Appl. Math. J. Chinese Univ.
2025, 40(3): 543-557

Higher-order optimality conditions for multiobjective optimization through a new type of directional derivatives

HUANG Zheng-gang

Abstract. This paper deals with extensions of higher-order optimality conditions for scalar optimization to multiobjective optimization. A type of directional derivatives for a multiobjective function is proposed, and with this notion characterizations of strict local minima of order k for a multiobjective optimization problem with a nonempty set constraint are established, generalizing the corresponding scalar case obtained by Studniarski [3]. Also necessary not sufficient and sufficient not necessary optimality conditions for this minima are derived based on our directional derivatives, which are generalizations of some existing scalar results and equivalent to some existing multiobjective ones. Many examples are given to illustrate them there.

§1 Introduction

The strict local minimizer of order k ($k \geq 1$ an integer) as one class of minima for scalar optimization problems was first used by Hestenes [1] for the values $k = 1$ and $k = 2$ to prove sufficient conditions. Let X, $\mathbb{R}^p$ with p a positive integer and $\mathbb{R}$, respectively, be a real normed vector space, the p-dimensional Euclidean space and the set of all real numbers. We recall that let a function $f : X \to \mathbb{R}$ and a subset $S \subset X$, a point $x^0 \in S$ is said to be a strict local minimizer of order k, denoted $x^0 \in \mathrm{Strl}(k, f, S)$, for the scalar optimization problem

$$\min\{f(x) : x \in S\} \tag{1}$$

if there exist $\alpha > 0$ and a neighborhood U of x^0 such that

$$f(x) > f(x^0) + \alpha \parallel x - x^0 \parallel^k, \ \ \forall x \in U \cap S \setminus \{x^0\}. \tag{2}$$

There are other names for strict local efficiency of order k in literatures: firm or isolated efficiency of order k (see, e.g., [2,8]). Under the assumption that these minimizers are often exactly those satisfied an kth directional derivatives test, Auslender [2] derived optimality conditions

Received: 2020-08-27. Revised: 2022-08-29.
MR Subject Classification: 90C46.
Keywords: strict local minima of order k, multiobjective optimization, higher-order optimality conditions, higher-order directional derivatives.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-4242-9.

characterizing such minima. Studniarski [3] generalized the Auslender's results to any extended real-valued function f and any subset S of $\mathbb{R}^n$ by using directional derivatives that are generalizations of lower and upper Hadamard directi onal derivatives. Applying lower and upper Studniarski derivatives, Jiménez [4] deduced optimality conditions of strict local minimizer of order k for problem (1) extending results of [3]. About fifteen years ago, Ginchev [5] extended the scalar notion of strict local minimizer of order k to unconstrained multiobjective problems. Recall that given $x^0 \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^p$ and $C \subset \mathbb{R}^p$ is a closed, convex and pointed cone, and $C^* \subset \mathbb{R}^p$ is the positive polar cone of C. x^0 is called a strict local (Pareto) minimizer of order k for f (on $\mathbb{R}^n$) if x^0 is a strict local minimizer of order k for the scalar function

$$\varphi(x) = \sup\{\langle \xi, f(x) - f(x^0)\rangle : \xi \in C^* \text{and} \parallel \xi \parallel = 1\},$$

i.e., there exist $\alpha > 0$ and a neighborhood U of x^0 such that

$$\sup\{\langle \xi, f(x) - f(x^0)\rangle : \xi \in C^* \text{and} \parallel \xi \parallel = 1\} > \alpha \parallel x - x^0 \parallel^k, \ \ \forall x \in U \setminus \{x^0\}. \tag{3}$$

Remark 1.1 Naturally, we define a strict local minima of order k for a multiobjective optimization problem with a set constraint (problem (4) below) by substituting U in (3) with $U \cap S$.

Jiménez [6] extended the notion (2) to vector optimization with an objective function defined on a normed space and with an arbitrary set constraint, then the stronger notion of superstrict local minimizer of order k with the final space $\mathbb{R}^p$ was introduced. Moreover, necessary conditions for strict and superstrict local minimizer of order k were established in [6]. Sufficient conditions for the latter were derived by the same author in [7]. Here we underline that these scalar and multiobjective (vector) optimality conditions [4,6,7] via directional derivatives used in these papers result from or are closely related to the scalar characterizations of strict local minima of order k [3]. Inspired and encouraged by works of [3,4,6], in this paper we first introduce a type of lower (upper) directional derivatives for a multiobjective function, which is the extension of lower (upper) Studniarski derivatives; then we establish both necessary and sufficient conditions of strict local minimizer of order k ($k \geq 1$) for (4) that extend the scalar characterizations [3]; finally we derive necessary not sufficient and sufficient not necessary conditions for this minima for (4) extending scalar conditions [3,4,6]. Of course, there exist some other good papers about this topic via directional derivatives in recent years, e.g., [13,14].

This paper is structured as follows. In the next section (Section 2), some preliminaries and notations needed in the sequel are presented. In Section 3, the notions of indicator functions and lower (upper) directional derivatives of order k for a multiobjective function are introduced. In Section 4, necessary and sufficient conditions for strict local minimizer of order k ($k \geq 1$) for (4) are developed. Finally in Section 5, our results are compared with some before results of scalar and multiobjective optimization.

§2 Preliminaries and notations

Let $\mathbb{R}^p_+$ and $\mathbb{N}^+$ be the nonnegative orthant of $\mathbb{R}^p$ and the set of all positive integers respectively. Denote by C a closed, convex and pointed cone in $\mathbb{R}^p$ and C^* the positive polar cone of

C, i.e., $C^* = \{\xi \in \mathbb{R}^p : \langle \xi, y \rangle \geq 0 \text{ for all } y \in C\}$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $\Gamma = \{\xi \in C^* : \| \xi \| = 1\}$, and let $k \geq 1$ be an integer unless distinguishing the cases $k > 1$ and $k = 1$. As it is usual, $\mathrm{intcone}M$ and $\mathrm{cl}M$ denote the interior of the cone generated by and the closure of a set $M \subset \mathbb{R}^n$, respectively.

In this paper, we consider the multiobjective optimization problem

$$\min\{f(x) : x \in S\} \tag{4}$$

where $f = (f_1, f_2, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ is an arbitrary mapping, $S \subset \mathbb{R}^n$ is a nonempty set, the point $x^0 \in S$ unless otherwise stated, and $C = \mathbb{R}^p_+$.

We now recall some well-known notions and results needed in the sequel.

Definition 2.1 Let M be a nonempty set of $\mathbb{R}^n$. The tangent cone to M at $x^0 \in \mathrm{cl}M$ is

$$T(M, x^0) := \{v \in X : \exists t_n > 0, \exists x_n \in M, x_n \to x^0 \text{ such that } t_n(x_n - x^0) \to v\}.$$

Definition 2.2 Let M be a nonempty set of $\mathbb{R}^n$. Then a mapping $I_M : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$I_M(x) = \begin{cases} 0 & if\ x \in M, \\ +\infty & if\ x \notin M \end{cases}$$

is said to be the indicator function of the set M.

