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New perspectives on Milne's rule inequalities with their computational analysis via quantum calculus

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Abstract. In this study, we propose a novel method for establishing Milne's rule-type inequalities within the context of quantum calculus applied to differentiable convex functions. Initially, we obtain a quantum integral identity, which serves as the foundation for deriving several new Milne's rule inequalities tailored for quantum differentiable convex functions. These inequalities are particularly relevant in Open-Newton's Cotes formulas, facilitating the determination of bounds for Milne's rule in both classical and q-calculus domains. Additionally, we conduct computational analysis on these inequalities for convex functions and present mathematical examples and graphical representation to demonstrate the validity of our newly established results within the realm of q-calculus.

§1 Introduction

Quantum calculus, an area of mathematics, has gained prominence for its applications in number theory, special functions, and quantum physics. Quantum calculus, often known as q-calculus, is the study of calculus without limits. This field was pioneered by the renowned mathematician Euler in the eighteenth century when he introduced the parameter q into Newton's work on infinite series. Based on Euler's foundational work, Jackson (1910) introduced q-definite integrals and began a symmetric analysis, which greatly advanced the study of q-calculus [23]. Researchers have made significant contributions to the comprehension of mathematical phenomena by investigating q-calculus, a crucial subfield of quantum calculus, in order to establish q-analogues of classical inequalities. These q-analogues provide new viewpoints and ideas, broadening the mathematical landscape and enabling progress in domains such as optimisation and mathematical analysis. Quantum calculus integrates mathematics and physics. Many physicists utilise quantum calculus because it has numerous applications in quantum

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group theory. Additional research is recommended for people interested in contemporary quantum calculus trends [3, 12, 17–19, 24].

Agarwal reported the q-fractional derivative for the first time [1]. Al-Salam [2] defined q-analogues of the Riemann–Liouville fractional integral operator. Tariboon and Ntouyas [32] have recently disclosed an extensive investigation of q-integrals and q-derivatives in the domain $[\sigma, \rho] \subset \mathcal{R}$. By developing quantum analogues of well-known mathematical findings, including Ostrowski, Hölder, and Hermite-Hadamard inequality, as well as other integral inequalities employing classical convexity, their study has yielded notable advancements. Alp et al. [4] have established a novel variant of Hermite-Hadamard inequality utilizing the left quantum integral. By leveraging the right quantum integral, Bermudo et al. [7] unveiled a new version of Hermite-Hadamard inequality. By involving Jensen-Mercer inequality, Budak and Kara [8] have formulated an extended version of the Hermite-Jensen-Mercer inequality for quantum integrals. Sitthiwirattham et al. [27] have obtained integral identity to find error bounds for Maclaurin's formulas in term of q-calculus. By combining q-derivatives/q-integrals with h-derivatives/h-integrals, Shi et al. [31] have investigated a range of inequalities related to $q - h$-integrals and revealed their connections to well-known results in different scientific and engineering disciplines. On Milne-type inequality for co-ordinated convex functions, Shehzadi et al. [30] have established a novel identity. Also, they presented some new inequalities for Milne-type co-ordinated convex functions. Alqudah et al. [6] have proposed novel definitions of partial q-operators. By leveraging these operators they derived new variants of Hermite–Hadamard inequalities on the co-ordinated convex functions. Vivas-Cortez et al. [34] proposed new quantum Simpson's and Newton's type inequalities. Readers can find some recent interesting findings in quantum calculus in [11, 14, 20, 25, 36, 37].

Recently, numerous researchers have turned their focus to Milne's type inequalities across diverse fractional integrals such as Riemann–Liouville fractional integral [10], Conformable fractional integral [13], Tempered fractional integrals [15, 16, 21, 22, 35]. Mateen et al. [26] have provided the computational analysis of some Milne's rule type inequalities in q-calculus and obtained better bounds to some existing ones. In [29], authors identified a novel quantum integral identity. By involving the newly established integral identity, they proved several new inequalities of Milne's rule type for quantum differentiable convex functions.

Inspired by current research endeavours, we formulate Milne-type inequalities by exploiting the convexity property of the function within the context of q-calculus. We prove that the inequalities obtained in this study extend certain existing ones. It is important to highlight that our main findings can be reduced to classical calculus simply by setting $q \to 1^-$. The study is divided into five sections, with the introduction being the first, in which we introduce the fundamentals of quantum calculus and Milne-type inequalities along with recent advancements in the field. Section 2 presents the necessary preliminaries of quantum calculus. In Section 3, we outline the main results of our study, detailing the newly derived inequalities. Section 4 provides numerical examples and graphical representations to validate the newly established inequalities. Finally, Section 5 concludes with a summary of our findings and discusses potential directions for future research.

§2 Preliminaries of q-Calculus

In the following section, we begin with examining the definitions and properties of q-derivatives and q-integrals. In 2013, Tariboon and Ntouyas [32] introduced the q_σ-derivative and integral along with a comprehensive exploration of their properties. We now recall the following definitions from their seminal work.

