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Fixed point results for asymptotically regular mappings in abstract metric spaces

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Abstract. In this paper, various extended contractions are introduced as generalizations of some existing contractions given by Kannan, Ćirić, Reich and Górnicki, et al. Then, several meaningful results about asymptotically regular mappings in cone metric spaces over Banach algebras are obtained, weakening the completeness of the spaces and the continuity of the mappings. Moreover, some nontrivial examples are showed to verify the innovation of the new concepts and our fixed point theorems.

§1 Introduction

The fixed point theorems of nonlinear operators are important content of the rapidly developing nonlinear functional analysis theory, which are closely connected with a large number of modern mathematics. In particular, they play a major role in finding the solutions of various equations (including various linear or nonlinear, deterministic or nondeterministic differential equations, and integral equations). Since the classic Banach contraction principle [2] was introduced in 1922, lots of meaningful extension versions were investigated including the fixed point of the Kannan type contraction mapping [20] as follows.

Theorem 1.1. *Suppose (M, d) is a complete metric space. The mapping $\Gamma : M \to M$ satisfies*

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq K\{d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)\}, \tag{1}$$

for all $\varsigma, \upsilon \in M$ and $K \in [0, \frac{1}{2})$. Then Γ has a unique fixed point $\varpi \in M$ and $\Gamma^n\varsigma \to \varpi$ for $\varsigma \in M$.

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If equation (1) is replaced by

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq \alpha d(\varsigma, \upsilon) + \beta d(\varsigma, \Gamma\varsigma) + \gamma d(\upsilon, \Gamma\upsilon), \tag{2}$$

where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$, then it is called Reich type contraction mapping [25, 26]. Likewise, if equation (1) is replaced by

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq \alpha d(\varsigma, \upsilon) + \beta d(\varsigma, \Gamma\varsigma) + \gamma d(\upsilon, \Gamma\upsilon) + \lambda d(\varsigma, \Gamma\upsilon) + \kappa d(\upsilon, \Gamma\varsigma), \tag{3}$$

where $\alpha, \beta, \gamma, \lambda, \kappa \in [0, 1)$ and $\alpha + \beta + \gamma + \lambda + \kappa < 1$, then it is called Hardy-Rogers type contraction mapping [15] and the conclusions are also true. In 2007, Huang and Zhang [17] reintroduced the notion of cone metric space by substituting the real number set $\mathbb{R}$ for a Banach space E and showed several fixed point results under normal cone. A year later, Rezapour and Hamlbarani [28] great improved these fixed point results by omitting the assumption of normality in [17]. Afterwards, Liu and Xu [21] gave the definition of cone metric space over a Banach algebra by taking the place of the Banach space E with a Banach algebra $\mathcal{A}$ in 2013. It is worth to mention that they proved the nonequivalence of fixed point theorems between metric spaces and cone metric spaces over Banach algebras by some valuable examples, which have attracted considerable attention in the fixed point field (see [12, 13, 16] and their references).

Asymptotic regularity is a basically significant notion of fixed point theory in metric spaces (see ([1, Chapter IX], [7, Chapter 9] and [4])), which is defined as follows.

Definition 1.2. *A mapping $\Gamma : M \to M$ in a metric space (M, d) is named asymptotically regular at $\varsigma \in M$ if $\lim_{n\to\infty} d(\Gamma^n\varsigma, \Gamma^{n+1}\varsigma) = 0$. If Γ is asymptotically regular at every $\varsigma \in M$, then Γ is said to be asymptotically regular.*

Asymptotic regularity is a useful tool to establish the fixed point theorems and has been studied in a great deal of papers about different contents (see [6, 9, 22, 27, 30]). Recently, Górnicki [8] has pointed out that any asymptotically regular self-mapping Γ at every $\varsigma \in M$ may not imply the sequence $\{\Gamma^n\varsigma\}$ to be a Cauchy sequence. There is no necessary connection between them. Moreover, the asymptotic regularity and continuity are independent. So, how to establish the fixed point theorems for asymptotically regular mappings? The author in [8] obtained an interesting fixed point theorem about asymptotically regular mapping when the contraction constant $k \in [0, \infty)$ in Kannan type contraction under the condition of continuity, which is as follows.

Theorem 1.3. *Let (M, d) be a complete metric space. If $\Gamma : M \to M$ is a continuous and asymptotically regular mapping which satisfies*

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq \alpha d(\varsigma, \upsilon) + K\{d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)\}, \tag{4}$$

for all $\varsigma, \upsilon \in M$ and $\alpha \in [0, 1)$, $K \geq 0$. Then Γ has a unique fixed point $\varpi \in M$ and $\Gamma^n\varsigma \to \varpi$ for $\varsigma \in M$.

Later on, Bisht [3] gave the following meaningful and significant result about asymptotically regular mapping in metric spaces, which weakened the condition of continuity in the above theorem.

Theorem 1.4. *Let (M, d) be a complete metric space. If $\Gamma : M \to M$ is an asymptotically regular mapping which satisfies*

$$d(\Gamma\varsigma, \Gamma\upsilon) \leq \alpha d(\varsigma, \upsilon) + K\{d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)\},$$

for all $\varsigma, \upsilon \in M$ and $\alpha \in [0, 1)$, $K \geq 0$. Then Γ has a unique fixed point $\varpi \in M$ and $\Gamma^n\varsigma \to \varpi$ for $\varsigma \in M$ provided Γ is either k-continuous for $k \geq 1$ or orbitally continuous.