Definition 2.3 Let a function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $x^0 \in \mathbb{R}^n$. We call the following limits

$$\underline{d}h(x^0, v) = \liminf_{(t,u) \to (0^+, v)} \frac{h(x^0 + tu) - h(x^0)}{t}$$

and

$$\overline{d}h(x^0, v) = \limsup_{(t,u) \to (0^+, v)} \frac{h(x^0 + tu) - h(x^0)}{t}$$

the lower and upper Hadamard directional derivative of h at x^0 in the direction $v \in \mathbb{R}^n$, respectively.

Moreover, we call

$$\lim_{(t,u) \to (0^+, v)} \frac{h(x^0 + tu) - h(x^0)}{t}$$

the Hadarmard directional derivative of h at x^0 in the direction $v \in \mathbb{R}^n$, denoted by $dh(x^0, v)$, when this limit exists.

When $dh(x^0, v)$ exists for all $v \in \mathbb{R}^n$, we say that h is Hadamard directional differentiable at x^0.

Definition 2.4 [3,4] Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $x^0 \in \mathbb{R}^n$. The lower (resp. upper) Studniarski derivative of order k for h at x^0 in the direction $v \in \mathbb{R}^n$ is defined by

$$\underline{d}^k h(x^0, v) = \liminf_{(t,u) \to (0^+, v)} \frac{h(x^0 + tu) - h(x^0)}{t^k},$$

$$(resp.\ \overline{d}^k h(x^0, v) = \limsup_{(t,u) \to (0^+, v)} \frac{h(x^0 + tu) - h(x^0)}{t^k}).$$

If $k = 1$, $\underline{d}^1 h(x^0, v)$ and $\overline{d}^1 h(x^0, v)$ are the lower and upper Hadamard derivative, respectively.

Definition 2.5 [9] Let $\overline{\mathbb{R}^p}$ be the extended Euclidean space of $\mathbb{R}^p$ which is defined as the cartesian product of p copies of $\overline{\mathbb{R}}$. The operations of addition and scalar multiplication in $\overline{\mathbb{R}^p}$

are performed componentwise whenever the respective operations in $\overline{\mathbb{R}}$ are defined. There, we adopt the conventions $0 \cdot (+\infty) = (+\infty) \cdot 0 = 0 \in \mathbb{R}$.

Studniarski derived characterizations of strict local minima of order k for (1) as follows.

Theorem 2.1 [3, Theorem 2.1] Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x^0 \in S \subset \mathbb{R}^n$.

(i) If $k > 1$, then the following three conditions are equivalent:

(a) $x^0 \in \mathrm{Strl}(k, f, S)$;

(b) for all $v \in \mathbb{R}^n \setminus \{0\}$, we have

$$\underline{d}^k(f + I_S)(x^0, v) > 0;$$

(c) the previous inequality holds for all $v \in K(x^0) \setminus \{0\}$;

(ii) If $k = 1$, the analogous equivalences are true with condition (c) replaced by the following one:

(c$'$) the previous inequality holds for all $v \in T(S, x^0) \setminus \{0\}$.

Here, $K(x^0) = T(S, x^0) \cap \{v \in \mathbb{R}^n : \underline{d}f(x^0, v) \leq 0\}$.

Sufficient conditions of superstrict local minima of order k for (4) were deduced by Jiménez as follows.

Theorem 2.2 [7, Theorem 3.1(b)$\Rightarrow$ (a)] If $\forall v \in \mathbb{R}^n \setminus \{0\}, \exists i, 1 \leq i \leq p$, such that $\underline{d}^k(f_i + I_S)(x^0, v) > 0$, then $x^0 \in \mathrm{SStrl}(k, f, S)$.

For strict and superstrict efficiency of order k for (4), the relation between them is as follows.

Theorem 2.3 [6, Corollary 3.10] Let $f : \mathbb{R}^n \to \mathbb{R}^p$ and $x^0 \in S \subset \mathbb{R}^n$. Then

(i) Let $k > 1$. $x^0 \in \mathrm{SStrl}(k, f, S) \Rightarrow x^0 \in \mathrm{Strl}(k, f, S)$; the converse is not true.

(ii) Let $k = 1$. $x^0 \in \mathrm{SStrl}(1, f, S) \Leftrightarrow x^0 \in \mathrm{Strl}(1, f, S)$.

Theorem 2.4 [11, Charpter 2, Proposition 1.1] Let function $\gamma : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$, $\zeta(v) := \liminf\limits_{(t,u) \to (0^+, v)} \gamma(t, u)$ and $\theta(v) := \limsup\limits_{(t,u) \to (0^+, v)} \gamma(t, u)$. Then the functions $\zeta : \mathbb{R}^n \to \mathbb{R}$ and $\theta : \mathbb{R}^n \to \mathbb{R}$ are lower and upper semicontinuous, respectively.

§3 Indicator function and directional derivatives

In this section, we introduce a concept of a indicator function with values in $\overline{\mathbb{R}^p}$ extending the usual indicator function (Definition 2.2), and a type of directional derivatives for a multiobjective function extending the scalar lower (upper) Studniarski derivatives (Definition 2.4).

Definition 3.1 Let M be a nonempty set of $\mathbb{R}^n$. We call the mapping $\mathbb{I}_M = (I_M^{(1)}, I_M^{(2)}, \cdots, I_M^{(p)}) : \mathbb{R}^n \to \overline{\mathbb{R}^p}$ with each component $I_M^{(i)}$ defined as Definition 2.2, i.e., $I_M^{(i)} = I_M, i = 1, 2, \cdots, p$, the indicator function of the set M.

Obviously, this concept reduces to Definition 2.2 when $p = 1$.

From Definitions 2.2 and 3.1, we can easily deduce that for any $\xi \in \Gamma$,

$$\xi \circ \mathbb{I}_M = I_M. \tag{5}$$

We next give a notion of directional derivatives for a multiobjective function.

Definition 3.2 Let a function $h : \mathbb{R}^n \to \mathbb{R}^p$ and $x^0 \in \mathbb{R}^n$. Then the lower and upper directional derivatives of order k for h at x^0 in the direction $v \in \mathbb{R}^n$ are defined by

$$d_l^k h(x^0, v) = \sup_{\xi\in\Gamma} \liminf_{(t,u)\to(0^+,v)} \frac{\langle \xi, h(x^0+tu)-h(x^0)\rangle}{t^k}, \tag{6}$$

and

$$d_u^k h(x^0, v) = \sup_{\xi\in\Gamma} \limsup_{(t,u)\to(0^+,v)} \frac{\langle \xi, h(x^0+tu)-h(x^0)\rangle}{t^k}, \tag{7}$$

respectively.

Remark 3.1 (i) Clearly, since $\xi \in \Gamma = \{1\}$ when $p = 1$, those derivatives reduce to lower and upper Studniarski derivatives of order 1 (lower and upper Hadmard directional derivatives) respectively.

