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Slant kink-wave solutions and spreading of the free boundary of the inhomogeneous pressureless Euler equations

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Abstract. In this paper, we consider the inhomogeneous pressureless Euler equations. First, we present a class of self-similar analytical solutions to the 1D Cauchy problem and investigate the large-time behavior of the solutions, and particularly, we obtain slant kink-wave solutions for the inhomogeneous Burgers (InhB) type equation. Next, we prove the integrability of the InhB equation in the sense of Lax pair. Furthermore, we study the spreading rate of the moving domain occupied by mass for the 1D Cauchy problem with compact support initial density. We find that the expanding domain grows exponentially in time, provided that the solutions exist and smooth at all time. Finally, we extend the corresponding results of the inhomogeneous pressureless Euler equations to the radially symmetric multi-dimensional case.

§1 Introduction

In this paper, we consider the following inhomogeneous pressureless Euler equations (see [8]):

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2)_x = \rho x, \tag{1.2}$$

where the fluid density $\rho = \rho(x, t)$ and the fluid velocity $u = u(x, t)$ are unknown variables, ρx is the external force. When the external force ρx is absent, system (1.1)-(1.2) is the standard pressureless Euler equations, which is studied as the sticky particle model in cosmology [31] or plasma physics [2].

As mentioned in [18], for the general case, the density $\rho(x, t)$ is no longer a function, but a measure. So researchers introduced various strongly related notions of weak solutions, such as

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measure solutions [1,17], duality solutions [3], duality solutions obtained by vanishing viscosity [4], mass and momentum potentials [5,7,12], together with generalized characteristics [28], generalized potentials and variational principles [10,13,27]. In this paper, we consider the C^1 solution of system (1.1)-(1.2). Although this is less natural than measure solution in general, the results obtained in this paper for smooth solution are still important in mathematical sense.

In this paper, we first present a class of analytical solutions for system (1.1)-(1.2) with C^1 smooth initial data. These solutions are of the form

$$\rho(x,t) = \frac{f(\frac{x}{a(t)})}{a(t)}, \quad u(x,t) = \frac{a'(t)}{a(t)}x, \tag{1.3}$$

where $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$. In fact, such analytical solutions are widely studied for the compressible Euler and Navier-Stokes equations (see [11,15,16,29] for instance). However, the analytical solutions of the pressureless Euler equations are rarely discussed because the measure solutions are more natural than the analytical ones. In [30], Yuen constructed the analytical solutions to the N-dimensional pressureless Euler equations in the form of

$$\rho(\overrightarrow{x},t) = \frac{f\left(\frac{1}{a(t)^s}\sum_{i=1}^{N} x_i^s\right)}{a(t)^N}, \quad \overrightarrow{u}(\overrightarrow{x},t) = \frac{a'(t)}{a(t)}\overrightarrow{x}, \quad a(t) = a_0 + a_1 t, \tag{1.4}$$

where the arbitrary function $f \geq 0 \in C^1$; $s \geq 1$, $a_0 > 0$ and a_1 are arbitrary constants; $\overrightarrow{x} = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$. Obviously, when $s = 1$ and $N = 1$, we obtain a class of special solutions (1.3) with $a(t) = a_0 + a_1 t$ for system (1.1)-(1.2) without the external force ρx. Using the form (1.3), we can obtain a class of self-similar analytical solutions to system (1.1)-(1.2) (see Theorem 2.1 below). For system (1.1)-(1.2), Ding and Huang [8] constructed an explicit weak solution with the form of

$$\rho(x,t) = -\frac{\partial^2}{\partial x^2}\min_y F(y;x,t), \quad \rho(x,t)u(x,t) = \frac{\partial^2}{\partial x \partial t}\min_y F(y;x,t) \tag{1.5}$$

in the sense of distributions, where $F(y;x,t)$ is a function depending on the initial data, and they further proved that the solution $u(x,t)$ converges to x as t tends to infinity. From Theorem 2.1 below, we can see that the solution $u(x,t)$ we constructed for system (1.1)-(1.2) converges to x or $-x$ as t tends to infinity. When $\rho > 0$, (1.2) is equivalent to an inhomogeneous Burgers (InhB) type equation (see (2.3) below), we obtain slant kink-wave solutions for this equation.

Next in the paper, we provide a Lax pair for the InhB equation and therefore it is integrable. We show the integrability through considering the negative WKI flows and finally the InhB model is a special case in the flow, and actually the two-component short pulse system is also provided with the integrability (see the detailed derivation in Section 3).

The third topic of this paper is the spreading rate of the moving domain occupied by mass for system (1.1)-(1.2) with $\rho(-a(t),t) = \rho(a(t),t) = 0$, where $(-a(t), a(t))$ is the moving domain occupied by mass. By the form (1.3), we can easily give a special solution $\rho(x,t) = \frac{1-\frac{x^2}{a(t)^2}}{a(t)}$, which satisfies $\rho(-a(t),t) = \rho(a(t),t) = 0$. For some special analytical solutions to the spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity coefficients, Guo and Xin [11] proved that the free boundary tends to infinity at an algebraic rate. For the non-radially symmetric compressible Euler equations, Sideris [25] proved that the diameter of a region occupied by the fluid surrounded by vacuum grows linearly in time provided that the pressure is positive and there are no singularities. Both systems in [11,25] contain the positive pressure, to our knowledge, there are no results about the spreading of the moving domain occupied by mass for the system (1.1)-(1.2) in the literature. In this paper, we will give such a result. For this, we need the following averaged quantities

$$m(t) = \int_{-a(t)}^{a(t)} \rho(x,t)dx, \tag{1.6}$$

$$I(t) = \int_{-a(t)}^{a(t)} x^2\rho(x,t)dx, \tag{1.7}$$

$$F(t) = \int_{-a(t)}^{a(t)} x\rho(x,t)u(x,t)dx, \tag{1.8}$$

$$E_k(t) = \frac{1}{2}\int_{-a(t)}^{a(t)} \rho(x,t)u^2(x,t)dx, \tag{1.9}$$

where $E_k(t)$ denotes the kinetic energy and k is the abbreviation of kinetic. In fact, such kind of averaged quantities were first introduced by Sideris [24] and further explored in [6] for investigating the formation of singularities of smooth solutions to the compressible Euler equations. By exploring the relations among the above averaged quantities, we find that $I(t), F(t)$ and $E_k(t)$ can be solved exactly if the initial data satisfy $I(0), F(0), E_k(0) < +\infty$. This is crucial to investigate the spreading rate of the moving domain since $I(t) \le m(0)a(t)^2$ (see the proof of Theorem 4.1 below).

