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A sharp lower bound for the weighted Lehmer mean involving complete p-elliptic integrals

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Abstract. In the article, we provide a sharp lower bound for the weighted Lehmer mean of the complete p-elliptic integrals of the first and second kinds, which is the extension of the previous results for complete p-elliptic integrals.

§1 Introduction

For $p \in (1, \infty)$ and $x \in [0, 1]$, the generalized sine function $\sin_p x$ and its half-period π_p are defined as the inverse function of

$$\operatorname{arcsin}_p x := \int_0^x \frac{dt}{(1-t^p)^{\frac{1}{p}}}$$

and

$$\frac{\pi_p}{2} := \operatorname{arcsin}_p 1 = \int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}} = \frac{\pi}{p\sin(\pi/p)} = \frac{1}{p} B\left(\frac{1}{p}, 1-\frac{1}{p}\right),$$

respectively, where B is the beta function defined by $B(a, b) = [\Gamma(a)\Gamma(b)]/\Gamma(a+b)$ $(a, b > 0)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the classical Euler gamma function. Clearly, $\sin_p = \sin$ and $\pi_p = \pi$ in the case when $p = 2$.

The Legendre's complete elliptic integrals of the first and second kind [1,2] are respectively defined by, for $r \in (0, 1)$,

$$\begin{cases} \mathscr{K}(r) = \displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \\ \mathscr{K}(0) = \pi/2, \quad \mathscr{K}(1^-) = \infty, \end{cases}$$

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and

$$\begin{cases} \mathscr{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \theta} d\theta = \int_0^1 \sqrt{\frac{1 - r^2 t^2}{1 - t^2}} dt, \\ \mathscr{E}(0) = \pi/2, \quad \mathscr{E}(1) = 1. \end{cases}$$

In 2016, Takeuchi [3] introduced the complete p-elliptic integrals $\boldsymbol{K}_p$ and $\boldsymbol{E}_p$ of the first and second kind by applying the generalized trigonometric functions to Legendres complete elliptic integrals, which are defined by

$$\boldsymbol{K}_p =: \boldsymbol{K}_p(r) = \int_0^{\pi_p/2} \frac{d\theta}{(1 - r^p \sin_p^p \theta)^{1-1/p}} = \int_0^1 \frac{dt}{(1 - t^p)^{1/p} (1 - r^p t^p)^{1-1/p}},$$

$$\boldsymbol{E}_p =: \boldsymbol{E}_p(r) = \int_0^{\pi_p/2} (1 - r^p \sin_p^p \theta)^{1/p} d\theta = \int_0^1 \left(\frac{1 - r^p t^p}{1 - t^p} \right)^{1/p} dt.$$

Moreover, the complete p-elliptic integrals can be represented by [3, Proposition 2.8]

$$\boldsymbol{K}_p(r) = \frac{\pi_p}{2} F\left(1 - \frac{1}{p}, \frac{1}{p}; 1; r^p\right) \quad \text{and} \quad \boldsymbol{E}_p(r) = \frac{\pi_p}{2} F\left(-\frac{1}{p}, \frac{1}{p}; 1; r^p\right), \tag{1.1}$$

where $F(a, b; c; x)$ is the Gaussian hypergeometric function [2,3] defined as

$$F(a, b; c; x) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad (|x| < 1),$$

for real parameters a, b, c with $c \neq 0, -1, -2, \cdots$ and $(a, 0) = 1$, $(a, n) = a(a+1)\cdots(a+n-1)$ is the shifted factorial function. Recently, the Legendres complete elliptic integrals and their generalizations (or related functions) have attracted the attention of many researchers. For their recent research progress, we recommend the literatrue [4–21] to readers.

Let $t \in \mathbb{R}$ and $x, y > 0$. Then the weighted t order Hölder mean [20,21] of a and b with weight $w \in (0, 1)$ is defined as

$$H_t(x, y; w) = \left[w x^t + (1 - w) y^t \right]^{1/t} \quad (t \neq 0) \quad \text{and} \quad H_0(x, y; w) = x^w y^{1-w}. \tag{1.2}$$

For $w = 1/2$, it reduces to the classical t-th Hölder mean $H_t(x, y) = H_t(x, y; 1/2)$. The s-th Lehmer mean $L_s(x, y)$ [22] of x and y is defined by

$$L_s(x, y) = \frac{x^{s+1} + y^{s+1}}{x^s + y^s}. \tag{1.3}$$

It is worth pointing out that some properties of $H_t(a, b)$ and $L_s(a, b)$ are very close as the *Sisters*, in particular, several classical bivariate means are the special cases of Hölder mean and Lehmer mean, such as

$$H_{-1}(x, y) = L_{-1}(x, y) = \frac{2xy}{x + y}, \quad H_0(x, y) = L_{-\frac{1}{2}}(x, y) = \sqrt{xy},$$

$$H_1(x, y) = L_0(x, y) = \frac{x + y}{2},$$

are the harmonic, geometric and arithmetic means of x and y, respectively. Furthermore, the inequalities between Hölder mean and Lehmer means were established by Liu [23] and we refer to [24–26] for more properties.