(ii) It immediately holds from Definition 3.2 that

$$d_l^k h(x^0, v) = \sup_{\xi\in\Gamma} \underline{d}^k(\xi\circ h)(x^0, v) \tag{8}$$

and

$$d_u^k h(x^0, v) = \sup_{\xi\in\Gamma} \overline{d}^k(\xi\circ h)(x^0, v) \tag{9}$$

for any $v \in \mathbb{R}^n$.

(iii) We have that

$$d_l^k h(x^0, v) \neq \min_{1\le i\le p} \underline{d}^k h_i(x^0, v)$$

and

$$d_u^k h(x^0, v) \neq \max_{1\le i\le p} \overline{d}^k h_i(x^0, v)$$

for all $v \in \mathbb{R}^n \setminus \{0\}$.

The reason is that $\Gamma := \{\xi \in \mathbb{R}^p_+ : \parallel \xi \parallel = 1\} \neq \{e_i \in \mathbb{R}^p : e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with its ith component 1 and others 0, $i = 1, 2, \cdots, p\}$.

(iv) If h is of C^2 class and $\nabla h(x^0) = 0$, it is easy to derive that

$$d_l^2 h(x^0, v) = d_u^2 h(x^0, v) = \nabla^2 h(x^0)(v, v)$$

since $\Gamma = \{1\}$.

We also have the following properties of the above new derivatives.

Proposition 3.1 $d_l^k h(x^0, \cdot)$ (resp. $d_u^k h(x^0, \cdot)$) is lower (resp. upper) semicontinous and positively homogeneous of degree k on $\mathbb{R}^n$.

Proof Let $\xi \in \Gamma$ and $v \in \mathbb{R}^n$. Take any $\epsilon > 0$, by Theorem 2.4, $\underline{d}^k(\xi\circ h)(x^0, \cdot)$ is lower semicontinuous at v, i.e., taking any ϵ' $(0 < \epsilon' < \epsilon)$, there exists $\delta > 0$ such that for all $u \in B(v, \delta)$

$$\underline{d}^k(\xi\circ h)(x^0, u) > \underline{d}^k(\xi\circ h)(x^0, v) - \epsilon'.$$

Consequently

$$\sup_{\xi\in\Gamma} \underline{d}^k(\xi\circ h)(x^0, u) \ge \sup_{\xi\in\Gamma} \underline{d}^k(\xi\circ h)(x^0, v) - \epsilon'.$$

Hence

$$\sup_{\xi\in\Gamma} \underline{d}^k(\xi\circ h)(x^0, u) > \sup_{\xi\in\Gamma} \underline{d}^k(\xi\circ h)(x^0, v) - \epsilon.$$

It follows from (8) that

$$d_l^k h(x^0, u) > d_l^k h(x^0, v) - \epsilon.$$

So $d_l^k h(x^0, \cdot)$ is lower semicontinous at $v \in \mathbb{R}^n$.

Let any $\lambda > 0$, we can easily to see from Definition 3.2 that $d_l^k h(x^0, \lambda v) = \lambda^k d_l^k h(x^0, v)$. So $d_l^k h(x^0, \cdot)$ is positively homogeneous of degree k on $\mathbb{R}^n$.

Analogously, $d_u^k h(x^0, \cdot)$ is upper semicontinous and positively homogenously of degree k on $\mathbb{R}^n$.

Observing that $\xi \circ (h + \mathbb{I}_S) = \xi \circ h + I_S$, using (8) and the fact that $\underline{d}^k(\varphi + \mathbb{I}_S)(x^0, v) \geq \underline{d}^k \varphi(x^0, v)$ for a scalar function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$, we have the following immediate consequence.

Proposition 3.2 Let function $f : \mathbb{R}^n \to \mathbb{R}^p$ and $x^0 \in S \subset \mathbb{R}^n$. Then it holds that

$$d_l^k(f + \mathbb{I}_S)(x^0, v) \geq d_l^k f(x^0, v), \quad \forall v \in \mathbb{R}^n.$$

We have the following relations between our directional derivatives and Studniarski derivatives.

Proposition 3.3 Suppose $f = (f_1, f_2, \cdots, f_p) : \mathbb{R}^n \to \overline{\mathbb{R}^p}$, $k > 1$, and $x^0 \in \mathbb{R}^n$. Then, for any given $v \in \mathbb{R}^n \setminus \{0\}$, we have

(i) $\exists i = i(v), 1 \leq i \leq p$, $\underline{d}^k f_i(x^0, v) > 0 \Rightarrow d_l^k f(x^0, v) > 0$.

(ii) $\exists i = i(v), 1 \leq i \leq p$, $\overline{d}^k f_i(x^0, v) > 0 \Leftrightarrow d_u^k f(x^0, v) > 0$.

Proof (i) Let $v \in \mathbb{R}^n \setminus \{0\}$, there exists $i = i(v), 1 \leq i \leq p$, such that

$$\underline{d}^k f_i(x^0, v) > 0.$$

Put $\hat{\xi} = (0, \cdots, 0, 1, 0, \cdots, 0) \in \Gamma$, with the ith component $\hat{\xi}_i = 1$ and each of the rest equal to 0, then we immediately obtain the conclusion from (8).

(ii) " $\Rightarrow$ " is similar to that in (i).

" $\Leftarrow$ ": If $d_u^k f(x^0, v) > 0$, then from the definition (7) there exists $\hat{\xi} \in \Gamma$ satisfying

$$\limsup_{(t,u)\to(0^+,v)} \frac{\langle \hat{\xi}, f(x^0 + tu) - f(x^0) \rangle}{t^k} > 0$$

i.e.,

$$\limsup_{(t,u)\to(0^+,v)} \frac{\sum_{j=1}^{p} \hat{\xi}_j \left(f_j(x^0 + tu) - f_j(x^0) \right)}{t^k} > 0. \tag{10}$$

Since

$$\limsup_{(t,u)\to(0^+,v)} \frac{\sum_{j=1}^{p} \hat{\xi}_j \left(f_j(x^0 + tu) - f_j(x^0) \right)}{t^k} \leq \sum_{j=1}^{p} \left(\limsup_{(t,u)\to(0^+,v)} \frac{\hat{\xi}_j \left(f_j(x^0 + tu) - f_j(x^0) \right)}{t^k} \right),$$

then noticing (10) there exists $i = i(v), 1 \leq i \leq p$, such that

$$\limsup_{(t,u)\to(0^+,v)} \frac{\hat{\xi}_i (f_i(x^0 + tu) - f_i(x^0))}{t^k} > 0.$$

Thus, we have

$$\overline{d}^k f_i(x^0, v) = \limsup_{(t,u)\to(0^+,v)} \frac{f_i(x^0 + tu) - f_i(x^0)}{t^k} > 0$$

due to that $\hat{\xi}_i \geq 0$.

Notice that the implications in (i) and (ii) in Proposition 3.3 are different (since the converse in (i) is not true as showed in Remark 4.1 in Sect. 4). Indeed, it is because of the difference between "liminf" and "limsup".