In fact, our above results for the 1D pressureless Euler equations can be extended to the radially symmetric multi-dimensional case. In [29], Yuen obtained the explicit functions of the density and the velocity in N-dimensional radial symmetry for the equation of conservation of mass, i.e.,

$$\rho(r,t) = \frac{f(\frac{r}{a(t)})}{a(t)^N}, \quad u(r,t) = \frac{a'(t)}{a(t)}r, \tag{1.10}$$

where $f \ge 0 \in C^1$, $a(t) > 0 \in C^1$, and $r = |\overrightarrow{x}|$. With the aid of (1.10), we can obtain the self-similar analytical solutions to the radially symmetric multi-dimensional pressureless Euler equations. In order to study the spreading rate of the moving domain occupied by mass for the N-dimensional radially symmetric pressureless Euler equations with $\rho(a(t),t) = 0$ and $u(0,t) = 0$, we need the following averaged quantities

$$m_1(t) = \int_0^{a(t)} \rho(r,t)r^{N-1}dr, \tag{1.11}$$

$$I_1(t) = \int_0^{a(t)} \rho(r,t)r^{N+1}dr, \tag{1.12}$$

$$F_1(t) = \int_0^{a(t)} \rho(r,t)u(r,t)r^N dr, \tag{1.13}$$

$$E_{1k}(t) = \frac{1}{2}\int_0^{a(t)} \rho(r,t)u^2(r,t)r^{N-1}dr. \tag{1.14}$$

We should remark that the averaged quantities (1.12)-(1.14) have been used in [9] for studying the blowup of smooth solutions to the compressible Euler and Euler-Poisson equations with radial symmetry and free boundary. For the averaged quantities (1.12)-(1.14), we can obtain the same relationships among them to the one-dimensional case and we can solve them exactly.

The plan of this paper is as follows. In Section 2, we give a class of self-similar analytical solutions to the Cauchy problem of system (1.1)-(1.2) and investigate the large-time behavior of the solutions. In Section 3, we prove that the InhB equation is still integrable in the sense of Lax pair. In Section 4, we will study the spreading rate of the moving domain occupied by mass for system (1.1)-(1.2) with compact support initial density. In Section 5, we extend the corresponding results of the inhomogeneous pressureless Euler equations to the radially symmetric multi-dimensional case.

§2 Analytical solutions

In this section, we consider the following Cauchy problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \rho x, \\ (\rho, u)|_{t=0} = (\rho_0(x), u_0(x)). \end{cases} \tag{2.1}$$

Our result is stated as follows.

Theorem 2.1 For the problem (2.1) with $\rho_0(x), u_0(x) \in C^1(\mathbb{R})$ and $\rho_0(x) > 0$, there exist solutions of the form

$$\rho(x,t) = \frac{f(\frac{x}{a(t)})}{a(t)}, \quad u(x,t) = \frac{a'(t)}{a(t)}x, \quad a(t) = \frac{a_0 + a_1}{2}e^t + \frac{a_0 - a_1}{2}e^{-t}, \tag{2.2}$$

where $f \geq 0 \in C^1$, $a_0 > 0$ and a_1 are two constants satisfying $\rho_0(x) = \frac{f(\frac{x}{a_0})}{a_0}$, $u_0(x) = \frac{a_1}{a_0}x$. Moreover, we have the following large-time behavior of the solutions (2.2).

Case (i) If $a_0 + a_1 > 0$, then $\lim_{t\to+\infty} a(t) = +\infty$, and for any fixed $x \in \mathbb{R}$, one has $\lim_{t\to+\infty} \rho(x,t) = 0$ and $\lim_{t\to+\infty} u(x,t) = x$;

Case (ii) If $a_0 + a_1 = 0$, then $\lim_{t\to+\infty} a(t) = 0$, and if we further assume that $\rho_0(x) \geq C > 0$ for some constant C, then we have $\lim_{t\to+\infty} \rho(x,t) = +\infty$ and $\lim_{t\to+\infty} u(x,t) = -x$ for any $x \in \mathbb{R}$;

Case (iii) If $a_0 + a_1 < 0$, then $\lim_{t\to t_0} a(t) = 0$, and if we further assume that $\rho_0(x) \geq C > 0$ for some constant C, then we have $\lim_{t\to t_0} \rho(x,t) = +\infty$ for any $x \in \mathbb{R}$, where $t_0 = \frac{1}{2}\ln\frac{a_1 - a_0}{a_1 + a_0}$.

Proof. It is not difficult to verify that the solutions (1.3) satisfy $(2.1)_1$. When $\rho(x,t) > 0 \in C^1(\mathbb{R})$, by $(2.1)_1$ we know that $(2.1)_2$ is equivalent to

$$u_t + uu_x = x. \tag{2.3}$$

Plugging $u(x,t) = \frac{a'(t)}{a(t)}x$ into (2.3), we obtain

$$\frac{a''(t)}{a(t)}x = x, \tag{2.4}$$

i.e.,

$$a''(t) - a(t) = 0, \tag{2.5}$$

which can be solved as

$$a(t) = \frac{a_0 + a_1}{2}e^t + \frac{a_0 - a_1}{2}e^{-t}, \tag{2.6}$$

where $a_0 > 0$ and a_1 are two constants satisfying $\rho_0(x) = \frac{f(\frac{x}{a_0})}{a_0}$, $u_0(x) = \frac{a_1}{a_0}x$.