Definition 2.1 (See [32])**.** Let $\mathsf{F}:[\sigma,\rho]\to\mathcal{R}$ be a continuous function. Then the q_σ-derivative of F at $\varkappa\in(\sigma,\rho]$ is outlined by

$$ {}_\sigma D_q\mathsf{F}(\varkappa)=\frac{\mathsf{F}(\varkappa)-\mathsf{F}(\sigma+q(\varkappa-\sigma))}{(1-q)(\varkappa-\sigma)}. \tag{1}$$

The q_σ-integral is described as

$$\int_\sigma^\varkappa \mathsf{F}(\zeta)_\sigma d_q\zeta=(1-q)(\varkappa-\sigma)\sum_{n=0}^{\infty}q^n\mathsf{F}\left(\sigma+q^n(\varkappa-\sigma)\right). \tag{2}$$

Bermudo et al. [7] unveiled a novel approach involving the q^ρ derivative and integral. They also examined several fundamental properties of these operators. Here, we recapitulate the definitions provided in their study

Definition 2.2 (See [7])**.** Let $\mathsf{F}:[\sigma,\rho]\to\mathcal{R}$ be a continuous function. Then the q^ρ-derivative of F at $\varkappa\in[\sigma,\rho)$ is outlined by

$$ {}^\rho D_q\mathsf{F}(\varkappa)=\frac{\mathsf{F}(\rho+q(\varkappa-\rho))-\mathsf{F}(\varkappa)}{(1-q)(\rho-\varkappa)}. \tag{3}$$

The q^ρ-integral is described as

$$\int_x^\rho \mathsf{F}(\zeta)^\rho d_q\zeta=(1-q)(\rho-\varkappa)\sum_{n=0}^{\infty}q^n\mathsf{F}\left(\rho+q^n(\varkappa-\rho)\right). \tag{4}$$

In [5] and [28], the authors provide the following formulas of q-integration by parts

Lemma 2.1 (See [5])**.** For continuous functions $h,\mathsf{F}:[\sigma,\rho]\to\mathcal{R}$, the subsequent equality is valid

$$\begin{aligned}&\int_0^c h(\zeta)_\sigma D_q\mathsf{F}(\zeta\rho+(1-\zeta)\sigma)d_q\zeta \qquad (5)\\ &=\left.\frac{h(\zeta)\mathsf{F}(\zeta\rho+(1-\zeta)\sigma)}{\rho-\sigma}\right|_0^c-\frac{1}{\rho-\sigma}\int_0^c D_qh(\zeta)\mathsf{F}(q\zeta\rho+(1-q\zeta)\sigma)d_q\zeta.\end{aligned}$$

Lemma 2.2 (See [28])**.** For continuous functions $h,\mathsf{F}:[\sigma,\rho]\to\mathcal{R}$, the subsequent equality is valid

$$\begin{aligned}&\int_0^c h(\zeta)^\rho D_q\mathsf{F}(\zeta\sigma+(1-\zeta)\rho)d_q\zeta \qquad (6)\\ &=\frac{1}{\rho-\sigma}\int_0^c D_qh(\zeta)\mathsf{F}(q\zeta\sigma+(1-q\zeta)\rho)d_q\zeta-\left.\frac{h(\zeta)\mathsf{F}(\zeta\sigma+(1-\zeta)\rho)}{\rho-\sigma}\right|_0^c.\end{aligned}$$

Lemma 2.3 (See [33])**.** The subsequent equality valid:

$$\int_a^b(\zeta-a)^\alpha\sigma d_q\zeta=\frac{(\rho-\sigma)^{\alpha+1}}{[\alpha+1]_q},$$

where $\alpha\in\mathcal{R}-\{-1\}$.

§3 Main Results

First, we formulate the fundamental identity essential for obtaining the desired results by employing quantum differentiable functions. In this investigation, q is considered as a constant with $0 < q < 1$, and $[\sigma, \rho] \subseteq \mathcal{R}$ represents an interval with $\sigma < \rho$. The q-number is defined as follows

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathcal{N}.$$

Lemma 3.1. Assume $\mathsf{F} : [\sigma, \rho] \to \mathcal{R}$ be a q-differentiable function. ${}_\sigma D_q \mathsf{F}(\zeta)$ and ${}_\rho D_q \mathsf{F}(\zeta)$ are q-integrable on $[\sigma, \rho]$, in this case, the following equality is valid

$$\begin{aligned}&\frac{1}{3}\left[2\mathsf{F}(\sigma) - \mathsf{F}\left(\frac{\sigma+\rho}{2}\right) + 2\mathsf{F}(\rho)\right] - \frac{1}{\rho-\sigma}\left[\int_\sigma^{\frac{\sigma+\rho}{2}} \mathsf{F}(\varkappa)_\sigma d_q\varkappa + \int_{\frac{\sigma+\rho}{2}}^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa\right] \qquad (7)\\ &= \frac{\rho-\sigma}{4}\int_0^1 \left(q\zeta - \frac{4}{3}\right)\left[{}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) - {}^\rho D_q\mathsf{F}\left((1-\zeta)\rho + \zeta\frac{\sigma+\rho}{2}\right)\right] d_q\zeta.\end{aligned}$$

Proof. Through the utilization of the q-integral definition, we attain

$$\begin{aligned}I_1 &= \int_0^1 \left(q\zeta - \frac{4}{3}\right) {}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta\\ &= \int_0^1 q\zeta\, {}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta - \frac{4}{3}\int_0^1 {}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta.\end{aligned}$$