Rahmo and Studniarski [10] gave higher-order directional derivatives $\underline{d}^k h(x^0,v)$ and $\overline{d}^k h(x^0,v) \subset \overline{\mathbb{R}^p}$ (with the help of the symbols of scalar derivatives of order k above for simplicity) for multiobjective programming to derive higher-order optimality conditions, and stated

$$\underline{d}^k h(x^0,v) = (\underline{d}^k h_1(x^0,v), \underline{d}^k h_2(x^0,v), \cdots, \underline{d}^k h_p(x^0,v)) \tag{11}$$

and

$$\overline{d}^k h(x^0,v) = (\overline{d}^k h_1(x^0,v), \overline{d}^k h_2(x^0,v), \cdots, \overline{d}^k h_p(x^0,v)).$$

We deduce the relation between these derivatives and our directional derivatives as follows.

Proposition 3.4 For any $v \in \mathbb{R}^n$, it follows that

(i) $d_l^k h(x^0,v) \geq \sup\limits_{\xi\in\Gamma} [\xi \circ \underline{d}^k h(x^0,v)]$;

(ii) $d_u^k h(x^0,v) \leq \sup\limits_{\xi\in\Gamma} [\xi \circ \overline{d}^k h(x^0,v)]$.

Proof (i) For any $v \in \mathbb{R}^n$, it holds from (8) that

$$\begin{aligned} & d_l^k h(x^0,v) \\ = \ & \sup_{\xi\in\Gamma} \underline{d}^k (\xi \circ h)(x^0,v) \\ = \ & \sup_{\xi\in\Gamma} \liminf_{(t,u)\to(0^+,v)} \left(\frac{\xi_1 \cdot [h_1(x^0+tu) - h_1(x^0)]}{t^k} + \cdots + \frac{\xi_p \cdot [h_p(x^0+tu) - h_p(x^0)]}{t^k} \right) \geq \sup_{\xi\in\Gamma} \\ & \left(\liminf_{(t,u)\to(0^+,v)} \frac{\xi_1 \cdot [h_1(x^0+tu) - h_1(x^0)]}{t^k} + \cdots + \liminf_{(t,u)\to(0^+,v)} \frac{\xi_p \cdot [h_p(x^0+tu) - h_p(x^0)]}{t^k} \right) \\ = \ & \sup_{\xi\in\Gamma} \sum_{i=1}^p \xi_i \, \underline{d}^k h_i(x^0,v) \\ = \ & \sup_{\xi\in\Gamma} [\xi \circ \underline{d}^k h(x^0,v)]. \end{aligned}$$

(ii) Analogous to that in (i).

§4 Higher-order optimality conditions

In this section, we shall deduce characterizations of strict local minima of order k for function f in (4), which are the corresponding extensions of Theorem 2.1, and shall deduce necessary not sufficient and sufficient not necessary conditions for this minima for (4). From now on we let $K(x^0) := T(S,x^0) \cap \{v \in \mathbb{R}^n : d_l^1 f(x^0,v) \leq 0\}$.

Theorem 4.1 (i) If $k > 1$, then the following conditions are equivalent:

(a) $x^0 \in \mathrm{Strl}(k,f,S)$;

(b) for all $v \in \mathbb{R}^n \setminus \{0\}$, we have

$$d_l^k (f + \mathbb{I}_S)(x^0,v) > 0;$$

(c) for all $v \in K(x^0) \setminus \{0\}$, we have

$$d_l^k (f + \mathbb{I}_S)(x^0,v) > 0;$$

(d) for any $v \in \mathbb{R}^n \setminus \{0\}$, there exists $\beta = \beta(v) > 0$ such that

$$d_l^k (f + \mathbb{I}_S)(x^0,v) \geq \beta \parallel v \parallel^k;$$

(e) for any $v \in K(x^0) \setminus \{0\}$, there exists $\beta = \beta(v) > 0$ such that

$$d_l^k (f + \mathbb{I}_S)(x^0,v) \geq \beta \parallel v \parallel^k;$$

(ii) If $k=1$, then those conditions above are equivalent when $K(x^0)$ is replaced by $T(S,x^0)$ in (c) and (e).

Proof (i) (a)$\Rightarrow$ (b): Contrary to the conclusion, suppose that (b) is false, then there exists $v\in\mathbb{R}^n\setminus\{0\}$ satisfying $d_l^k(f+\mathbb{I}_S)(x^0,v)\le 0$. Hence, for any $\xi\in\Gamma$ we have that for each $\delta>0$ and each $V\subset\mathbb{R}^n$,

$$\inf_{t\in(0,\delta),u\in V}\frac{\langle\xi,(f+\mathbb{I}_S)(x^0+tu)-(f+\mathbb{I}_S)(x^0)\rangle}{t^k}\le 0. \tag{12}$$

In particular, we may choose δ and V such that

$$V\subset\{u\in X:\parallel u-v\parallel<\frac{\parallel v\parallel}{2}=c\} \tag{13}$$

and

$$x^0+tu\in U \text{ for all } t\in(0,\delta) \text{ and all } u\in V, \tag{14}$$

where U is such as in *Remark* 1.1 (or (3)).

Take any $\varepsilon>0$. By (12), there exists $t'\in(0,\delta)$ and $u'\in V$ satisfying

$$\frac{\langle\xi,(f+\mathbb{I}_S)(x^0+t'u')-(f+\mathbb{I}_S)(x^0)\rangle}{t'^k}\le\varepsilon. \tag{15}$$

Then

$$x^0+t'u'\in S. \tag{16}$$

By (13)

$$\parallel u'-v\parallel<c.$$

So

$$\parallel u'\parallel>c. \tag{17}$$

Hence, from (14) and (16) we have

$$x^0+t'u'\in U\cap S. \tag{18}$$

From (5), (16), (17) and the assumption of (a), we obtain

$$\varepsilon\ge\frac{\langle\xi,(f+\mathbb{I}_S)(x^0+t'u')-(f+\mathbb{I}_S)(x^0)\rangle}{t'^k}>\alpha\cdot c^k>0.$$

Since $\varepsilon>0$ is arbitrary, then it follows that

$$0\ge\alpha\cdot c^k>0,$$

which is a contradiction.

(b)$\Rightarrow$ (c) is trivial.