We are concerned in this paper with the fixed point theorems of asymptotically regular mappings in the abstract metric spaces. These conclusions develop and extend some meaningful work in the literature [8, 22, 24]. It should be pointed that all results we obtain are established by weakening the completeness of the spaces and the continuity of the mappings. We also expand the concept of asymptotic regularity in metric spaces to the abstract metric spaces. Furthermore, there are some examples to verify that our new notions and main conclusions are genuine improvements and extensions of the corresponding notions and works in the literature, so they have a much wider range of applications.

§2 Preliminaries

First, let us recall some preliminary concepts of Banach algebras and cone metric spaces.

Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $\varsigma, \upsilon, \varpi \in \mathcal{A}, a \in \mathbb{R}$,

(1) $\varsigma(\upsilon\varpi) = (\varsigma\upsilon)\varpi$;

(2) $\varsigma(\upsilon + \varpi) = \varsigma\upsilon + \varsigma\varpi$ and $(\varsigma + \upsilon)\varpi = \varsigma\varpi + \upsilon\varpi$;

(3) $a(\varsigma\upsilon) = (a\varsigma)\upsilon = \varsigma(a\upsilon)$;

(4) $\|\varsigma\upsilon\| \leq \|\varsigma\|\|\upsilon\|$.

In this paper, we shall assume that the Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $e\varsigma = \varsigma e = \varsigma$ for all $\varsigma \in \mathcal{A}$, then $\mathcal{A}$ is named a unital Banach algebra. An element $\varsigma \in \mathcal{A}$ is said to be invertible if there is an inverse element $\upsilon \in \mathcal{A}$ such that $\varsigma\upsilon = \upsilon\varsigma = e$. The inverse of ς is denoted by ς^{-1}. For more details, we refer to [29].

A subset P of $\mathcal{A}$ is called a cone if

(i) P is non-empty, closed and $\{\theta, e\} \subset P$, where θ denotes the zero element of $\mathcal{A}$;

(ii) $\alpha P + \beta P \in P$ for all non-negative real numbers α, β;

(iii) $P^2 = PP \subset P$;

(iv) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $\varsigma \preceq \upsilon$ if and only if $\upsilon - \varsigma \in P$. We shall write $\varsigma \prec \upsilon$ if $\varsigma \preceq \upsilon$ and $\varsigma \neq \upsilon$, while $\varsigma \ll \upsilon$ will stand for $\upsilon - \varsigma \in \text{int}P$, where $\text{int}P$ denotes the interior of P.

A cone P is called normal if there is a number $K > 0$ such that for all $\varsigma, \upsilon \in \mathcal{A}$,

$$\theta \preceq \varsigma \preceq \upsilon \quad \text{implies} \quad \| \varsigma \| \leq K \| \upsilon \| .$$

The least positive number satisfying the above inequality is called the normal constant of P. Indeed, the number K cannot be less than 1 (see [28]). A cone P is called regular if every increasing sequence bounded from above is convergent. In other words, if there is a $\upsilon \in \mathcal{A}$ such that

$$\varsigma_1 \preceq \varsigma_2 \preceq \cdots \preceq \varsigma_n \preceq \cdots \preceq \upsilon,$$

then there exists $\varsigma \in \mathcal{A}$ such that $\lim_{n\to\infty} \|\varsigma_n - \varsigma\| = 0$. Equivalently, a cone P is regular if and only if every decreasing sequence bounded from below is convergent. It is well known that every regular cone is normal.

Throughout this paper, we always assume that P is a cone over Banach algebra $\mathcal{A}$ with $\mathrm{int}P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 2.1. *([17], [21]) Assume M is a nonempty set. Suppose that the mapping $d : M \times M \to \mathcal{A}$ satisfies*

(d1) $\theta \preceq d(\varsigma, \upsilon)$ for all $\varsigma, \upsilon \in M$ and $d(\varsigma, \upsilon) = \theta$ if and only if $\varsigma = \upsilon$;
(d2) $d(\varsigma, \upsilon) = d(\upsilon, \varsigma)$ for all $\varsigma, \upsilon \in M$;
(d3) $d(\varsigma, \upsilon) \preceq d(\varsigma, \varpi) + d(\varpi, \upsilon)$ for all $\varsigma, \upsilon, \varpi \in M$.

Then d is called a cone metric on M and (M, d) is called a cone metric space over Banach algebra $\mathcal{A}$.

Definition 2.2. *([17], [21]) Assume (M, d) is a cone metric space over a Banach algebra $\mathcal{A}$, $\varsigma \in M$ and $\{\varsigma_n\}$ be a sequence in M. Then*

(i) $\{\varsigma_n\}$ converges to ς if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n > N$, $d(\varsigma_n, \varsigma) \ll c$;
(ii) $\{\varsigma_n\}$ is a Cauchy sequence if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n, m > N$, $d(\varsigma_n, \varsigma_m) \ll c$;
(iii) (M, d) is a complete cone metric space if every Cauchy sequence is convergent in M.

Definition 2.3. *([19]) Assume P is a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.*

Lemma 2.4. *([16]) Assume P is a solid cone in a Banach algebra $\mathcal{A}$ and $\{\varsigma_n\}$, $\{\upsilon_n\}$ are two sequences in P. If $\{\varsigma_n\}$ and $\{\upsilon_n\}$ are c-sequences and $\alpha, \beta \in P$, then $\{\alpha\varsigma_n + \beta\upsilon_n\}$ is a c-sequence.*

Lemma 2.5. *([29]) Assume $\mathcal{A}$ is a unital Banach algebra and $\varsigma \in \mathcal{A}$. If the spectral radius $r(\varsigma)$ of ς is less than 1, i.e.,*

$$r(\varsigma) = \lim_{n\to\infty} \|\varsigma^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|\varsigma^n\|^{\frac{1}{n}} < 1,$$

then $e - \varsigma$ is invertible. Actually, $(e - \varsigma)^{-1} = \sum_{i=0}^{\infty} \varsigma^i$.