Likewise,

$$\begin{aligned}I_2 &= \int_0^1 \left(q\zeta - \frac{4}{3}\right) {}^\rho D_q\mathsf{F}\left((1-\zeta)\rho + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta\\ &= \int_0^1 q\zeta\, {}^\rho D_q\mathsf{F}\left((1-\zeta)\rho + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta - \frac{4}{3}\int_0^1 {}^\rho D_q\mathsf{F}\left((1-\zeta)\rho + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta.\end{aligned}$$

Therefore,

$$\begin{aligned}I_1 - I_2 = &\int_0^1 q\zeta\, {}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta - \frac{4}{3}\int_0^1 {}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta \qquad (8)\\ &- \int_0^1 q\zeta\, {}^\rho D_q\mathsf{F}\left((1-\zeta)\rho + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta + \frac{4}{3}\int_0^1 {}^\rho D_q\mathsf{F}\left((1-\zeta)\rho + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta.\end{aligned}$$

By Lemma 2.2, we attain

$$\begin{aligned}&\int_0^1 q\zeta\, {}_\sigma D_q\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right) d_q\zeta \qquad (9)\\ &= \frac{2q\zeta\mathsf{F}\left((1-\zeta)\sigma + \zeta\frac{\sigma+\rho}{2}\right)}{\rho-\sigma}\Bigg|_0^1 - \frac{2}{\rho-\sigma}\int_0^1 q\mathsf{F}\left((1-q\zeta)\sigma + q\zeta\frac{\sigma+\rho}{2}\right) d_q\zeta\\ &= \frac{2q}{\rho-\sigma}\mathsf{F}\left(\frac{\sigma+\rho}{2}\right) - \frac{2(1-q)}{\rho-\sigma}\sum_{n=0}^\infty q^{n+1}\mathsf{F}\left((1-q^{n+1})\sigma + q^{n+1}\frac{\sigma+\rho}{2}\right)\\ &= \frac{2}{\rho-\sigma}\mathsf{F}\left(\frac{\sigma+\rho}{2}\right) - \frac{2(1-q)}{\rho-\sigma}\sum_{n=0}^\infty q^{n}\mathsf{F}\left((1-q^{n})\sigma + q^{n}\frac{\sigma+\rho}{2}\right)\end{aligned}$$

$$=\frac{2}{\rho-\sigma}\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)-\frac{4}{(\rho-\sigma)^2}\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa.$$

Similarly,

$$\int_0^1 {}_{\sigma}D_q\mathsf{F}\left((1-\zeta)\sigma+\zeta\frac{\sigma+\rho}{2}\right)d_q\zeta=\frac{2}{\rho-\sigma}\left[\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)-\mathsf{F}(\sigma)\right], \tag{10}$$

$$\int_0^1 q\zeta^{\rho}D_q\mathsf{F}\left((1-\zeta)\rho+\zeta\frac{\sigma+\rho}{2}\right)d_q\zeta=\frac{4}{(\rho-\sigma)^2}\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa-\frac{2}{\rho-\sigma}\mathsf{F}\left(\frac{\sigma+\rho}{2}\right), \tag{11}$$

$$\int_0^1 {}^{\rho}D_q\mathsf{F}\left((1-\zeta)\rho+\zeta\frac{\sigma+\rho}{2}\right)d_q\zeta=-\frac{2}{\rho-\sigma}\left[\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)-\mathsf{F}(\rho)\right]. \tag{12}$$

Substituting equalities (9)-(12) into equation (8), we arrive at

$$\frac{\rho-\sigma}{4}[I_1-I_2]$$
$$=\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]. \tag{13}$$

The proof of Lemma 3.1 is concluded. □

Theorem 3.2. Assume that the conditions outlined in Lemma 3.1 hold. If $|{}^{\rho}D_q\mathsf{F}(\zeta)|$ and $|{}_{\sigma}D_q\mathsf{F}(\zeta)|$ are convex on $[\sigma,\rho]$, then the subsequent inequality is valid

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right| \tag{14}$$
$$\leq\frac{\rho-\sigma}{4}\left[\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|\right)+\left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|\right)\right],$$

where

$$\mathcal{A}_1(q)=\int_0^1\frac{2-\zeta}{2}\left|q\zeta-\frac{4}{3}\right|d_q\zeta=\frac{4+9q+9q^2+2q^3}{6[2]_q[3]_q},$$
$$\mathcal{A}_2(q)=\int_0^1\frac{\zeta}{2}\left|q\zeta-\frac{4}{3}\right|d_q\zeta=\frac{4+q+q^2}{6[2]_q[3]_q}.$$

Proof. Applying the absolute value in Lemma 3.1, we obtain

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right| \tag{15}$$
$$\leq\frac{\rho-\sigma}{4}\int_0^1\left|q\zeta-\frac{4}{3}\right|\left[\left|{}_{\sigma}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|+\left|{}^{\rho}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|\right]d_q\zeta.$$

Since $|{}^{\rho}D_q\mathsf{F}(\zeta)|$ and $|{}_{\sigma}D_q\mathsf{F}(\zeta)|$ are convex on $[\sigma,\rho]$, it yields

$$\left|{}_{\sigma}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|\leq\frac{2-\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|+\frac{\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right| \tag{16}$$

and

$$\left|{}^{\rho}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|\leq\frac{2-\zeta}{2}\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\frac{\zeta}{2}\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|. \tag{17}$$