(c)$\Rightarrow$ (a): If the set $W:=\{u\in K(x^0):\parallel u\parallel=1\}\ne\emptyset$, then since $W\subset\mathbb{R}^n$ is compact and $d_l^k(f+\mathbb{I}_S)(x^0,\cdot)$ is lower semicontinuous by Proposition 3.1, we deduce from (c) that there exists $m=\min\{d_l^k(f+\mathbb{I}_S)(x^0,v):v\in W\}>0$. Suppose $x^0\notin\mathrm{Strl}(k,f,S)$, then we can choose a sequence $\{x_n\}\subset S$ satisfying $x_n\to x^0,x_n\ne x^0$, for any $\xi\in\Gamma$, such that

$$\langle\xi,f(x_n)-f(x^0)\rangle\le\frac{m}{2}\parallel x_n-x^0\parallel^k \text{ for all } n\in\mathbb{N}^+. \tag{19}$$

For each $n\in\mathbb{N}^+$, let $t_n:=\parallel x_n-x^0\parallel$ and $v_n:=(x_n-x^0)/\parallel x_n-x^0\parallel$. Then $t_n\to 0^+$, and we may assume, without loss of generality, that $v_n \to v$ with $\| v \|= 1$. Thus, $v \in T(S, x^0) \setminus \{0\}$. Moreover, it follows from the definition of lower limit, $k > 1$ and (18) that

$$\begin{aligned} d_l^1 f(x^0, v) &\leq \sup_{\xi\in\Gamma}\liminf_{n\to+\infty} \frac{\langle \xi, f(x^0 + t_n u_n) - f(x^0)\rangle}{t_n} \\ &= \sup_{\xi\in\Gamma}\liminf_{n\to+\infty} \frac{\langle \xi, f(x_n) - f(x^0)\rangle}{t_n} \\ &\leq \sup_{\xi\in\Gamma}\lim_{n\to+\infty} (\frac{m}{2} \cdot t_n^{k-1}) \\ &= 0. \end{aligned}$$

So $v \in \{v \in \mathbb{R}^n : d_l^1 f(x^0, v) \leq 0\}$.
Hence $v \in W$, which implies $d_l^k(f + \mathbb{I}_S)(x^0, v) \geq m$.
Then, there exists from (8) $\hat{\xi} \in \Gamma$ such that $\overline{d}^k(\hat{\xi} \circ (f + \mathbb{I}_S))(x^0, v) > \frac{m}{2}$. So, there exist $\delta > 0$ and a neighborhood V of v such that

$$\frac{\langle \hat{\xi}, (f + \mathbb{I}_S)(x^0 + tu) - (f + \mathbb{I}_S)(x^0)\rangle}{t^k} > \frac{m}{2} \tag{20}$$

for all $t \in (0, \delta)$ and all $u \in V$ satisfying $x^0 + tu \in S$.
For $x_n \in S$, $t_n < \delta$ and $v_n \in S$ for all $n \in \mathbb{N}^+$ large enough and (20), we have

$$\frac{\langle \hat{\xi}, f(x_n) - f(x^0)\rangle}{t_n^k} > \frac{m}{2},$$

which contradicts (19).
Therefore, (a)$\Leftrightarrow$ (b)$\Leftrightarrow$ (c).

(c)$\Rightarrow$ (e): For $v \in K(x^0) \setminus \{0\}$, by (c) there exists $\eta \in \mathbb{R}$ such that

$$\sup_{\xi\in\Gamma}\liminf_{(t,u)\to(0^+,v)} \frac{\langle \xi, (f + \mathbb{I}_S)(x^0 + tu) - (f + \mathbb{I}_S)(x^0)\rangle}{t^k} = \eta > 0.$$

So, for any β satisfying $0 < \beta < \eta$ we have

$$\sup_{\xi\in\Gamma}\liminf_{(t,u)\to(0^+,v)} \frac{\langle \xi, (f + \mathbb{I}_S)(x^0 + tu) - (f + \mathbb{I}_S)(x^0)\rangle}{t^k} > \beta.$$

Then, there exists $\hat{\xi} \in \Gamma$ such that

$$\liminf_{(t,u)\to(0^+,v)} \frac{\langle \hat{\xi}, (f + \mathbb{I}_S)(x^0 + tu) - (f + \mathbb{I}_S)(x^0)\rangle}{t^k} \geq \beta.$$

It follows from $\beta = \beta \| v \|^k$ that

$$\liminf_{(t,u)\to(0^+,v)} \frac{\langle \hat{\xi}, (f + \mathbb{I}_S)(x^0 + tu) - (f + \mathbb{I}_S)(x^0)\rangle}{t^k} \geq \beta \| v \|^k .$$

Consequently

$$d_l^k(f + \mathbb{I}_S)(x^0, v) \geq \beta \| v \|^k$$

by the definition (6).

(e)$\Rightarrow$ (c) is trivial.

(b)$\Rightarrow$ (d): Similar to that (c)$\Rightarrow$ (e).

(d)$\Rightarrow$ (b) is trivial.

Thus, $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$.

If $W=\emptyset$, the proof is valid for any $m>0$.
(ii) Analogous to that in (i).

Remark 4.1 (i) When $p=1$ this theorem reduces to the scalar characterizations (Theorem 2.1) since our derivatives reduce to lower (upper) Studniarski derivatives.

(ii) By means of this theorem, we can infer that the converse of Proposition 3.3 (i) is not true.

Proof For any $v\in\mathbb{R}^n\backslash\{0\}$, if $d_l^k f(x^0,v)>0$, then

$$d_l^k(f+\mathbb{I}_S)(x^0,v)>0 \tag{21}$$

for all $v\in\mathbb{R}^n\setminus\{0\}$ due to Proposion 3.2.
By Theorem 4.1 (a)$\Leftrightarrow$(b), we have

$$x^0\in \mathrm{Strl}(k,f,S)\Leftrightarrow \text{the inequality (21) holds.}$$

Suppose on the contrary that the converse of Proposition 3.3 (i) is true, i.e., there exists $i=i(v), 1\le i\le p$, such that

$$\underline{d}^k(f_i+I_S)(x^0,v)>0.$$

Then $x^0\in \mathrm{SStrl}(k,f,S)$ by Theorem 2.2.
Thus, $x^0\in \mathrm{Strl}(k,f,S)\Rightarrow x^0\in \mathrm{SStrl}(k,f,S)$, which contradicts Theorem 2.3 (i).

Obviously, Theorem 4.1 can be employed as a sufficiency of $x^0\in \mathrm{Strl}(k,f,S)$ for (4). But it is difficult to check from the view of practical point since it involves the indicator function $\mathbb{I}_S$ with values in $\overline{\mathbb{R}^p}$. This drawback can be eliminated as the following corollary shows due to Proposition 3.2.

Corollary 4.1 (Sufficient conditions of strict local minima of order k for (4)) (i) If $k>1$, then each of the following conditions is a sufficiency for $x^0\in \mathrm{Strl}(k,f,S)$:
(a) $d_l^k f(x^0,v)>0$ for all $v\in\mathbb{R}^n\setminus\{0\}$;
(b) $d_l^k f(x^0,v)>0$ for all $v\in K(x^0)\setminus\{0\}$;
(c) for any $v\in\mathbb{R}^n\setminus\{0\}$, there exists $\beta=\beta(v)>0$ such that

$$d_l^k f(x^0,v)\ge\beta\parallel v\parallel^k;$$

(d) for any $v\in K(x^0)\setminus\{0\}$, there exists $\beta=\beta(v)>0$ such that

$$d_l^k f(x^0,v)\ge\beta\parallel v\parallel^k .$$

(ii) If $k=1$, then each of the conditions above is a sufficiency of $x^0\in \mathrm{Strl}(1,f,S)$ when $K(x^0)$ in (b) and (d) is replaced by $T(S,x^0)$.