Lemma 2.6. *([31]) Assume $\mathcal{A}$ is a unital Banach algebra and $\varsigma, \upsilon \in \mathcal{A}$.*

(i) If ς and υ commute, then $r(\varsigma\upsilon) \leq r(\varsigma)r(\upsilon)$.
(ii) If $0 \leq r(\varsigma) < 1$, then we have $r((e - \varsigma)^{-1}) \leq (1 - r(\varsigma))^{-1}$.

§3 Main results

In this section, we obtain some fixed point theorems in orbitally complete cone metric spaces over Banach algebras. The regularity or normality of the cone, the continuity of the mapping are not necessary. At first, inspired by the notions of orbitally continuous, Γ-orbitally complete [5] and k-continuous [23] in metric spaces, we have given the similar concepts in cone b-metric space over a Banach algebra $\mathcal{A}$ in our paper and show some valuable examples to dedicate the relationships of these concepts (see [13, 14]). If we take $b = 1$, then these notions are in cone metric space over a Banach algebra $\mathcal{A}$ in the following.

Definition 3.1. *Suppose (M, d) is a cone metric space over a Banach algebra $\mathcal{A}$ and the mapping $\Gamma : M \to M$. Let $\varsigma \in M$ and $O_\Gamma(\varsigma) = \{\varsigma, \Gamma\varsigma, \Gamma^2\varsigma, \Gamma^3\varsigma, \ldots\}$.*

(i) The mapping Γ is named orbitally continuous at an element $\varpi \in M$ if for each sequence $\{\varsigma_n\} \subset O_\Gamma(\varsigma)$ (for all $\varsigma \in M$), $\varsigma_n \to \varpi$ as $n \to \infty$ implies $\Gamma\varsigma_n \to \Gamma\varpi$ as $n \to \infty$. Notice that each continuous mapping is orbitally continuous, but not the converse.

(ii) The mapping Γ is said to be k-continuous, $k = 1, 2, \ldots,$ if $\Gamma^{k-1}\varsigma_n \to \varpi$ as $n \to \infty$ implies $\Gamma^k\varsigma_n = \Gamma\varpi$ as $n \to \infty$. It is clear that 1-continuity is equivalent to continuity, and k-continuity implies $(k+1)$-continuity for any $k = 1, 2, \ldots,$ but not the converse. Furthermore, continuity of Γ^k and k-continuity of Γ are independent when $k > 1$.

(iii) The space (M, d) is named Γ-orbitally complete, if for some $\varsigma \in M$, each Cauchy sequence in $O_\Gamma(\varsigma)$ converges in M. Every complete cone metric space over Banach algebra $\mathcal{A}$ is Γ-orbitally complete for any self-mapping Γ, but a Γ-orbitally complete cone metric space over Banach algebra $\mathcal{A}$ need not to be complete.

Now, let us see the notion of asymptotic regularity in cone metric spaces over Banach algebras, which is a great improvement of the corresponding in metric spaces.

Definition 3.2. *Suppose (M, d) is a cone metric space over a Banach algebra $\mathcal{A}$. The mapping $\Gamma : M \to M$ is said to be asymptotically regular, if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a number $N \in \mathbb{N}$ satisfying for all $n \geq N$, $d(\Gamma^{n+1}\varsigma, \Gamma^n\varsigma) \ll c$ for $\varsigma \in M$. This yields that $\{d(\Gamma^{n+1}\varsigma, \Gamma^n\varsigma)\}$ is a c-sequence for any $\varsigma \in M$.*

In this way, we observe that the concept of sequence c-sequence, which is a development of that $\lim_{n\to\infty} d(\Gamma^{n+1}\varsigma, \Gamma^n\varsigma) = 0$ in Definition 1.2. The latter is very useful only in normal cones (see Proposition 2.5 in [18]) or usual metric spaces (see [3, 8, 10-11]), while our results are discussed in non-normal cone metric spaces over Banach algebras. Before giving the main results, we start with a generalized Kannan-Górnicki type mapping in the abstract metric spaces.

Definition 3.3. *(Generalized Kannan-Górnicki type mapping) Suppose (M, d) is a cone metric space over a Banach algebra $\mathcal{A}$. The mapping $\Gamma : M \to M$ is named a generalized Kannan-Górnicki type mapping if there exists some $h \in P$ such that, for all $\varsigma, \upsilon \in M$,*

$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq h\{d(\varsigma, \Gamma\varsigma) + d(\upsilon, \Gamma\upsilon)\}. \tag{5}$$

It is not difficult to observe that every generalized Kannan-Górnicki type mapping is a sharp extension of Kannan type contraction mapping. The next example fully illustrates this point.

