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Controlled proximal contractions with an application to a class of integral equations

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Abstract. In this study, we explore some of the best proximity point results for generalized proximal contractions in the setting of double-controlled metric-type spaces. A non-trivial example is given to elucidate our analysis, and some novel results are derived. The discovered results generalize previously known results in the context of a double controlled metric type space environment. This article's proximity point results are the first of their kind in the realm of controlled metric spaces. To build on the results achieved in this article, we present an application demonstrating the usability of the given results.

§1 Introduction and preliminaries

Fixed point theory serves as a significant method for solving the equation $Tx = x$, where T denotes a mapping defined on a subset of a metric space, a simplified linear space, or a topological vector space. A mapping $T : A \to B$ that does not map elements to themselves might not possess a fixed point, but it is certain that there always exists an element x closer to Tx. From this perspective, the best approximation theorem and the best proximity point are relevant. A classical best approximation theorem, due to Fan [10], claims that if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X with a semi-norm p and $T : A \to X$ is a continuous mapping, then there is an element x in A satisfying the condition that $d(x, Tx) = d(Tx, A)$. Many subsequent extensions and variations of Fan's theorem have occurred including [3–5, 11, 16].

Although the best approximation theorems are sufficient for the delivery of an approximate solution to the equation $Tx = x$, such conclusions exclude an ideal approximate solution. Moreover, the theorem of the best proximity point specifies adequate criteria for the presence of an element x to reduce the error $d(x, Tx)$. A best proximity point theorem is concerned

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with the global minimization of the real-valued function $x \to d(x,Tx)$ and is an indication of the error involved in the solution of the approximated operator $Tx = x$. Since, for a nonself-mapping $T : A \to B$, $d(x,Tx)$ is at least $d(A,B)$ for all x in A, the best proximity point theorem establishes a globally optimal solution of error $d(x,Tx)$ by constraining an approximate solution x of the equation $Tx = x$ to the condition that $d(x,Tx) = d(A,B)$. Such an ideal approximate solution $Tx = x$ is the best proximity point of nonself-mapping $T : A \to B$. In fact, the best proximity point theorem can be used as a logical extension of fixed point theorems, since the best proximity point comes down to a fixed point as the underlying mapping turns out to be a self-mapping.

The basic theory of the best proximity point results have been demonstrated in [6]. Anuradha and Veeramani have tested the presence of the best proximity point for proximal pointwise contractions [3]. Generally, several the best proximity point theorems were analyzed for multiple variants of contractions in [1, 13, 17, 19, 20, 22, 24–26]. The best proximity point theorem for contraction mappings was presented in [7]. Some interesting common best proximity theorems have been discussed in [29] and [30]. The best proximity point theorems for various forms of multi-valued mappings have also been obtained in [4, 5, 12, 21].

Moreover, some recent papers deal with complementary aspects of analysis and stability of approximation theory via fixed-point methods and best approximation theory. The authors have studied applications of convergence results to solve variational inequalities and fixed point problems in the setting of real Hilbert spaces (see, e.g., [18, 27]). For application part related to best proximity point see [35–38].

Recently, according to the parameters of the left-hand side of the triangular inequality in extended $b-$metric space, the authors in [1] introduced a type of extended $b-$metric spaces by substituting the constant s by functions $\alpha(x,y)$ depending on the parameters of the left-hand side of the triangular inequality. The primary purpose of this article is to include the best proximity point theorems for generalized and modified proximal contractions in the context of double-controlled metric type spaces. Thus provide an optimal approximate solution to the equation $Tx = x$. It is acknowledged that the previous best proximity point theorems also include the well-known Banach contraction principle and some of its generalizations and variations are special cases.

Definition 1.1. *(cf. Abdeljawad at al. [1]) Let X be a nonempty set and $\alpha, \mu : X \times X \to [1,+\infty)$ be two noncomparable functions. Suppose that a function $d : X \times X \to [0,+\infty)$ satisfies:*

(1) $d(x_1,x_2) = 0$ *if and only if* $x_1 = x_2$,

(2) $d(x_1,x_2) = d(x_2,x_1)$,

(3) $d(x_1,x_2) \le \alpha(x_1,x_3)\, d(x_1,x_3) + \mu(x_3,x_2)\, d(x_3,x_2)$,

for all $x_1,x_2,x_3 \in X$, then (X,d) is called a double controlled metric type space.

Let A and B be two nonempty subsets of a double controlled metric type space (X,d).

Define

$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}(\text{distance of a set } A \text{ to a set } B).$$

Remark 1.1. *(cf. M. Gabeleh [33]) If (A,B) is a nonempty, bounded and closed pair in a reflexive Banach space X, then its proximal pair (A_0,B_0) is also nonempty, closed and convex.*

Definition 1.2. *(cf. Abdeljawad at al. [1]) Let (X,d) be a double controlled metric type space. A sequence $\{x_n\}$ converges to some x in X, if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$, such that $d(x_n, x) < \epsilon$ for each $n \geq N$. In this case, we write*

$$\lim_{n\to+\infty} x_n = x.$$

Definition 1.3. *(cf. Abdeljawad at al. [1]) The sequence $\{x_n\}$ in a double controlled metric type space (X,d) is called a Cauchy sequence, if for every $\epsilon > 0$, $d(x_n, x_m) < \epsilon$ for all $m,n \geq N_\epsilon$ where $N_\epsilon \in \mathbb{N}$.*

Definition 1.4. *(cf. Abdeljawad at al. [1]) A double controlled metric type space (X,d) is called complete if every Cauchy sequence is convergent in X.*

Definition 1.5. *(cf. Raj [22]) Let A and B be the nonempty subsets of a metric space such that A_0 is nonempty. Then a pair (A,B) has the P-property if and only if*

$$d(x_1,y_1) = d(A,B),$$
$$d(x_2,y_2) = d(A,B),$$

implies that

$$d(x_1,x_2) = d(y_1,y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

For Further information about P-property researchers can read [31] and [32].

Lemma 1.1. *(cf. M. Gabeleh & O. Olela Otafudu [34]) Let (A,B) be a nonempty, closed and convex pair in a reflexive and Busemann convex space X so that B is bounded. Then (A,B) has the P-property.*

Definition 1.6. *(cf. Jleli & Samet [15]) Let (X,d) be a double controlled metric type space. Given $T : A \to B$ and $\eta : A \times A \to [0,+\infty)$. The mapping T is said to be η-proximal admissible if*

$$\eta(x_1,x_2) \geq 1,$$
$$d(u_1,Tx_1) = d(A,B),$$
$$d(u_2,Tx_2) = d(A,B),$$

implies that

$$\eta(u_1,u_2) \geq 1,$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.7. *(cf. Rockafellar & Wet [23]) Let (X,d) be a double-controlled metric-type space. Let $\mathcal{CB}(X)$ represent the closed and bounded subsets of X. Let H be the Pompeiu–Hausdroff metric induced by metric d defined by*

$$\mathcal{H}(A,B)=\max\{\sup_{a\in A}\mathcal{D}(a,B),\sup_{b\in B}\mathcal{D}(b,A)\},$$

for $A,B\in\mathcal{CB}(X)$, where

$$\mathcal{D}(a,B)=\inf\{d(a,b):\ \textit{for all } b\in B\},$$

and we will denote

$$\mathcal{D}^*(a,b)=\{\mathcal{D}(a,b)-d(A,B),\ \textit{for all } a\in A \textit{ and } b\in B\}.$$

Definition 1.8. *Let (X,d) be a double-controlled metric-type space. Let (A,B) be a pair of nonempty subsets of double controlled metric type space such that A_0 is nonempty. An element $x^*\in A$ is said to be best proximity point of the mapping $T:A\to B$ if $d(x^*,Tx^*)=d(A,B)$.*

§2 Proximal contractions

Within the realm of double controlled metric type space (X,d), certain best proximity point theorems will be covered for multi-valued mappings in this section.
Define $\mathfrak{p}:X\times X\to[1,+\infty)$ by the following

$$\mathfrak{p}(x,B)=\inf\{\alpha(x,y):\ \text{for all } y\in B\},$$
$$\mathfrak{p}(A,B)=\inf\{\alpha(x,y):\ \text{for all } x\in A \text{ and } y\in B\}.$$

and $\mathfrak{q}:X\times X\to[1,+\infty)$ as

$$\mathfrak{q}(x,B)=\inf\{\mu(x,y):\ \text{for all } y\in B\},$$
$$\mathfrak{q}(A,B)=\inf\{\mu(x,y):\ \text{for all } x\in A \text{ and } y\in B\},$$

where $\alpha,\mu:X\times X\to[1,+\infty)$ for all $x,y\in X$ and (A,B) be a pair of nonempty subsets of double controlled metric type space (X,d).
From now and onward consider F as the collection of all non-decreasing functions $\lambda:[0,+\infty)\to[0,1)$ where for every bounded sequence $\{t_n\}$ of positive real numbers, $\lambda\{t_n\}\to 1$ implies $t_n\to 0$.