Substituting (16) and (17) in (15), we acquire

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right|$$

$$\leq \frac{\rho-\sigma}{4}\int_0^1\left|q\zeta-\frac{4}{3}\right|\left[\frac{2-\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|+\frac{\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|+\frac{2-\zeta}{2}\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\frac{\zeta}{2}\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|\right]d_q\zeta.$$

Computing the quantum integrals, we have

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right|$$
$$\leq\frac{\rho-\sigma}{4}\left[\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|\right)+\left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|\right)\right].$$

Hence, the proof is concluded. □

Remark 3.3. If we assign $q\to1^-$ in Theorem 3.2, then we attain the subsequent inequality

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)d\varkappa\right|\leq\frac{5(\rho-\sigma)}{24}\left[\left|\mathsf{F}'(\sigma)\right|+\left|\mathsf{F}'(\rho)\right|\right],$$

which is identified in [9, Remark 1].

Theorem 3.4. Assume that the conditions outlined in Lemma 3.1 hold. If $\left|{}^{\rho}D_q\mathsf{F}(\zeta)\right|^s$ and $\left|{}_{\sigma}D_q\mathsf{F}(\zeta)\right|^s$ are convex on $[\sigma,\rho]$ and $\frac{1}{p}+\frac{1}{s}=1$ with $p,s>1$, then the subsequent inequality is valid

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right| \tag{18}$$
$$\leq\frac{\rho-\sigma}{4}\left(\left(\frac{4}{3}\right)^p-\frac{q^p}{[p+1]_q}\right)^{\frac{1}{p}}\left[\left(\frac{(1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{(1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s+\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right].$$

Proof. By utilizing q-Hölder's inequality in (15), it yields

$$\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right| \tag{19}$$
$$\leq\frac{\rho-\sigma}{4}\left(\int_0^1\left|q\zeta-\frac{4}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|{}_{\sigma}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|^s d_q\zeta\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left|{}^{\rho}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|^s d_q\zeta\right)^{\frac{1}{s}}\right].$$

Since $\left|{}^{\rho}D_q\mathsf{F}\right|^s$ and $\left|{}_{\sigma}D_q\mathsf{F}\right|^s$ are convex on $[\sigma,\rho]$, we acquire

$$\int_0^1\left|{}_{\sigma}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|^s d_q\zeta\leq\int_0^1\left[\frac{2-\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\frac{\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s\right]d_q\zeta \tag{20}$$
$$=\frac{1+2q}{2[2]_q}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\frac{1}{2[2]_q}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s.$$

Similarly,

$$\int_0^1\left|{}^{\rho}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|^s d_q\zeta\leq\frac{1+2q}{2[2]_q}\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s+\frac{1}{2[2]_q}\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s. \tag{21}$$

Here we use the fact that

$$|\mathcal{B}_1-\mathcal{B}_2|<\mathcal{B}_1^p-\mathcal{B}_2^p \tag{22}$$

for $\mathcal{B}_1 > \mathcal{B}_2 > 0$ and $p > 1$. Applying (20) and (21) to (19), we achieve

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right|\\&\leq\frac{\rho-\sigma}{4}\left(\left(\frac{4}{3}\right)^p-\frac{q^p}{[p+1]_q}\right)^{\frac{1}{p}}\left[\left(\frac{(1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.\\&\left.+\left(\frac{(1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s+\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right].\end{aligned}$$

Thus, the proof is concluded. □

Remark 3.5. Setting $q \to 1^-$ in Theorem 3.4, we attain the subsequent inequality

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)d\varkappa\right|\\&\leq\frac{\rho-\sigma}{4}\left(\left(\frac{4}{3}\right)^p-\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathsf{F}'(\sigma)\right|^s+\left|\mathsf{F}'(\rho)\right|^s}{4}\right)^{\frac{1}{s}}+\left(\frac{3\left|\mathsf{F}'(\rho)\right|^s+\left|\mathsf{F}'(\sigma)\right|^s}{4}\right)^{\frac{1}{s}}\right],\end{aligned}$$

which is obtained in [10, Corollary 1].

Theorem 3.6. Assume that the conditions outlined in Lemma 3.1 hold. If $\left|{}^{\rho}D_q\mathsf{F}(\zeta)\right|^s$ and $\left|{}_{\sigma}D_q\mathsf{F}(\zeta)\right|^s$ are convex on $[\sigma,\rho]$ for $s \geq 1$, then the subsequent inequality is valid:

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right| \qquad (23)\\&\leq\frac{\rho-\sigma}{4}\left(\mathcal{A}_3(q)\right)^{1-\frac{1}{s}}\left[\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s\right)^{\frac{1}{s}}\right.\\&\left.+\left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s\right)^{\frac{1}{s}}\right],\end{aligned}$$

where

$$\mathcal{A}_3(q)=\int_0^1\left|q\zeta-\frac{4}{3}\right|d_q\zeta=\frac{4+q}{3[2]_q}.$$

$\mathcal{A}_1(q)$ and $\mathcal{A}_2(q)$ are defined as in Theorem 3.2.