In 1990, Anderson, Vamanamurthy and Vuorinen [2, Theorem 3.31] proved that the inequality

$$\sqrt{\boldsymbol{K}(r)\boldsymbol{E}(r)} > \frac{\pi}{2}, \tag{1.4}$$

holds for all $r \in (0,1)$.

Later, Wang et al. [27] gave a generalization of inequality (1.4) and proved that the inequality

$$H_t(\boldsymbol{K}(r), \boldsymbol{E}(r)) > \frac{\pi}{2}, \tag{1.5}$$

holds for all $r \in (0,1)$ if and only if $t \geq -1/2$.

Recently, Wang et al. [20, Theorem 1.1] also extended (1.5) to the case of complete p-elliptic integrals and proved the following theorem.

Theorem 1.1. *([20]). Let $p \in (1,\infty)$. Then the inequalities*

$$H_s\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) > \frac{\pi_p}{2}, \tag{1.6}$$

and

$$H_t\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) < \frac{\pi_p}{2}, \tag{1.7}$$

hold for all $r \in (0,1)$ if and only if $s \geq s_0 = (1-p)/2$ and $t \leq t_0 = \log(1-1/p)/\log(\pi_p/2)$.

As shown in (1.2), we extend the definition of Lehmer mean (1.3) to the weighted s-order Lehmer mean of x and y as

$$L_s(x,y;w) = \frac{wx^{s+1} + (1-w)y^{s+1}}{wx^s + (1-w)y^s}. \tag{1.8}$$

It is easy to verify that $L_s(x,y;w)$ is continuous and strictly increasing with respect to $s \in \mathbb{R}$ for fixed $w \in (0,1)$ and $x, y > 0$ with $x \neq y$.

Motived by Theorem 1.1, it makes sense to establish the weighted Lehmer mean inequality for complete p-elliptic integrals. As a tool of proof, we first present the inequalities between the weighted Hölder mean and weighted Lehmer mean, which is an extension of [23, Theorem 1]. Our main result is the following.

Theorem 1.2. *Let $p \geq 2$. Then the inequality*

$$L_s\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) > \frac{\pi_p}{2}, \tag{1.9}$$

holds for all $r \in (0,1)$ if and only if $s \geq s_*(p)$, where

$$s_*(p) = \begin{cases} -\dfrac{p+1}{4}, & 2 \leq p \leq 3, \\ -1, & p > 3. \end{cases}$$

§2 Preliminaries

Throughout this paper, we denote $r' = \sqrt[p]{1-r^p}$ for $r \in (0,1)$ with $p > 1$. Recall the derivative formulas [3, Propositions 2.1 and 2.2] for the complete p-elliptic integrals of the first and second kind

$$\frac{d\boldsymbol{K}_p}{dr} = \frac{\boldsymbol{E}_p - r'^p \boldsymbol{K}_p}{r r'^p}, \qquad \frac{d\boldsymbol{E}_p}{dr} = \frac{\boldsymbol{E}_p - \boldsymbol{K}_p}{r},$$
$$\frac{d(\boldsymbol{E}_p - r'^p \boldsymbol{K}_p)}{dr} = (p-1) r^{p-1} \boldsymbol{K}_p, \qquad \frac{d(\boldsymbol{K}_p - \boldsymbol{E}_p)}{dr} = \frac{r^{p-1}\boldsymbol{E}_p}{r'^p}.$$

To facilitate the presentation, we denote by

$$\begin{aligned} f(r) &=: \frac{\boldsymbol{K}_p}{\boldsymbol{E}_p}, \qquad g(r) =: \frac{\boldsymbol{E}_p - r'^p \boldsymbol{K}_p}{(p-1) r'^p (\boldsymbol{K}_p - \boldsymbol{E}_p)}, \\ h_s(r) &=: \frac{\boldsymbol{K}_p^{s+1} + (s+1)(p-1)\boldsymbol{K}_p \boldsymbol{E}_p^s - s(p-1)\boldsymbol{E}_p^{s+1}}{(p-1)\boldsymbol{E}_p^{s+1} + (s+1)\boldsymbol{K}_p^s \boldsymbol{E}_p - s\boldsymbol{K}_p^{s+1}}, \end{aligned} \tag{2.1}$$

for $r \in (0,1)$ with $p \in (1,\infty)$ and $s \in \mathbb{R}$.