Definition 2.1. *A mapping $T:X\to\mathcal{CB}(X)$ is continuous in a double controlled metric type space (X,d) at $x\in X$ if, for all $\epsilon>0$, there exists $\delta>0$ such that*

$$T(K(x,\delta))\subseteq K(Tx,\epsilon),$$

where $K(x,\epsilon)$ is given as

$$K(x,\epsilon)=\{y\in X,d(x,y)<\epsilon\}.$$

Clearly, if T is continuous at x, then $x_n\to x$ implies that $Tx_n\to Tx$ as $n\to\infty$.

Definition 2.2. *Let (X,d) be a double-controlled metric-type space. Suppose that (A,B) be a pair of nonempty subsets of a double controlled metric type space (X,d) such that A_0 is nonempty. Then a pair (A,B) has the P-property if*

$$d(x_1,y_1)=d(A,B),$$
$$d(x_2,y_2)=d(A,B),$$

implies that

$$d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.3. *Let (X, d) be a double controlled metric type space. A pair of mappings (g, T), where $T : A \to \mathcal{CB}(B)$ and $g : A \to A$ is said to be $(\lambda - \eta)_M-$proximal Geraghty contraction, if $\eta : A \times A \to [0, +\infty)$, satisfying*

$$\begin{gathered}\eta(x, y) \geq 1,\\ \mathcal{D}(gu, Tx) = d(A, B),\\ \mathcal{D}(gv, Ty) = d(A, B),\end{gathered}$$

implies that

$$\eta(x, y)\mathcal{H}(Tx, Ty) \leq \lambda(M(u, v, x, y))M(u, v, x, y),$$

where

$$\begin{aligned}M(u, v, x, y) = \max\Bigg\{&d(gx, gy), \frac{\mathcal{D}(gx, Tx) - \mathfrak{q}(gy, Tx)d(A, B)}{\mathfrak{p}(gx, gy)},\\ &\mathcal{D}^*(gu, Tx), \frac{\mathcal{D}(gu, Ty) - \mathfrak{q}(gv, Ty)d(A, B)}{\mathfrak{p}(gu, gv)}\Bigg\},\end{aligned}$$

for all u, v, x and $y \in A$, where $\lambda \in F$.

Definition 2.4. *Let (X, d) be a double controlled metric type space. A mapping $T : A \to \mathcal{CB}(B)$ is said to be $(\lambda - \eta)_M-$generalized proximal Geraghty contraction, if $\eta : A \times A \to [0, +\infty)$, satisfying*

$$\begin{gathered}\eta(x, y) \geq 1\\ \mathcal{D}(u, Tx) = d(A, B)\\ \mathcal{D}(v, Ty) = d(A, B)\end{gathered}$$

implies that

$$\eta(x, y)\mathcal{H}(Tx, Ty) \leq \lambda(M(u, v, x, y))M(u, v, x, y),$$

where

$$\begin{aligned}M(u, v, x, y) = \max\Bigg\{&d(x, y), \frac{\mathcal{D}(x, Tx) - \mathfrak{q}(y, Tx)d(A, B)}{\mathfrak{p}(x, y)},\\ &\mathcal{D}^*(u, Tx), \frac{\mathcal{D}(u, Ty) - \mathfrak{q}(v, Ty)d(A, B)}{\mathfrak{p}(u, v)}\Bigg\},\end{aligned}$$

for all $u, v, x, y \in A$, and $\lambda \in F$.

Note that, if we take $g = I_A$ (g as an identity mapping on A), then every $(\lambda - \eta)_M-$proximal Geraghty contraction will reduce to $(\lambda - \eta)_M-$generalized proximal Geraghty contraction.

The following theorem provides the coincidence best proximity point for the pair of mapping (g, T), where T is a **multi-valued** mapping.

Theorem 2.1. *Let (X, d) be a complete double-controlled metric type space. Suppose that $T : A \to \mathcal{CB}(B)$, $g : A \to A$, and $\eta : A \times A \to [0, +\infty)$ are mappings, where A is a closed subset and the pair (A, B) satisfying the P-property with $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$. If a pair of continuous mappings (g, T) satisfying $(\lambda - \eta)_M-$proximal Geraghty contraction, where T is*

η-proximal admissible then there exist elements $x_0, x_1 \in A_0$ such that $D(gx_1, Tx_0) = d(A, B)$ and $\eta(x_0, x_1) \geq 1$. Conceding that $\{x_n\}$ is a sequence in A such that $\eta(x_n, x_{n+1}) \geq 1$ and

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})}\mu(x_i, x_m) < \frac{1}{k}, \text{ where } k \in (0,1),$$

then the pair of mapping (g, T) has a unique coincidence best proximity point $x^* \in A$.

Proof. From the given condition there exist $x_0, x_1 \in A_0$ such that $\mathcal{D}(gx_1, Tx_0) = d(A, B)$ and $\mu(x_0, x_1) \geq 1$. As $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $\mathcal{D}(gx_2, Tx_1) = d(A, B)$. Since T is η-proximal admissible, and $\eta(x_0, x_1) \geq 1$

$$\mathcal{D}(gx_1, Tx_0) = d(A, B),$$
$$\mathcal{D}(gx_2, Tx_1) = d(A, B),$$

making use of the P-property, we acquire

$$d(gx_1, gx_2) = \mathcal{H}(Tx_0, Tx_1).$$

Since the pair of mapping (g, T) satisfy $(\lambda-\eta)_M-$proximal Geraghty contraction with $\eta(x_1, x_2) \geq 1$, we have

$$d(gx_1, gx_2) \leq \lambda(M(x_0, x_1, x_1, x_2))M(x_0, x_1, x_1, x_2), \tag{2.1}$$

where

$$\begin{aligned}
M(x_0, x_1, x_1, x_2) \leq \max\Bigg\{& d(gx_0, gx_1), \frac{\mathcal{D}(gx_0, Tx_0) - \mathfrak{q}(gx_1, Tx_0)d(A, B)}{\mathfrak{p}(gx_0, gx_1)}, \\
& \mathcal{D}^*(gx_1, Tx_0), \frac{\mathcal{D}(gx_1, Tx_1) - \mathfrak{q}(gx_2, Tx_1)d(A, B)}{\mathfrak{p}(gx_1, gx_2)}\Bigg\}, \\
\leq \max\Bigg\{& d(gx_0, gx_1), \frac{\mathfrak{p}(gx_0, gx_1)d(gx_0, gx_1) + \mathfrak{q}(gx_1, Tx_0)\mathcal{D}(gx_1, Tx_0)}{\mathfrak{p}(gx_0, gx_1)} \\
& - \frac{\mathfrak{q}(gx_1, Tx_0)d(A, B)}{\mathfrak{p}(gx_0, gx_1)}, \mathcal{D}(gx_1, Tx_0) - d(A, B), \frac{\mathfrak{p}(gx_1, gx_2)d(gx_1, gx_2)}{\mathfrak{p}(gx_1, gx_2)} \\
& + \frac{\mathfrak{q}(gx_2, Tx_1)\mathcal{D}(gx_2, Tx_1) - \mathfrak{q}(gx_2, Tx_1)d(A, B)}{\mathfrak{p}(gx_1, gx_2)}\Bigg\}, \\
\leq \max\{& d(gx_0, gx_1), d(gx_0, gx_1), 0, d(gx_1, gx_2)\}.
\end{aligned}$$