Case (i) If $a_0+a_1>0$, by (2.6) we know that $\lim\limits_{t\to+\infty} a(t)=+\infty$. In view of (2.2) and $(2.1)_3$, one has $\rho_0(x)=\frac{f(\frac{x}{a_0})}{a_0}$, which together with $\rho_0(x)>0$ implies that $f>0$, so by (2.2) we have $\rho(x,t)>0$, $\lim\limits_{t\to+\infty}\rho(x,t)=0$ and $\lim\limits_{t\to+\infty}u(x,t)=x$ for any fixed $x\in\mathbb{R}$.

Case (ii) If $a_0+a_1=0$, then by (2.6) it holds $a(t)=a_0e^{-t}$, which leads to $\lim\limits_{t\to+\infty}a(t)=0$ and $\lim\limits_{t\to+\infty}u(x,t)=-x$ by (2.2). If we further assume that $\rho_0(x)\geq C>0$ for some constant C, then by $\rho_0(x)=\frac{f(\frac{x}{a_0})}{a_0}$ we know that $f\geq a_0,C>0$, which together with (2.2) and $\lim\limits_{t\to+\infty}a(t)=0$ implies that $\lim\limits_{t\to+\infty}\rho(x,t)=+\infty$.

Case (iii) If $a_0+a_1<0$, then by (2.6) we know that $\lim\limits_{t\to t_0}a(t)=0$, where $t_0=\frac{1}{2}\ln\frac{a_1-a_0}{a_1+a_0}$. If we further assume that $\rho_0(x)\geq C>0$ for some constant C, then by $\rho_0(x)=\frac{f(\frac{x}{a_0})}{a_0}$ we know that $f\geq a_0C>0$, which together with (2.2) and $\lim\limits_{t\to t_0}a(t)=0$ implies that $\lim\limits_{t\to t_0}\rho(x,t)=+\infty$.

We complete the proof of Theorem 2.1. □

Remark 2.1 Case (i) of Theorem 2.1 implies that the C^1 solutions of the problem (2.1) can exist globally in time if $\rho_0(x),u_0(x)\in C^1(\mathbb{R})$, $\rho_0(x)>0$ and $a_0+a_1>0$. The result $\lim\limits_{t\to+\infty}u(x,t)=x$ is identical to [8].

Remark 2.2 Case (ii) of Theorem 2.1 indicates that the density will blow up everywhere at infinite time if $\rho_0(x),u_0(x)\in C^1(\mathbb{R})$, $a_0+a_1=0$ and $\rho_0(x)\geq C>0$ for some constant C. The result $\lim\limits_{t\to+\infty}u(x,t)=-x$ is not identical to [8].

Remark 2.3 Case (iii) of Theorem 2.1 means that the density will blow up everywhere at a finite time if $\rho_0(x),u_0(x)\in C^1(\mathbb{R})$, $a_0+a_1<0$ and $\rho_0(x)\geq C>0$ for some constant C.

Remark 2.4 We remark that the solutions constructed in Theorem 2.1 are not unique because f is not a concrete function. This is a class of solutions. However, we don't know whether this is the only class of solutions or not.

It is not hard to check the InhB equation (2.3) possesses the following explicit solutions

$$u(x,t)=x\tanh(t+A),\ \forall A\in\mathbb{R}, \tag{2.7}$$

$$u(x,t)=x\coth(t+B),\ \forall B\in\mathbb{R}, \tag{2.8}$$

which indicate that the solution (2.7) approaches a kink-wave along x-direction – called 45°-slant kink-wave when t goes to $\pm\infty$, while the solution (2.8) is more approximate to a blow-up kink-wave along x-direction – called 45°-blow-up-slant kink-wave when t goes to $\pm\infty$ since it is discontinuous at $t=-B$.

However, the interesting thing is that the InhB equation is still integrable in the sense of Lax pair. Let us discuss this in next section.

§3 Integrability of InhB equation (2.3) in the sense of Lax pair

A nonlinear PDE $u_t=K(u,u_x,u_{xx},\ldots)$ is called 'integrable' in the sense of Lax pair [14] if the PDE is the compatibility condition of two operators L,M satisfying $L_t=[M,L]$ where $[\cdot,\cdot]$ refers to the commutator of two operators, and a pair of operators L and M usually correspond to the spectral problem and the time auxiliary problem, respectively. Our strategy to derive the

Lax pair for the InhB equation (2.3) is from an eigenvalue problem studied by Wadati, Konno and Ichikowa [26] (see also [19]) and the negative order flows proposed by Qiao [21-23].

In spirit, the InhB equation (2.3) is able to be generated from the negative order flow of the Wadati-Konno-Ichikowa (WKI) hierarchy. Let us consider the WKI spectral problem [26,19]

$$L\cdot\psi=\lambda\psi,\ L=L\left(u,v\right)=\frac{1}{1-uv}\left(\begin{array}{cc} i & -u\\ -v & -i\end{array}\right)\partial,\quad \psi=\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right), \tag{3.1}$$

which has its spectral gradients $\frac{\delta\lambda}{\delta u}=\lambda\psi_2^2$, $\frac{\delta\lambda}{\delta v}=-\lambda\psi_1^2$, that satisfy the following Lenard's eigenvalue problem [20]

$$K\cdot\left(\begin{array}{c}\frac{\delta\lambda}{\delta u}\\ \frac{\delta\lambda}{\delta v}\end{array}\right)=\lambda J\cdot\left(\begin{array}{c}\frac{\delta\lambda}{\delta u}\\ \frac{\delta\lambda}{\delta v}\end{array}\right) \tag{3.2}$$

with the Lenard's operators pair

$$K=\frac{1}{2i}\left(\begin{array}{cc} -\frac12\partial^2\frac{u}{w}\partial^{-1}\frac{u}{w}\partial^2 & \partial^3+\frac12\partial^2\frac{u}{w}\partial^{-1}\frac{v}{w}\partial^2\\ \partial^3+\frac12\partial^2\frac{v}{w}\partial^{-1}\frac{u}{w}\partial^2 & -\frac12\partial^2\frac{v}{w}\partial^{-1}\frac{v}{w}\partial^2\end{array}\right), \tag{3.3}$$

$$J=\left(\begin{array}{cc} 0 & -\partial^2\\ \partial^2 & 0\end{array}\right), \tag{3.4}$$

where $w=\sqrt{1-uv}$. The recursion operator $\mathcal{L}=J^{-1}K$ is

$$\mathcal{L}=\frac{1}{2i}\left(\begin{array}{cc} \partial+\frac{v}{2w}\partial^{-1}\frac{u}{w}\partial^2 & -\frac{v}{2w}\partial^{-1}\frac{v}{w}\partial^2\\ \frac{u}{2w}\partial^{-1}\frac{u}{w}\partial^2 & -\partial-\frac{u}{2w}\partial^{-1}\frac{v}{w}\partial^2\end{array}\right). \tag{3.5}$$