Example 3.4. *Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1] \times C^1_{\mathbb{R}}[0,1]$ with the norm*

$$\|(\varsigma_1, \varsigma_2)\| = \|\varsigma_1\|_\infty + \|\varsigma_2\|_\infty + \|\varsigma_1'\|_\infty + \|\varsigma_2'\|_\infty.$$

Define the multiplication by

$$\varsigma\upsilon = (\varsigma_1, \varsigma_2)(\upsilon_1, \upsilon_2) = (\varsigma_1\upsilon_1, \varsigma_1\upsilon_2 + \varsigma_2\upsilon_1),$$

where $\varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2) \in \mathcal{A}$. Thus, $\mathcal{A}$ is a unital Banach algebra with $e = (1, 0)$. Let $P = \{(\varsigma_1(t), \varsigma_2(t)) \in \mathcal{A} : \varsigma_1(t) \geq 0, \varsigma_2(t) \geq 0, t \in [0,1]\}$. Let $M = [0,1] \times [0,1]$ and define $d : M \times M \to \mathcal{A}$ by $d((\varsigma_1, \varsigma_2), (\upsilon_1, \upsilon_2))(t) = (|\varsigma_1 - \upsilon_1|, |\varsigma_2 - \upsilon_2|) \cdot \alpha^t \in P$, $\forall \varsigma = (\varsigma_1, \varsigma_2), \upsilon = (\upsilon_1, \upsilon_2) \in M, \alpha > 0$. If $\Gamma : M \to M$ is given as $\Gamma\varsigma = \Gamma(\varsigma_1, \varsigma_2) = \left(\frac{2}{3}\varsigma_1, \frac{2}{3}\varsigma_2\right)$, it can be easily checked that Γ is not usual Kannan type mapping. In fact, if $\varsigma = (0,0), \upsilon = (1,1)$, there is not any $h \in [0, \frac{1}{2})$ such that

$$d(\Gamma\varsigma, \Gamma\upsilon) = (\frac{2}{3}, \frac{2}{3}) \preceq h(\frac{1}{3}, \frac{1}{3}).$$

In the sequel, a new fixed point result of asymptotically regular generalized Kannan-Górnicki type mapping in orbitally complete cone metric spaces over Banach algebras is discussed, without depending on regularity or normality of the cone. The mapping is not required to be continuous.

Theorem 3.5. *Suppose $\Gamma : M \to M$ is an asymptotically regular generalized Kannan-Górnicki type mapping in the Γ-orbitally complete cone metric space (M, d) over a unital Banach algebra $\mathcal{A}$. If the mapping Γ is k-continuous or orbitally continuous, then Γ has a unique fixed point $\varpi \in M$ and for each $\varsigma \in M$ the iterated sequence $\{\Gamma^n\varsigma\}$ converges to ϖ.*

Proof. For any $\varsigma_0 \in M$, define $\varsigma_n = \Gamma\varsigma_{n-1} = \Gamma^n\varsigma_0, n \geq 1$. If $\varsigma_n = \varsigma_{n+1}$ for some n, the proof is completed, so we assume that $\varsigma_n \neq \varsigma_{n+1}, \forall n \in \mathbb{N}$. By the asymptotical regularity of Γ, the sequence $\{d(\varsigma_n, \varsigma_{n+1})\}$ is a c-sequence. For any $m, n \in \mathbb{N}$ and $m > n$, we derive

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &= d(\Gamma\varsigma_{n-1}, \Gamma\varsigma_{m-1}) \\ &\preceq h[d(\varsigma_{n-1}, \Gamma\varsigma_{n-1}) + d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})] \\ &= h[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_{m-1}, \varsigma_m)]. \end{aligned}$$

Since Γ is asymptotically regular, it follows that $\{d(\varsigma_n, \varsigma_m)\}$ is a c-sequence by Lemma 2.4. So, $\{\varsigma_n\}$ is a Cauchy sequence in M. As (M, d) is Γ-orbitally complete, we see $\varsigma_n \to \varpi$ as $n \to \infty$ for some $\varpi \in M$. If Γ is k-continuous, then $\lim_{n\to\infty} \Gamma^{k-1}\varsigma_n = \varpi$ implies $\lim_{n\to\infty} \Gamma^k\varsigma_n = \Gamma\varpi$. By the uniqueness of the limitation, we have $\varpi = \Gamma\varpi$. If Γ is orbitally continuous, then $\lim_{n\to\infty} \varsigma_n = \varpi$ implies $\lim_{n\to\infty} \Gamma\varsigma_n = \Gamma\varpi$. By the uniqueness of the limitation, we also have $\varpi = \Gamma\varpi$.

Now, it remains to present the uniqueness of ϖ. If υ is another fixed point of Γ, then

$$d(\upsilon, \varpi) = d(\Gamma\upsilon, \Gamma\varpi) \preceq h[d(\upsilon, \Gamma\upsilon) + d(\varpi, \Gamma\varpi)] = \theta,$$

which yields $\upsilon = \varpi$. Thus, ϖ is the unique fixed point of Γ. □

Remark 3.6. *We can prove that if* Γ *is a Kannan type mapping in a cone metric space* (M,d) *over Banach algebra* $\mathcal{A}$ *and* $h\in P, r(h)\in[0,\frac{1}{2})$, *then* Γ^m *is also a Kannan type mapping with vector* $(e-h)^{-(m-1)}h^m$ *by Lemmas 2.5 and 2.6, where* $r((e-h)^{-(m-1)}h^m)\le\left(\frac{r(h)}{1-r(h)}\right)^{m-1}r(h)$ *for any* $m\in\mathbb{N}$ *and* $m\ge 2$. *In fact,*

$$d(\Gamma^m\varsigma,\Gamma^m\upsilon)=d(\Gamma(\Gamma^{m-1}\varsigma),\Gamma(\Gamma^{m-1}\upsilon))\preceq h[d(\Gamma^{m-1}\varsigma,\Gamma^m\varsigma)+d(\Gamma^{m-1}\upsilon,\Gamma^m\upsilon)],$$

while

$$\begin{aligned}d(\Gamma^{m-1}\varsigma,\Gamma^m\varsigma)+d(\Gamma^{m-1}\upsilon,\Gamma^m\upsilon)\preceq{}& h[d(\Gamma^{m-2}\varsigma,\Gamma^{m-1}\varsigma)+d(\Gamma^{m-1}\varsigma,\Gamma^m\varsigma)\\&+d(\Gamma^{m-2}\upsilon,\Gamma^{m-1}\upsilon)+d(\Gamma^{m-1}\upsilon,\Gamma^m\upsilon)].\end{aligned}$$