More precisely, we can write

$$M(x_0, x_1, x_1, x_2) \leq \max\{d(gx_0, gx_1), d(gx_1, gx_2)\}.$$

If $\max\{d(gx_0, gx_1), d(gx_1, gx_2)\} = d(gx_1, gx_2)$, then (2.1) becomes

$$d(gx_1, gx_2) \leq \lambda(d(gx_1, gx_2))d(gx_1, gx_2),$$

which is a contradiction. So we conclude that

$$d(gx_1, gx_2) \leq \lambda(d(gx_0, gx_1))d(gx_0, gx_1).$$

Further, by the fact that $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that $\mathcal{D}(gx_3, Tx_2) = d(A, B)$. As T is η-proximal admissible where $\eta(x_2, x_3) \geq 1$,

$$\mathcal{D}(gx_2, Tx_1) = d(A, B),$$
$$\mathcal{D}(gx_3, Tx_2) = d(A, B),$$

utilizing $P-$Property, we arrive at

$$d(gx_2, gx_3) = \mathcal{H}(Tx_1, Tx_2).$$

Since the pair of mapping (g, T) satisfying $(\lambda - \eta)_M-$proximal Geraghty contraction with $\eta(x_2, x_3) \geq 1$, we have

$$d(gx_2, gx_3) \leq \lambda(M(x_2, x_3, x_1, x_2))M(x_2, x_3, x_1, x_2), \tag{2.2}$$

where

$$\begin{aligned}
M(x_2, x_3, x_1, x_2) \leq \max\{&d(gx_1, gx_2), \frac{\mathcal{D}(gx_1, Tx_1) - \mathfrak{q}(gx_2, Tx_1)d(A, B)}{\mathfrak{p}(gx_1, gx_2)}, \\
&\mathcal{D}^*(gx_2, Tx_1), \frac{\mathcal{D}(gx_2, Tx_2) - \mathfrak{q}(gx_3, Tx_2)d(A, B)}{\mathfrak{p}(gx_2, gx_3)}\}, \\
&\leq \max\{d(gx_1, gx_2), \frac{\mathfrak{p}(gx_1, gx_2)d(gx_1, gx_2) + \mathfrak{q}(gx_2, Tx_1)\mathcal{D}(gx_2, Tx_1)}{\mathfrak{p}(gx_1, gx_2)} \\
&- \frac{\mathfrak{q}(gx_2, Tx_1)d(A, B)}{\mathfrak{p}(gx_1, gx_2)}, \mathcal{D}(gx_2, Tx_1) - d(A, B), \frac{\mathfrak{p}(gx_2, gx_3)d(gx_2, gx_3)}{\mathfrak{p}(gx_2, gx_3)} \\
&+ \frac{\mathfrak{q}(gx_3, Tx_2)\mathcal{D}(gx_3, Tx_2) - \mathfrak{q}(gx_3, Tx_2)d(A, B)}{\mathfrak{p}(gx_2, gx_3)}\}, \\
&\leq \max\{d(gx_1, gx_2), d(gx_1, gx_2), 0, d(gx_2, gx_3)\},
\end{aligned}$$

hence, we have

$$M(x_2, x_3, x_1, x_2) \leq \max\{d(gx_1, gx_2), d(gx_2, gx_3)\}.$$

If $\max\{d(gx_1, gx_2), d(gx_2, gx_3)\} = d(gx_2, gx_3)$, then (2.2) becomes

$$d(gx_2, gx_3) \leq \lambda(d(gx_2, gx_3))d(gx_2, gx_3),$$

which is a contradiction. So we conclude that

$$d(gx_2, gx_3) \leq \lambda(d(gx_1, gx_2))d(gx_1, gx_2).$$

Similarly, we can construct a sequence $\{x_n\} \subseteq A_0$, where $\mu(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$

$$\mathcal{D}(gx_n, Tx_{n-1}) = d(A, B),$$
$$\mathcal{D}(gx_{n+1}, Tx_n) = d(A, B),$$

using $P-$property, $d(gx_n, gx_{n+1}) = \mathcal{H}(Tx_n, Tx_{n-1})$. Since the pair of mapping (g, T) satisfying $(\lambda - \eta)_M-$ proximal Geraghty contraction with $\eta(x_n, x_{n+1}) \geq 1$, then

$$d(gx_n, gx_{n+1}) \leq \lambda(M(x_n, x_{n+1}, x_{n-1}, x_n))M(x_n, x_{n+1}, x_{n-1}, x_n), \tag{2.3}$$

where

$$\begin{aligned}
M(x_n, x_{n+1}, x_{n-1}, x_n) \leq \max\{&d(gx_{n-1}, gx_n), \frac{\mathcal{D}(gx_{n-1}, Tx_{n-1}) - \mathfrak{q}(gx_n, Tx_{n-1})d(A, B)}{\mathfrak{p}(gx_{n-1}, gx_n)}, \\
&\mathcal{D}^*(gx_n, Tx_{n-1}), \frac{\mathcal{D}(gx_n, Tx_n) - \mathfrak{q}(gx_{n+1}, Tx_n)d(A, B)}{\mathfrak{p}(gx_n, gx_{n+1})}\} \\
&\leq \max\{d(gx_{n-1}, gx_n), \\
&\frac{\mathfrak{p}(gx_{n-1}, gx_n)d(gx_{n-1}, gx_n) + \mathfrak{q}(gx_n, Tx_{n-1})\mathcal{D}(gx_n, Tx_{n-1})}{\mathfrak{p}(gx_{n-1}, gx_n)} \\
&- \frac{\mathfrak{q}(gx_n, Tx_{n-1})d(A, B)}{\mathfrak{p}(gx_{n-1}, gx_n)}, \mathcal{D}(gx_n, Tx_{n-1}) - d(A, B),
\end{aligned}$$

$$\frac{\mathfrak{p}(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + \mathfrak{q}(gx_{n+1}, Tx_n)\mathcal{D}(gx_{n+1}, Tx_n)}{\mathfrak{p}(gx_n, gx_{n+1})}$$
$$- \frac{\mathfrak{q}(gx_{n+1}, Tx_n)d(A, B)}{\mathfrak{p}(gx_n, gx_{n+1})}\}$$
$$\leq \max\{d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), 0, d(gx_n, gx_{n+1})\},$$

so we can write

$$M(x_n, x_{n+1}, x_{n-1}, x_n) \leq \max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}.$$

If $\max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\} = d(gx_n, gx_{n+1})$, then (2.3) becomes

$$d(gx_n, gx_{n+1}) \leq \lambda(d(gx_n, gx_{n+1}))d(gx_n, gx_{n+1}),$$

which is a contradiction. This amounts to say that

$$d(gx_n, gx_{n+1}) \leq \lambda(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n). \tag{2.4}$$

Further, we can write

$$d(gx_n, gx_{n+1}) \leq \lambda(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n) \leq d(gx_{n-1}, gx_n), \tag{2.5}$$

which shows $\{d(gx_n, gx_{n+1})\}$ is a decreasing sequence. Since $\lambda \in F$, from (2.5), we have

$$\lambda(d(gx_n, gx_{n+1})) \leq \lambda(d(gx_{n-1}, gx_n)),$$

and

$$\lambda(d(gx_{n-1}, gx_n)) \leq \lambda(d(gx_{n-2}, gx_{n-1})).$$

Continuing on the same lines, we can write

$$\lambda(d(gx_{n-1}, gx_n)) \leq \lambda(d(gx_{n-2}, gx_{n-1})) \leq \ldots \leq \lambda(d(gx_0, gx_1)).$$

Using inequality (2.4), we obtain

$$d(gx_{n-1}, gx_n) \leq \lambda(d(gx_{n-2}, gx_{n-1}))d(gx_{n-2}, gx_{n-1}) \tag{2.6}$$

From inequalities (2.4) and (2.6), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \lambda(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n), \\ &\leq \lambda(d(gx_{n-1}, gx_n))\lambda(d(gx_{n-2}, gx_{n-1}))d(gx_{n-2}, gx_{n-1}). \end{aligned}$$

Continuing in the same way as above, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \lambda(d(gx_{n-1}, gx_n))\lambda(d(gx_{n-2}, gx_{n-1}))\ldots\lambda(d(gx_0, gx_1))d(gx_0, gx_1) \\ &= \lambda^n(d(gx_0, gx_1))d(gx_0, gx_1), \end{aligned}$$

i.e.,

$$d(gx_n, gx_{n+1}) \leq \lambda^n(d(gx_0, gx_1))d(gx_0, gx_1).$$

Suppose that $d(gx_{n-1}, gx_n) > 0$, from (2.5) we obtain

$$\frac{d(gx_n, gx_{n+1})}{d(gx_{n-1}, gx_n)} \leq \lambda(d(gx_{n-1}, gx_n)) \leq 1 \text{ for all } n \geq 1.$$

