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On the synchronizable system by groups and the generalized synchronizable system

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Abstract. In this paper, the synchronizable system by groups and the generalized synchronizable system are studied for a coupled system of wave equations. Moreover, situations possessing different groupings are also discussed.

§1 Introduction

In the researches of the exact boundary synchronization for hyperbolic systems (see [2]–[3]), Lei et al. [1] defined and studied the synchronizable system, which has synchronization solutions in the absence of boundary controls. Then for the synchronizable system, the exact boundary synchronization can be further investigated. Li et al. [4]–[5] extended this work to the partially synchronizable system. This paper aims to study the corresponding synchronizable system by groups and the generalized synchronizable system, respectively.

Consider the following coupled system of wave equations with homogeneous Dirichlet boundary condition

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma, \end{cases} \tag{1.1}$$

in which, $U = (u^{(1)}, \dots, u^{(N)})^{\mathrm{T}}$ denotes the state variable, " $'$ " is the partial derivative with respect to the time variable t, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator with respect to the space variable x, $A = (a_{ij})$ is the coupled matrix of order N with constant components, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary Γ. The initial data is given by

$$t = 0: \ (U, U') = (U_0(x), U_1(x)). \tag{1.2}$$

Correspondingly, we will consider the following coupled system of wave equations with

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Dirichlet boundary controls

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0,+\infty)\times\Omega, \\ U = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ U = DH & \text{on } (0,+\infty)\times\Gamma_1, \end{cases} \tag{1.3}$$

in which, $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\mathrm{mes}(\Gamma_1) \neq 0$, $H = (h^{(1)}, \ldots, h^{(M)})^{\mathrm{T}}$ is the boundary control ($M \leq N$), D is the boundary control matrix of order $N \times M$ with full column-rank and with constant components.

In what follows, we first recall in Section 2 the properties of system (1.1) to be a synchronizable system, and the application on the exact boundary synchronization for the corresponding system (1.3) with Dirichlet boundary controls. Then we consider the synchronizable system by groups and the generalized synchronizable system in Section 3 and 5, respectively. In particular, we discuss furthermore in Section 4 the situations that system (1.1) possesses different groupings.

§2 Synchronizable system

Definition 2.1 (cf. [1]). System (1.1) is a synchronizable system if there exists an initial data $(U_0(x), U_1(x))$ such that the solution $U = U(t,x)$ to the corresponding problem (1.1)–(1.2) is a synchronization solution

$$u^{(1)}(t,x) \equiv \cdots \equiv u^{(N)}(t,x) \triangleq u(t,x), \tag{2.1}$$

where $u(t,x)$ is called the synchronizable state, and $u(t,x) \not\equiv 0$.

The initial data (1.2) of a synchronization solution must be the synchronization one, that is, $U_0(x) = (u_0^{(1)}(x), \ldots, u_0^{(N)}(x))^{\mathrm{T}}$ and $U_1(x) = (u_1^{(1)}(x), \ldots, u_1^{(N)}(x))^{\mathrm{T}}$ must satisfy

$$u_0^{(1)} \equiv \cdots \equiv u_0^{(N)} \triangleq \hat{u}_0,\ u_1^{(1)} \equiv \cdots \equiv u_1^{(N)} \triangleq \hat{u}_1. \tag{2.2}$$

For non-trivial synchronization solution, it is required that $(U_0(x), U_1(x)) \not\equiv (0,0)$.

Let $e = (1, \ldots, 1)^{\mathrm{T}}$ be the synchronization basis, and C_1 be the corresponding synchronization matrix of order $(N-1) \times N$ with full row-rank: $\mathrm{Ker}(C_1) = \mathrm{Span}\{e\}$. Then the synchronization (2.1) can be rewritten as

$$C_1 U(t,x) \equiv 0, \tag{2.3}$$

and for the synchronizable system, there exists a synchronizable state $u(t,x)$ ($\not\equiv 0$) such that

$$U(t,x) = u(t,x)e. \tag{2.4}$$

Theorem 2.1 (cf. [1]). *System* (1.1) *is a synchronizable system if and only if the coupling matrix A satisfies the following condition of C_1-compatibility*

$$A\mathrm{Ker}(C_1) \subseteq \mathrm{Ker}(C_1), \tag{2.5}$$

or equivalently, there exists a constant a such that $Ae = ae$, that is, the following row-sum condition holds

$$\sum_{j=1}^{N} a_{ij} = a, \ i = 1, \dots, N,$$

where a is independent of $i = 1, \dots, N$.