Lemma 2.1. *(*[2, Monotone l'Hôpital Rule]*) Let $-\infty < a < b < \infty$, and $f, g : [a,b] \to \mathbb{R}$ be continuous that are differentiable on (a,b) such that $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a,b)$. If f'/g' is (strictly) increasing (decreasing) on (a,b), then so is f/g.*

Lemma 2.2. *For $p > 1$, then the functions*

(i) $(\boldsymbol{E}_p - r'^p \boldsymbol{K}_p)/r^p$ is strictly increasing from $(0,1)$ onto $((p-1)\pi_p/(2p), 1)$;

(ii) $(\boldsymbol{K}_p - \boldsymbol{E}_p)/r^p$ is strictly increasing from $(0,1)$ onto $(\pi_p/(2p), \infty)$;

(iii) $r'^{-p}\boldsymbol{E}_p$ is strictly increasing from $(0,1)$ onto $(\pi_p/2, \infty)$;

(iv) $[p(\boldsymbol{K}_p - \boldsymbol{E}_p) - r^p \boldsymbol{E}_p]/r^{2p}$ is strictly increasing from $(0,1)$ onto $\big((p^2+1)\pi_p/(4p^2), +\infty\big)$.

Proof. Parts (i), (ii) and (iii) follow from [28, Lemma 3.4 (1), (4) and (7)]. For part (iv), we denote $\mu_1(r) = p(\boldsymbol{K}_p - \boldsymbol{E}_p) - r^p \boldsymbol{E}_p$ and $\mu_2(r) = r^{2p}$. Then by Lemma 2.1, Lemma 2.2 (iv) can be derived from $\mu_1(0) = \mu_2(0) = 0$ and Lemma 2.2 (ii), (iii) together with

$$\frac{\mu_1'(r)}{\mu_2'(r)} = \frac{1}{2}\left(\frac{\boldsymbol{E}_p}{r'^p} + \frac{\boldsymbol{K}_p - \boldsymbol{E}_p}{p r^p}\right) \quad \text{and} \quad \lim_{r\to 0^+} \frac{\mu_1(r)}{\mu_2(r)} = \lim_{r\to 0^+} \frac{\mu_1'(r)}{\mu_2'(r)} = \frac{\pi_p}{4}\left(1 + \frac{1}{p^2}\right).$$

□

Lemma 2.3. *(See* [20, Lemma 2.7]*) For $p > 1$ and $s \in \mathbb{R}$, the function $f^{s-1}(r)g(r)$ is strictly increasing on $(0,1)$ if and only if $s \geq (1-p)/2$. In particular, the inequality $g(r) > f^{(p+1)/2}(r)$ holds for all $r \in (0,1)$.*

Lemma 2.4. *For $p > 1$ and $s \in \mathbb{R}$, then we have*

$$(1) \lim_{r\to 0^+} \frac{\log f(r)}{r^p} = \frac{1}{p}; \quad (2) \lim_{r\to 0^+} \frac{\log g(r)}{r^p} = \frac{p+1}{2p}; \quad (3) \lim_{r\to 0^+} \frac{\log h_s(r)}{r^p} = \frac{s+1}{p}.$$

Proof. (1) By l'Hôpital Rule, it follows from Lemma 2.2 (1) and (2) that

$$\lim_{r\to 0^+} \frac{\log f(r)}{r^p} = \frac{1}{p} \lim_{r\to 0^+} \left(\frac{\boldsymbol{E}_p - r'^p \boldsymbol{K}_p}{r^p r'^p \boldsymbol{K}_p} + \frac{\boldsymbol{K}_p - \boldsymbol{E}_p}{r^p \boldsymbol{E}_p}\right) = \frac{1}{p}\left(\frac{p-1}{p} + \frac{1}{p}\right) = \frac{1}{p}.$$

(2) An elementary calculation together with Lemma 2.2 (2) and (4) leads to

$$\begin{aligned}
&\lim_{r\to0^+}\frac{\boldsymbol{E}_p^2-(p-1)r'^p\boldsymbol{K}_p^2+(p-2)r'^p\boldsymbol{K}_p\boldsymbol{E}_p}{r^{2p}}\\
=&\lim_{r\to0^+}\frac{(\boldsymbol{K}_p-\boldsymbol{E}_p)^2+(p-1)r^p\boldsymbol{K}_p(\boldsymbol{K}_p-\boldsymbol{E}_p)-\boldsymbol{K}_p\left[p(\boldsymbol{K}_p-\boldsymbol{E}_p)-r^p\boldsymbol{E}_p\right]}{r^{2p}}\\
=&\lim_{r\to0^+}\left(\frac{\boldsymbol{K}_p-\boldsymbol{E}_p}{r^p}\right)^2+\frac{(p-1)\pi_p}{2}\lim_{r\to0^+}\frac{\boldsymbol{K}_p-\boldsymbol{E}_p}{r^p}-\frac{\pi_p}{2}\lim_{r\to0^+}\frac{p(\boldsymbol{K}_p-\boldsymbol{E}_p)-r^p\boldsymbol{E}_p}{r^{2p}}\\
=&\left(\frac{\pi_p}{2p}\right)^2+\frac{(p-1)\pi_p}{2}\cdot\frac{\pi_p}{2p}-\frac{\pi_p}{2}\cdot\frac{\pi_p(p^2+1)}{4p^2}=\frac{\pi_p^2}{8}\left(1-\frac{1}{p}\right)^2.
\end{aligned}$$