Let $p = \lim_{n\to+\infty} d(gx_{n-1}, gx_n)$. Using equation (2.5) and letting $n \to +\infty$, we acquire

$$\frac{p}{p} = 1 \leq \lim_{n\to+\infty} \lambda(d(gx_{n-1}, gx_n)) \leq 1.$$

Thus $\lim_{n\to+\infty} \lambda(d(gx_{n-1}, gx_n)) = 1$. Using the definition of the λ, we conclude that

$$\lim_{n\to+\infty} d(gx_{n-1}, gx_n) = 0.$$

Now, we have to show that $\{gx_n\}$ is a Cauchy sequence. Since (X, d) is a double controlled metric type space for all natural numbers $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
&d(gx_n, gx_m)\\
&\le \alpha(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + \mu(gx_{n+1}, gx_m)d(gx_{n+1}, gx_m)\\
&\le \alpha(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + \alpha(gx_{n+1}, gx_{n+2})\mu(gx_{n+1}, gx_m)d(gx_{n+1}, gx_{n+2})\\
&+ \mu(gx_{n+1}, gx_m)\mu(gx_{n+2}, gx_m)d(gx_{n+2}, gx_m)\\
&\le \alpha(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + \alpha(gx_{n+1}, gx_{n+2})\mu(gx_{n+1}, gx_m)d(gx_{n+1}, gx_{n+2})\\
&+ \alpha(gx_{n+2}, gx_{n+3})\mu(gx_{n+1}, gx_m)\mu(gx_{n+2}, gx_m)d(gx_{n+2}, gx_{n+3})\\
&+ \mu(gx_{n+1}, gx_m)\mu(gx_{n+2}, gx_m)\mu(gx_{n+3}, gx_m)d(gx_{n+3}, gx_m)\\
&\le \alpha(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})d(gx_i, gx_{i+1})\\
&+ \prod_{k=n+1}^{m-1}\mu(gx_k, gx_m)d(gx_{m-1}, gx_m),\\
&\le \alpha(gx_n, gx_{n+1})\lambda^n d(gx_0, gx_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})\lambda^i d(gx_0, gx_1)\\
&+ \prod_{i=n+1}^{m-1}\mu(gx_i, gx_m)\lambda^{m-1}d(gx_0, gx_1),\\
&\le \alpha(gx_n, gx_{n+1})\lambda^n d(gx_0, gx_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})\lambda^i d(gx_0, gx_1)\\
&+ \prod_{i=n+1}^{m-1}\mu(gx_i, gx_m)\alpha(gx_{m-1}, gx_m)\lambda^{m-1}d(gx_0, gx_1),\\
&= \alpha(gx_n, gx_{n+1})\lambda^n d(gx_0, gx_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})\lambda^i d(gx_0, gx_1),\\
&\le \alpha(gx_n, gx_{n+1})\lambda^n d(gx_0, gx_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})\lambda^i d(gx_0, gx_1),\\
&\le \alpha(gx_n, gx_{n+1})\lambda^n d(gx_0, gx_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})\lambda^i d(gx_0, gx_1).
\end{aligned}$$

Assume that

$$S_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\mu(gx_j, gx_m)\right)\alpha(gx_i, gx_{i+1})\lambda^i d(gx_0, gx_1).$$

The above inequality reduces to

$$d(gx_n, gx_m) \leq d(gx_0, gx_1)[\lambda^n(d(gx_0, gx_1))\alpha(gx_n, gx_{n+1}) + (S_{m-1} - S_n)]. \tag{2.7}$$

Using ratio test, we have

$$a_i = \left(\prod_{j=0}^{i} \mu(gx_j, gx_m)\right) \alpha(gx_i, gx_{i+1})\lambda^i(d(gx_0, gx_1)), \text{ where } \frac{a_{i+1}}{a_i} < \frac{1}{k},$$

and taking limit $n, m \to +\infty$, (2.7) becomes

$$\lim_{n,m\to+\infty} d(gx_n, gx_m) = 0.$$

This implies that $\{gx_n\}$ is a Cauchy sequence in a complete double controlled metric type space (X, d), hence it is convergent and suppose that it converges to some x^* in $A_0 \subseteq A$ (as set A is closed), which assures that the sequence $\{x_n\} \subseteq A_0$ since $x_n \to x^*$. As (g, T) is a pair of continuous mapping, which shows that

$$\mathcal{D}(gx^*, Tx^*) = d(A, B).$$

Therefore, x^* is a coincidence best proximity point of the pair of mapping (g, T).

For uniqueness, suppose that there are two distinct coincidence best proximity points of (g, T) such that $x^* \neq y^*$. Thus $q = d(x^*, y^*) > 0$. Since $d(gx^*, Tx^*) = d(gy^*, Ty^*) = d(A, B)$, using the $P-$property, we conclude that $q = \mathcal{H}(Tx^*, Ty^*)$. Moreover, the pair of mapping (g, T) satisfies $(\lambda - \eta)_M-$ proximal Geraghty contraction, we obtain $q \leq \lambda(q)q$. Thus, $\lambda(q) \geq 1$. Since $\lambda(q) \leq 1$, we conclude that $\lambda(q) = 1$ and therefore $q = 0$, which is contradiction. Hence, x^* is a unique coincidence best proximity point of the pair of mapping (g, T). □

Corollary 2.1. *Let (X, d) be a complete double controlled metric type space. Suppose that $T : A \to \mathcal{CB}(B)$, $\eta : A \times A \to [0, +\infty)$ be mappings and (A, B) be a pair of nonempty subsets of a double controlled metric type space (X, d) satisfying the P-property with $T(A_0) \subseteq B_0$. Assuming that a continuous mapping T satisfying $(\lambda - \eta)_M-$generalized proximal Geraghty contraction, where T is η-proximal admissible, then there exist elements $x_0, x_1 \in A_0$ such that $\mathcal{D}(x_1, Tx_0) = d(A, B)$ and $\eta(x_0, x_1) \geq 1$. If $\{x_n\}$ is a sequence in A such that $\eta(x_n, x_{n+1}) \geq 1$ and*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})}\mu(x_i, x_m) < \frac{1}{k}, \text{ where } k \in (0, 1),$$

then T has a unique best proximity point $x^* \in A$.

Proof. If we take identity mapping $g = I_A$ (g is identity on A), the remaining proof is same as in Theorem 2.1. □

Definition 2.5. *Let (X, d) be a double controlled metric type space. Suppose that $T : A \to B$, $g : A \to A$ and $\eta : A \times A \to [0, +\infty)$ be mappings. Then a pair of mapping (g, T) is said to be $(\lambda - \eta)_S-$modified proximal Geraghty contraction, if*

$$\eta(x, y) \geq 1,$$
$$d(gu, Tx) = d(A, B),$$
$$d(gv, Ty) = d(A, B),$$

implies that

$$\eta(x, y)d(Tx, Ty) \leq \lambda(M(u, v, x, y))M(u, v, x, y),$$

where

$$M(u,v,x,y) = \max\left\{d(gx,gy), \frac{d(gx,Tx) - \alpha(gy,Tx)d(A,B)}{\alpha(gx,gy)},\right.$$
$$\left. d^*(gu,Tx), \frac{d(gu,Ty) - \alpha(gv,Ty)d(A,B)}{\alpha(gu,gv)}\right\},$$

for all u, v, x and $y \in A$, where $\lambda \in F$.