Apparently, the Gateaux derivative $L_*\left(\xi\right)$ of the spectral operator L in the direction $\xi=\left(\xi_1,\xi_2\right)^T\in\mathcal{B}^2$ is

$$L_*\left(\xi\right)=\frac{1}{1-uv}\left(\begin{array}{cc} u\xi_2 & -i\xi_1\\ i\xi_2 & v\xi_1\end{array}\right)L, \tag{3.6}$$

which is an injective homomorphism.

Through lengthy calculations, one can obtain the inverse operators of $L,\ J,\ K$ and $\mathcal{L}$ [21-23]

$$L^{-1}=\left(\begin{array}{cc} -i\partial^{-1} & \partial^{-1}u\\ \partial^{-1}v & i\partial^{-1}\end{array}\right), \tag{3.7}$$

$$J^{-1}=\left(\begin{array}{cc} 0 & \partial^{-2}\\ -\partial^{-2} & 0\end{array}\right), \tag{3.8}$$

$$K^{-1}=2i\left(\begin{array}{cc} \frac12\partial^{-1}v\partial^{-1}v\partial^{-1} & \partial^{-3}-\frac12\partial^{-1}v\partial^{-1}u\partial^{-1}\\ \partial^{-3}-\frac12\partial^{-1}u\partial^{-1}v\partial^{-1} & \frac12\partial^{-1}u\partial^{-1}u\partial^{-1}\end{array}\right), \tag{3.9}$$

$$\mathcal{L}^{-1}=2i\left(\begin{array}{cc} \partial^{-1}-\frac12\partial^{-1}v\partial^{-1}u\partial & -\frac12\partial^{-1}v\partial^{-1}v\partial\\ \frac12\partial^{-1}u\partial^{-1}u\partial & -\partial^{-1}+\frac12\partial^{-1}u\partial^{-1}v\partial\end{array}\right). \tag{3.10}$$

Therefore, the entire (both positive and negative orders) WKI hierarchy of NLEEs is produced through

$$\left(\begin{array}{c}u\\ v\end{array}\right)_t=J\cdot G_m,\quad m\in\mathbb{Z}, \tag{3.11}$$

$$G_m=\left\{\begin{array}{ll}\mathcal{L}^m\cdot G_0, & m=0,1,2\ldots,\\ \mathcal{L}^m\cdot G_{-1}, & m=-1,-2,\ldots,\end{array}\right. \tag{3.12}$$

where $G_0 \in \mathrm{Ker}J$, $G_{-1} \in \mathrm{Ker}K$, J, $\mathcal{L}$ and $\mathcal{L}^{-1}$ are defined by (3.4), (3.5) and (3.10), respectively.

For any $G = \left(G^{[1]}, G^{[2]}\right)^T \in \mathcal{B}^2$, the equation $[V, L] = L_*(K \cdot G) L^{-1} - L_*(J \cdot G)$ has the following operator solution

$$V = V(G) = \begin{pmatrix} 0 & \bar{B} \\ \bar{C} & 0 \end{pmatrix} + \bar{A} \begin{pmatrix} -i & u \\ v & i \end{pmatrix} L, \tag{3.13}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ are given by

$$\begin{aligned} \bar{A} = \bar{A}(G) &= \frac{1}{2w}\left(\frac{u}{w}G^{[1]}_{xx} - \frac{v}{w}G^{[2]}_{xx}\right)^{(-1)}, \quad w = \sqrt{1-uv}, \\ \bar{B} = \bar{B}(G) &= \frac{1}{4i}\left(2G^{[2]}_{xx} - \partial\frac{u}{w}\cdot\left(\frac{u}{w}G^{[1]}_{xx} - \frac{v}{w}G^{[2]}_{xx}\right)^{(-1)}\right), \\ \bar{C} = \bar{C}(G) &= \frac{1}{4i}\left(2G^{[1]}_{xx} + \partial\frac{v}{w}\cdot\left(\frac{u}{w}G^{[1]}_{xx} - \frac{v}{w}G^{[2]}_{xx}\right)^{(-1)}\right). \end{aligned}$$

Thus, the WKI hierarchy (3.11) has the following Lax form [21, 22]

$$L_t = [W_m, L], m \in \mathbb{Z}, \tag{3.14}$$

with the Lax operator

$$W_m = \sum V(G_j) L^{m-j}, \ m \in \mathbb{Z}. \tag{3.15}$$

Here L, L^{-1} and $V(G_j)$ are given by (3.1), (3.7) and (3.13) with $G = G_j$ defined by (3.12), respectively.

Let us discuss reductions of (3.11) below.