This in conjunction with Lemma 2.2 (1) and (2) together with l'Hôpital Rule implies that

$$\begin{aligned}
&\lim_{r\to0^+}\frac{\log g(r)}{r^p}=\frac{1}{p}\lim_{r\to0^+}\left[\frac{(p-1)\boldsymbol{K}_p}{\boldsymbol{E}_p-r'^p\boldsymbol{K}_p}-\frac{\boldsymbol{E}_p}{r'^p(\boldsymbol{K}_p-\boldsymbol{E}_p)}\right]+1\\
=&-\frac{1}{p}\lim_{r\to0^+}\left[\frac{\boldsymbol{E}_p^2-(p-1)r'^p\boldsymbol{K}_p^2+(p-2)r'^p\boldsymbol{K}_p\boldsymbol{E}_p}{r^{2p}}\cdot\frac{r^p}{\boldsymbol{E}_p-r'^p\boldsymbol{K}_p}\cdot\frac{r^p}{\boldsymbol{K}_p-\boldsymbol{E}_p}\right]+1\\
=&-\frac{1}{p}\cdot\frac{\pi_p^2}{8}\left(1-\frac{1}{p}\right)^2\cdot\frac{2p}{(p-1)\pi_p}\cdot\frac{2p}{\pi_p}+1=\frac{p+1}{2p}.
\end{aligned}$$

(3) The l'Hôpital Rule together with Lemma 2.2(1) and (2) enables us to know that

$$\begin{aligned}
&\lim_{r\to0^+}\frac{\log h_s(r)}{r^p}\\
=&\lim_{r\to0^+}\left\{\frac{(s+1)\left(\boldsymbol{E}_p^2-2r'^p\boldsymbol{K}_p\boldsymbol{E}_p+r'^p\boldsymbol{K}_p^2\right)}{pr^pr'^p}\right.\\
&\left.\times\frac{\left[\left(\boldsymbol{K}_p^s+(p-1)\boldsymbol{E}_p^s\right)^2+(p-1)s^2\boldsymbol{E}_p^{s-1}\boldsymbol{K}_p^{s-1}(\boldsymbol{K}_p-\boldsymbol{E}_p)^2\right]}{\left[\boldsymbol{K}_p^{s+1}+(s+1)(p-1)\boldsymbol{K}_p\boldsymbol{E}_p^s-s(p-1)\boldsymbol{E}_p^{s+1}\right]\left[(p-1)\boldsymbol{E}_p^{s+1}+(s+1)\boldsymbol{K}_p^s\boldsymbol{E}_p-s\boldsymbol{K}_p^{s+1}\right]}\right\}\\
=&\frac{4(s+1)}{p\pi_p^2}\lim_{r\to0^+}\left[\frac{\boldsymbol{E}_p(\boldsymbol{E}_p-r'^p\boldsymbol{K}_p)}{r^p}+\frac{r'^p\boldsymbol{K}_p\left(\boldsymbol{K}_p-\boldsymbol{E}_p\right)}{r^p}\right]\\
=&\frac{4(s+1)}{p\pi_p^2}\cdot\frac{\pi_p}{2}\left[\frac{(p-1)\pi_p}{2p}+\frac{\pi_p}{2p}\right]=\frac{s+1}{p}.
\end{aligned}$$

□

As mentioned in the introduction, the inequalities between $H_t(x,y;1/2)$ and $L_s(x,y;1/2)$ had been studied in [23]. For $w\neq1/2$, it is clear that $H_t(x,y;w)$ and $L_s(x,y;w)$ are non-symmetric but the relations $H_t(x,y;w)=H_t(y,x;1-w)$ and $L_s(x,y;w)=L_s(y,x;1-w)$ allow us to only consider the case that $x>y>0$ with $w\in(0,1)$. We now present the comparison inequalities between $H_t(x,y;w)$ and $L_s(x,y;w)$ in Proposition 2.5.

Proposition 2.5. *Let $s\in\mathbb{R}$ and $w\in(0,1)$ with $w\neq1/2$. The following statements are true.*

- *In the case $w\in(0,1/2)$,*
  (1) $H_{2s+1}(x,y;w)\le L_s(x,y;w)$ holds for $x>y>0$ if and only if $s\in[-1,-1/2]$;
  (2) $H_{2s+1}(x,y;w)\ge L_s(x,y;w)$ holds for $x>y>0$ if and only if $s\in(-\infty,-1]$.