Definition 2.6. *Let (X, d) be a double controlled metric type space. Assume that $T : A \to B$ and $\eta : A \times A \to [0, +\infty)$ be mappings. Then, mapping T is said to be $(\lambda - \eta)_S-$proximal Geraghty contraction, if*

$$\eta(x,y) \geq 1,$$
$$d(u,Tx) = d(A,B),$$
$$d(v,Ty) = d(A,B),$$

implies that

$$\eta(x,y)d(Tx,Ty) \leq \lambda(M(u,v,x,y))M(u,v,x,y),$$

where

$$M(u,v,x,y) = \max\left\{d(x,y), \frac{d(x,Tx) - \alpha(y,Tx)d(A,B)}{\alpha(x,y)},\right.$$
$$\left. d^*(u,Tx), \frac{d(u,Ty) - \alpha(v,Ty)d(A,B)}{\alpha(u,v)}\right\},$$

for all $u, v, x, y \in A$, where $\lambda \in F$. Note that, if we take $g = I_A$ (g as an identity mapping on A), then every $(\lambda - \eta)_S-$modified proximal Geraghty contraction will reduce to $(\lambda - \eta)_S-$proximal Geraghty contraction.

The following result guarantees the existence of unique coincidence best proximity point for single valued pair of mapping (g, T).

Theorem 2.2. *Let (X, d) be a complete double controlled metric type space. Suppose that $T : A \to B$, $g : A \to A$, $\mu : A \times A \to [0, +\infty)$ be mappings and (A, B) be a pair of nonempty subsets of a double controlled metric type space (X, d) satisfying the P-property, where A is closed subset with $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$. If a pair of continuous mappings (g, T) satisfy $(\lambda - \eta)_S-$modified proximal Geraghty contraction, where T is η-proximal admissible then there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$ and $\eta(x_0, x_1) \geq 1$. Conceding that, $\{x_n\}$ is a sequence in A such that $\eta(x_n, x_{n+1}) \geq 1$ and*

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})}\mu(x_i, x_m) < \frac{1}{k}, \text{ where } k \in (0,1),$$

then the pair of mapping (g, T) has a unique coincidence best proximity point $x^* \in A$.

Proof. From the given condition there exists $x_0, x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$ and $\eta(x_0, x_1) \geq 1$. As $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A, B)$. As T is η-proximal admissible $\eta(x_0, x_1) \geq 1$

$$d(gx_1, Tx_0) = d(A,B),$$
$$d(gx_2, Tx_1) = d(A,B),$$

using the P-property $d(gx_1, gx_2) = d(Tx_0, Tx_1)$. Since the pair of mapping (g, T) satisfy

$(\lambda-\eta)_S-$*modified* proximal Geraghty contraction with $\eta(x_1,x_2)\geq 1$, then

$$d(gx_1,gx_2)\leq \lambda(M(x_0,x_1,x_1,x_2))M(x_0,x_1,x_1,x_2), \tag{2.8}$$

where

$$\begin{aligned}
M(x_0,x_1,x_1,x_2) &\leq \max\{d(gx_0,gx_1), \frac{d(gx_0,Tx_0)-\mu(gx_1,Tx_0)d(A,B)}{\alpha(gx_0,gx_1)},\\
&d^*(gx_1,Tx_0), \frac{d(gx_1,Tx_1)-\mu(gx_2,Tx_1)d(A,B)}{\alpha(gx_1,gx_2)}\},\\
&\leq \max\{d(gx_0,gx_1), \frac{\alpha(gx_0,gx_1)d(gx_0,gx_1)+\mu(gx_1,Tx_0)d(gx_1,Tx_0)}{\alpha(gx_0,gx_1)}\\
&-\frac{\mu(gx_1,Tx_0)d(A,B)}{\alpha(gx_0,gx_1)}, d(gx_1,Tx_0)-d(A,B), \frac{\alpha(gx_1,gx_2)d(gx_1,gx_2)}{\alpha(gx_1,gx_2)}\\
&+\frac{\mu(gx_2,Tx_1)d(gx_2,Tx_1)-\mu(gx_2,Tx_1)d(A,B)}{\alpha(gx_1,gx_2)}\},\\
&\leq \max\{d(gx_0,gx_1), d(gx_0,gx_1), 0, d(gx_1,gx_2)\},
\end{aligned}$$

we can write

$$M(x_0,x_1,x_1,x_2)\leq \max\{d(gx_0,gx_1), d(gx_1,gx_2)\}.$$

If $\max\{d(gx_0,gx_1), d(gx_1,gx_2)\}=d(gx_1,gx_2)$, then (2.8) becomes

$$d(gx_1,gx_2)\leq \lambda(d(gx_1,gx_2))d(gx_1,gx_2),$$

which is a contradiction. So we conclude that

$$d(gx_1,gx_2)\leq \lambda(d(gx_0,gx_1))d(gx_0,gx_1).$$

Further, by the fact that $T(A_0)\subseteq B_0$, there exists $x_3\in A_0$ such that $d(gx_3,Tx_2)=d(A,B)$. As T is η-proximal admissible where $\eta(x_2,x_3)\geq 1$,

$$d(gx_2,Tx_1)=d(A,B),$$
$$d(gx_3,Tx_2)=d(A,B),$$

using the $P-$Property $d(gx_2,gx_3)=d(Tx_1,Tx_2)$. Since the pair of mapping (g,T) satisfy $(\lambda-\eta)_S-$*modified* proximal Geraghty contraction with $\eta(x_2,x_3)\geq 1$, then

$$d(gx_2,gx_3)\leq \lambda(M(x_2,x_3,x_1,x_2))M(x_2,x_3,x_1,x_2), \tag{2.9}$$

where

$$\begin{aligned}
M(x_2,x_3,x_1,x_2) &\leq \max\{d(gx_1,gx_2), \frac{d(gx_1,Tx_1)-\mu(gx_2,Tx_1)d(A,B)}{\alpha(gx_1,gx_2)},\\
&d^*(gx_2,Tx_1), \frac{d(gx_2,Tx_2)-\mu(gx_3,Tx_2)d(A,B)}{\alpha(gx_2,gx_3)}\},\\
&\leq \max\{d(gx_1,gx_2), \frac{\alpha(gx_1,gx_2)d(gx_1,gx_2)+\mu(gx_2,Tx_1)d(gx_2,Tx_1)}{\alpha(gx_1,gx_2)}\\
&-\frac{\mu(gx_2,Tx_1)d(A,B)}{\alpha(gx_1,gx_2)}, d(gx_2,Tx_1)-d(A,B), \frac{\alpha(gx_2,gx_3)d(gx_2,gx_3)}{\alpha(gx_2,gx_3)}\\
&+\frac{\mu(gx_3,Tx_2)d(gx_3,Tx_2)-\mu(gx_3,Tx_2)d(A,B)}{\alpha(gx_2,gx_3)}\},\\
&\leq \max\{d(gx_1,gx_2), d(gx_1,gx_2), 0, d(gx_2,gx_3)\},
\end{aligned}$$

so we can say that

$$M(x_2, x_3, x_1, x_2) \leq \max\{d(gx_1, gx_2), d(gx_2, gx_3)\}.$$

If $\max\{d(gx_1, gx_2), d(gx_2, gx_3)\} = d(gx_2, gx_3)$, then (2.9) becomes

$$d(gx_2, gx_3) \leq \lambda(d(gx_2, gx_3))d(gx_2, gx_3),$$

which is a contradiction. So we concluded that

$$d(gx_2, gx_3) \leq \lambda(d(gx_1, gx_2))d(gx_1, gx_2).$$

Similarly, we can construct a sequence $\{x_n\} \subseteq A_0$, where $\eta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$

$$d(gx_n, Tx_{n-1}) = d(A, B),$$
$$d(gx_{n+1}, Tx_n) = d(A, B),$$

using the $P-$property $d(gx_n, gx_{n+1}) = d(Tx_n, Tx_{n-1})$. Since the pair of mapping (g, T) satisfy $(\lambda - \eta)_S-$*modified* proximal Geraghty contraction with $\eta(x_n, x_{n+1}) \geq 1$, then

$$d(gx_n, gx_{n+1}) \leq \lambda(M(x_n, x_{n+1}, x_{n-1}, x_n))M(x_n, x_{n+1}, x_{n-1}, x_n), \tag{2.10}$$

where

$$\begin{aligned} M(x_n, x_{n+1}, x_{n-1}, x_n) &\leq \max\{d(gx_{n-1}, gx_n), \frac{d(gx_{n-1}, Tx_{n-1}) - \mu(gx_n, Tx_{n-1})d(A, B)}{\alpha(gx_{n-1}, gx_n)}, \\ &\quad d^*(gx_n, Tx_{n-1}), \frac{d(gx_n, Tx_n) - \mu(gx_{n+1}, Tx_n)d(A, B)}{\alpha(gx_n, gx_{n+1})}\}, \\ &\leq \max\{d(gx_{n-1}, gx_n) \\ &\quad \frac{\alpha(gx_{n-1}, gx_n)d(gx_{n-1}, gx_n) + \mu(gx_n, Tx_{n-1})d(gx_n, Tx_{n-1})}{\alpha(gx_{n-1}, gx_n)} \\ &\quad - \frac{\mu(gx_n, Tx_{n-1})d(A, B)}{\alpha(gx_{n-1}, gx_n)}, d(gx_n, Tx_{n-1}) - d(A, B), \\ &\quad \frac{\alpha(gx_n, gx_{n+1})d(gx_n, gx_{n+1}) + \mu(gx_{n+1}, Tx_n)d(gx_{n+1}, Tx_n)}{\alpha(gx_n, gx_{n+1})} \\ &\quad - \frac{\mu(gx_{n+1}, Tx_n)d(A, B)}{\alpha(gx_n, gx_{n+1})}\}, \\ &\leq \max\{d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), 0, d(gx_n, gx_{n+1})\}. \end{aligned}$$