Then the synchronizable state $u = u(t,x)$ satisfies

$$\begin{cases} u'' - \Delta u + au = 0 & \text{in } (0,+\infty) \times \Omega, \\ u = 0 & \text{on } (0,+\infty) \times \Gamma, \end{cases} \tag{2.6}$$

with the initial data

$$t = 0: \ (u, u') = (\hat{u}_0, \hat{u}_1), \tag{2.7}$$

where $(\hat{u}_0, \hat{u}_1)$ is given by (2.2), and $(\hat{u}_0, \hat{u}_1) \not\equiv (0,0)$ for non-trivial synchronizable state.

Since the condition of C_1-compatibility (2.5) is independent of the initial data, we have

Theorem 2.2 (cf. [1]). *If system (1.1) is a synchronizable system, then it always possesses a synchronization solution for any given synchronization initial data.*

For system (1.3) with Dirichlet boundary controls, by definition, it is exactly synchronizable (see [3]) if there exists $T > 0$, such that for any given initial data $(U_0, U_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, there exists a boundary control $H \in L^2_{\text{loc}}(0, +\infty; (L^2(\Gamma_1))^M)$ with compact support in $[0, T]$, such that the solution $U = U(t,x)$ to problem (1.3) and (1.2) satisfies the synchronization requirement

$$t \geq T: \ C_1 U(t,x) \equiv 0, \ x \in \Omega. \tag{2.8}$$

Thus by Theorem 2.2, we have

Corollary 2.3 (cf. [1]). *If system (1.1) is a synchronizable system, then the exact boundary synchronization (2.8) of system (1.3) is equivalent to that for any given initial data $(U_0, U_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, there exists a boundary control H, such that the corresponding solution $U = U(t,x)$ attains a synchronizable state at time $t = T$, that is,*

$$C_1(U(T,x), U'(T,x)) \equiv (0,0), \ x \in \Omega. \tag{2.9}$$

§3 Synchronizable system by groups

Correspondingly, the synchronizable system by p groups can be defined to be the system possessing synchronization solutions by p linearly independent groups. Let $n_0, n_1, \dots, n_p$ be a series of integers satisfying

$$0 = n_0 < n_1 < \cdots < n_p = N. \tag{3.1}$$

After a suitable arrangement of the components of the state variable, we can give the following

Definition 3.1. System (1.1) is a synchronizable system by p groups if there exists an initial data $(U_0(x), U_1(x))$ such that the solution $U = U(t,x)$ to problem (1.1)–(1.2) is a synchronization solution by p groups

$$\begin{cases} u^{(1)}(t,x) \equiv \cdots \equiv u^{(n_1)}(t,x) \triangleq u_1(t,x), \\ u^{(n_1+1)}(t,x) \equiv \cdots \equiv u^{(n_2)}(t,x) \triangleq u_2(t,x), \\ \cdots \\ u^{(n_{p-1}+1)}(t,x) \equiv \cdots \equiv u^{(n_p)}(t,x) \triangleq u_p(t,x), \end{cases} \tag{3.2}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), \dots, u_p(t,x))^{\mathrm{T}}$ is called the synchronizable state by p groups, and the p components of $u(t,x)$ are linearly independent.

Clearly, the initial data of a synchronization solution by groups must be the corresponding synchronization initial data by groups, that is, $U_0(x) = (u_0^{(1)}(x), \dots, u_0^{(N)}(x))^{\mathrm{T}}$ and $U_1(x) = (u_1^{(1)}(x), \dots, u_1^{(N)}(x))^{\mathrm{T}}$ must satisfy

$$\begin{cases} u_i^{(1)}(x) \equiv \cdots \equiv u_i^{(n_1)}(x) \triangleq u_{i1}(x), \\ u_i^{(n_1+1)}(x) \equiv \cdots \equiv u_i^{(n_2)}(x) \triangleq u_{i2}(x), \\ \cdots \\ u_i^{(n_{p-1}+1)}(x) \equiv \cdots \equiv u_i^{(n_p)}(x) \triangleq u_{ip}(x) \end{cases} \tag{3.3}$$

for $i = 0, 1$, respectively.