Due to the fact that $r(h)\in[0,\frac{1}{2})$ *and Lemma 2.5, we know* $e-h$ *is invertible, then*

$$d(\Gamma^{m-1}\varsigma,\Gamma^m\varsigma)+d(\Gamma^{m-1}\upsilon,\Gamma^m\upsilon)\preceq(e-h)^{-1}h[d(\Gamma^{m-2}\varsigma,\Gamma^{m-1}\varsigma)+d(\Gamma^{m-2}\upsilon,\Gamma^{m-1}\upsilon)].$$

By iteration, we immediately have

$$d(\Gamma^m\varsigma,\Gamma^m\upsilon)\preceq(e-h)^{-(m-1)}h^m[d(\varsigma,\Gamma\varsigma)+d(\upsilon,\Gamma\upsilon)].$$

By Lemma 2.6,

$$r((e-h)^{-(m-1)}h^m)\le(r(e-h)^{-1}h)^{m-1}r(h)\le\left(\frac{r(h)}{1-r(h)}\right)^{m-1}r(h).$$

Thus, Γ^m *is a Kannan type mapping, but this is not true if* Γ *is a generalized Kannan-Górnicki type mapping. See the next example.*

Example 3.7. *Let* $\mathcal{A}$ *and* P *be defined as the same as in Example 3.4. Let* $M=[\frac{1}{4},4]\times[\frac{1}{4},4]$ *and* $\Gamma:M\to M$ *be given as* $\Gamma(\varsigma_1,\varsigma_2)=(\frac{1}{\varsigma_1},\frac{1}{\varsigma_2})$ *for all* $\varsigma=(\varsigma_1,\varsigma_2)\in M$. *Let*

$$d(\varsigma,\upsilon)(t)=\begin{cases}(2^t,\ 2^t), & \text{if } \varsigma\ne\upsilon,\\ (0,\ 0), & \text{if } \varsigma=\upsilon,\end{cases}$$

where $\varsigma=(\varsigma_1,\varsigma_2),\upsilon=(\upsilon_1,\upsilon_2)$ *and* $\varsigma=\upsilon$ *implies* $\varsigma_1=\upsilon_1,\varsigma_2=\upsilon_2$. *We can check that* Γ *is a generalized Kannan-Górnicki type mapping with* $h=(1,1)$. *However,* Γ^2 *is not a generalized Kannan-Górnicki type mapping for any* $h\in P$ *since* $\Gamma^2\varsigma=\varsigma$ *and* $(2^t,2^t)\preceq h(0,0)$ *does not hold for any* $h\in P,\varsigma\ne\upsilon$.

Next, we extend Kannan's and Górnicki's results into some other more general theorems. At first, we define the class Ψ of such functions $\psi:P\times P\to P$ satisfying the following conditions

(i) $\psi(\theta,\theta)=\theta$;

(ii) ψ is continuous at (θ,θ).

Definition 3.8. *Suppose* (M,d) *is a cone metric space over a Banach algebra* $\mathcal{A}$. *The mapping* $\Gamma:M\to M$ *is named a*

(i) generalized Ćirić-Proinov-Górnicki type mapping if there is some $h\in P$ *with* $r(h)<1$ *such that*

$$d(\Gamma\varsigma,\Gamma\upsilon)\preceq hu(\varsigma,\upsilon)+\psi(d(\varsigma,\Gamma\varsigma),d(\upsilon,\Gamma\upsilon)),\tag{6}$$

where $u(\varsigma,\upsilon)\in\{d(\varsigma,\upsilon),d(\varsigma,\Gamma\upsilon),d(\upsilon,\Gamma\varsigma)\}$;

(ii) generalized Hardy-Rogers-Proinov-Górnicki type mapping if there are $k, l, s \in P$ with $r(k+l+s) < 1$ such that

$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq kd(\varsigma, \upsilon) + ld(\varsigma, \Gamma\upsilon) + sd(\upsilon, \Gamma\varsigma) + \psi(d(\varsigma, \Gamma\varsigma), d(\upsilon, \Gamma\upsilon)); \tag{7}$$

(iii) generalized Reich-Proinov-Górnicki type mapping if there is some $h \in P$ with $r(h) < 1$ such that

$$d(\Gamma\varsigma, \Gamma\upsilon) \preceq hd(\varsigma, \upsilon) + \psi(d(\varsigma, \Gamma\varsigma), d(\upsilon, \Gamma\upsilon)) \tag{8}$$

for all $\varsigma, \upsilon \in M$ and for some $\psi \in \Psi$.

Inspired by the results in [28], we would like to give the following results without the assumption of normality according to the above definitions.