Consequently, we can write

$$M(x_n, x_{n+1}, x_{n-1}, x_n) \leq \max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}.$$

If $\max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\} = d(gx_n, gx_{n+1})$, then (2.10) becomes

$$d(gx_n, gx_{n+1}) \leq \lambda(d(gx_n, gx_{n+1}))d(gx_n, gx_{n+1}),$$

which is a contradiction. Hence, we conclude that

$$d(gx_n, gx_{n+1}) \leq \lambda(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n), \tag{2.11}$$

further, we can write

$$d(gx_n, gx_{n+1}) \leq \lambda(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n) \leq d(gx_{n-1}, gx_n), \tag{2.12}$$

which shows $\{d(gx_n, gx_{n+1})\}$ is a decreasing sequence. Since $\lambda \in F$ then, from (2.12), we have

$$\lambda(d(gx_n, gx_{n+1})) \leq \lambda(d(gx_{n-1}, gx_n)),$$

and

$$\lambda(d(gx_{n-1}, gx_n)) \le \lambda(d(gx_{n-2}, gx_{n-1})).$$

Continuing on the same lines, we can write

$$\lambda(d(gx_{n-1}, gx_n)) \le \lambda(d(gx_{n-2}, gx_{n-1})) \le \ldots \le \lambda(d(gx_0, gx_1)).$$

Using inequality (2.11), we can write

$$d(gx_{n-1}, gx_n) \le \lambda(d(gx_{n-2}, gx_{n-1}))d(gx_{n-2}, gx_{n-1}). \tag{2.13}$$

From inequalities (2.11) and (2.13), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\le \lambda(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n) \\ &\le \lambda(d(gx_{n-1}, gx_n))\lambda(d(gx_{n-2}, gx_{n-1}))d(gx_{n-2}, gx_{n-1}). \end{aligned}$$

In the same way as before, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\le \lambda(d(gx_{n-1}, gx_n))\lambda(d(gx_{n-2}, gx_{n-1}))\ldots\lambda(d(gx_0, gx_1))d(gx_0, gx_1) \\ &= \lambda^n(d(gx_0, gx_1))d(gx_0, gx_1), \end{aligned}$$

i.e.,

$$d(gx_n, gx_{n+1}) \le \lambda^n(d(gx_0, gx_1))d(gx_0, gx_1).$$

From (2.12) suppose that $d(gx_{n-1}, gx_n) > 0$, so we conclude

$$\frac{d(gx_n, gx_{n+1})}{d(gx_{n-1}, gx_n)} \le \lambda(d(gx_{n-1}, gx_n)) \le 1 \text{ for all } n \ge 1.$$

Let $p = \lim_{n\to+\infty} d(gx_{n-1}, gx_n)$. Using equation (2.12) and letting $n \to +\infty$, we obtain

$$\frac{p}{p} = 1 \le \lim_{n\to+\infty} \lambda(d(gx_{n-1}, gx_n)) \le 1.$$

Thus $\lim_{n\to+\infty} \lambda(d(gx_{n-1}, gx_n)) = 1$. Using the definition of λ, we obtain

$$\lim_{n\to+\infty} d(gx_{n-1}, gx_n) = 0.$$

The next part of the proof is similar to the proof of the Theorem 2.1. □

Corollary 2.2. *Let (A, B) be a pair of nonempty subsets of a complete double controlled metric type space (X, d) satisfying the P-property. Suppose that $T : A \to B$ and $\eta : A \times A \to [0, +\infty)$ be mappings, where A is closed subset with $T(A_0) \subseteq B_0$. If a continuous mapping T satisfies $(\lambda - \eta)_S-$proximal Geraghty contraction, where T is η-proximal admissible then there exist elements $x_0, x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$ and $\eta(x_0, x_1) \ge 1$. Conceding that $\{x_n\}$ is a sequence in A such that $\eta(x_n, x_{n+1}) \ge 1$ and*

$$\sup_{m\ge1} \lim_{i\to+\infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})}\mu(x_i, x_m) < \frac{1}{k}, \ \textit{where } k \in (0, 1),$$

then the pair of mapping (g, T) has a unique best proximity point $x^* \in A$.

Proof. If we take $g = I_A$ (g is identity on A), proof will remain same as in Theorem 2.2. □

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ and suppose that the function d given as $d(x, y) = d(y, x)$ and $d(x, x) = 0$, where*

d	0	1	2	3	4	5
0	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{6}$
1	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{10}$	$\frac{3}{4}$
2	$\frac{1}{3}$	$\frac{1}{4}$	0	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{1}{10}$
3	$\frac{1}{10}$	$\frac{2}{3}$	$\frac{6}{7}$	0	$\frac{1}{2}$	$\frac{1}{3}$
4	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{7}{8}$	$\frac{1}{2}$	0	$\frac{1}{4}$
5	$\frac{1}{6}$	$\frac{3}{4}$	$\frac{1}{10}$	$\frac{1}{3}$	$\frac{1}{4}$	0

Take $\alpha, \mu : X \times X \to [1, +\infty)$ *to be symmetric defined as* $\alpha(x, y) = 16x + 18y + 1$ *and* $\mu(x, y) = 20x + 22y + 2$. *It is easy to see that* (X, d) *is double controlled metric type space. Suppose* $A = \{0, 1, 2\}$ *and* $B = \{3, 4, 5\}$ *are nonempty subsets of double controlled metric type space* (X, d). *After simple calculation* $d(A, B) = \frac{1}{10}$, *satisfying the* $P-$*property and* $A_0 = A$ *and* $B_0 = B$,

$$Tx = \begin{cases} 3, & \text{if } x = 2 \\ \{3, 4\}, & \text{if } x = \{0, 1\} \end{cases}$$

and

$$gx = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 2 \\ 2, & \text{if } x = 1 \end{cases}$$

and clearly $T(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$. *Now, we have to show that the pair* (g, T) *satisfies* $(\lambda - \eta)_M-$*proximal Geraghty contraction*

$$\eta(x, y)\mathcal{H}(Tx, Ty) \leq \lambda(M(u, v, x, y))M(u, v, x, y),$$

where

$$M(u, v, x, y) = \max\Bigg\{d(gx, gy), \frac{\mathcal{D}(gx, Tx) - \mathfrak{q}(gy, Tx)d(A, B)}{\mathfrak{p}(gx, gy)}, \\ \mathcal{D}^*(gu, Tx), \frac{\mathcal{D}(gu, Ty) - \mathfrak{q}(gv, Ty)d(A, B)}{\mathfrak{p}(gu, gv)}\Bigg\},$$

for all u, v, x *and* $y \in A$ *and for function* $\eta : A \times A \to [0, +\infty)$ *defined as:*

$$\eta(x, y) = d(x, y) + 1.$$

Since

$$\mathcal{D}(g0, T2) = d(A, B)$$
$$\mathcal{D}(g0, T1) = d(A, B)$$
$$\mathcal{D}(g2, T0) = d(A, B)$$
$$\mathcal{D}(g2, T1) = d(A, B).$$