- *In the case $w\in(1/2,1)$,*

(3) $H_{2s+1}(x,y;w) \leq L_s(x,y;w)$ *holds for* $x > y > 0$ *if and only if* $s \in [0,\infty)$*;*

(4) $H_{2s+1}(x,y;w) \geq L_s(x,y;w)$ *holds for* $x > y > 0$ *if and only if* $s \in [-1/2, 0]$*.*

Equalities hold for all $x, y > 0$ *when* $s = -1$ *or* 0*. Moreover, inequalities (1) and (3) are best possible in the sense that* $2s+1$ *cannot be replaced by any larger function of* s*, and inequalities (2) and (4) are best possible in the sense that* $2s+1$ *cannot be replaced by any smaller function of* s*.*

Proof. For the special values $s = -1$ or $s = 0$, it is verified directly from (1.2) and (1.8) that $H_{2s+1}(x,y;w) = L_s(x,y;w)$ for all $x, y > 0$.

To see that this is the best possible observe that

$$H_t(1+\varepsilon, 1; w) = 1 + w\varepsilon + \frac{1}{2}w(1-w)(t-1)\varepsilon^2 + \cdots$$

and

$$L_s(1+\varepsilon, 1; w) = 1 + w\varepsilon + w(1-w)s\varepsilon^2 + \cdots .$$

Thus $H_t(1+\varepsilon,1;w) \leq (\geq) L_s(1+\varepsilon,1;w)$ for small enough $\varepsilon > 0$ implies that $t \leq (\geq) 2s+1$.

By homogeneity it suffices to prove the inequalities of Proposition 2.5 for $x > 1$ and $y = 1$. For $x > 1$, we denote

$$\begin{aligned}
\varphi(x) &= \frac{1}{2s+1}\log\left(wx^{2s+1}+1-w\right) - \log\left(\frac{wx^{s+1}+1-w}{wx^s+1-w}\right),\\
\varphi_1(x) &= w(1+s-sx)x^{2s+1} + (1-2w)x^{s+1} + (1-w)(s-x-sx),\\
\varphi_2(x) &= (s+1)\left[w(1+2s-2sx)x^{2s} + (1-2w)x^s + w - 1\right],\\
\varphi_3(x) &= s(s+1)\left[2w(2s+1)(1-x)x^s + 1 - 2w\right],\\
\varphi_4(x) &= ws(s+1)(2s+1)\left[s-(1+s)x\right].
\end{aligned}$$

By differentiating $\varphi(x)$ several times, one has

$$\varphi'(x) = \frac{w(1-w)x^{s-1}\varphi_1(x)}{(wx^s+1-w)(wx^{s+1}+1-w)(wx^{2s+1}+1-w)}, \tag{2.2}$$

$$\varphi_1'(x) = \varphi_2(x), \tag{2.3}$$

$$\varphi_2'(x) = x^{s-1}\varphi_3(x), \tag{2.4}$$

$$\varphi_3'(x) = 2x^{s+1}\varphi_4(x), \tag{2.5}$$

$$\varphi_4'(x) = -ws(2s+1)(s+1)^2. \tag{2.6}$$

We now divide the proof into two cases.

Case 2.1 $w \in (0, 1/2)$.

(1) $s \in (-1, -1/2]$. Then it follows from (2.6) that $\varphi_4(x)$ is decreasing on $(1,\infty)$, which in conjunction with (2.5) and $\varphi_4(1) = -ws(s+1)(2s+1) \leq 0$ yields $\varphi_3(x)$ is decreasing on $(1,\infty)$. By the same argument, $\varphi_3(1) = (1-2w)s(s+1) < 0$ and $\varphi_1(1) = \varphi_2(1) = 0$ together with (2.2)-(2.4) lead to the conclusion that $\varphi(x) < 0$ for $x \in (1,\infty)$ and so

$$H_{2s+1}(x,y;w) < L_s(x,y;w), \tag{2.7}$$

holds for all $x > y > 0$.

(2) $s \in (-\infty, -1)$. Then it follows from (2.6) that $\varphi_4(x)$ is decreasing on $(1, \infty)$, which in conjunction with (2.5), $\varphi_4(1) = -ws(s+1)(2s+1) > 0$ and $\varphi_4(\infty) = -\infty$ implies that there exists $x_1 \in (1, \infty)$ such that $\varphi_3(x)$ is strictly increasing on $(1, x_1)$ and strictly decreasing on (x_1, ∞). According to this with $\varphi_3(1) = (1 - 2w)s(s + 1) > 0$ and $\varphi_3(\infty) = 0$, it can be easily seen that $\varphi_3(x) > 0$ for $x \in (1, \infty)$. The same argument as (1) together with (2.2)-(2.4) and $\varphi_1(1) = \varphi_2(1) = 0$ leads to the conclusion that

$$H_{2s+1}(x, y; w) > L_s(x, y; w), \tag{2.8}$$

holds for all $x > y > 0$.