Let $\{e_1, \dots, e_p\}$ be the synchronization basis by p groups, defined by

$$(e_k)_i = \begin{cases} 1, & n_{k-1} + 1 \le i \le n_k, \\ 0, & \text{others} \end{cases} \tag{3.4}$$

for $k = 1, \dots, p$. And let C_p be the synchronization matrix by groups, a full row-rank matrix of order $(N-p) \times N$, determined by

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \dots, e_p\}. \tag{3.5}$$

Then, the requirement (3.2) for the synchronization by groups can be rewritten as

$$C_p U(t,x) \equiv 0, \tag{3.6}$$

and for the synchronizable system by p groups, the corresponding requirement becomes that there exists a synchronizable state by p groups $u(t,x) = (u_1(t,x), \dots, u_p(t,x))^{\mathrm{T}}$ with p linearly independent components, such that

$$U(t,x) = u_1(t,x)e_1 + \cdots + u_p(t,x)e_p = (e_1, \dots, e_p)u(t,x). \tag{3.7}$$

Theorem 3.1. *System* (1.1) *is a synchronizable system by p groups if and only if the coupling matrix A satisfies the following condition of C_p-compatibility*

$$A\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p), \tag{3.8}$$

namely, there exists a square matrix $\tilde{A}_p$ of order p, such that the following row-sum condition by blocs holds

$$A(e_1, \dots, e_p) = (e_1, \dots, e_p)\tilde{A}_p, \tag{3.9}$$

where $e_1, \dots, e_p$ and C_p are given by (3.4) *and* (3.5), *respectively.*

Then, the synchronizable state by p groups $u = u(t,x) = (u_1(t,x), \dots, u_p(t,x))^{\mathrm{T}}$ satisfies

$$\begin{cases} u'' - \Delta u + \tilde{A}_p u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \Gamma, \end{cases} \tag{3.10}$$

with the initial data

$$t=0: \ u=(u_{01},\dots,u_{0p})^{\mathrm{T}}, \ u'=(u_{11},\dots,u_{1p})^{\mathrm{T}}, \tag{3.11}$$

given by (3.3).

Proof. Assume that $U=U(t,x)$ is a synchronization solution by p linearly independent groups to system (1.1), satisfying (3.6). Multiplying the equations in (1.1) by C_p from the left, we have

$$C_pAU=0,$$

then by (3.7), we get

$$C_pAU(t,x)=C_pAe_1u_1(t,x)+\cdots+C_pAe_pu_p(t,x)=0.$$

By assumption that $u_1(t,x),\dots,u_p(t,x)$ are linearly independent, we get

$$C_pAe_1=\cdots=C_pAe_p=0,$$

then from (3.5), we obtain the condition of C_p-compatibility (3.8).

Plugging $U=(e_1,\dots,e_p)u$ into (1.1), and noting the condition of C_p-compatibility (3.9), we get (3.10). Moreover, the initial condition (3.11) can be obtained by (3.3).

On the other hand, if A satisfies the condition of C_p-compatibility (3.8), namely, (3.9), we can take $u=u(t,x)$ satisfying (3.10) with $(u_{01},\dots,u_{0p})^{\mathrm{T}}$ being linearly independent, then the components of $u(t,x)$ are linearly independent. Therefore, $U=U(t,x)=(e_1,\dots,e_p)u(t,x)$ is a synchronization solution by p linearly independent groups to system (1.1). □

Remark 3.2. For the synchronizable system, the initial data of a non-trivial synchronizable state are not zeros. However, for the synchronizable system by p groups, the linear independence for the p components $u_1(t,x),\dots,u_p(t,x)$ of the synchronizable state $u(t,x)$ by p groups on the whole solvable domain does not imply the linear independence for their initial data $(u_{01},u_{11}),\dots,(u_{0p},u_{1p})$. On the contrary, the example below shows that their initial data can be linearly dependent.