We give an example to illustrate Corollary 4.1.

Example 4.1 (i) (The case $k>1$) Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x)=(f_1(x),f_2(x))$ where $f_1(x)=\begin{cases}-x^2 & \text{if } x\ge 0,\\ x^2 & \text{if } x<0,\end{cases}$ $f_2(x)=\begin{cases}x^2 & \text{if } x\ge 0,\\ -x^2 & \text{if } x<0,\end{cases}$ $x^0=0\in\mathbb{R}$ and let $S=[-1,1]\subset\mathbb{R}$.

We have $\sup\limits_{\xi\in\Gamma}\langle\xi,f(x)-f(x^0)\rangle=x^2>\frac{1}{2}x^2=\frac{1}{2}\parallel x-x^0\parallel^2$ for all $x\in U\cap S\setminus\{x^0\}$. By *Remark* 1.1, $x^0\in \mathrm{Strl}(2,f,S)$ [take $\alpha=\frac{1}{2}$ and $U=\mathbb{R}$]. On the other hand, for any $v\in\mathbb{R}\setminus\{0\}$

we have

$$
\begin{aligned}
d_l^2 f(x^0, v) &= \sup_{\xi\in\Gamma} \liminf_{(t,u)\to(0^+,v)} \frac{\xi_1 f_1(tu) + \xi_2 f_2(tu)}{t^2} \\
&= \sup_{\xi\in\Gamma} \begin{cases} (-\xi_1+\xi_2)v^2 & \text{if } v>0, \\ (\xi_1-\xi_2)v^2 & \text{if } v<0, \end{cases} \\
&= v^2 > 0,
\end{aligned}
$$

which satisfies the sufficient conditions (a), (b), (c) and (d) in Corollary 4.1 (take $\beta = 1$). Hence the same conclusion as the above is obtained.

(ii) (The case $k = 1$) Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by

$$
f(x) = \begin{cases} (0, x(2-\sin\frac{1}{x})) & \text{if } x \neq 0, \\ (0,0) & \text{if } x = 0, \end{cases}
$$

$x^0 = 0$ and $S = \mathbb{R}_+$. We can easily obtain that $x^0 \in \mathrm{Strl}(1, f, S)$ from *Remark* 1.1 (take $\alpha = 1$ and U is an arbitrary neighborhood of x^0). Let $\hat{\xi} = (0, 1)$. By computation, we have that $T(S, x^0) = \mathbb{R}_+$, and for any $v \in T(S, x^0)\backslash\{0\}$ it follows that

$$
\begin{aligned}
d_l^1 f(0, v) &= \liminf_{(t,u)\to(0^+,v)} \tfrac{1}{t}\langle \hat{\xi}, f(0+tu)\rangle \\
&= \liminf_{(t,u)\to(0^+,v)} u(2-\sin\tfrac{1}{tu}) \\
&= v > 0.
\end{aligned}
$$

By Corollary 4.1, we have the same conclusion as the above.

Remark 4.2 Each of the sufficient conditions in this corollary is not necessary as the following example shows.

Example 4.2 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$
f(x_1, x_2) = \begin{cases} (\mid x_1 \mid^k, -\mid x_1 \mid) & \text{if } x_2 = 0, \\ (0,0) & \text{if } x_2 \neq 0 \end{cases}
$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let $x^0 = (0, 0) \in \mathbb{R}^2$ and $S = \mathbb{R} \times \{0\}$.

For all $x = (x_1, 0) \in U \cap S \setminus \{x^0\} = \{(x_1, 0) : x_1 \in \mathbb{R} \text{ and } x_1 \neq 0\}$, where U is an arbitrary neighborhood of x^0, we can compute that $\sup\limits_{\xi\in\Gamma}\langle \xi, f(x) - f(x^0)\rangle = \sup\limits_{\xi\in\Gamma}(\xi_1 \mid x_1 \mid^k - \xi_2 \mid x_1 \mid) = \mid x_1 \mid^k > \frac{1}{3} \mid x_1 \mid^k = \frac{1}{3} \parallel x - x^0 \parallel^k$. So, $x^0 \in \mathrm{Strl}(k, f, S)$ due to *Remark* 1.1. However, from easy computation we have that $K(x^0) = \mathbb{R} \times \{0\}$ and $d_l^k f(x^0, v) = 0$ for all $v \in K(x^0) \setminus \{(0, 0)\}$, which doesn't satisfy each of the sufficient conditions in Corollary 4.1.

We give the necessary optimality condition for strict local minimizer of order k for (4) as follows.

Theorem 4.2 (Necessary condition of strict local minima of order k for (4)) If $x^0 \in \mathrm{Strl}(k, f, S)$, then

$$
d_u^k f(x^0, v) > 0
$$

for all $v \in T(S, x^0) \setminus \{0\}$.

Proof Contrary to the conclusion, suppose that there exists $v \in T(S, x^0) \setminus \{0\}$ satisfying $d_u^k f(x^0, v) \leq 0$. By the definition (7), for all $\xi \in \Gamma$ and any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and a neighborhood V of v such that

$$\frac{\langle \xi, f(x^0+tu)-f(x^0)\rangle}{t^k} \leq \varepsilon \tag{22}$$

for all $t \in (0,\delta)$ and all $u \in V$.

We may assume, without loss of generality, that δ and V satisfy conditions (13) and (14). Since $v \in T(S,x^0) \setminus \{0\}$, it follows that

$$(x^0+tV) \cap S \neq \emptyset.$$

Thus, there exist $t^{'} \in (0,\delta)$ and $u^{'} \in V$ satisfying (13), (14) and (22). The remaining part of the proof is the same as in Theorem 4.1 (a)⇒ (b).

We present an example to illustrate Theorem 4.2.

Example 4.3 (i) (The case $k>1$) Let $x^0 = 0 \in \mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x) = \begin{cases} (x^2/2, \int_0^x t^2\sin(1/t^2)dt & \text{if } x\neq 0,\\ (0,0) & \text{if } x=0,\end{cases}$ and let $S=[0,1)$.

By usual calculus, we can verify that $\frac{x^2}{2} \geq \int_0^x t^2\sin(1/t^2)dt$ for all $x\in S$. So it is easy to obtain $x^0 \in \mathrm{Strl}(2,f,S)$. Moreover, taking $\hat{\xi}=(1,0)\in\Gamma\subset\mathbb{R}^2$ we have from the definition (7) that

$$\begin{aligned} d_u^2 f(x^0,v) &\geq \limsup_{(t,u)\to(0^+,v)} \frac{\langle\hat{\xi}, f(x^0+tu)-f(x^0)\rangle}{t^2} \\ &= \lim_{(t,u)\to(0^+,v)} \frac{\frac{t^2}{2}u^2}{t^2} \\ &= \frac{1}{2}v^2 > 0 \end{aligned}$$

for all $v\in T(S,x^0)\backslash\{0\} = \mathbb{R}_+\setminus\{0\}$. By Theorem 4.2, we obtain the same conclusion as the above.

