f'(\xi)g'(\xi)+g''(\xi)\frac{L_{\varrho}^{(\alpha_{\varrho})}\left((y-\xi)^{2};\xi\right)}{2}\left(f(\xi)-L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)\right)$$
$$+g'(\xi)L_{\varrho}^{(\alpha_{\varrho})}(y-\xi;\xi)\left(f(\xi)-L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)\right)\Big\}.$$

By Theorem 1 for each $\xi\in(0,\infty)$, $L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)\to f(\xi)$ as $\varrho\to\infty$ and by Theorem 2 for each $f\in C_{1+\xi^2}[0,\infty)$, we obtain

$$\varrho\left(L_{\varrho}^{(\alpha_{\varrho})}(f;\xi)-f(\xi)-f'(\xi)L_{\varrho}^{(\alpha_{\varrho})}(y-\xi;\xi)-\frac{L_{\varrho}^{(\alpha_{\varrho})}\left((y-\xi)^{2};\xi\right)}{2}f''(\xi)\right)\to 0,$$

as $\varrho\to\infty$. Hence, using Lemma 2, we get the desired result. □

§4 Examples

In this part, we present some graphical and numerical examples about the convergence of the $L_{\varrho}^{(\alpha)}$ operators to the function $\phi(\xi)=2\xi e^{-3\xi}$ for $\vartheta(u)=1$.

Example 1. *The convergence of the operators is given in Figure 1 for $\alpha=0.001$, $\varrho=80$, $\varrho=120$ and $\varrho=160$. The convergence improves for increasing values of ϱ.*

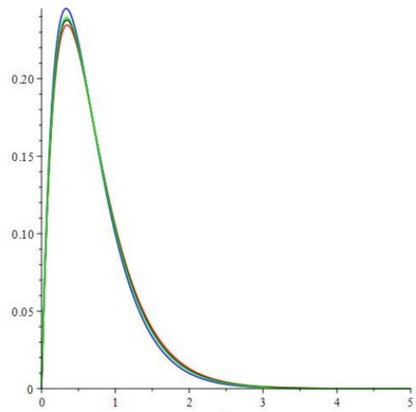

Figure 1. The convergenge of $L_{\varrho}^{(\alpha)}(\phi;\xi)$ to $\phi(\xi)=2\xi e^{-3\xi}$ (blue) for $L_{80}^{(0.001)}(\phi;\xi)$ (red), $L_{120}^{(0.001)}(\phi;\xi)$ (black) and $L_{160}^{(0.001)}(\phi;\xi)$ (green).

Example 2. *The convergence of the $L_{\varrho}^{(\alpha)}$ operators is given in Figure 2 for $\varrho=100$, $\alpha=0.003$ and $\alpha=0.0003$. The convergence improves for decreasing values of α.*

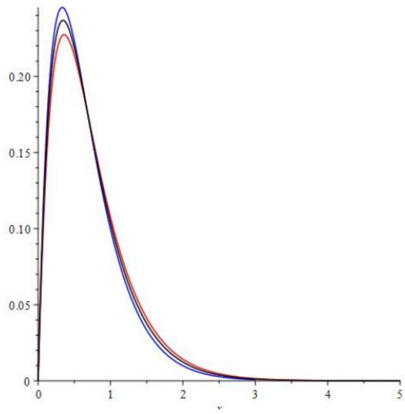

Figure 2. The convergenge of $L_\varrho^{(\alpha)}(\phi;\xi)$ to $\phi(\xi)=2\xi e^{-3\xi}$ (blue) for $L_{100}^{(0.003)}(\phi;\xi)$ (red) and $L_{100}^{(0.0003)}(\phi;\xi)$ (black).

Example 3. *Let $\phi(\xi)=2\xi e^{-3\xi}$ as in the graphs. For $\vartheta(u)=e^u$, $(\alpha_\varrho(\xi))=\left(\dfrac{1}{\varrho\xi}\right)$, $(a_\varrho)=\left(\dfrac{1}{\varrho}\right)$ and $\xi\in(0,1]$, we compute the error estimation of function ϕ by using the first modulus of continuity for $L_\varrho^{(\alpha)}$ operators. Considering the values in the Table 1, it is seen that the error bound of ϕ decreases for incresing ϱ values.*

Table 1. Error estimation of the function $\phi(\xi)=2\xi e^{-3\xi}$ for $L_\varrho^{(\alpha)}$ operators.