Example 3.3. Let $u(t,x)=(u_1(t,x),u_2(t,x))^{\mathrm{T}}$ be the synchronizable state by 2 groups. Assume that $u_1(t,x)$ and $u_2(t,x)$ satisfy

$$\begin{cases} u_1''-\Delta u_1=0 & \text{in } (0,+\infty)\times\Omega,\\ u_2''-\Delta u_2+u_2=0 & \text{in } (0,+\infty)\times\Omega,\\ u_1=u_2=0 & \text{on } (0,+\infty)\times\Gamma \end{cases} \tag{3.12}$$

with the same initial data

$$t=0: \ u_1=u_2\triangleq\hat{u}_0, \ u_1'=u_2'\triangleq\hat{u}_1, \tag{3.13}$$

where $(\hat{u}_0,\hat{u}_1)\neq(0,0)$. Then we still have that $u_1(t,x)$ and $u_2(t,x)$ are linearly independent.

In fact, if $u_1(t,x)$ and $u_2(t,x)$ are linearly dependent, noting that they have the same initial data, we have $u_1(t,x)\equiv u_2(t,x)$. Then by (3.12) we have $u_1(t,x)\equiv u_2(t,x)\equiv 0$, and this contradicts to $(\hat{u}_0,\hat{u}_1)\neq(0,0)$. □

Noting that the condition of C_p-compatibility (3.8) is independent of the initial data, from Theorem 3.1 we have

Theorem 3.2. *If system* (1.1) *is a synchronizable system by p groups, then it always possesses a corresponding synchronization solution by p groups for any given synchronization initial data* (3.3) *by p groups.*

Thus, for a given grouping, the synchronizable system by groups can be equivalently defined as the system that always possesses a synchronization solution by groups for any given synchronization initial data by groups.

For the corresponding system (1.3) with Dirichlet boundary controls, the exact boundary synchronization by p groups (see [3]) means that there exists a moment $T > 0$, such that for any given initial data $(U_0, U_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, there exists a boundary control $H \in L^2_{\text{loc}}(0, +\infty; (L^2(\Gamma_1))^M)$ with compact support in $[0, T]$, such that the solution $U = U(t, x)$ to the problem (1.3) and (1.2) satisfies the following synchronization by p groups requirement

$$t \geq T: \ C_p U(t, x) \equiv 0, \ x \in \Omega. \tag{3.14}$$

Then by Theorem 3.2 we have

Corollary 3.3. *If system* (1.1) *is a synchronizable system by p groups, then the exact boundary synchronization by p groups* (2.8) *of system* (1.3) *is equivalent to that for any given initial data $(U_0, U_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, there exists a boundary control H, such that the solution $U = U(t, x)$ satisfies at time $t = T$,*

$$C_p(U(T, x), U'(T, x)) \equiv (0, 0), \ x \in \Omega, \tag{3.15}$$

that is, there exists $(u(T, x), u'(T, x)) \in (L^2(\Omega) \times H^{-1}(\Omega))^p$ such that

$$(U(T, x), U'(T, x)) = (e_1, \ldots, e_p)(u(T, x), u'(T, x)), \ x \in \Omega. \tag{3.16}$$

§4 Synchronizable systems by groups with different groupings

Different from the synchronizable system, for the synchronizable system by groups, since there are many ways of groupings, we can get different kinds of synchronization solutions by groups.

Example 4.1. Let $N = 8$, and state variable $U = (u^{(1)}, \ldots, u^{(8)})^{\mathrm{T}}$. The synchronization solution by 3 groups to system (1.1) can be

$$u^{(1)} \equiv u^{(2)}, \ u^{(3)} \equiv u^{(4)} \equiv u^{(5)}, \ u^{(6)} \equiv u^{(7)} \equiv u^{(8)}, \tag{4.1}$$

corresponding to the synchronization basis by 3 groups

$$e_1 = (1,1,0,0,0,0,0,0)^{\mathrm{T}}, \ e_2 = (0,0,1,1,1,0,0,0)^{\mathrm{T}}, \ e_3 = (0,0,0,0,0,1,1,1)^{\mathrm{T}};$$