Theorem 3.9. *Suppose $\Gamma : M \to M$ is an asymptotically regular generalized Ćirić-Proinov-Górnicki type mapping in the Γ-orbitally complete cone metric space (M, d) over a unital Banach algebra $\mathcal{A}$. If Γ is k-continuous or orbitally continuous, then Γ has a unique fixed point $\varpi \in M$ and for each $\varsigma \in M$ the iterated sequence $\{\Gamma^n\varsigma\}$ converges to ϖ.*

Proof. For any $\varsigma_0 \in M$, define $\varsigma_n = \Gamma\varsigma_{n-1} = \Gamma^n\varsigma_0, n \geq 1$. If $\varsigma_n = \varsigma_{n+1}$ for some n, then the proof is completed, so we assume that $\varsigma_n \neq \varsigma_{n+1}, \forall n \in \mathbb{N}$. Since Γ is asymptotically regular, the sequence $\{d(\varsigma_n, \varsigma_{n+1})\}$ is a c-sequence. For any natural number m, n and $m > n$, we have

$$d(\varsigma_n, \varsigma_m) = d(\Gamma\varsigma_{n-1}, \Gamma\varsigma_{m-1}) \preceq hu(\varsigma_{n-1}, \varsigma_{m-1}) + \psi[d(\varsigma_{n-1}, \Gamma\varsigma_{n-1}), d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})],$$

while

$$u(\varsigma_{n-1}, \varsigma_{m-1}) \in \{d(\varsigma_{n-1}, \varsigma_{m-1}), d(\varsigma_{n-1}, \Gamma\varsigma_{m-1}), d(\varsigma_{m-1}, \Gamma\varsigma_{n-1})\}.$$

Denote $\delta_n = d(\varsigma_{n-1}, \Gamma\varsigma_{n-1})$, then $\delta_m = d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})$. This gives

$$\psi[d(\varsigma_{n-1}, \Gamma\varsigma_{n-1}), d(\varsigma_{m-1}, \Gamma\varsigma_{m-1})] = \psi(\delta_n, \delta_m).$$

Now the proof will be divided in three cases.

Case 1. If $u(\varsigma_{n-1}, \varsigma_{m-1}) = d(\varsigma_{n-1}, \varsigma_{m-1})$, then

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\preceq hd(\varsigma_{n-1}, \varsigma_{m-1}) + \psi(\delta_n, \delta_m) \\ &\preceq h[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_n, \varsigma_m) + d(\varsigma_m, \varsigma_{m-1})] + \psi(\delta_n, \delta_m). \end{aligned}$$

According to $r(h) < 1$, we see $e - h$ is invertible and

$$d(\varsigma_n, \varsigma_m) \preceq (e-h)^{-1}h[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_m, \varsigma_{m-1})] + (e-h)^{-1}\psi(\delta_n, \delta_m).$$

Case 2. If $u(\varsigma_{n-1}, \varsigma_{m-1}) = d(\varsigma_{n-1}, \Gamma\varsigma_{m-1}) = d(\varsigma_{n-1}, \varsigma_m)$, then

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\preceq hd(\varsigma_{n-1}, \varsigma_m) + \psi(\delta_n, \delta_m) \\ &\preceq h[d(\varsigma_{n-1}, \varsigma_n) + d(\varsigma_n, \varsigma_m)] + \psi(\delta_n, \delta_m), \end{aligned}$$

which yields

$$d(\varsigma_n, \varsigma_m) \preceq (e-h)^{-1}hd(\varsigma_{n-1}, \varsigma_n) + (e-h)^{-1}\psi(\delta_n, \delta_m).$$

Case 3. If $u(\varsigma_{n-1}, \varsigma_{m-1}) = d(\varsigma_{m-1}, \Gamma\varsigma_{n-1}) = d(\varsigma_{m-1}, \varsigma_n)$, then

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\preceq hd(\varsigma_{m-1}, \varsigma_n) + \psi(\delta_n, \delta_m) \\ &\preceq h[d(\varsigma_{m-1}, \varsigma_m) + d(\varsigma_m, \varsigma_n)] + \psi(\delta_n, \delta_m), \end{aligned}$$

which implies

$$d(\varsigma_n,\varsigma_m) \preceq (e-h)^{-1}hd(\varsigma_{m-1},\varsigma_m)+(e-h)^{-1}\psi(\delta_n,\delta_m).$$

In all cases, we always have

$$d(\varsigma_n,\varsigma_m) \preceq (e-h)^{-1}h(\delta_n+\delta_m)+(e-h)^{-1}\psi(\delta_n,\delta_m).$$

Since Γ is asymptotically regular, it follows that δ_n is a c-sequence as $n\to\infty$, as well as δ_m for $m>n$. By the condition that ψ is continuous at (θ,θ), we know $\{\varsigma_n\}$ is a Cauchy sequence in M. Due to the fact (M,d) is Γ-orbitally complete, we conclude that $\varsigma_n\to\varpi$ as $n\to\infty$ for some $\varpi\in M$. If Γ is k-continuous, then $\lim_{n\to\infty}\Gamma^{k-1}\varsigma_n=\varpi$ implies $\lim_{n\to\infty}\Gamma^k\varsigma_n=\Gamma\varpi$. By the uniqueness of the limitation, we have $\varpi=\Gamma\varpi$. If Γ is orbitally continuous, then $\lim_{n\to\infty}\varsigma_n=\varpi$ implies $\lim_{n\to\infty}\Gamma\varsigma_n=\Gamma\varpi$. We also have $\varpi=\Gamma\varpi$.