Proof. Employing q-power mean inequality in (15), we acquire

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{\rho-\sigma}\left[\int_{\sigma}^{\frac{\sigma+\rho}{2}}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa+\int_{\frac{\sigma+\rho}{2}}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa\right]\right| \qquad (24)\\&\leq\frac{\rho-\sigma}{4}\left(\int_0^1\left|q\zeta-\frac{4}{3}\right|d_q\zeta\right)^{1-\frac{1}{s}}\left[\left(\int_0^1\left|q\zeta-\frac{4}{3}\right|\left|{}_{\sigma}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|^s d_q\zeta\right)^{\frac{1}{s}}\right.\\&\left.+\left(\int_0^1\left|q\zeta-\frac{4}{3}\right|\left|{}^{\rho}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|^s d_q\zeta\right)^{\frac{1}{s}}\right].\end{aligned}$$

Since $\left|{}^{\rho}D_q\mathsf{F}\right|^s$ and $\left|{}_{\sigma}D_q\mathsf{F}\right|^s$ are convex on $[\sigma,\rho]$, we acquire

$$\begin{aligned}\int_0^1\left|q\zeta-\frac{4}{3}\right|\left|{}_{\sigma}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|^s d_q\zeta&\leq\int_0^1\left|q\zeta-\frac{4}{3}\right|\left[\frac{2-\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\frac{\zeta}{2}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s\right]d_q\zeta\\&=\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s. \qquad (25)\end{aligned}$$

Similarly,

$$\int_0^1 \left|q\zeta - \frac{4}{3}\right| \left|{}^{\rho}D_q\mathsf{F}\left(\frac{2-\zeta}{2}\rho + \frac{\zeta}{2}\sigma\right)\right|^s d_q\zeta \leq \mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s + \mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s. \tag{26}$$

Applying (25) and (26) to (24), we achieve the desired result. □

Remark 3.7. Setting $q \to 1^-$ in Theorem 3.6, we attain the subsequent inequality

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathsf{F}(\sigma) - \mathsf{F}\left(\frac{\sigma+\rho}{2}\right) + 2\mathsf{F}(\rho)\right] - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathsf{F}(\varkappa)d\varkappa\right| \\ &\leq \frac{5(\rho-\sigma)}{24}\left[\left(\frac{4|\mathsf{F}'(\sigma)|^s + |\mathsf{F}'(\rho)|^s}{5}\right)^{\frac{1}{s}} + \left(\frac{4|\mathsf{F}'(\rho)|^s + |\mathsf{F}'(\sigma)|^s}{5}\right)^{\frac{1}{s}}\right],\end{aligned}$$

which is identified in [9, Remark 2].

§4 Examples

In this section, we provide examples that serve to illustrate our results and showcase the applications of theorems.

Example 4.1. Let us assume the function $\mathsf{F} : [0,2] \to \mathcal{R}$ defined as $\mathsf{F}(\varkappa) = \varkappa^2$. Then F is q-differentiable. Based on these assumptions, we observe

$${}^{\rho}D_q\mathsf{F}(\varkappa) = {}^{2}D_q\mathsf{F}(\varkappa) = [2]_q\varkappa + 2(1-q)$$

and

$${}_{\sigma}D_q\mathsf{F}(\varkappa) = {}_{0}D_q\mathsf{F}(\varkappa) = [2]_q\varkappa.$$

These functions are convex on $[0,2]$. By utilizing Theorem 3.2 to the function $\mathsf{F}(\varkappa) = \varkappa^2$, it yields

$$\frac{1}{3}\left[2\mathsf{F}(\sigma) - \mathsf{F}\left(\frac{\sigma+\rho}{2}\right) + 2\mathsf{F}(\rho)\right] = \frac{7}{3}$$

and

$$\frac{1}{\rho-\sigma}\left[\int_\sigma^{\frac{\sigma+\rho}{2}} \mathsf{F}(\varkappa)\,{}_{\sigma}d_q\varkappa + \int_{\frac{\sigma+\rho}{2}}^{\rho} \mathsf{F}(\varkappa)\,{}^{\rho}d_q\varkappa\right] = 2 - \frac{2}{[2]_q} + \frac{1}{[3]_q}.$$

So, the left-hand side of (14) is

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathsf{F}(\sigma) - \mathsf{F}\left(\frac{\sigma+\rho}{2}\right) + 2\mathsf{F}(\rho)\right] - \frac{1}{\rho-\sigma}\left[\int_\sigma^{\frac{\sigma+\rho}{2}} \mathsf{F}(\varkappa)\,{}_{\sigma}d_q\varkappa + \int_{\frac{\sigma+\rho}{2}}^{\rho} \mathsf{F}(\varkappa)\,{}^{\rho}d_q\varkappa\right]\right| \\ &= \left|\frac{7}{3} - \left[2 - \frac{2}{[2]_q} + \frac{1}{[3]_q}\right]\right|.\end{aligned} \tag{27}$$

Now, we let

$$\begin{aligned}&\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right| = \left|{}^{2}D_q\mathsf{F}(0)\right| = 2(1-q), \quad &&\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right| = \left|{}_{0}D_q\mathsf{F}(2)\right| = 2[2]_q, \\ &\left|{}^{\rho}D_q\mathsf{F}(\rho)\right| = \left|{}^{2}D_q\mathsf{F}(2)\right| = 4, &&\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right| = \left|{}_{0}D_q\mathsf{F}(0)\right| = 0.\end{aligned}$$