(3) $s \in (-1/2, 0)$. Then it follows from $\varphi_3(1) = (1 - 2w)s(s + 1) < 0$ and (2.4) that there exists $x_2 \in (1, \infty)$ such that $\varphi_2(x)$ is strictly decreasing on $(1, x_2)$. This in conjunction with $\varphi_1(1) = \varphi_2(1) = 0$ and (2.2), (2.3) yields $\varphi(x) < 0$ for $x \in (1, x_2)$. On the other hand, $\varphi(\infty) = \infty$ enables us to know that $\varphi(x) > 0$ for $x \in (x_3, \infty)$ with enough large $x_3 > 1$.

(4) $s \in (0, \infty)$. Then it is easy to see that $\varphi_3(1) > 0$ and $\varphi(\infty) = \frac{\log w}{2s+1} < 0$. According this with the same approach as (3), it follows that there exist $x_4, x_5 \in (1, \infty)$ such that $\varphi(x) > 0$ for $x \in (1, x_4)$ and $\varphi(x) < 0$ for $x \in (x_5, \infty)$.

Case 2.2 $w \in (1/2, 1)$.

(1) $s \in [-1/2, 0)$. Then it follows from (2.6) and $\varphi_4(1) = -ws(s + 1)(2s + 1) \geq 0$ that $\varphi_3(x)$ is strictly increasing on $(1, \infty)$. This in conjunction with (2.2)-(2.4) and $\varphi_3(1) = (1 - 2w)s(s + 1) > 0$, $\varphi_1(1) = \varphi_2(1) = 0$ implies that $\varphi(x) > 0$ for $x \in (1, \infty)$, that is

$$H_{2s+1}(x, y; w) > L_s(x, y; w), \tag{2.9}$$

for all $x > y > 0$.

(2) $s \in (0, \infty)$. In this case, $\varphi_3(1) = (1-2w)s(s+1) < 0$ and $\varphi_4(1) = -ws(s+1)(2s+1) < 0$. This in conjunction with (2.2)-(2.6) and $\varphi_1(1) = \varphi_2(1) = 0$ yields $\varphi(x) < 0$ for $x \in (1, \infty)$, which is equivalently

$$H_{2s+1}(x, y; w) < L_s(x, y; w), \tag{2.10}$$

holds for all $x > y > 0$.

(3) $s \in (-\infty, -1) \cup (-1, -1/2)$. Then we claim that $\varphi_3(1) \cdot \varphi(\infty) < 0$. Indeed, if $s \in (-\infty, -1)$, then $\varphi_3(1) = (1 - 2w)s(s + 1) < 0$ and $\varphi(\infty) = \frac{\log(1-w)}{2s+1} > 0$; if $s \in (-1, -1/2)$, then $\varphi_3(1) = (1 - 2w)s(s + 1) > 0$ and $\varphi(\infty) = -\infty$. In the case $w \in (0, 1/2)$, the same approach to (3) and (4) leads to the conclusion that there are different signs of $\varphi(x)$ as x close to 1 and ∞.

Therefore, inequalities (2.7)-(2.10) complete the proof of Proposition 2.5. $\square$

§3 Proof of Theorem 1.2

Proof of Theorem 1.1. Let

$$\Phi_s(r)=\log\left[\frac{\frac{1}{p}\boldsymbol{K}_p^{s+1}+\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s+1}}{\frac{1}{p}\boldsymbol{K}_p^{s}+\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s}}\right]-\log\frac{\pi_p}{2}.$$

By the monotonicity of $L_s(x,y;w)$ with respect to s, it suffices to prove

$$\Phi_{s_*(p)}(r)>0 \tag{3.1}$$

for $r\in(0,1)$ with $p\geq 2$ instead of (1.9).

We divide into two cases $2\leq p\leq 3$ and $p>3$ to complete the proof of (3.1).

Case 1 : $2\leq p\leq 3$. In this case, we denote $s_*=-(p+1)/4$. Differentiation of $\Phi_s(r)$ yields

$$\begin{aligned}\Phi_s'(r)=&\frac{\frac{s+1}{p}\boldsymbol{K}_p^{s}\frac{\boldsymbol{E}_p-r'^p\boldsymbol{K}_p}{rr'^p}-(s+1)\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s}\frac{\boldsymbol{K}_p-\boldsymbol{E}_p}{r}}{\frac{1}{p}\boldsymbol{K}_p^{s+1}+\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s+1}}\\&-\frac{\frac{s}{p}\boldsymbol{K}_p^{s-1}\frac{\boldsymbol{E}_p-r'^p\boldsymbol{K}_p}{rr'^p}-s\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s-1}\frac{\boldsymbol{K}_p-\boldsymbol{E}_p}{r}}{\frac{1}{p}\boldsymbol{K}_p^{s}+\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s}}\\=&\frac{(p-1)\boldsymbol{E}_p^{s-1}(\boldsymbol{K}_p-\boldsymbol{E}_p)\left[(p-1)\boldsymbol{E}_p^{s+1}+(s+1)\boldsymbol{K}_p^{s}\boldsymbol{E}_p-s\boldsymbol{K}_p^{s+1}\right]}{p^2r\left[\frac{1}{p}\boldsymbol{K}_p^{s}+\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s}\right]\left[\frac{1}{p}\boldsymbol{K}_p^{s+1}+\left(1-\frac{1}{p}\right)\boldsymbol{E}_p^{s+1}\right]}\Big[\phi_s(r)-1\Big],\end{aligned} \tag{3.2}$$