I. Positive case $(m = 0, 1, 2, \ldots)$

- With $G_0 = (ax+b, \ cx+d)^T \in \mathrm{Ker}J$ $(a, b, c, d$ are independent of $x)$, the positive order category of (3.11) reads as the WKI hierarchy (corresponding to the isospectral case $\lambda_t = 0$)

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = J\mathcal{L}^m \cdot \begin{pmatrix} ax+b \\ cx+d \end{pmatrix}, \ m = 0, 1, 2, \ldots, \tag{3.16}$$

which has the following representative equations

$$\begin{cases} u_t = -i\left(\frac{u}{w}\right)_{xx}, \\ v_t = i\left(\frac{v}{w}\right)_{xx}, \end{cases} \quad m = 1, \tag{3.17}$$

$$\begin{cases} u_t = \frac{1}{2}\left(\frac{u_x}{w^3}\right)_{xx}, \\ v_t = \frac{1}{2}\left(\frac{v_x}{w^3}\right)_{xx}, \end{cases} \quad m = 2. \tag{3.18}$$

They possess the standard Lax operators

$$W_1 = \begin{pmatrix} 0 & -i\left(\frac{u}{w}\right)_x \\ i\left(\frac{v}{w}\right)_x & 0 \end{pmatrix} L - \frac{2}{w}\begin{pmatrix} -i & u \\ v & i \end{pmatrix} L^2, \tag{3.19}$$

$$W_2 = \frac{1}{2w^3}\begin{pmatrix} 0 & u_x \\ v_x & 0 \end{pmatrix} L + \frac{1}{w^3}\begin{pmatrix} uv_x - u_xv & -iu_x \\ iv_x & u_xv - uv_x \end{pmatrix} L^2 - \frac{2}{w}\begin{pmatrix} -i & u \\ v & i \end{pmatrix} L^3, \tag{3.20}$$

respectively.

Apparently, the hierarchy (3.16) has the standard Lax representation $L_t = [W_m, L]$, $W_m = \sum_{j=0}^{m-1} V(G_j) L^{m-j}$, where $V(G_j)$ are given by (3.13) with $G = G_j = \mathcal{L}^j \cdot (ax+b, cx+d)^T$, $j \geq 0$.

The well-known Harry-Dym equation is included in the special case of $a = c = 0$,

$$(u,v)_t^T = JG_1 = KG_0,$$

which can be reduced to

$$s_t = -(\frac{1}{\sqrt{s}})_{xxx}, s = 1+u, u = v.$$

II. Negative case $(m = -1, -2, \ldots)$

- The negative order generator G_{-1} has the following two seed functions [22, 23]

$$G_{-1}^1 = i\begin{pmatrix} -v^{(-1)} \\ -u^{(-1)} \end{pmatrix}, \tag{3.21}$$

$$G_{-1}^2 = i\begin{pmatrix} a_1x^2 - \left(v\left(a_1u+b_1v\right)^{(-1)}\right)^{(-1)} \\ -b_1x^2 + \left(u\left(a_1u+b_1v\right)^{(-1)}\right)^{(-1)} \end{pmatrix}, \tag{3.22}$$

where $a_1 = a_1(t)$ and $b_1 = b_1(t)$ are two arbitrarily given C^∞-functions, and superscript $f^{(-1)} = \int f dx$. They generate two isospectral $(\lambda_t = 0)$ negative order hierarchies of (3.1)

$$(u,v)_t^T = J\mathcal{L}^{m+1} \cdot G_{-1}^k,\ m<0,\ m\in\mathbb{Z},\ k=1,2, \tag{3.23}$$

which have the standard Lax representation $L_t = [W_m^k, L]$ with $W_m^k = -\sum_{j=m}^{-1} V\left(G_j^k\right) L^{m-j}$, $k = 1,2$, where $V\left(G_j^k\right)$ and L^{-1} are given by (3.13) with $G = G_j^k = \mathcal{L}^{j+1} \cdot G_j^k$ and by (3.1), respectively. Thus, the hierarchies (3.23) are integrable.

(3.23) has the following representative equations

$$\begin{cases} u_t = 4iu^{(-1)} - 2i\left(uu^{(-1)}v^{(-1)}\right)_x, \\ v_t = 4iv^{(-1)} - 2i\left(vv^{(-1)}u^{(-1)}\right)_x, \end{cases} \quad m=-3, \quad k=1, \tag{3.24}$$

and

$$\begin{cases} u_t = 2ib_1 - i\left(u\left(a_1u+b_1v\right)^{(-1)}\right)_x, \\ v_t = 2ia_1 - i\left(v\left(a_1u+b_1v\right)^{(-1)}\right)_x, \end{cases} \quad m=-1, \quad k=2, \tag{3.25}$$

which can respectively be changed to

$$\begin{cases} Q_{xt} = 4iQ - 2i\left(RQQ_x\right)_x, \\ R_{xt} = 4iR - 2i\left(QRR_x\right)_x, \end{cases} \tag{3.26}$$

and

$$\begin{cases} Q_t = 2ib_1x - iQ_x\left(a_1Q+b_1R\right), \\ R_t = 2ia_1x - iR_x\left(a_1Q+b_1R\right), \end{cases} \tag{3.27}$$

via the transformations $u^{(-1)} = Q,\ v^{(-1)} = R$.

Equation (3.26) is actually the two-component short pulse equation (2SPE) [22,23]

$$\begin{cases} Q_{xt} = Q - \frac{1}{2}\left(RQQ_x\right)_x, \\ R_{xt} = R - \frac{1}{2}\left(QRR_x\right)_x, \end{cases} \quad \textit{here just } t \to -\frac{1}{4}it, \tag{3.28}$$

which is directly reduced to the regular short pulse equation

$$u_{xt} = u - \frac{1}{6}(u^3)_{xx}, \ u = R = Q.$$

If we put $R = Q^*$ (complex conjugate), then we obtain the complex short pulse equation:

$$Q_{xt} = Q - \frac{1}{2}\left(|Q|^2 Q_x\right)_x, \ \textit{integrable}.$$

Equation (3.27) is actually the two-component inhomogeneous Burgers equation (2IBE)

$$\begin{cases} Q_t = b_1 x - \frac{1}{2} Q_x \left(a_1 Q + b_1 R\right), \\ R_t = a_1 x - \frac{1}{2} R_x \left(a_1 Q + b_1 R\right), \end{cases} \tag{3.29}$$

which is directly reduced to the new inhomogeneous Burgers equation (2.3)

$$u_t = x - u u_x, u = R = Q, a_1(t) = b_1(t) = 1.$$

(3.26) and (3.27) possess the standard Lax operators

$$W^1_{-3} = -2iQR \begin{pmatrix} -i & Q_x \\ R_x & i \end{pmatrix} L - 2 \begin{pmatrix} 0 & Q \\ -R & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} L^{-1}, \tag{3.30}$$

$$W^2_{-1} = -i\left(a_1 Q + b_1 R\right) \begin{pmatrix} -i & Q_x \\ R_x & i \end{pmatrix} L + \begin{pmatrix} 0 & -b_1 \\ a_1 & 0 \end{pmatrix}, \tag{3.31}$$

where L and L^{-1} are given by (3.1) and (3.7) with $u = Q_x, \ v = R_x$, respectively.