Therefore, the right-hand side of (14) is

$$\frac{\rho-\sigma}{4}\left[\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right| + \mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|\right) + \left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right| + \mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|\right)\right]$$

$$= \frac{8+10q+10q^2+2q^3}{3[2]_q[3]_q}.$$

With reference to inequality (14), we acquire

$$\left|\frac{7}{3} - \left[2 - \frac{2}{[2]_q} + \frac{1}{[3]_q}\right]\right| \leq \frac{8+10q+10q^2+2q^3}{3[2]_q[3]_q}. \tag{28}$$

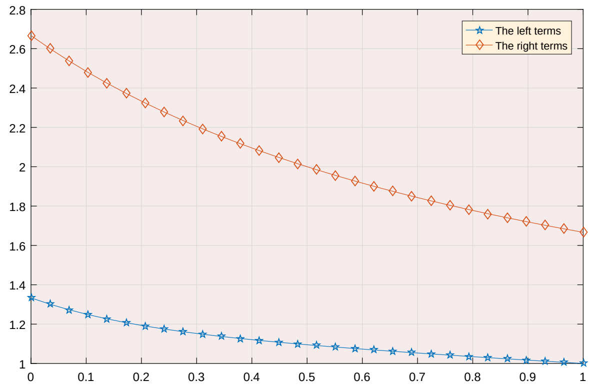

Figure 1. In Example 4.1, depending on $q \in [0,1]$, MATLAB has been used to compute and plot the graph of both sides of (28). Therefore, the validity of inequality (28) has been verified.

Example 4.2. Let us assume the function $\mathsf{F} : [0,2] \to \mathcal{R}$ defined as $\mathsf{F}(\varkappa) = \varkappa^2$ and $p = s = 2$. Then F is q-differentiable. Based on these assumptions, we observe

$$\left|{}^{\rho}D_q\mathsf{F}(\varkappa)\right|^s = \left|{}^{2}D_q\mathsf{F}(\varkappa)\right|^2 = \left([2]_q\varkappa + 2(1-q)\right)^2$$

and

$$\left|{}_{\sigma}D_q\mathsf{F}(\varkappa)\right|^2 = \left|{}_{0}D_q\mathsf{F}(\varkappa)\right|^2 = [2]_q^2\varkappa^2.$$

These functions are convex on $[0,2]$. By utilizing Theorem 3.4, the left-hand side of inequality (18) is similar to (27).

On the other hand, by (18), we have

$$\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s = \left|{}^{2}D_q\mathsf{F}(0)\right|^2 = 4(1-q)^2, \quad \left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s = \left|{}_{0}D_q\mathsf{F}(2)\right|^2 = 4[2]_q^2,$$
$$\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s = \left|{}^{2}D_q\mathsf{F}(2)\right|^2 = 16, \quad \left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s = \left|{}_{0}D_q\mathsf{F}(0)\right|^2 = 0.$$

Therefore, the right-hand side of (18) is

$$\frac{\rho-\sigma}{4}\left(\left(\frac{4}{3}\right)^p - \frac{q^p}{[p+1]_q}\right)^{\frac{1}{p}} \left[\left(\frac{(1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s + \left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left. + \left(\frac{(1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s + \left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right]$$
$$= \frac{1}{2}\left(\frac{16}{9} - \frac{q^2}{[3]_q}\right)^{\frac{1}{2}} \left[(2[2]_q)^{\frac{1}{2}} + \left(\frac{10+12q+2q^2}{[2]_q}\right)^{\frac{1}{2}}\right].$$

With reference to inequality (18), we have

$$\left|\frac{7}{3}-\left[2-\frac{2}{[2]_q}+\frac{1}{[3]_q}\right]\right|\leq\frac{1}{2}\left(\frac{16}{9}-\frac{q^2}{[3]_q}\right)^{\frac{1}{2}}\left[(2[2]_q)^{\frac{1}{2}}+\left(\frac{10+12q+2q^2}{[2]_q}\right)^{\frac{1}{2}}\right].\tag{29}$$

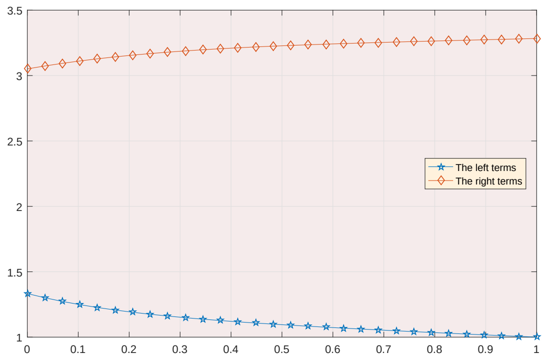

Figure 2. The graph of both sides of (28) in Example 4.2 has been computed and plotted using MATLAB, contingent on $q\in[0,1]$. Consequently, it has been confirmed that inequality (29) is satisfied.

Example 4.3. Let us assume the function $\mathsf{F}:[0,2]\to\mathcal{R}$ defined as $\mathsf{F}(\varkappa)=\varkappa^2$ and $s=2$. Then F is q-differentiable. Based on these assumptions, we observe

$$\left|{}^{\rho}D_q\mathsf{F}(\varkappa)\right|^s=\left|{}^{2}D_q\mathsf{F}(\varkappa)\right|^2=([2]_q\varkappa+2(1-q))^2$$

and

$$\left|{}_{\sigma}D_q\mathsf{F}(\varkappa)\right|^2=\left|{}_{0}D_q\mathsf{F}(\varkappa)\right|^2=[2]_q^2\varkappa^2.$$

These functions are convex on $[0,2]$. By utilizing Theorem 3.6, the left-hand side of inequality (23) is similar to (27).