where $f(r)$, $g(r)$, $h_s(r)$ are defined as in (2.1) and $\phi_s(r)=\left[f(r)\right]^{s-1}g(r)h_s(r)$.

Let $x=f(r)$ for short. Then $x>1$ for $r\in(0,1)$. It follows from Lemma 2.3 that $g(r)>x^{(p+1)/2}$. Moreover, $h_s(r)$ can be rewritten as

$$h_s(r)=\frac{x^{s+1}+(s+1)(p-1)x-s(p-1)}{(p-1)+(s+1)x^s-sx^{s+1}}.$$

This gives

$$\phi_{s_*}(r)>\frac{x^{\frac{p-3}{4}}\left[4x^{\frac{3-p}{4}}+(3-p)(p-1)x+(p^2-1)\right]}{4(p-1)+(3-p)x^{-\frac{p+1}{4}}+(p+1)x^{\frac{3-p}{4}}}:=\hat{\phi}_p(x), \tag{3.3}$$

for $x>1$.

A simple calculation yields

$$\hat{\phi}_p(1)=1, \tag{3.4}$$

$$\begin{aligned}\hat{\phi}_p'(x)=&\frac{(p+1)(3-p)(x-1)x^{\frac{p-7}{4}}\left[2x\left((p-1)^2x^{\frac{p-1}{2}}-1\right)+(p-1)^2(x-1)x^{\frac{p+1}{4}}\right]}{2\left[3-p+(p+1)x+4(p-1)x^{\frac{p+1}{4}}\right]^2}\\\geq&\ 0,\end{aligned} \tag{3.5}$$

for $x>1$. It follows from (3.4) and (3.5) that $\hat{\phi}_p(x)\geq 1$ for $x>1$. This in conjunction with (3.2) and (3.3) implies that $\Phi_{s_*}(r)$ is strictly increasing on $(0,1)$.

Therefore, inequality (3.1) follows from the monotonicity of $\Phi_{s_*}(r)$ and $\Phi_{s_*}(0^+;p)=0$.

Case 2 : $p > 3$. In this case, $s_* = -1$. Then inequality (1.9) reduces to

$$L_{-1}\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) = H_{-1}\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) > \frac{\pi_p}{2},$$

which is valid from (1.6).

Now it remains to prove $s_*(p)$ is the best possible parameter.

From Lemma 2.4 we clearly see that

$$\begin{aligned}\lim_{r\to 0^+}\frac{\log\phi_s(r)}{r^p} &= (s-1)\lim_{r\to 0^+}\frac{\log f(r)}{r^p} + \lim_{r\to 0^+}\frac{\log g(r)}{r^p} + \lim_{r\to 0^+}\frac{\log h_s(r)}{r^p}\\ &= \frac{s-1}{p} + \frac{p+1}{2p} + \frac{s+1}{p} = \frac{2}{p}\left(s + \frac{p+1}{4}\right). \end{aligned} \tag{3.6}$$

For $2 \le p \le 3$, if $s < s_*(p) = -(p+1)/4$, then it follows from (3.6) that there exists $\delta_1 \in (0,1)$ such that $\phi_s(r) < 1$ for $r \in (0, \delta_1)$. This in conjunction with (3.2) and $\Phi_s(0^+; p) = 0$ yields $\Phi_s(r) < 0$ for $r \in (0, \delta_1)$.

For $p > 3$, if $s < -1$, then we clearly see that

$$\lim_{r\to 1^-}\Phi_s(r) = -\log\frac{\pi_p}{2} < -\log\frac{\pi_\infty}{2} = 0, \tag{3.7}$$

where the last inequality follows from the monotonicity of π_p and $\pi_\infty = 2$. Inequality (3.7) shows that there exists $\delta_2 \in (0,1)$ such that $\Phi_s(r) < 0$ for $r \in (\delta_2, 1)$.

Remark 3.1. For $2 < p \le 3$, then $1/p \in (0, 1/2)$. As an application, Proposition 2.5 and (1.6) enable us to know that

$$L_{-\frac{p+1}{4}}\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) \ge H_{\frac{1-p}{2}}\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) > \frac{\pi_p}{2}$$

for all $r \in (0,1)$.

