Appl. Math. J. Chinese Univ.
2025, 40(3): 523-542

New criteria on the existence and global exponential stability of periodic solutions for quaternion-valued cellular neural networks

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Abstract. In this paper, a class of quaternion-valued cellular neural networks (QVCNNS) with time-varying delays are considered. Combining graph theory with the continuation theorem of Mawhin's coincidence degree theory as well as Lyapunov functional method, we establish new criteria on the existence and exponential stability of periodic solutions for QVCNNS by removing the assumptions for the boundedness on the activation functions and the assumptions that the values of the activation functions are zero at origin. Hence, our results are less conservative and new.

§1 Introduction

Because of the fact that neural networks (NNS) can be applied to signal processing, patten recognition, optimization and associative memories and image processing, the dynamical behaviors of NNS have been widely investigated both in theory and application ([1-4], [33], [34]).

Complex-valued neural networks (CVNNS) as an extension of real-valued neural networks (RVNNS) also can be applied to signal and information processing in complex-valued states, up to now, the dynamical behaviors of CVNNS have been widely investigated, for example, see ([5-8]) and their references therein.

On the other hand, quaternions (QVNNS) which were invented in 1843 by Hamilton [9] as an extension of RVNNS and CVNNS have many practical applications such as the 3D geometrical affine transformation, especially spatial rotation ([10, 11]), image impression, color night vision [12] etc. It is well known that the dynamics of Quaternions plays an important part in their

Received: 2020-05-18. Revised: 2024-07-18.
MR Subject Classification: 45M10, 45M15.
Keywords: the existence of periodic solutions, exponential stability, quaternion-valued cellular neural networks, combining graph theory with Mawhin's continuation theorem of coincidence degree theory, Lyapunov function method, inequality techniques.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-025-4149-5.
Supported by the Innovation Platform Open Fund in Hunan Province Colleges and Universities of China (201485).
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implementation and applications. Hence, the study of the dynamics of Quaternions is essential necessary. Up to now, the dynamical behaviors, such as the stability and synchronization of QVNNS have been widely investigated, and as far as the study of the synchronization of QVNNS is concerned, we can refer to these papers ([13]-[15], [29]-[32]). As for as the study of stability of equilibrium point of QVNNS is concerned, we can refer to ([16]-[18], [29], [35]-[37]). In [16], the linear threshold discrete-time quaternion-valued neural network with time-delays was discussed and sufficient criteria of boundedness and periodicity for the neural network were obtained. In [17], global μ-stability criteria for a class of quaternion-valued neural networks with unbounded time-varying delays were established. In [18], without using Lyapunov function, some sufficient conditions were established to guarantee the global exponential stability for a class of QVNNS based on Halanay inequality. In [29], the existence-uniqueness and global asymptotic stability of equilibrium point of a kind of quaternion-valued BAM fuzzy NNS were discussed. By using Homeomorphism theorem and the properties of unary high degree inequality, a criterion assuring the existence and uniqueness of equilibrium point of the considered system was obtained. Then by applying integral inequality method, a criteria on asymptotic stability of equilibrium point for the system was attained. In [35], the globally exponential stability in Lagrange sense of BAM quaternion-valued inertial NNS was concerned by non-reduced order and un-decomposed method. Auxiliary function-based inequalities and reciprocally convex inequality were applied to the set of quaternion and several criteria for the system were acquired in the form of LMIS. In [36], the commutative QVNNS were established on time scales, which can bring two different forms of discrete time and continuous time QVNNS into a single framework. Some criteria for globally exponential stability of QVNNS were studied mainly by using matrix measure and some inequalities on time scales. In [37], the article was dedicated to studying the stability of QVNNS. The direct quaternion method was used to analyze the QVNNS. By establishing their reciprocally convex inequality and wirtinger-based inequality in quaternion domain, the existence, uniqueness and global stability criteria in the form of LMIS for QVNNS with several freedom matrices were derived.

However, so far, the studies on the existence and the stability of periodic solutions for the QVNNS have been very rare. We only find four articles which investigated the existence and global stability of periodic solutions ([19], [38]-[40]). In [38], the periodic solutions of the QVC-NNS were discussed. By using Schauder fixed point theorem and by constructing an appropriate Lyapunov function, the existence and globally exponential stability of periodic solutions of the networks were attained. In [39], the anti-periodic solutions of a class of QVCNNS were discussed (the existence of an ω anti-periodic solution can implies the existence of an 2ω periodic solution-s). Applying the continuation theorem of coincidence degree theory and inequality techniques, some criteria on the existence and globally exponential stability of periodic solutions of the networks were presented. In [40], the existence and global exponential stability of anti-periodic solutions for QVNNS were concerned. By using a continuation theorem of coincidence degree theory and the wirtinger inequality, some criteria were obtained on the existence and globally exponential stability of QVNNS. In [19], by combining the continuation theorem of Mawhin's coincidence degree theory with the priori estimate method of periodic solutions, the existence of periodic solutions for QVCNNS was obtained under the assumptions that the activation functions satisfy boundedness conditions and the values of the activation functions at origin are zero. By constructing a Lyapunov functional, a sufficient condition was derived to guarantee the global exponential stability of periodic solution for QVNNS. Since the QVCNNs support a complete representation and has a great capacity for more practical problems and the results of the existence and globally stability for QVNNS are very rare, this inspires us to study the existence and globally exponential stability of periodic solutions of QVNNS.