(ii) (The case k=1) Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $f(x_1,x_2) = \begin{cases}(\sin x_2, x_2) & \text{if } x_2\neq 0,\\ (0,0) & \text{if } x_2 = 0\end{cases}$ for all $x=(x_1,x_2)\in\mathbb{R}^2, x^0=(0,0)\in\mathbb{R}^2$ and $S=\{0\}\times[0,\frac{\pi}{4}]$.

Since $\sup\limits_{\xi\in\Gamma}\langle\xi, f(x)-f(x^0)\rangle = \sup\limits_{\xi\in\Gamma}(\xi_1\sin x_2+\xi_2 x_2) = x_2 = \mid x_2\mid = \parallel x-x^0\parallel$ for all $x=(0,x_2)\in S\cap U$, where U is a small enough neighborhood of x^0. So, $x^0\in\mathrm{Strl}(1,f,S)$. On the other hand, we can compute $d_u^1 f(x^0,v) = v_2 > 0$ for all $v\in T(S,x^0)\setminus\{(0,0)\} = \{0\}\times\mathbb{R}_+\backslash\{(0,0)\}$, which satisfies the necessary condition in Theorem 4.2.

Remark 4.3 The necessary condition in this theorem is not sufficient as the following example shows.

Example 4.4 Let $x^0=(0,0)\in\mathbb{R}^2, f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$f(x) = \begin{cases}(-\mid x_1\mid, \mid x_2\mid^k) & \text{if } x_1\neq 0,\\ (0,0) & \text{if } x_1 = 0,\end{cases}$$

for all $x=(x_1,x_2)\in\mathbb{R}^2$, and let $S=\{0\}\times\mathbb{R}$.

It is easy to verify that $x^0\notin\mathrm{Strl}(k,f,S)$. But for all $v=(0,v_2)\in T(S,x^0)\setminus\{(0,0)\} = \{0\}\times\mathbb{R}\setminus\{(0,0)\}$ we have $d_u^k f(x^0,v) = \mid v_2\mid^k > 0$, which satisfies the necessary condition in

Theorem 4.2.

§5 Comparison with some previous results

In this section, we compare our results with some previous necessary and sufficient optimality conditions for strict local minimizer of order k in scalar and multiobjective optimization.

Studniarski [3] and Jiménez and Novo [4], respectively, established the necessary and sufficient optimality conditions of strict minimizer of order k for the scalar problem (1) as follows

Proposition 5.1 (3, Theorem 2.2) If $x^0 \in \mathrm{Strl}(k, f, S)$, then $\overline{d}^k f(x^0, v) > 0$ for all $v \in T(S, x^0) \setminus \{0\}$.

Proposition 5.2 (3, Corollary 2.1) (i) If $k = 1$ and $\underline{d} f(x^0, v) > 0$ for all $v \in T(S, x^0)\setminus\{0\}$, then $x^0 \in \mathrm{Strl}(1, f, S)$; (ii) If $k > 1$ and $\underline{d}^k f(x^0, v) > 0$ for all $v \in K(x^0) \setminus \{0\}$, then $x^0 \in \mathrm{Strl}(k, f, S)$.

Proposition 5.3 (4, Theorem 2.2) If $x^0 \in \mathrm{Strl}(k, f, G \cap Q)$ then $\underline{d}^k(f + I_Q)(x^0, v) > 0,\ \forall v \in C_0(G, x^0) \cap T(Q, x^0)\setminus\{0\}$. Here, $C_0(G, x^0) := \{v \in X : dg(x^0, v) \in \mathrm{intcone}(-C - g(x^0))\}$.

Proposition 5.4 (4, Theorem 3.1) Let $k > 1$. If $\forall v \in C(G, x^0) \cap T(Q, x^0) \cap C(f, x^0)\setminus\{0\}$ we have $\underline{d}^k(f + I_Q)(x^0, v) > 0$, then $x^0 \in \mathrm{Strl}(k, f, G\cap Q)$. Where, $C(f, x^0) := \{v \in X : \underline{d} f(x^0, v) \leq 0\}$.

Proposition 5.5 (4, Theorem 3.2) If $\forall v \in C(G, x^0)\cap T(Q, x^0)\setminus\{0\}$ we have $\underline{d}(f + I_Q)(x^0, v) > 0$, then $x^0 \in \mathrm{Strl}(1, f, G \cap Q)$. Here, $C(G, x^0) := \{v \in X : dg(x^0, v) \in \mathrm{clcone}(-C - g(x^0))\}$.

We can easily see that if $X = \mathbb{R}^n$ in [4], there are conclusions as follows:

(i) Theorem 4.2 reduces to Proposition 5.1 (see also [4, Theorem 2.1]);

(ii) When $p = 1$, Corollary 4.1 (b) in (i) and (ii) is advantageous over Proposition 5.2 (ii) due to Proposition 3.3 (i) in which the converse is not true;

(iii) When $p = 1$ and do not consider the constraint g in [4], Theorem 4.1 (a)⇒(c) reduces to Proposition 5.3, Theorem 4.1 (i) (c)⇒(a) reduces to Proposition 5.4, and Theorem 4.1 (ii) (c)⇒(a) reduces to Proposition 5.5.

Jiménez established the following necessary conditions for multiobjective problems in terms of upper Studniarski derivatives.

Proposition 5.6 (6, Theorem 4.1 (ii)) If $x^0 \in \mathrm{Strl}(k, f, C)$, then $\forall v \in T(C, x^0) \setminus \{0\}\, \exists i \in \{1, 2, \cdots, p\}$ such that $\overline{d}^k f_i(x^0, v) > 0$. Here, C represents the constraint set of the multiobjective problem in [6].

Obviously, our result (Theorem 4.2) is equivalent to this proposition by Proposition 3.3 (ii).

Luu established the necessary conditions for multiobjective problems as follows.

Proposition 5.7 (8, Theorem 5.1) Let $x^0 \in \mathrm{Strl}(k, f, M)$. Assume that $\mathrm{int}\, S \neq \emptyset$ and there exists the derivative $d_S g(x^0, u)$ at x^0 in all directions $u \in X$. Then, for every $v \in T(C, x^0)\cap\{u : d_S g(x^0, u) \in -\mathrm{int}\, S\} \setminus \{0\}$, there exists i, $i \in \{1, 2, \cdots, p\}$, such that $\overline{d}^k_S f_i(x^0, v) > 0$. Here, M represents the constraint set of the multiobjective problem in [8].

If $X = \mathbb{R}^n$ and do not consider the constraint g in [8], then $T(C, x^0) = T(C, x^0) \cap \{u : d_S g(x^0, u) \in -\mathrm{int}\, S\}$. So, Proposition 5.6 is Proposition 5.7.

Rahmo and Studniarski derived the necessary conditions for multiobjective optimization in terms of upper Studniarski derivatives.