Thus, the InhB equation (2.3) has the Lax pair $L_t = [W^2_{-1}, L]$ with the following two operators

$$\begin{aligned} L &= \frac{1}{1-u^2} \begin{pmatrix} i & -u \\ -u & -i \end{pmatrix} \partial, \\ W^2_{-1} &= -u^{(-1)} \begin{pmatrix} -i & u \\ u & i \end{pmatrix} L + \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \end{aligned}$$

§4 Spreading of the moving domain occupied by mass

In this section, we consider the following Cauchy problem with compact support initial density

$$\begin{cases} \rho_t + (\rho u)_x = 0, \quad x \in (-a(t), a(t)), \\ (\rho u)_t + (\rho u^2)_x = \rho x, \quad x \in (-a(t), a(t)), \\ (\rho(x,t), u(x,t))|_{t=0} = (\rho_0(x), u_0(x)), \quad \mathrm{supp}\rho_0(x) \subset (-a_0, a_0), \\ \rho(-a(t), t) = \rho(a(t), t) = 0, \end{cases} \tag{4.1}$$

where $(-a(t), a(t))$ is the moving domain occupied by mass and $a_0 = a(0) > 0$ is a constant. Similar to Theorem 2.1, we can construct a special solution to (4.1) as $\rho(x,t) = \frac{1-\frac{x^2}{a(t)^2}}{a(t)}$, $u(x,t) = \frac{a'(t)}{a(t)} x$, where $a(t) = \frac{a_0+a_1}{2} e^t + \frac{a_0-a_1}{2} e^{-t}$, $a_0 > 0$ and a_1 are two constants satisfying $\rho_0(x) = \frac{1-\frac{x^2}{a_0^2}}{a_0}$, $u_0(x) = \frac{a_1}{a_0} x$. Obviously, the moving domain $(-a(t), a(t))$ constructed above grows exponentially in time when $a_0 + a_1 > 0$. However, for the general problem (4.1), such a result is not obvious. In the following, we study the spreading rate of the moving domain for

(4.1) by using the method of averaged quantities.

Theorem 4.1 Assume that $0 < m(0), I(0), E_k(0) < +\infty$ and

$$\frac{I(0)}{4} + \frac{E_k(0) + F(0)}{2} > 0. \tag{4.2}$$

Let (ρ, u) be a C^1 solution of (4.1), then we have

$$a(t) \geq \frac{1}{\sqrt{m(0)}} \left\{ \left[\frac{I(0)}{4} + \frac{E_k(0) + F(0)}{2} \right] e^{2t} - \left[\frac{F(0) - E_k(0)}{2} - \frac{I(0)}{4} \right] e^{-2t} + \frac{I(0)}{2} - E_k(0) \right\}^{\frac{1}{2}}. \tag{4.3}$$

Proof. By (1.6), $(4.1)_4$ and $(4.1)_1$, we obtain

$$\begin{aligned} m'(t) &= \int_{-a(t)}^{a(t)} \rho_t dx + \rho(a(t), t)a'(t) - \rho(-a(t), t)(-a'(t)) \\ &= \int_{-a(t)}^{a(t)} \rho_t dx = -\int_{-a(t)}^{a(t)} (\rho u)_x dx = 0, \end{aligned} \tag{4.4}$$

which implies that

$$m(t) = m(0). \tag{4.5}$$

Similarly, we have

$$I'(t) = \int_{-a(t)}^{a(t)} x^2 \rho_t dx = -\int_{-a(t)}^{a(t)} x^2 (\rho u)_x dx = 2\int_{-a(t)}^{a(t)} x\rho u dx = 2F(t). \tag{4.6}$$

In view of (1.8), $(4.1)_2$, $(4.1)_4$, integration by parts and (1.9), one has

$$\begin{aligned} F'(t) &= \int_{-a(t)}^{a(t)} x(\rho u)_t = -\int_{-a(t)}^{a(t)} x(\rho u^2)_x dx + \int_{-a(t)}^{a(t)} x^2 \rho dx \\ &= \int_{-a(t)}^{a(t)} \rho u^2 dx + \int_{-a(t)}^{a(t)} x^2 \rho dx = 2E_k(t) + I(t). \end{aligned} \tag{4.7}$$

We multiply $(4.1)_1$ and $(4.1)_2$ by $-\frac{1}{2}u^2$ and u, respectively, and sum up the two resultant equations to have

$$\frac{1}{2}(\rho u^2)_t + \frac{1}{2}(\rho u^3)_x = x\rho u. \tag{4.8}$$

Integrating (4.8) over $[-a(t), a(t)]$ and using (1.9), $(4.1)_4$ and (1.8), we get

$$E_k'(t) = F(t). \tag{4.9}$$

By (4.7), (4.9) and (4.6) we know that

$$F''(t) = 2E_k'(t) + I'(t) = 2F(t) + 2F(t) = 4F(t), \tag{4.10}$$

which can be solved as

$$F(t) = c_1 e^{2t} + c_2 e^{-2t}, \tag{4.11}$$

where c_1 and c_2 are two constants to be determined later. In view of (4.6), (4.9) and (4.11), we obtain

$$I(t) = c_1 e^{2t} - c_2 e^{-2t} + c_3, \tag{4.12}$$

$$E_k(t) = \frac{c_1}{2} e^{2t} - \frac{c_2}{2} e^{-2t} + c_4, \tag{4.13}$$

where c_3 and c_4 are two constants to be determined later. We plug (4.11)-(4.13) into (4.7) to have

$$2c_1 e^{2t} - 2c_2 e^{-2t} = c_1 e^{2t} - c_2 e^{-2t} + 2c_4 + c_1 e^{2t} - c_2 e^{-2t} + c_3, \tag{4.14}$$