Now, graph theory has been used to research global asymptotic stability of discrete-time Cohen-Grossberg NNS with finite and infinite delays in [37]. The existence and global stability of periodic solutions for coupled networks also were studied in [22-25].

Recently, without applying the priori estimate method of periodic solutions, some criteria to guarantee the existence of periodic solutions for NNS have been established by combining coincidence degree theory with Lyapunov functional method or linear matrix inequality method [26-28].

However, so far, the results on the existence and global exponential stability of periodic solutions for delayed QVNNS have been rare by combining coincidence degree theory with graph theory as well as Lyapunov functional method. This inspires us to study the periodic solutions for QVNNS.

In this article, our main purpose is to establish novel sufficient conditions on the existence and global exponential stability of periodic solutions for system (1) by combining graph theory with continuation theorem as well as Lyapunov functional method, and removing the assumptions for boundedness on the activation functions and the assumptions of the values of the activation functions at origin being zero in [19]. In the proof of our main theorems, some novel inequalities are used to obtain the boundness of periodic solutions of operator equations. Hence the contributions of this paper include the following three aspects:

(a) A novel study method of periodic solutions for QVCNNS is introduced, that is combining graph theory with continuation theorem of coincidence degree theory as well as Lyapunov functional method studies periodic solutions for QVCNNS.

(b) Inequalities techniques are used to study the periodic solutions for QVCNNS.

(c) Novel criteria to guarantee the existence and global exponential stability of periodic solutions for QVCNNS are derived by removing the assumption for the boundedness on the activation functions in [19] and removing the assumption that the value of the activation function is zero at origin.

This paper is organized as follows. Some preliminaries and lemmas are introduced in Section 2. In Section 3, a sufficient condition is derived to guarantee the existence of periodic solutions of system (1). In Section 4, a sufficient condition is established on the global exponential stability of periodic solutions for system (1). In Section 5, an illustrative example is given to prove the effectiveness of the proposed theory. In Section 6, a conclusion is given.

§2 Preliminaries

In [19], the QVNNS discussed are described by the following differential equations

$$u'_w(t) = -a_w u_w(t) + \sum_{v=1}^{l} b_{wv}(t) g_v(u_v(t)) + \sum_{v=1}^{l} c_{wv}(t) g_v(u_v(t - \sigma_{wv}(t))) + r_w(t), \tag{1}$$

where $w = 1, 2, \cdots, l, u_w(t) \in Q$ stands for the state of the wth unit at time $t, g_v(u_v(t)) \in Q$ denotes the output of the vth unit at time $t, b_{wv}(t) \in Q$ denotes the strength of the vth unit on the wth unit at time $t, c_{wv}(t)$ denotes the strength of the vth unit on the vth unit at time $t - \sigma_{wv}(t), r_w(t) \in Q$ is the external input on the vth at time $t, \sigma_{wv}(t) \geq 0$ denotes to the transmission delay along the axon of the vth unit on the wth unit at time t, and $a_w > 0$ denotes the rate with which the wth unit will reset its potential to the reserving state when disconnected from the network and external inputs.

The initial value of system (1) is given by

$$u_w(s) = \phi_w(s) \in Q, s \in [-\sigma, 0], \sigma = \max_{1 \leq w, v \leq l} \{ \max_{0 \leq t \leq \omega} |\sigma_{wv}(t)| \}. \tag{2}$$

Let $|.|$ be the Euclidean norm for R and $L = \{1, 2, \cdots, l\}$. We cite the following notation

$$\overline{f} = \max_{t \in [0, \omega]} \{ |f(t)| \},$$

where $f(t)$ is a continuous ω periodic function with $\omega > 0$.