Case (i): *If* $\mathcal{D}(g0, T2) = \mathcal{D}(g0, T1) = d(A, B)$, *then* $u = v = 0$, $x = 2$ *and* $y = 1$. *After simple calculations,* $\mathcal{H}(Tx, Ty) = \mathcal{H}(3, \{3, 4\}) = 0$ *and* $\eta(x, y) = d(3, \{3, 4\}) + 1 = 1$. *Now, we have to* *show that the pair (g,T) satisfying $(\lambda-\eta)_M-$proximal Geraghty contraction*

$$(1)(0) \leq \lambda(M(0,0,2,1))M(0,0,2,1)$$

$$0 \leq \lambda(M(0,0,2,1))(\frac{1}{4}).$$

Case (ii): *If $\mathcal{D}(g0,T2)=\mathcal{D}(g2,T0)=d(A,B)$, then $u=0$, $v=x=2$ and $y=0$. Subsequentially, $\mathcal{H}(Tx,Ty)=\mathcal{H}(3,\{3,4\})=0$ and $\eta(x,y)=d(3,\{3,4\})+1=1$. Since, the pair (g,T) fulfilling $(\lambda-\eta)_M-$proximal Geraghty contraction*

$$(1)(0) \leq \lambda(M(0,2,2,0))M(0,2,2,0)$$

$$0 \leq \lambda(M(0,2,2,0))(\frac{1}{2}).$$

Case (iii): *If $\mathcal{D}(g0,T2)=\mathcal{D}(g2,T1)=d(A,B)$, then $u=0$, $v=x=2$ and $y=1$. We inquire, $\mathcal{H}(Tx,Ty)=\mathcal{H}(3,\{3,4\})=0$ and $\eta(x,y)=d(3,\{3,4\})+1=1$. From the fact that the pair (g,T) adequating $(\lambda-\eta)_M-$proximal Geraghty contraction*

$$(1)(0) \leq \lambda(M(0,2,2,1))M(0,2,2,1)$$

$$0 \leq \lambda(M(0,2,2,1))(\frac{1}{4}).$$

Case (iv): *If $\mathcal{D}(g0,T1)=\mathcal{D}(g2,T0)=d(A,B)$, then $u=0$, $v=2$, $x=1$ and $y=0$. After that, $\mathcal{H}(Tx,Ty)=\mathcal{H}(\{3,4\},\{3,4\})=0$ and $\eta(x,y)=d(\{3,4\},\{3,4\})+1=1$. Though we know that the pair (g,T) achieving $(\lambda-\eta)_M-$proximal Geraghty contraction*

$$(1)(0) \leq \lambda(M(0,2,1,0))M(0,2,1,0)$$

$$0 \leq \lambda(M(0,2,1,0))(\frac{1}{3}).$$

Case (v): *If $\mathcal{D}(g0,T1)=\mathcal{D}(g2,T1)=d(A,B)$, then $u=0$, $v=2$ and $x=y=1$. Calculating the terms we get, $\mathcal{H}(Tx,Ty)=\mathcal{H}(\{3,4\},\{3,4\})=0$ and $\eta(x,y)=d(\{3,4\},\{3,4\})+1=1$. Given that the pair (g,T) efficiently $(\lambda-\eta)_M-$proximal Geraghty contraction*

$$(1)(0) \leq \lambda(M(0,2,1,1))M(0,2,1,1)$$

$$0 \leq \lambda(M(0,2,1,1))(0).$$

Case (vi): *If $\mathcal{D}(g2,T0)=\mathcal{D}(g2,T1)=d(A,B)$, then $u=v=2$, $x=0$ and $y=1$. We infer, $\mathcal{H}(Tx,Ty)=\mathcal{H}(\{3,4\},\{3,4\})=0$ and $\eta(x,y)=d(\{3,4\},\{3,4\})+1=1$. As previously stated, that the pair (g,T) accomplishing $(\lambda-\eta)_M-$proximal Geraghty contraction*

$$(1)(0) \leq \lambda(M(2,2,0,1))M(2,2,0,1)$$

$$0 \leq \lambda(M(2,2,0,1))(\frac{1}{3}).$$

for every $\lambda:[0,+\infty)\to[0,1)$ the pair (g,T) satisfied $(\lambda-\eta)_M-$proximal Geraghty contraction. Hence, all the conditions of the Theorem 2.1 is satisfied and 0 is the unique coincidence point of the pair of the mapping (g,T).

Corollary 2.3. *Let (X,d) be a complete double controlled metric type space and $T:A\to A$ be a self mapping. If $\sup_{m\geq 1}\lim_{i\to+\infty}\frac{\alpha(x_{i+1},x_{i+2})}{\alpha(x_i,x_{i+1})}\mu(x_i,x_m)<\frac{1}{k}$, where $k\in(0,1)$, T is continuous and there exists a real number $\lambda\in[0,1)$ such that the following λ_S-modified proximal type contraction is satisfied*

$$d(Tx,Ty) \leq \lambda M(u,v,x,y),$$

where

$$M(u,v,x,y) = \max\left\{d(Tx,Ty), \frac{d(Tx,Tx)-\alpha(Ty,Tx)d(x,y)}{\alpha(Tx,Ty)}, \right.$$
$$\left. d^*(Tu,Tx), \frac{d(Tu,Ty)-\alpha(Tv,Ty)d(x,y)}{\alpha(Tu,Tv)}\right\},$$

for all $u,v,x,y \in A$. Then there exists a unique fixed point of the mapping T.

Remark 2.1. *Applying the example 2.1 to the proximal conditions enunciated in [2, 14], we can see that (g,T) does not satisfy the respective contractions. Hence the results of [2, 14] can not be applied on (g,T).*

§3 Application to operators of integral type

As an application of the proximal results proved in the previous section, a specific non-linear integral equation is examined for the existence of a solution in this section. See notable publications [8, 9] for more analysis on the applications of generalized contractions.

Let $X = \mathcal{C}[a,b]$ be a set of all real valued continuous functions on $[a,b]$. Define the mappings $d : X \times X \to [0,+\infty)$ by

$$d(x,y) = \sup_{c\in[a,b]} |x(c)-y(c)|^p,$$

for all $x,y \in X$ and $\alpha(x,y) = x+7y+5, \mu(x,y) = 5x+y+2$. Then, (X,d) is a complete double controlled metric type space. Consider the Fredholm integral equation given by

$$x(t) = f(t) + \xi\int_a^b \mathcal{I}(t,s,x(s))ds, \tag{3.1}$$

where $t \in [a,b], |\xi| > 0$ and $\mathcal{I} : [a,b]\times[a,b]\times X \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are continuous functions. Let $T : X \to X$ be an integral operator defined by

$$Tx(t) = f(t) + \xi\int_a^b \mathcal{I}(t,s,x(s))ds. \tag{3.2}$$

Then $x(t)$ is a fixed point of T if and only it is a solution of the Fredholm integral equation (3.1).

To demonstrate the existence of a solution to the Fredholm integral equation, we now propose the following subsequent theorem.

Theorem 3.1. *Let $T : X \to X$ be an integral operator defined in 3.2. Suppose that the following assumptions hold.*

For any $x,y \in X, x \neq y$, $\mathcal{I} : [a,b]\times[a,b]\times X \to \mathbb{R}$ satisfying

$$|\mathcal{I}(t,s,x(s)) - \mathcal{I}(t,s,y(s))| \leq \xi(t,s) \mid x(s)-y(s) \mid,$$

where $(s,t) \in [a,b]\times[a,b]$ and $\xi : [a,b]\times[a,b] \to \mathbb{R}$ is a continuous function such that

$$\sup_{t\in[a,b]}\int_a^b \xi^p(t,s)|x(s)-y(s)|^p ds \leq \frac{1}{2^p|\xi|^p(b-a)^{p-1}}M(t,s,x,y), \tag{3.3}$$

where $M(t,s,x,y)$ is defined as in Corollary 2.3, then, the integral operator T has a unique solution in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0, n \geq 1$. From 3.2, we obtain

$$x_{n+1} = Tx_n(t) = f(t) + \xi \int_a^b \mathcal{I}\left(t, s, x_n(s)\right) ds. \tag{3.4}$$