which implies that

$$c_3 = -2c_4. \tag{4.15}$$

Let $t = 0$ in (4.11)-(4.13), then we get

$$\begin{cases} F(0) = c_1 + c_2, \\ I(0) = c_1 - c_2 + c_3, \\ E_k(0) = \frac{c_1}{2} - \frac{c_2}{2} + c_4, \end{cases} \tag{4.16}$$

which together with (4.15) lead to

$$\begin{cases} c_1 = \frac{I(0)}{4} + \frac{E_k(0)+F(0)}{2}, \\ c_2 = \frac{F(0)-E_k(0)}{2} - \frac{I(0)}{4}, \\ c_3 = \frac{I(0)}{2} - E_k(0), \\ c_4 = \frac{2E_k(0)-I(0)}{4}. \end{cases} \tag{4.17}$$

By (4.12) and (4.17), one has

$$I(t) = \left[\frac{I(0)}{4} + \frac{E_k(0) + F(0)}{2}\right] e^{2t} - \left[\frac{F(0) - E_k(0)}{2} - \frac{I(0)}{4}\right] e^{-2t} + \frac{I(0)}{2} - E_k(0). \tag{4.18}$$

In view of (1.7), (1.6) and (4.5), we obtain

$$I(t) \leq a(t)^2 \int_{-a(t)}^{a(t)} \rho dx = a(t)^2 m(t) = m(0)a(t)^2, \tag{4.19}$$

which together with (4.18) lead to (4.3). The proof of Theorem 4.1 is finished. □

Remark 4.1 The inequality (4.3) in Theorem 4.1 means that the moving domain occupied by mass for the problem (4.1) grows exponentially in time.

Remark 4.2 We explain the reasonableness of the condition (4.2) in Theorem 4.1. On one hand, by the Schwarz inequality, it holds

$$F(0)^2 = \left(\int_{-a(0)}^{a(0)} x\rho_0 u_0 dx\right)^2 \leq \int_{-a(0)}^{a(0)} x^2 \rho_0 dx \cdot \int_{-a(0)}^{a(0)} \rho_0 u_0^2 dx = 2I(0)E_k(0), \tag{4.20}$$

where the equation holds if and only if $kx\sqrt{\rho_0} = \sqrt{\rho_0}u_0$, i.e., $u_0 = kx$ for some constant k. On the other hand, by the Cauchy inequality, one has

$$\frac{I(0)}{4} + \frac{E_k(0)}{2} \geq \frac{\sqrt{2I(0)E_k(0)}}{2}, \tag{4.21}$$

where the equation holds if and only if $I(0) = 2E_k(0)$. In view of (4.20) and (4.21), we obtain

$$\frac{I(0)}{4} + \frac{E_k(0)}{2} \geq \frac{|F(0)|}{2},$$

which leads to

$$\frac{I(0)}{4} + \frac{E_k(0) + F(0)}{2} \geq 0. \tag{4.22}$$

So the condition (4.2) always holds except the case that all the three conditions $I(0) = 2E_k(0)$, $F(0) < 0$ and $u_0 = kx$ hold, where k is a constant.

Remark 4.3 From the proof of Theorem 4.1, we can obtain the exact formulations of $F(t)$ and $E_k(t)$ for the problem (4.1)

$$F(t) = \left[\frac{I(0)}{4} + \frac{E_k(0) + F(0)}{2}\right] e^{2t} - \left[\frac{F(0) - E_k(0)}{2} - \frac{I(0)}{4}\right] e^{-2t}, \tag{4.23}$$

$$E_k(t) = \left[\frac{I(0)}{8} + \frac{E_k(0) + F(0)}{4}\right] e^{2t} - \left[\frac{F(0) - E_k(0)}{4} - \frac{I(0)}{8}\right] e^{-2t} + \frac{2E_k(0) - I(0)}{4}. \tag{4.24}$$

§5 Radially symmetric multi-dimensional case

The multi-dimensional case of (1.1)-(1.2) is as follows

$$\rho_t + \mathrm{div}(\rho\overrightarrow{u}) = 0, \tag{5.1}$$

$$(\rho\overrightarrow{u})_t + \mathrm{div}(\rho\overrightarrow{u}\otimes\overrightarrow{u}) = \rho\overrightarrow{x}, \tag{5.2}$$

where $\overrightarrow{u} = (u_1, u_2, \cdots, u_N)$ and $\overrightarrow{x} = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$ $(N \geq 2)$. Let $r = |\overrightarrow{x}|$, $\rho(\overrightarrow{x}, t) = \rho(r,t)$ and $\overrightarrow{u}(\overrightarrow{x}, t) = u(r,t)\frac{\overrightarrow{x}}{r}$. Then, the system (5.1)-(5.2) becomes

$$\rho_t + (\rho u)_r + \frac{N-1}{r}\rho u = 0, \tag{5.3}$$

$$(\rho u)_t + (\rho u^2)_r + \frac{N-1}{r}\rho u^2 = r\rho. \tag{5.4}$$

When $\rho(r,t) > 0$, by (5.3) we know that (5.4) is equivalent to

$$u_t + uu_r = r. \tag{5.5}$$

Using the form (1.10) and performing the similar procedures to the proof of Theorem 2.1, we can easily obtain the following theorem.