For $u = u^R + iu^I + ju^J + ku^K \in Q, g : Q \to Q, g(u)$ can be expressed as $g(u) = g^R(u^R, u^I, u^J, u^K) + ig^I(u^R, u^I, u^J, u^K) + jg^J(u^R, u^I, u^J, u^K) + kg^K(u^R, u^I, u^J, u^K)$. For $u_w = u_w^R + iu_w^I + ju_w^J + ku_w^K \in Q$, we denote the activation functions g_q in (1) as follows

$$\begin{aligned} g_v(u_v) \quad = \quad & g_v^R(u^R, u^I, u^J, u^K) + ig_v^I(u^R, u^I, u^J, u^K) + jg_v^J(u^R, u^I, u^J, u^K) \\ & + kg_v^K(u^R, u^I, u^J, u^K), v = 1, 2, \cdots, l. \end{aligned}$$

Throughout this paper, we assume that

(h_1) $\sigma_{wv} \in C(R, R^+)$, b_{wv}, c_{wv} and $r_w \in C(R, Q)$ are ω periodic functions, $w, v = 1, 2, \cdots, l$.

(h_2) $\sigma_{wv} \in C(R, R)$ and $\sigma'_{wv} < 1, w, v = 1, 2, \cdots, l$.

(h_3) $g_v^r \in C(R^4, R)$ and there exists a positive constant δ such that for $v = 1, 2, \cdots, l; r = R, I, J, K; u_v^R, u_v^I, u_v^J, u_v^K, x_v^R, x_v^I, x_v^J, x_v^K \in R$, and
$|g_v^r(u_v^R, u_v^I, u_v^J, u_v^K) - g_v^r(x_v^R, x_v^I, x_v^J, x_v^K)| \leq \delta \Big(|u_v^R - x_v^R| + |u_v^I - x_v^I|| + |u_v^J - x_v^J| + |u_v^K - x_v^K| \Big)$,

(h_4)

$$4\delta(b_{wv} + \frac{4c_{wv}}{1 - \sigma'_{wv}}) < \frac{a_v}{l}.$$

For $w, v = 1, 2, \cdots, l$, we denote

$$\begin{gathered} b_{wv}(t) = b_{wv}^R(t) + ib_{wv}^I(t) + jb_{wv}^J(t) + kb_{wv}^K(t), \\ c_{wv}(t) = c_{wv}^R(t) + ic_{wv}^I(t) + jc_{wv}^J(t) + kc_{wv}^K(t), \\ r_w(t) = r_w^R(t) + ir_w^I(t) + jr_w^J(t) + kr_w^K(t), \\ \sigma = \max_{1 \leq w, v \leq l} \{ \max_{0 \leq t \leq \omega} |\sigma_{wv}(t)| \}, \\ \sigma'_{wv} = \max_{t \in [0, \omega]} \{ \sigma'_{wv}(t) \}, b_{wv} = \max\{ \overline{b_{wv}^R}, \overline{b_{wv}^I}, \overline{b_{wv}^J}, \overline{b_{wv}^K} \}, c_{wv} = \max\{ \overline{c_{wv}^R}, \overline{c_{wv}^I}, \overline{c_{wv}^J}, \overline{c_{wv}^K} \}. \end{gathered}$$

Obviously, by $(h_1), b^r_{wv}(t), c^r_{wv}(t)$, and $r^r_w(w, v = 1, 2, \cdots, l, r = R, I, J, K)$ are all continuous ω-periodic functions on R.

Denote $u'_w(t) = u'^R_w(t) + u'^I_w(t) + u'^J_w(t) + u'^K_w(t), g^r_v = g^r_v(u^R_v(t), u^I_v(t), u^J_v(t), u^K_v(t))$, and $G^r_v = g^r_v(u^R_v(t-\sigma_{wv}(t)), u^I_v(t-\sigma_{wv}(t)), u^J_v(t-\sigma_{wv}(t)), u^K_v(t-\sigma_{wv}(t)))(w, v = 1, 2, \cdots, l; r = R, I, J, K.)$ Therefore, system (1) can be changed as the following real valued system (see [19]) for $w = 1, 2, \cdots, l$:

$$\begin{aligned} u'^R_w(t) &= -a_w u^R_w(t) + \sum_{v=1}^{l} \Big[b^R_{wv}(t)g^R_v - b^I_{wv}(t)g^I_v - b^J_{wv}(t)g^J_v - b^K_{wv}(t)g^K_v\Big] + \sum_{v=1}^{l} \Big[c^R_{wv}(t)G^R_v \\ &\quad -c^I_{wv}(t)G^I_v - c^J_{wv}(t)G^J_v - c^K_{wv}(t)G^K_v\Big] + r^R_w(t) \\ &= V_{wR}(t, u(t), u(t-\sigma_{w1}(t)), \cdots, u(t-\sigma_{wl}(t))), \end{aligned} \tag{3}$$