Finally, we only need to show the uniqueness of ϖ. If υ is another fixed point of Γ, then

$$d(\upsilon,\varpi)=d(\Gamma\upsilon,\Gamma\varpi)\preceq hu(\upsilon,\varpi)+\psi(d(\upsilon,\Gamma\upsilon),d(\varpi,\Gamma\varpi))\preceq hd(\upsilon,\varpi)+\psi(\theta,\theta)=hd(\upsilon,\varpi),$$

where

$$u(\upsilon,\varpi)\in\{d(\upsilon,\varpi),d(\upsilon,\Gamma\varpi),d(\varpi,\Gamma\upsilon)\}=d(\upsilon,\varpi).$$

By $r(h)<1$, we see $\upsilon=\varpi$. Therefore, ϖ is the unique fixed point of Γ. □

By Theorem 3.9, we immediately deduce the fixed point results about equations (7) and (8) in the following. We omit their proofs.

Theorem 3.10. *Suppose $\Gamma: M\to M$ is an asymptotically regular generalized Hardy-Rogers-Proinov-Górnicki type mapping in the Γ-orbitally complete cone metric space (M,d) over a unital Banach algebra $\mathcal{A}$. If Γ is k-continuous or orbitally continuous, then Γ has a unique fixed point $\varpi\in M$ and for each $\varsigma\in M$ the iterated sequence $\{\Gamma^n\varsigma\}$ converges to ϖ.*

Theorem 3.11. *Suppose $\Gamma: M\to M$ is an asymptotically regular generalized Reich-Proinov-Górnicki type mapping in the Γ-orbitally complete cone metric space (M,d) over a unital Banach algebra $\mathcal{A}$. If Γ is k-continuous or orbitally continuous, then Γ has a unique fixed point $\varpi\in M$ and for each $\varsigma\in M$ the iterated sequence $\{\Gamma^n\varsigma\}$ converges to ϖ.*

Remark 3.12. *The main theorems in our paper are extensions of the results in [8, 22, 24]. The results in these literature always rely strongly on the completeness of the metric space. Moreover, Theorem 2.6 in [8], Theorem 2.3 in [22] and Theorems 4.1 and 4.2 in [24] all need the continuity of the mapping and the completeness of the metric space. However, the continuity of the mapping is weakened by k-continuity or orbital continuity and the completeness of the space is weakened by orbital completeness in our results.*

There is an example in which the mapping Γ has a fixed point since it satisfies our conditions (6)-(8). However, it does not satisfies the condition (5), which means that our results are more meaningful.

Example 3.13. *Let $\mathcal{A},P$ and d be defined as the same as in Example 3.4. Let the set $M=[0,+\infty)\times[0,+\infty)$ and $\Gamma: M\to M$ be given as*

$$\Gamma(\varsigma_1,\varsigma_2)=\left(\frac{\varsigma_1}{2\varsigma_1^2+1},\frac{\varsigma_2}{2\varsigma_2^2+1}\right)$$

for all $\varsigma \in M$. We can check that Γ satisfies the conditions (6)-(8) for any $h \in P$ with $r(h) < 1$ and $\psi(\varsigma, \upsilon) = (\varsigma + \upsilon)^{\frac{1}{3}}$ for all $\varsigma, \upsilon \in M$. The other conditions of Theorem 3.9 are also satisfied. Thus, the mapping Γ has a unique fixed point $\varsigma = (0,0)$ in M. However, Γ is not a usual Kannan type mapping (which does not satisfy (4)) by taking $\varsigma = (0,0)$ and $\upsilon = (\frac{1}{n_1}, \frac{1}{n_2})$. By a simple calculating, we get $\Gamma\varsigma = (0,0)$ and $\Gamma\upsilon = (\frac{n_1}{2+n_1^2}, \frac{n_2}{2+n_2^2})$. Then

$$d(\Gamma\varsigma, \Gamma\upsilon) = \left(\frac{n_1}{2+n_1^2}, \frac{n_2}{2+n_2^2}\right) \preceq \alpha \left(\frac{1}{n_1}, \frac{1}{n_2}\right) + K \left[(0,0) + \left(\frac{2}{n_1(2+n_1^2)}, \frac{2}{n_2(2+n_2^2)}\right)\right],$$

which is a contradiction for any $\alpha \in [0,1)$, $K \geq 0$ and large enough n_1, n_2.

The last example is intended to support the main results of this paper. It satisfies all the conditions of Theorem 3.11, so the fixed point exists and is unique. However, the results in the references do not lead to corresponding conclusions since they require some additional conditions.

Example 3.14. *Let $\mathcal{A} = C^1_{\mathbb{R}}[0,2]$ and $M = [0,2)$. For each $\varsigma \in \mathcal{A}$, $\|\varsigma\| = \|\varsigma\|_\infty + \|\varsigma'\|_\infty$. The multiplication is defined by its usual pointwise multiplication, then $\mathcal{A}$ is a unital Banach algebra and $e = 1$. Define $d(\varsigma, \upsilon)(t) = |\varsigma - \upsilon|\phi$ for all $\varsigma, \upsilon \in M$ and $\phi \in P = \{f(t) \in \mathcal{A} : f(t) \geq 0, t \in [0,1]\}$. Then P is a non-normal cone in cone metric space (M, d) over Banach algebra $\mathcal{A}$. Choose $h(t) = \frac{t}{4} + \frac{2}{3}$ and $\psi(\varsigma(t), \upsilon(t)) = \varsigma(t) + \upsilon(t)$ for all $\varsigma(t), \upsilon(t) \in P$. We deduce that*

$$h^n(t) = (\frac{t}{4} + \frac{2}{3})^n, \quad (h^n(t))' = \frac{n}{4}(\frac{t}{4} + \frac{2}{3})^{n-1},$$

hence that (t=1)

$$\|h^n\| = \|h^n\|_\infty + \|(h^n)'\|_\infty = (\frac{11}{12})^n + \frac{n}{4}(\frac{11}{12})^{n-1} = \frac{n}{4}(\frac{11}{12})^{n-1}(\frac{4}{n} \cdot \frac{11}{12} + 1) = \frac{n}{4}(\frac{11}{12})^{n-1}(1 + \frac{11}{3n}),$$

and finally that

$$r(h) = \lim_{n\to\infty} \|h^n\|^{\frac{1}{n}} = \lim_{n\to\infty} (\frac{n}{4})^{\frac{1}{n}} (\frac{11}{12})^{\frac{n-1}{n}} (1 + \frac{11}{3n})^{\frac{1}{n}} = \frac{11}{12}.$$