On the other hand, by (23), we have

$$\begin{aligned}&\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s=\left|{}^{2}D_q\mathsf{F}(0)\right|^2=4(1-q)^2, &&\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s=\left|{}_{0}D_q\mathsf{F}(2)\right|^2=4[2]_q^2,\\ &\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s=\left|{}^{2}D_q\mathsf{F}(2)\right|^2=16, &&\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s=\left|{}_{0}D_q\mathsf{F}(0)\right|^2=0.\end{aligned}$$

Therefore, the right-hand side of (23) is

$$\frac{\rho-\sigma}{4}\left(\mathcal{A}_3(q)\right)^{1-\frac{1}{s}}\times$$

$$\left[\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s\right)^{\frac{1}{s}}+\left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s\right)^{\frac{1}{s}}\right]$$

$$=\frac{1}{2}\left(\frac{4+q}{3[2]_q}\right)^{\frac{1}{2}}\left[\left(\frac{2(4+q+q^2)[2]_q^2}{3[2_q][3]_q}\right)^{\frac{1}{2}}+\left(\frac{8(4+9q+9q^2+2q^3)}{3[2]_q[3]_q}+\frac{2(4+q+q^2)(1-q)^2}{3[2]_q[3]_q}\right)^{\frac{1}{2}}\right].$$

With reference to inequality (23), we have

$$\left|\frac{7}{3}-\left[2-\frac{2}{[2]_q}+\frac{1}{[3]_q}\right]\right|\leq\frac{1}{2}\left(\frac{4+q}{3[2]_q}\right)^{\frac{1}{2}}\left[\left(\frac{2(4+q+q^2)[2]_q^2}{3[2_q][3]_q}\right)^{\frac{1}{2}}\right.$$

$$+\left(\frac{8(4+9q+9q^2+2q^3)}{3[2]_q[3]_q}+\frac{2(4+q+q^2)(1-q)^2}{3[2]_q[3]_q}\right)^{\frac{1}{2}}\Bigg]. \tag{30}$$

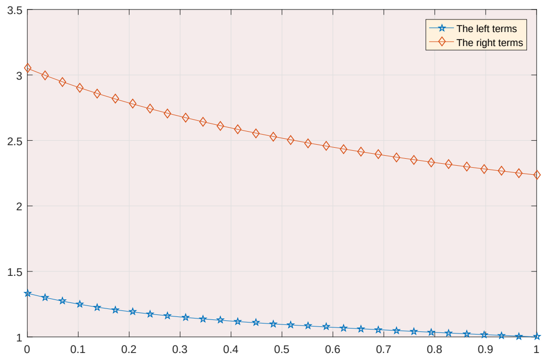

Figure 3. The graph of both sides of (30) in Example 4.3 has been computed and plotted using MATLAB, contingent on $q \in [0, 1]$. As a result, inequality (30) has been shown to be valid.

§5 Conclusion

This research has contributed novel findings to the realm of Milne's rule-type inequalities applied for differentiable convex functions in the framework of quantum calculus. First, we revealed a novel integral identity. Leveraging this integral identity and convexity properties, we examined these inequalities, crucial for Open-Newton's Cotes formulas. We reported innovative inequalities aimed at establishing error bounds in Milne's rule, both within classical and quantum calculus domains. Furthermore, we have provided numerical examples and graphical analysis of these newly established inequalities. Our study demonstrates that these findings not only refine but also expand previous findings in the field of integral inequalities. These inequalities will be helpful for researchers who are working in the field of optimization theory and mathematical inequalities in the context of quantum calculus. In the future studies, the authors can generalize the newly obtained results for the other type of convexities or for fractional quantum integrals.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] R P Agarwal. *A propos d'une note de M Pierre Humbert*, Comptes Rendus de l'Académie des Sciences-Paris, 1953, 236(21): 2031-2032.

[2] W A Al-Salam. *Some fractional q-integrals and q-derivatives*, Proceedings of the Edinburgh Mathematical Society, 1966, 15(2): 135-140.

[3] M Anwar, R Bibi, M Bohner, et al. *Integral inequalities on time scales via the theory of isotonic linear functionals*, Abstract and Applied Analysis, 2011, 2011(1): 483595.

[4] N Alp, M Z Sarıkaya, M Kunt, et al. *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University-Science, 2018, 30(2): 193-203.

[5] M A Ali, H Budak, K Nanlaopon, et al. *Simpson's and Newton's inequalities for* (α, m)*-convex functions via quantum calculus*, Symmetry, 2021, 14(4): 736.

[6] M A Alqudah, A Kashuri, P O Mohammed, et al. *Hermite–Hadamard integral inequalities on co-ordinated convex functions in quantum calculus*, Advances in Difference Equations, 2021, 2021(1): 264.

[7] S Bermudo, P Kórus, Nápoles Valdés. *On q-Hermite-Hadamard inequalities for general convex functions*, Acta Mathematica Hungarica, 2020, 162: 364-374.

[8] H Budak, H Kara. *On Quantum Hermite-Jensen-Mercer Inequalities*, Miskolc Math Notes, 2020, DOI: 10.18514/MMN.2020.3053.