Let $q > 1$ be a constant with $\frac{1}{p} + \frac{1}{q} = 1$. By the Holder's inequality, we speculate that

$$\begin{aligned}
&|Tx(t) - Ty(t)|^p \\
&= \left|\xi \int_a^b \mathcal{I}(t, s, x(s))ds - \xi \int_a^b \mathcal{I}(t, s, y(s))ds\right|^p \\
&\leq \left(\int_a^b |\xi||\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))|ds\right)^p \\
&\leq \left(\int_a^b |\xi|^q ds\right)^{\frac{p}{q}} \left(\left(\int_a^b |\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))|^p ds\right)^{\frac{1}{p}}\right)^p \\
&= |\xi|^p (b-a)^{p-1} \left(\int_a^b |\mathcal{I}(t, s, x(s)) - \mathcal{I}(t, s, y(s))|^p ds\right) \\
&\leq |\xi|^p (b-a)^{p-1} \int_a^b \xi^p(t, s)|x(s) - y(s)|^p ds,
\end{aligned}$$

and we deduce that

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[a,b]} |Tx(t) - Ty(t)|^p \\
&\leq |\xi|^p (b-a)^{p-1} \sup_{t\in[a,b]} \left[\int_a^b \xi^p(t, s)|x(s) - y(s)|^p ds\right] \\
&\leq \frac{1}{2^p} M(t, s, x, y).
\end{aligned}$$

Setting $\lambda = \frac{1}{2^p} < 1$, we obtain that

$$d(Tx, Ty) \leq \lambda\, M(t, s, x, y). \tag{3.5}$$

Thus, all the conditions of Corollary 2.3 are contended, and hence T possesses a unique fixed point in X. □

Future scope.

- Can the results presented in this article be extended to the framework of Controlled-metric type spaces endowed with graphs or in graphical metric spaces?

- Can the findings shown in this article be used to generate a solution to the following semi-linear operator system of the form:

$$\begin{cases} \mathcal{U}(\alpha, \beta) = \alpha, \\ \mathcal{V}(\alpha, \beta) = \beta, \end{cases}$$

  where $\mathcal{U}, \mathcal{V} : P \times P \to P$ are nonlinear operators defined on a Banach space $(P, ||.||)$?

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] T Abdeljawad, N Mlaiki, H Aydi, et al. *Double controlled metric type spaces and some fixed point results*, Mathematics, 2018, 6(12), DOI: 10.3390/math6120320.

[2] I Altun, M Aslantas, H Sahin. *Best proximity point results for p-proximal contractions*, Acta Math Hungar, 2020, 162: 393-402.

[3] J Anuradha, P Veeramani. *Proximal pointwise contraction*, Topol Appl, 2009, 156(18): 2942-2948.

[4] S Basha, P Veeramani. *Best approximations and best proximity pairs*, Acta Sci Math, 1997, 63: 289-300.

[5] S Basha, P Veeramani. *Best proximity pair theorems for multifunction with open fibers*, J Approx Theory, 2000, 103(1): 119-129.

[6] S Basha. *Extensions of Banach's contraction principle*, Numer Funct Anal Optim, 2010, 31: 569-576.

[7] S Basha, N Shahzad, R Jeyaraj. *Common best proximity points global optimization of multi-objective functions*, Appl Math Lett, 2011, 24(6): 883-886.

[8] L Chen, X Liu, Y Zhao. *Exponential stability of a class of nonlinear systems via fixed point theory*, Nonlinear Anal, 2021, 196, DOI: 10.1016/j.na.2020.111784.

[9] L Chen, N Yang, Y Zhao. *Fixed point theorems for the Mann's iteration scheme in convex graphical rectangular b-metric spaces*, Optim, 2022, 70(5-6): 1359-1373.

[10] K Fan. *Extensions of two fixed point theorems of F.E. Browder*, Math Z, 1969, 112(3): 234-240.

[11] M Gabeleh, C Vetro. *A note on best proximity point theory using proximal contractions*, J Fixed Point Theory Appl, 2018, 20: 149.

[12] M Gabeleh, J Markin. *A note on the paper "Best proximity point results for p-proximal contractions"*, Acta Math Hungar, 2021, 164(1): 326-329.

[13] M Gabeleh, J Markin. *Some notes on the paper "On best proximity points of interpolative proximal contractions"*, Quaest Math, 2021, 45(10): 1539-1544.

[14] N Hussain, A Latif, P Salimi. *Best proximity point results for modified Suzuki α-ψ-proximal contractions*, Fixed Point Theory Appl, 2014, 10, DOI: 10.1186/1687-1812-2014-10.

[15] M Jleli, B Samet. *Best proximity points for $\alpha-\psi-$proximal contractive type mappings and applications*, Bull Sci Math, 2013, 137(8): 977-995.

[16] H Ahmad, M Younis, M E Koksal. *Double Controlled Partial Metric Type Spaces and Convergence Results*, J Math, 2021, 2021(1): 7008737.

[17] W A Kirk, S Reich, P Veeramani. *Proximinal retracts and best proximity pair theorems*, Numer Funct Anal Optim, 2003, 24(7-8): 851-862.

[18] N Pakkaranang, H Rehman, W Kumam. *Two strongly convergent self-adaptive iterative schemes for solving pseudo-monotone equilibrium problems with applications*, Demonstr Math, 2021, 54: 280-298.

[19] S Radenović, H Aydi, H Lakzian, et al. *Best proximity points of MT-cyclic contractions with property UC*, Numer Funct Anal Optim, 2020, 41(7): 871-82.

[20] A Kostić, V Rakočević, S Radenović. *Best proximity points involving simulation functions with* w_0*-distance*, RACSAM, 2019, 113: 715-727.

[21] N Saleem, H Ahmad, H Aydi, et al. *On Some Coincidence Best Proximity Point Results*, J Math, 2021, 2021(1): 8005469.

[22] V S Raj. *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal, 2011 74: 4804-4808.

[23] R T Rockafellar, R J V Wets. *Variational Analysis*, Heidelberg: Springer Berlin, 1998.

[24] P S Srinivasan. *Analysis-Best proximity pair theorems*, Acta Sci Math, 2001, 67(1-2): 421-430.

[25] T Suzuki, M Kikkawa, C Vetro. *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal, 2009, 71(7-8): 2918-2926.

[26] C Vetro, T Suzuki. *Three existence theorems for weak contractions of Matkowski type*, Int J Math Stat, 2010, 6(10): 110-220.

[27] N Wairojjana, N Pakkaranang, N Pholasa. *Strong convergence inertial projection algorithm with self-adaptive step size rule for pseudomonotone variational inequalities in Hilbert spaces*, Demonstr Math, 2021 54: 110-128.

[28] S Z Wang, B Y Li, Z M Gao, et al. *Some fixed point theorems on expansion mappings*, Math Japon, 1984, 29: 631-636.

[29] H Sahin, M Aslantas, I Altun. *Best proximity and best periodic points for proximal nonunique contractions*, J Fixed Point Theory Appl, 2021, 23(4), DOI: 10.1007/s11784-021-00889-7.

[30] M Aslantas, H Sahin, U Sadullah. *Some generalizations for mixed multivalued mappings*, Appl Gen Topol, 2022, 23(1): 169-178.

[31] H Sahin. *A new kind of* F*-contraction and some best proximity point results for such mappings with an application*, Turkish J Math, 2022, 46(6): 2151-2166.

[32] J Zhang, Y Su, Q Cheng. *A note on "A best proximity point theorem for Geraghty-contractions"*, Fixed Point Theory Appl, 2013, 99(2013), DOI: 10.1186/1687-1812-2013-99.

[33] M Gabeleh. *Convergence of Picards iteration using projection algorithm for noncyclic contractions*, Indag Math, 2019, 30(1): 227-239.

[34] M Gabeleh, O O Otafudu. *Markov Kakutani's theorem for best proximity pairs in Hadamard spaces*, Indag Math, 2017, 28(3): 680-693.

[35] M Gabeleh, C Vetro. *A new extension of Darbo's fixed point theorem using relatively Meir Keeler condensing operators*, Bull Aust Math Soc, 2018, 98(2): 286-297.

[36] M Younis, H Ahmad, W Shahid. *Best proximity points for multivalued mappings and equation of motion*, J Appl Anal Comput, 2024, 14(1): 298-316.

[37] M Aslantas. *Finding a solution to an optimization problem and an application*, J Optim Theory Appl, 2022, 194(1): 121-141.

[38] I Altun, M Aslantas, H Sahin. *KW-type nonlinear contractions and their best proximity points*, Numer Funct Anal Optim, 2021, 42(8): 935-954.

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