On the other hand, it can also be

$$u^{(1)} \equiv u^{(2)}, \ u^{(3)} \equiv u^{(5)}, \ u^{(4)} \equiv u^{(6)} \equiv u^{(7)} \equiv u^{(8)}, \tag{4.2}$$

corresponding to the synchronization basis by 3 groups

$$\hat{e}_1 = (1,1,0,0,0,0,0,0)^{\mathrm{T}}, \ \hat{e}_2 = (0,0,1,0,1,0,0,0)^{\mathrm{T}}, \ \hat{e}_3 = (0,0,0,1,0,1,1,1)^{\mathrm{T}}.$$

□

Definition 4.2. $\{e_1, \ldots, e_p\}$ is called as a synchronization basis by p groups if each component of e_k $(k = 1, \ldots, p)$ is either 0 or 1, and $e_1 + \cdots + e_p = (1, \ldots, 1)^{\mathrm{T}}$.

Clearly, the aforementioned results, true for $\{e_1,\dots,e_p\}$ given by (3.4), still hold for a general synchronization basis by p groups $\{e_1,\dots,e_p\}$.

Then for the synchronizable system (1.1) by 2 groups, we have

Theorem 4.1. *If system* (1.1) *is a synchronizable system by* 2 *groups under two different groupings, then it is a synchronizable system.*

Proof. Assume that system (1.1) is a synchronizable system by 2 groups with respect to $\{e_1,e_2\}$ and with respect to $\{\hat{e}_1,\hat{e}_2\}$, and

$$\mathrm{Span}\{e_1,e_2\}\neq\mathrm{Span}\{\hat{e}_1,\hat{e}_2\}.$$

It is easy to see that

$$\mathrm{Span}\{e_1,e_2\}\cap\mathrm{Span}\{\hat{e}_1,\hat{e}_2\}=\mathrm{Span}\{e\},$$

where $e=(1,\dots,1)^{\mathrm{T}}$. By Theorem 3.1, both $\mathrm{Span}\{e_1,e_2\}$ and $\mathrm{Span}\{\hat{e}_1,\hat{e}_2\}$ are invariant for A, thus $\mathrm{Span}\{e\}$ is also invariant for A, that is, A satisfies the condition of C_1-compatibility (2.5), then system (1.1) is a synchronizable system. □

For general cases of groupings, we have

Theorem 4.2. *If system* (1.1) *is a synchronizable system by p groups with respect to $\{e_1,\dots,e_p\}$ and a synchronizable system by $\hat{p}$ groups with respect to $\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\}$, then it is a synchronizable system by q groups with respect to $\{e_1^*,\dots,e_q^*\}$, where $\{e_1^*,\dots,e_q^*\}$ is a synchronization basis by q groups, determined by*

$$\mathrm{Span}\{e_1^*,\dots,e_q^*\}=\mathrm{Span}\{e_1,\dots,e_p\}\cap\mathrm{Span}\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\},\tag{4.3}$$

and $1\le q\le\min(p,\hat{p})$.

Proof. Since system (1.1) is a synchronizable system by p groups with respect to $\{e_1,\dots,e_p\}$ and a synchronizable system by $\hat{p}$ groups with respect to $\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\}$, by Theorem 3.1, both $\mathrm{Span}\{e_1,\dots,e_p\}$ and $\mathrm{Span}\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\}$ are invariant for A, thus $\mathrm{Span}\{e_1^*,\dots,e_q^*\}$ given by (4.3) is also invariant for A. In what follows, we need to show that (4.3) indeed determines a synchronization basis by groups, namely, $\{e_1^*,\dots,e_q^*\}$ is a series of non-zero vectors consisted of 0 and 1, such that $e_1^*+\cdots+e_q^*=(1,\dots,1)^{\mathrm{T}}$, therefore system (1.1) is a synchronizable system by q groups with respect to $\{e_1^*,\dots,e_q^*\}$.

We apply the induction with respect to $\hat{p}$.

For $\hat{p}=1$, obviously, $\mathrm{Span}\{e_1,\dots,e_p\}\cap\mathrm{Span}\{e\}=\mathrm{Span}\{e\}$, in which $e=(1,\dots,1)^{\mathrm{T}}$ is a synchronization basis.

Supposing that it is true for $\hat{p}-1$, we now consider the case for $\hat{p}$.