[9] H Budak, P Kösem, H Kara. *On new Milne-type inequalities for fractional integrals*, Journal of Inequalities and Applications, 2023, 2023(1): 1-15.

[10] H Budak, P Karagözoğlu. *Fractional Milne type inequalities*, Acta Mathematica Universitatis Comenianae, 2024, 93(1): 1-15.

[11] S I Butt, H Budak, K Nonlaopon. *New quantum Mercer estimates of Simpson-Newton-like inequalities via convexity*, Symmetry, 2022, 14(9): 1935.

[12] A Bakht, M Anwar. *Hermite-Hadamard and Ostrowski type inequalities via* α*-exponential type convex functions with applications*, AIMS Mathematics, 2024, 9(4): 9519-9535.

[13] B Celik, H Budak, E Set. *On generalized Milne type inequalities for new conformable fractional integrals*, Filomat, 2024, 38(5): 1807-1823.

[14] T S Du, C Y Luo, B Yu. *Certain quantum estimates on the parameterized integral inequalities and their applications*, Journal of Mathematical Inequalities, 2021, 15(1): 201-228.

[15] I Demir. *A new approach of Milne-type inequalities based on proportional Caputo-Hybrid operator: A new approach for Milne-type inequalities*, Journal of Advances in Applied and Computational Mathematics, 2023, 10: 102-119.

[16] H D Desta, H Budak, H Kara. *New perspectives on fractional Milne-type inequalities: Insights from twice-differentiable functions*, Universal Journal of Mathematics and Applications, 2023, 7(1): 30-37.

[17] T Ernst. *A comprehensive treatment of q-calculus*, Springer Science & Business Media, 2012.

[18] J W Green. *Recent applications of convex functions*, The American Mathematical Monthly, 1954, 61(7P1): 449-454.

[19] H Gauchman. *Integral inequalities in q-calculus*, Computers and Mathematics with Applications, 2004, 47(2-3): 281-300.

[20] G Gulshan, H Budak, R Hussain, et al. *New Quantum Hermite–Hadamard-Type Inequalities for p-Convex Functions Involving Recently Defined Quantum Integrals*, Ukrainian Mathematical Journal, 2024, 1-17.

[21] F Hezenci, H Budak, H Kara, et al. *Novel results of Milne-type inequalities involving tempered fractional integrals*, Boundary Value Problems, 2024, 2024(1): 12.

[22] W Haider, H Budak, A Shehzadi, et al. *A comprehensive study on Milne-type inequalities with tempered fractional integrals*, Boundary Value Problems, 2024, 2024, DOI: 10.1186/s13661-024-01855-1.

[23] F H Jackson. *On q-definite integrals*, The Quarterly Journal of Pure and Applied Mathematics, 1910, 41(15): 193-203.

[24] V G Kac, P Cheung. *Quantum calculus*, New York: Springer, 2002.

[25] H Kalsoom, M A Ali, M Abbas, et al. *Generalized quantum Montgomery identity and Ostrowski type inequalities for preinvex functions*, TWMS Journal of Pure and Applied Mathematics, 2022, 13(1): 72-90.

[26] A Mateen, Z Zhang, M A Ali. *Some Milne's Rule Type Inequalities for Convex Functions with Their Computational Analysis On Quantum Calculus*, Filomat, 2024, 38(10): 3329-3345.

[27] T Sitthiwirattham, M A Ali, H Budak. *On some new Maclaurin's type inequalities for convex functions in q-calculus*, Fractal and Fractional, 2023, 7(8): 572.

[28] I B Sial, S Mei, M A Ali, et al. *On some generalized Simpson's and Newton's inequalities for (α, m)-convex functions in q-calculus*, Mathematics, 2021, 9(24): 3266.

[29] I B Sial, H Budak, M A Ali. *Some Milne's rule type inequalities in quantum calculus*, Filomat, 2023, 37(27): 9119-9134.

[30] A Shehzadi, H Budak, W Haider, et al. *Milne-type Inequalities for Co-ordinated Convex Functions*, Filomat, 2024, 38: 8295-8303.

[31] D Shi, G Farid, B A I Younis, et al. *A unified representation of q-and h-integrals and consequences in inequalities*, Axioms, 2024, 13(4): 278.

[32] J Tariboon, S K Ntouyas. *Quantum calculus on finite intervals and applications to impulsive difference equations*, Advances in Difference Equations, 2013, 2013: 1-19.

[33] J Tariboon, S K Ntouyas. *Quantum integral inequalities on finite intervals*, Journal of Inequalities and Applications, 2014, 2014: 1-13.

[34] M Vivas-Cortez, M A Aamir Ali, A Kashuri, et al. *Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus*, Symmetry, 2020, 12(9): 1476.

[35] R Ying, A Lakhdari, H Xu, et al. *On Conformable Fractional Milne-Type Inequalities*, Symmetry, 2024, 16(2): 196.

[36] B Yu, C Y Luo, T S Du. *On the refinements of some important inequalities via (p, q)-calculus and their applications*, Journal of Inequalities and Applications, 2021, 2021(1): 82.

[37] Y Zhang, T S Du, H Wang, et al. *Different types of quantum integral inequalities via (α, m)-convexity*, Journal of Inequalities and Applications, 2018, 2018(1): 264.

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