ϱ	Error estimate by $L_\varrho^{(\alpha)}(\phi;\xi)$ operators
10^6	0.008898281
10^7	0.002823813
10^8	0.000893965
10^9	0.000282796
10^{10}	0.000089438
10^{11}	0.000028283
10^{12}	0.8944×10^{-5}
10^{13}	0.2828×10^{-5}
10^{14}	0.894×10^{-6}
10^{15}	0.282×10^{-6}

Example 4. *Let us consider the function $\phi(\xi)=2\xi e^{-3\xi}$ and the sequences $(\alpha_\varrho(\xi))=\left(\dfrac{1}{\varrho\xi}\right)$ and $(\gamma_\varrho(\xi))=(\rho_\varrho(\xi))=\left(\dfrac{1}{\varrho^2\xi}\right)$. Using these positive numerical sequences, the (a_ϱ) sequences obtained by the inequality mentioned in Lemma 3 are $\left(\dfrac{1}{\varrho}\right)$ and $\left(\dfrac{1}{\varrho^2}\right)$, respectively. For the operators $L_\varrho^{(\alpha_\varrho(\xi))}$ and $L_\varrho^{(\gamma_\varrho(\xi))}$ the determining function ϑ is choosen as $\vartheta(u)=e^u$ and for the operators $L_\varrho^{(\rho_\varrho(\xi))}$ as $\vartheta(u)=1$. For $\varrho=10^8$, we compute the error estimations of function $\phi(\xi)$ by using the first modulus of continuity for the operators $L_\varrho^{(\alpha_\varrho(\xi))}$, $L_\varrho^{(\gamma_\varrho(\xi))}$ and $L_\varrho^{(\rho_\varrho(\xi))}$ at*

certain points from the interval $(0,1]$. *A comparison of the first and second columns of the Table 2 indicates that a better pointwise approximation can be achieved through the choise of* $(\alpha_\varrho(\xi))$. *Similarly, a comparison of the second and third columns demonstrates that a better pointwise approximation can be obtained through the choise of an alternative determining function* ϑ.

Table 2. Error estimation of the function $\phi(\xi) = 2\xi e^{-3\xi}$ for the operators $L_\varrho^{(\alpha_\varrho(\xi))}$, $L_\varrho^{(\gamma_\varrho(\xi))}$ and $L_\varrho^{(\rho_\varrho(\xi))}$ at certain points.

ξ	$\left\|L_\varrho^{(\alpha_\varrho(\xi))}(\phi;\xi) - \phi(\xi)\right\|$	$\left\|L_\varrho^{(\gamma_\varrho(\xi))}(\phi;\xi) - \phi(\xi)\right\|$	$\left\|L_\varrho^{(\rho_\varrho(\xi))}(\phi;\xi) - \phi(\xi)\right\|$
0.05	0.0003075306	0.0000936823	0.0000936809
0.10	0.0002290380	0.0000950433	0.0000950404
0.15	0.0001626253	0.0000796479	0.0000796434
0.20	0.0001067782	0.0000582325	0.0000582262
0.25	0.0000601814	0.0000354158	0.0000354076
0.30	0.0000216761	0.0000135005	0.0000134902
0.35	0.0000098297	0.0000063763	0.0000063636
0.40	0.0000351489	0.0000236383	0.0000236230
0.45	0.0000551632	0.0000381376	0.0000381194
0.50	0.0000706199	0.0000499231	0.0000499016
0.55	0.0000821840	0.0000591654	0.0000591402
0.60	0.0000904486	0.0000660988	0.0000660696
0.65	0.0000959411	0.0000709904	0.0000709566
0.70	0.0000991258	0.0000741116	0.0000740728
0.75	0.0001004095	0.0000757254	0.0000756810
0.80	0.0001001476	0.0000760792	0.0000760286
0.85	0.0000986485	0.0000753995	0.0000753420
0.90	0.0000961760	0.0000738879	0.0000738228
0.95	0.0000929575	0.0000717233	0.0000716498

Declarations

Conflict of interest The authors declare no conflict of interest.

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