If $\mathrm{Span}\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\}\subseteq\mathrm{Span}\{e_1,\dots,e_p\}$, then $\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\}$ is a synchronization basis by groups, determined by (4.3).

If not, then there exists a $\xi=(\xi_1,\dots,\xi_N)^{\mathrm{T}}\in\mathrm{Span}\{\hat{e}_1,\dots,\hat{e}_{\hat{p}}\}$, namely,

$$\xi_i=\xi_j,\ \forall(i,j)\in\hat{R},$$

where $\hat{R}=\{(i,j):(\hat{e}_k)_i=(\hat{e}_k)_j=1,\ 1\le k\le\hat{p},\ 1\le i,j\le N\}$, but $\xi\notin\mathrm{Span}\{e_1,\dots,e_p\}$, that is, there exist $i_0\neq j_0$,

$$(i_0,j_0)\in R,$$

where $R = \{(i,j) : (e_k)_i = (e_k)_j = 1,\ 1 \le k \le p,\ 1 \le i,j \le N\}$, such that $\xi_{i_0} \ne \xi_{j_0}$. Hence

$$(i_0, j_0) \notin \hat{R}.$$

Without loss of generality, assume that $(e_1)_{i_0} = (e_1)_{j_0} = 1$, while $(\hat{e}_1)_{i_0} = 1$, $(\hat{e}_2)_{j_0} = 1$. Then we can easily get

$$\mathrm{Span}\{e_1, \dots, e_p\} \cap \mathrm{Span}\{\hat{e}_1, \dots, \hat{e}_{\hat{p}}\} = \mathrm{Span}\{e_1, \dots, e_p\} \cap \mathrm{Span}\{\hat{e}_1 + \hat{e}_2, \hat{e}_3, \dots, \hat{e}_{\hat{p}}\},$$

in which $\{\hat{e}_1 + \hat{e}_2, \hat{e}_3, \dots, \hat{e}_{\hat{p}}\}$ is a synchronization basis by $(\hat{p} - 1)$ groups. By the induction assumption, the right-hand side of the above formula determines a synchronization basis by groups, then the left-hand side determines a synchronization basis by groups. Hence the conclusion is also true for $\hat{p}$.

Clearly, $1 \le q \le \min(p, \hat{p})$. In fact, (4.3) implies that $q \le \min(p, \hat{p})$, and $e \in \mathrm{Span}\{e_1^*, \dots, e_q^*\}$ shows that $q \ge 1$.

The proof is complete. $\square$

The proof shows that the grouping corresponding to the synchronization basis by groups $\{e_1^*, \dots, e_q^*\}$ given by (4.3) is a combination of the original two groupings. For instance, in Example 4.1,

$$\mathrm{Span}\{e_1, e_2, e_3\} \cap \mathrm{Span}\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \mathrm{Span}\{e_1^*, e_2^*\},$$

where

$$e_1^* = (1,1,0,0,0,0,0,0)^{\mathrm{T}},\ e_2^* = (0,0,1,1,1,1,1,1)^{\mathrm{T}},$$

then it follows from the synchronization (4.1) by 3 groups with respect to $\{e_1, e_2, e_3\}$ and the synchronization (4.2) by 3 groups with respect to $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ that system (1.1) possesses the synchronization by 2 groups with respect to $\{e_1^*, e_2^*\}$,

$$u^{(1)} \equiv u^{(2)},\ u^{(3)} \equiv u^{(4)} \equiv u^{(5)} \equiv u^{(6)} \equiv u^{(7)} \equiv u^{(8)}.$$

More generally, we have

Corollary 4.3. *If system* (1.1) *is a synchronizable system by p_k groups with respect to $\{e_1^{(k)}, \dots, e_{p_k}^{(k)}\}$ for all $k = 1, \dots, K$, then it is a synchronizable system by q groups with respect to $\{e_1^*, \dots, e_q^*\}$, where $\{e_1^*, \dots, e_q^*\}$ is a synchronization basis by groups determined by*

$$\mathrm{Span}\{e_1^*, \dots, e_q^*\} = \bigcap_{1 \le k \le K} \mathrm{Span}\{e_1^{(k)}, \dots, e_{p_k}^{(k)}\}, \tag{4.4}$$

and $1 \le q \le \min(p_1, \dots, p_K)$.