For $p > 1$, we now prove

$$\frac{\log(1-1/p)}{\log(\pi_p/2)} < -1 \Longleftrightarrow \frac{\pi_p}{2}\left(1-\frac{1}{p}\right) < 1, \tag{3.8}$$

since $\pi_p > 2$.

By substituting $u = \pi/p \in (0,\pi)$ into (3.8), it can be easily obtained that

$$\frac{\pi_p}{2}\left(1-\frac{1}{p}\right) = \frac{u(\pi-u)}{\pi\sin u} := \frac{\zeta_1(u)}{\zeta_2(u)}.$$

Clearly, one has $\zeta_1(0) = \zeta_2(0) = \zeta_1(\pi) = \zeta_2(\pi) = 0$, $\zeta_1'(\pi/2) = \zeta_2'(\pi/2) = 0$ and

$$\frac{\zeta_1'(u)}{\zeta_2'(u)} = \frac{\pi-2u}{\pi\cos u}, \quad \frac{\zeta_1''(u)}{\zeta_2''(u)} = \frac{2}{\pi\sin u}.$$

This in conjunction with Lemma 2.1 implies that $\zeta_1(u)/\zeta_2(u)$ is strictly decreasing on $(0,\pi/2)$ and $\zeta_1(u)/\zeta_2(u)$ is strictly increasing on $(\pi/2,\pi)$, which together with l'Hôpital Rule gives

$$\frac{\zeta_1(u)}{\zeta_2(u)} < \begin{cases}\displaystyle\lim_{u\to 0^+}\frac{\zeta_1(u)}{\zeta_2(u)} = \lim_{u\to 0^+}\frac{\zeta_1'(u)}{\zeta_2'(u)} = 1, & u \in (0,\pi/2),\\ \displaystyle\lim_{u\to \pi^-}\frac{\zeta_1(u)}{\zeta_2(u)} = \lim_{u\to \pi^-}\frac{\zeta_1'(u)}{\zeta_2'(u)} = 1, & u \in (\pi/2,\pi).\end{cases}$$

Proposition 2.5 together with (1.7) and (3.8) enables us to give the following corollary.

Corollary 3.2. *Let $p \in [2, \infty)$. Then the inequality*

$$L_s\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) < \frac{\pi_p}{2},$$

holds for all $r \in (0,1)$ if $s \le \log(1-1/p)/\log(\pi_p/2)$.

Remark 3.3. As in Lemma 2.4, we can compute

$$\lim_{r\to 0^+} \frac{\log f(r) - \frac{r^p}{p}}{r^{2p}} = \frac{p^2-p+1}{2p^3}, \quad \lim_{r\to 0^+} \frac{\log g(r) - \frac{p+1}{2p} r^p}{r^{2p}} = \frac{7p^3+4p^2-p+2}{24p^3},$$

$$\lim_{r\to 0^+} \frac{\log h_s(r) - \frac{s+1}{p} r^p}{r^{2p}} = \frac{(s+1)\left[p^2+1+2s-p(s+1)\right]}{2p^3},$$

which gives

$$\lim_{r\to 0^+} \frac{\log \phi_{-\frac{p+1}{4}}(r)}{r^{2p}} = \frac{(p+1)\left(p+\frac{\sqrt{865}+27}{2}\right)\left(p-\frac{\sqrt{865}-27}{2}\right)}{96p^3}. \tag{3.9}$$

When $1 < p < (\sqrt{865}-27)/2 = 1.20544\cdots$, it follows from (3.2) and (3.9) that $\Phi_s(r) < 0$ for $r \in (0, \tau)$ with small $\tau > 0$, which yields $s_*(p) > -(p+1)/4$. While $(\sqrt{865}-27)/2 \le p < 2$, computer experiments show that $\Phi_{-\frac{p+1}{4}}(r) > 0$ for $r \in (0,1)$. In other words, $s_*(p)$ in Theorem 1.2 can be extended as $s_*(p) = -(p+1)/4$ for $(\sqrt{865}-27)/2 \le p \le 3$.

Unfortunately, we don't establish the inequality as (1.9) for $p \in (1,2)$ but Remark 3.3 provides us some informations. A sufficient condition for the parameter s such that the inverse inequality of (1.9) in Corollary 3.2 holds has been obtained, but this is not the optimal constant from computer experiments. This allows us to pose the following problem.

Problem 3.4. *Find the best possible functions $s_*(p)$ for $p \in (1,2)$ and $s^*(p)$ for $p \in (1,\infty)$ such that the inequality*

$$L_s\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) > \frac{\pi_p}{2},$$

holds for all $r \in (0,1)$ if and only $s \ge s_*(p)$ and the inequality

$$L_s\left(\boldsymbol{K}_p(r), \boldsymbol{E}_p(r); 1/p\right) < \frac{\pi_p}{2},$$

holds for all $r \in (0,1)$ if and only $s \le s^*(p)$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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