Theorem 5.1 For system (5.3)-(5.4) and $\rho|_{t=0} = \rho_0(r), u|_{t=0} = u_0(r) \in C^1$ and $\rho_0(r) > 0$, there exists solutions of the form

$$\rho(r,t) = \frac{f(\frac{r}{a(t)})}{a(t)^N}, \quad u(r,t) = \frac{a'(t)}{a(t)}r, \quad a(t) = \frac{a_0+a_1}{2}e^t + \frac{a_0-a_1}{2}e^{-t}, \tag{5.6}$$

where $f \geq 0 \in C^1$, $a_0 > 0$ and a_1 are two constants satisfying $\rho_0(r) = \frac{f(\frac{r}{a_0})}{a_0^N}$, $u_0(r) = \frac{a_1}{a_0}r$. Moreover, we have the following large-time behavior of the solutions

Case (i) If $a_0+a_1 > 0$, then $\lim\limits_{t\to+\infty} a(t) = +\infty$, and for any fixed $r \geq 0$, one has $\lim\limits_{t\to+\infty}\rho(r,t) = 0$ and $\lim\limits_{t\to+\infty} u(r,t) = r$;

Case (ii) If $a_0 + a_1 = 0$, then $\lim\limits_{t\to+\infty} a(t) = 0$, and if we further assume that $\rho_0(r) \geq C > 0$ for some constant C, then we have $\lim\limits_{t\to+\infty}\rho(r,t) = +\infty$ and $\lim\limits_{t\to+\infty} u(r,t) = -r$ for any $r \geq 0$;

Case (iii) If $a_0 + a_1 < 0$, then $\lim\limits_{t\to t_0} a(t) = 0$, and if we further assume that $\rho_0(r) \geq C > 0$ for some constant C, then we have $\lim\limits_{t\to t_0}\rho(r,t) = +\infty$ for any $r \geq 0$, where $t_0 = \frac{1}{2}\ln\frac{a_1-a_0}{a_1+a_0}$.

In the following we study the spreading of the moving domain occupied by mass for system (5.3)-(5.4), and the corresponding problem is

$$\begin{cases} \rho_t + (\rho u)_r + \frac{N-1}{r}\rho u = 0, \quad r \in (0, a(t)), \\ (\rho u)_t + (\rho u^2)_r + \frac{N-1}{r}\rho u^2 = r\rho, \quad r \in (0, a(t)), \\ (\rho, u)|_{t=0} = (\rho_0(r), u_0(r)), \quad \mathrm{supp}\rho_0(r) \subset \{\overrightarrow{x} \in \mathbb{R}^N : |\overrightarrow{x}| < a_0\}), \\ \rho(a(t), t) = 0, \quad u(0,t) = 0, \end{cases} \tag{5.7}$$

where $\{\overrightarrow{x} \in \mathbb{R}^N : |\overrightarrow{x}| < a(t)\}$ is the moving domain occupied by mass and $a_0 = a(0) > 0$ is a constant. Obviously, (5.7) can be satisfied if we take $\rho(r,t) = \frac{1-\frac{r^2}{a(t)^2}}{a(t)^N}$ and $u(r,t) = \frac{a'(t)}{a(t)}r$, where $a(t) = \frac{a_0+a_1}{2}e^t + \frac{a_0-a_1}{2}e^{-t}$, $a_0 > 0$ and a_1 are two constants satisfying $\rho_0(r) = \frac{1-\frac{r^2}{a_0^2}}{a_0^N}$, $u_0(r) = \frac{a_1}{a_0}r$. Similar to Theorem 4.1, by using the averaged quantities (1.11)-(1.14), we can obtain the following theorem.

Theorem 5.2 Assume that $0 < m_1(0), I_1(0), E_{1k}(0) < +\infty$ and

$$\frac{I_1(0)}{4} + \frac{E_{1k}(0) + F_1(0)}{2} > 0. \tag{5.8}$$

Let (ρ, u) be a C^1 solution of (5.7). Then, we have

$$a(t) \geq \frac{1}{\sqrt{m_1(0)}}$$

$$\times \left\{ \left[\frac{I_1(0)}{4} + \frac{E_{1k}(0) + F_1(0)}{2} \right] e^{2t} - \left[\frac{F_1(0) - E_{1k}(0)}{2} - \frac{I_1(0)}{4} \right] e^{-2t} + \frac{I_1(0)}{2} - E_{1k}(0) \right\}^{\frac{1}{2}}. \tag{5.9}$$

Proof. By (1.11), $(5.7)_4$ and $(5.7)_1$, we obtain

$$m_1'(t) = \int_0^{a(t)} \rho_t r^{N-1} dr = -\int_0^{a(t)} (\rho u r^{N-1})_r dr = 0, \tag{5.10}$$

which implies that

$$m_1(t) = m_1(0). \tag{5.11}$$

Similarly,

$$\begin{aligned} I_1'(t) &= \int_0^{a(t)} \rho_t r^{N+1} dr = -\int_0^{a(t)} (\rho u)_r r^{N+1} dr - (N-1) \int_0^{a(t)} \rho u r^N dr \\ &= 2 \int_0^{a(t)} \rho u r^N dr = 2F_1(t). \end{aligned} \tag{5.12}$$

In view of (1.13), $(5.7)_2$, $(5.7)_4$, integration by parts and (1.14), one has

$$\begin{aligned} F_1'(t) &= \int_0^{a(t)} (\rho u)_t \cdot r^N dr = -\int_0^{a(t)} (\rho u^2)_r \cdot r^N dr - (N-1) \int_0^{a(t)} \rho u^2 r^{N-1} dr + \int_0^{a(t)} \rho r^{N+1} dr \\ &= \int_0^{a(t)} \rho u^2 r^{N-1} dr + \int_0^{a(t)} \rho r^{N+1} dr = 2E_{1k}(t) + I_1(t). \end{aligned} \tag{5.13}$$

We multiply $(5.7)_1$ and $(5.7)_2$ by $-\frac{1}{2}u^2 r^{N-1}$ and ur^{N-1}, respectively, and sum up the two resultant equations to have

$$\frac{1}{2}(\rho u^2)_t \cdot r^{N-1} + \frac{1}{2}(\rho u^3 r^{N-1})_r = \rho u r^N. \tag{5.14}$$

Integrating (5.14) over $[0, a(t)]$ and using (1.14) and $(5.7)_4$, we get

$$E_{1k}'(t) = F_1(t). \tag{5.15}$$

It follows from (1.12), (1.11) and (5.11) that

$$I_1(t) \leq a(t)^2 \int_0^{a(t)} \rho r^{N-1} dr = a(t)^2 m_1(t) = m_1(0) a(t)^2. \tag{5.16}$$

We omit the rest of the proof because it is the same as the one of Theorem 4.1. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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