§5 Generalized synchronizable system

Similarly to the synchronizable system by groups, we can define the generalized synchronizable system as the system possessing generalized synchronization solutions by p linearly independent groups.

According to [6], denote $\Theta_p \in \mathbb{M}^{(N-p)\times N}(\mathbb{R})$ $(0 \le p < N)$ as the generalized synchronization matrix that is a full row-rank matrix, and denote $\{\epsilon_1, \dots, \epsilon_p\}$ as the corresponding generalized synchronization basis given by

$$\mathrm{Ker}(\Theta_p) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\}. \tag{5.1}$$

Then, for system (1.1), the generalized synchronization solution $U = U(t,x)$ with respect to Θ_p is the solution satisfying

$$\Theta_p U(t,x) \equiv 0, \tag{5.2}$$

that is, there exists $u(t,x) = (u_1(t,x), u_2(t,x), \dots, u_p(t,x))^{\mathrm{T}}$ such that the solution

$$U(t,x) = u_1(t,x)\epsilon_1 + \cdots + u_p(t,x)\epsilon_p = (\epsilon_1, \dots, \epsilon_p)u(t,x), \tag{5.3}$$

which is also called the generalized synchronization solution with respect to $\{\epsilon_1, \dots, \epsilon_p\}$, and $u(t,x)$ is called the corresponding generalized synchronizable state.

Definition 5.1. System (1.1) is a generalized synchronizable system with respect to Θ_p, if there exists an initial data $(U_0(x), U_1(x))$ such that the corresponding solution $U = U(t,x)$ to problem (1.1)–(1.2) is a generalized synchronization solution (5.3) with respect to Θ_p, and the components of the corresponding generalized synchronizable state $u(t,x)$ are linearly independent.

Remark 5.2. Now we explain, from the standpoint of the generalized synchronization, that for the synchronizable system by groups (resp., generalized synchronizable system), the components of the synchronizable state by groups (resp., generalized synchronizable state) are supposed to be linearly independent. In fact, assume that system (1.1) possesses a generalized synchronization solution (5.3) with respect to $\{\epsilon_1, \dots, \epsilon_p\}$. If the components of the generalized synchronizable state $u(t,x) = (u_1(t,x), \dots, u_p(t,x))^{\mathrm{T}}$ are linearly dependent, then let $\tilde{u}_1(t,x), \dots, \tilde{u}_{\tilde{p}}(t,x)$ be the maximal linearly independent ones in $u_1(t,x), \dots, u_p(t,x)$, where $\tilde{p} < p$. There exists a full column-rank matrix Q of order $p \times \tilde{p}$, such that

$$u(t,x) = Q\tilde{u}(t,x),$$

where $\tilde{u}(t,x) = (\tilde{u}_1(t,x), \dots, \tilde{u}_{\tilde{p}}(t,x))^{\mathrm{T}}$. Denote $(\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{\tilde{p}}) = (\epsilon_1, \dots, \epsilon_p)Q$, then the generalized synchronization solution (5.3) with respect to $\{\epsilon_1, \dots, \epsilon_p\}$ turns into

$$U(t,x) = (\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{\tilde{p}})\tilde{u}(t,x),$$

which is actually a generalized synchronization solution with respect to $\{\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{\tilde{p}}\}$, and the components of the corresponding generalized synchronizable state $\tilde{u}(t,x)$ are linearly independent. Thus it is sufficient to consider the case that the components of the generalized synchronizable state $u(t,x)$ are linearly independent (at least for one initial data).

In what follows, we first consider the properties of system (1.1) to be a generalized synchronizable system with respect to Θ_p.