This means

$$r(h) = \frac{11}{12} < 1.$$

Define the mapping $\Gamma : M \to M$ by

$$\Gamma\varsigma = \begin{cases} \frac{\varsigma}{4} \arctan \varsigma, & \varsigma \in [0,1], \\ \ln(1 + \frac{\varsigma}{2}), & \varsigma \in (1,2). \end{cases}$$

Then Γ is asymptotically regular and orbitally continuous but not continuous. The space (M, d) is Γ-orbitally complete but not complete. Now, we will show that the inequality (8) is satisfied in the following three cases.

(1) For all $\varsigma, \upsilon \in [0,1]$,

$$\begin{aligned} d(\Gamma\varsigma, \Gamma\upsilon)(t) &= |\frac{\varsigma}{4} \arctan \varsigma - \frac{\upsilon}{4} \arctan \upsilon|\phi \\ &\preceq (\frac{t}{4} + \frac{1}{4})|\varsigma - \upsilon|\phi \\ &\preceq (\frac{t}{4} + \frac{2}{3})|\varsigma - \upsilon|\phi + |\varsigma - \frac{\varsigma}{4} \arctan \varsigma|\phi + |\upsilon - \ln(1 + \frac{\upsilon}{2})|\phi \\ &= hd(\varsigma, \upsilon)(t) + \psi(d(\varsigma, \Gamma\varsigma)(t), d(\upsilon, \Gamma\upsilon)(t)). \end{aligned}$$

(2) For all $\varsigma, \upsilon \in (1,2)$,

$$\begin{aligned} d(\Gamma\varsigma, \Gamma\upsilon)(t) &= |\ln(1+\frac{\varsigma}{2}) - \ln(1+\frac{\upsilon}{2})|\phi \\ &\preceq \frac{1}{3}|\varsigma - \upsilon|\phi \\ &\preceq (\frac{t}{4}+\frac{2}{3})|\varsigma-\upsilon|\phi + |\varsigma - \ln(1+\frac{\varsigma}{2})|\phi + |\upsilon - \ln(1+\frac{\upsilon}{2})|\phi \\ &= hd(\varsigma,\upsilon)(t) + \psi(d(\varsigma,\Gamma\varsigma)(t), d(\upsilon,\Gamma\upsilon)(t)). \end{aligned}$$

(3) For all $\varsigma \in [0,1]$, $\upsilon \in (1,2)$,

$$\begin{aligned} d(\Gamma\varsigma, \Gamma\upsilon)(t) &= |\frac{\varsigma}{4}\arctan\varsigma - \ln(1+\frac{\upsilon}{2})|\phi \\ &\preceq |\frac{\varsigma}{4}\arctan\varsigma|\phi + |\ln(1+\frac{\upsilon}{2})|\phi \\ &\preceq (\frac{\varsigma}{4}+\frac{\upsilon}{2})\phi \\ &\preceq (\frac{t}{4}+\frac{2}{3})|\varsigma-\upsilon|\phi + |\varsigma - \frac{\varsigma}{4}\arctan\varsigma|\phi + |\upsilon - \ln(1+\frac{\upsilon}{2})|\phi \\ &= hd(\varsigma,\upsilon)(t) + \psi(d(\varsigma,\Gamma\varsigma)(t), d(\upsilon,\Gamma\upsilon)(t)). \end{aligned}$$

Similarly, we can also prove that $d(\Gamma\varsigma,\Gamma\upsilon)(t) \preceq hd(\varsigma,\upsilon)(t) + \psi(d(\varsigma,\Gamma\varsigma)(t), d(\upsilon,\Gamma\upsilon)(t))$ *for all* $\varsigma \in (1,2)$, $\upsilon \in [0,1]$. *Therefore,* Γ *has a unique fixed point in* M *by Theorem 3.11.*

The mapping Γ in the above example has a fixed point while the Banach algebra is not $\mathbb{R}$, the cone is not normal, the space is not complete and the mapping is not continuous. Therefore, the conclusions from the literature which request one of these conditions are not applicable here.

Conclusion 3.15. *First of all, we would like to thank the reviewers for their valuable comments. Secondly, we find that there are some very good results in the latest paper [Ravindra K. Bisht, An overview of the emergence of weaker continuity notions, various classes of contractive mappings and related fixed point theorems, J. Fixed Point Theory Appl. (2023) 25: 11, 1-29]. The author in this paper mentioned several concepts related to continuity: orbital continuity,* x_0*-orbital continuity, almost orbital continuity,* T*-orbital lower semi-continuity and so on. However, as far as we know there is no definition of the lower limit in cone metric spaces over Banach algebras until now. Therefore, it will be a very challenging work to give the fixed point results for the lower semi-continuous mappings in cone metric spaces over Banach algebras, which will be a direction of our future research. At last, thanks again to the anonymous reviewers for their very constructive suggestions and important information.*

Declarations

Conflict of interest The authors declare no conflict of interest.

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