Proposition 5.8 (10, Theorem 11 (b)) Let $x^0 \in \mathrm{Strl}(k, f, S)$. Then there exists $\beta > 0$ such that $\overline{d}^k f(x^0, v) \notin B(0, \beta \parallel v \parallel^k) - \overline{\mathbb{R}}^p_+$ for all $v \in T(S, x^0) \cap \{u \in X : \overline{d}g(x^0, u) < 0\}$.

If $X = \mathbb{R}^n$ and do not consider the constraint g in [10], we can prove that Theorem 4.2 $\Leftrightarrow$ Proposition 5.8.

Proof Let $x^0 \in \mathrm{Strl}(k, f, S)$. Then by Theorem 4.2, $d^k_u f(x^0, v) > 0$ for all $v \in T(S, x^0) \setminus \{0\}$. Consequently, we have from Proposition 3.3 (ii) that for all $v \in T(S, x^0) \setminus \{0\}$, $d^k_u f(x^0, v) > 0 \Leftrightarrow \exists i = i(v), 1 \leq i \leq p$, $\overline{d}^k f_i(x^0, v) > 0 \Leftrightarrow \exists \beta > 0$, such that $\overline{d}^k f_i(x^0, v) > \beta \parallel v \parallel^k \Leftrightarrow \exists \beta > 0$, such that $\overline{d}^k f(x^0, v) \notin B(0, \beta \parallel v \parallel^k) - \overline{\mathbb{R}^p}_+$. Here, $\overline{d}^k f(x^0, v) = (\overline{d}^k f_1(x^0, v), \overline{d}^k f_2(x^0, v), \cdots, \overline{d}^k f_p$ $(x^0, v))$ by (11).

Thus, if $X = \mathbb{R}^n$ and do not consider the constraint g in multiobjective problems, we have Theorem 4.2 $\Leftrightarrow$ Proposition 5.6 $\Leftrightarrow$ Proposition 5.7 $\Leftrightarrow$ Proposition 5.8.

Luu established the following necessary condition for multiobjective problems in terms of lower Studniarski derivatives.

Proposition 5.9 (8, Theorem 5.2) Let x^0 be a feasible point for (P). Assume that there is the derivative $d_S g(x^0; v)$ in all directions $v \in X$, and there exits $i_0 \in \{1, 2, \cdots, p\}$ such that

$$\underline{d}^k_S {f_{i_0}}^C(x^0, v) > 0 \ \forall v \in K_C(x^0) \cap \{u : d_S g(x^0, u) \in -S_{g(x^0)}\},$$

where ${f_{i_0}}^C(\cdot) = f_{i_0}(\cdot) + I_C(\cdot)$, in which $I_C(\cdot)$ is the indicator function of C and $S_{g(x^0)} := \mathrm{clcone}(S + g(x^0))$. Then $x^0 \in \mathrm{Strl}(k, f, S)$.

If $X = \mathbb{R}^n$ and do not consider the constraint g in problem (P) in [8], we can easily to see that when $k > 1$, Proposition 5.9 is a consequence of Theorem 4.1 $(c) \Rightarrow (a)$. Indeed, by Proposition 3.3 (i) for any given $v \in \mathbb{R}^n \setminus \{0\}$, the requirement of our derivative $d^k_l(f + \mathbb{I})(x^0, v) > 0$ is weaker than that of the derivative $\underline{d}^k(f_i + I)(x^0, v) > 0$, where $i = i(v), i \in \{1, 2, \cdots, p\}$; when $k = 1$, the condition that $d^1_l f(x^0, v) > 0$ is weaker than or equivalent to that $\underline{d}f_i(x^0, v) > 0$ due to (8), where $i = i(v), i \in \{1, 2, \cdots, p\}$.

Jiménez and Novo studied problem (4) and obtained one of sufficient optimality conditions as follows.

Proposition 5.10 [12, Corollary 5.13] If $f = (f_1, f_2, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ is stable at $x^0 \in S \subset X$ and for any $v \in T(S, x^0) \setminus \{0\}$, $\exists i \in \{1, 2, \cdots, p\}$ such that

$$\underline{d}f_i(x^0, v) > 0$$

then $x^0 \in \mathrm{Strl}(1, f, S)$.

Clearly, our result (Corollary 4.1 (ii)) has weaker assumptions than this proposition because we do not need to require that f be stable at x^0, and because the condition that $d^1_l f(x^0, v) > 0$ is weaker than or equivalent to that $\underline{d}f_i(x^0, v) > 0$ by (8), where $i = i(v), i \in \{1, 2, \cdots, p\}$.

§6 Conclusions

Using a new type of directional derivatives, we have obtained some necessary and sufficient conditions of strict local minima of order k for a multiobjective optimization problem with a nonempty set constraint, which are generalization of some before scalar results and are equivalent to some before multiobjective ones. Of course, there are other works in future needed to do:

(i) to establish characterizations of superstrict local minima of order k for (4);

(ii) to analyze and get the gap between the strict and superstrict local minima of order k ($k > 1$) for (4) when results in (i) are obtained.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] M R Hestenes. *Optimization Theory: The Finite-Dimensional Case*, New York: Robert E Krieger Publishing Co, 1981.

[2] A Auslender. *Stability in mathematical programming with nondifferentiable data*, SIAM J Control Optim, 1984, 22: 239-254

[3] M Studniarski. *Necessary and sufficient conditions for isolated local minima of nonsmooth functions*, SIAM J Control Optim, 1986, 24: 1044-1049.

[4] B Jiménez, V Novo. *Higher-order optimality conditions for strict local minima*, Ann Oper Res, 2008, 157: 183-192.

[5] I Ginchev, A Guerraggio, M Rocca. *From scalar to vector optimization*, Appl Math, 2006, 51: 5-36.

[6] B Jiménez. *Strict efficiency in vector optimization*, J Math Anal Appl, 2002, 265: 264-284.

[7] B Jiménez. *Strict minimality conditions in nondifferentiable multiobjective programming*, J Optim Theory Appl, 2003, 116: 99-116.

[8] D V Luu. *Higher-order necessary and sufficient conditions for strict local Pareto minima in terms of Studniarski's derivatives*, Optim, 2008, 57(4): 593-605.

[9] R T Rockafellar, R J B Wets. *Variational Analysis*, Berlin: Springer, 1998.

[10] E D Rahmo, M Studniarski. *Higher-order conditions for strict local Pareto minima in terms of generalized lower and upper directional derivatives*, J Math Anal Appl, 2012, 393: 212-221.

[11] V F Demyanov, A M Rubinov. *Constructive nonsmooth analysis*, Peter Lang, Frankfurt and Main, 1995.

[12] B Jiménez, V Novo. *First order optimality conditions in vector optimization involving stable functions*, Optimization, 2008, 57(3): 449-471.

[13] V I Ivanov. *Higher-order optimality conditions for inequality-constrained problems*, Applic Anal, 2013, 92(12): 2600-2617.

[14] V I Ivanov. *Higher-order optimality conditions with an arbitrary nondifferentiable function*, Optimization, 2016, 65: 1909-1927.

School of Mathematical Sciences, Chongqing University of Technology, Chongqing 400054, China.

Email: hzg@cqut.edu.cn