First, the initial data of a generalized synchronization solution with respect to Θ_p must be the generalized synchronization initial data with respect to Θ_p, namely, the initial data $(U_0(x), U_1(x))$ must satisfy

$$\Theta_p(U_0(x), U_1(x)) \equiv 0,$$

that is, there exist $\hat{u}_0(x) = (u_{01}(x), \dots, u_{0p}(x))^{\mathrm{T}}$ and $\hat{u}_1(x) = (u_{11}(x), \dots, u_{1p}(x))^{\mathrm{T}}$ such that

$$(U_0(x), U_1(x)) = (\epsilon_1, \dots, \epsilon_p)(\hat{u}_0(x), \hat{u}_1(x)). \tag{5.4}$$

Then, similarly to Theorem 3.1, we have

Theorem 5.1. *System* (1.1) *is a generalized synchronizable system with respect to* Θ_p *if and only if the coupling matrix* A *satisfies the condition of* Θ_p*-compatibility*

$$A\mathrm{Ker}(\Theta_p) \subseteq \mathrm{Ker}(\Theta_p), \tag{5.5}$$

that is, there exists a square matrix $\tilde{A}_p$ *of order* p*, such that*

$$A(\epsilon_1, \dots, \epsilon_p) = (\epsilon_1, \dots, \epsilon_p)\tilde{A}_p. \tag{5.6}$$

Then the generalized synchronizable state $u = u(t, x)$ *given in* (5.3) *satisfies*

$$\begin{cases} u'' - \Delta u + \tilde{A}_p u = 0 & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{5.7}$$

with the initial data

$$t = 0: \ (u, u') = (\hat{u}_0(x), \hat{u}_1(x)), \tag{5.8}$$

where $(\hat{u}_0(x), \hat{u}_1(x))$ *are given by* (5.4).

Similarly to Theorem 3.2, we have

Theorem 5.2. *If system* (1.1) *is a generalized synchronizable system with respect to* Θ_p*, then it possesses a corresponding generalized synchronization solution for any given generalized synchronization initial data with respect to* Θ_p.

For the corresponding system (1.3) with Dirichlet boundary controls, the generalized exact boundary synchronization with respect to Θ_p (see [6]) means that there exists $T > 0$, such that for any given initial data $(U_0, U_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, there exists a boundary control $H \in L^2_{\mathrm{loc}}(0, +\infty; (L^2(\Gamma_1))^M)$ with compact support in $[0, T]$, such that the solution $U = U(t, x)$ to problem (1.3) and (1.2) satisfies the generalized synchronization requirement

$$t \geq T: \ \Theta_p U(t, x) \equiv 0, \ x \in \Omega. \tag{5.9}$$

Similarly to Corollary 3.3, we have

Corollary 5.3. *If system* (1.1) *is a generalized synchronizable system with respect to* Θ_p*, then the generalized exact boundary synchronization with respect to* Θ_p *for system* (1.3) *is equivalent to that for any given initial data* $(U_0, U_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$*, there exists a boundary control* H*, such that the solution satisfies at time* $t = T$*,*

$$\Theta_p(U(T, x), U'(T, x)) \equiv (0, 0), \ x \in \Omega, \tag{5.10}$$

that is, there exists $(u(T, x), u'(T, x)) \in (L^2(\Omega) \times H^{-1}(\Omega))^p$ *such that*

$$(U(T, x), U'(T, x)) = (\epsilon_1, \dots, \epsilon_p)(u(T, x), u'(T, x)), \ x \in \Omega. \tag{5.11}$$

For the case with different groupings, similarly to Corollary 4.3, we have

Corollary 5.4. *If system* (1.1) *is a generalized synchronizable system with respect to* $\Theta^{(k)}_{p_k}$ *for all* $k = 1, \dots, K$*, then it is a generalized synchronizable system with respect to* Θ^*_q*, where* Θ^*_q *satisfies*

$$\mathrm{Ker}(\Theta^*_q) = \bigcap_{1 \leq k \leq K} \mathrm{Ker}(\Theta^{(k)}_{p_k}), \tag{5.12}$$

and q *as the dimension of* $\mathrm{Ker}(\Theta^*_q)$ *satisfies* $0 \leq q \leq \min(p_1, \dots, p_K)$.

Remark 5.3. Different from Corollary 4.3 for the synchronization by groups, the above result for the generalized synchronization includes the case $q = 0$: $\mathrm{Ker}(\Theta_0^*) = \{0\}$ in (5.12), thus the generalized synchronization solution with respect to Θ_0^* is actually the zero solution. This is a trivial case.

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Declarations

Conflict of interest The author declares no conflict of interest.

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