$$\begin{aligned} u'^I_w(t) &= -a_w u^I_w(t) + \sum_{v=1}^{l} \Big[b^I_{wv}(t)g^R_v + b^R_{wv}(t)g^I_v - b^K_{wv}(t)g^J_v + b^J_{wv}(t)g^K_v\Big] + \sum_{v=1}^{l} \Big[c^I_{wv}(t)G^R_v \\ &\quad +c^R_{wv}(t)G^I_v - c^K_{wv}(t)G^J_v + c^J_{wv}(t)G^K_v\Big] + r^I_v(t) \\ &= V_{wI}(t, u(t), u(t-\sigma_{w1}(t)), \cdots, u(t-\sigma_{wl}(t))), \end{aligned} \tag{4}$$

$$\begin{aligned} u'^J_w(t) &= -a_w u^J_w(t) + \sum_{v=1}^{l} \Big[b^J_{wv}(t)g^R_v + b^K_{wv}(t)g^I_v + b^R_{wv}(t)g^J_v - b^I_{wv}(t)g^K_v\Big] + \sum_{v=1}^{l} \Big[c^J_{wv}(t)G^R_v \\ &\quad +c^K_{wv}(t)G^I_v + c^R_{wv}(t)G^J_v - c^I_{wv}(t)G^K_v\Big] + r^J_v(t) \\ &= V_{wJ}(t, u(t), u(t-\sigma_{w1}(t)), \cdots, u(t-\sigma_{wl}(t))), \end{aligned} \tag{5}$$

$$\begin{aligned} u'^K_w(t) &= -a_w u^K_w(t) + \sum_{v=1}^{l} \Big[b^K_{wv}(t)g^R_v - b^J_{wv}(t)g^I_v + b^I_{wv}(t)g^J_v + b^R_{wv}(t)g^K_v\Big] + \sum_{v=1}^{l} \Big[c^K_{wv}(t)G^I_v \\ &\quad -c^J_{wv}(t)G^I_v + c^I_{wv}(t)G^J_v + c^R_{wv}(t)G^K_v\Big] + r^K_v(t) \\ &= V_{wK}(t, u(t), u(t-\sigma_{w1}(t)), \cdots, u(t-\sigma_{wl}(t))). \end{aligned} \tag{6}$$

By putting $\phi_w(s) = \phi^R_w(s) + i\phi^I_w(s) + j\phi^J_w(s) + k\phi^K_w(s), \phi^r_w \in C([-\sigma, 0], R)$, from (2), it follows that the initial values of (3)-(6) are in the following form

$$u^r_w(s) = \phi^r_w(s), \phi^r_w \in C([-\sigma, 0], R), w = 1, 2, \cdots, l, r = R, I, J, K.$$

Lemma 2.1 (Gaines and Mawhin[20]). *Suppose that X_1 and Z_1 are two Banach spaces, and $L_1 : DomL_1 \subset X_1 \to Z_1$ is linear, $N_1 : X_1 \to Z_1$ is continuous. Assume that L_1 is one-to-one and $K_1 = L_1^{-1}N_1$ is compact. Furthermore, assume that there exists a bounded and open subset $\Omega_1 \subset X_1$ with $o \in \Omega$ such that operation equation $L_1u = \lambda N_1 u$ has no solutions in $\partial\Omega \cap DomL_1$ for any $\lambda \in (0, 1)$. Then the problem $L_1u = N_1u$ has at least one solution in $\overline{\Omega}$.*

Definition 2.1 (Graph theory [21]). A directed graph $g = (U, K)$ contains a set $U = \{1, 2, \cdots, l\}$ of vertices and a set K of arcs (w, v) leading from initial vertex w to terminal vertex v. A subgraph Γ of g is said to be spanning if Γ and g have the same vertex set. A subgraph Γ is unicyclic if it is a disjoint union of rooted trees whose roots from a directed cycle. For a weighted digraph g with l vertices, we define the weight matrix $B = (b_{wv})_{l\times l}$ whose entry $b_{wv} > 0$ is equal to the weight of arc (v, w) if it exists, and 0 otherwise. A digraph g is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other.

The Laplacian matrix of (g, B) is defined as $L = (\beta_{wv})_{l\times l}$, where $\beta_{wv} = -b_{wv}$ for $w \neq v$ and $\beta_{wv} = \sum\limits_{w\neq v} b_{wv}$ for $w = v$.

Lemma 2.2([21]). *Suppose that $l \geq 2$ and c_w denotes the cofactor of the w-th diagonal element of the Laplacian matrix of (g, B). Then $\sum\limits_{w,v=1}^{l} c_w b_{wv} G_{wv}(x_w, x_v) = \sum\limits_{Q\in\Omega} W(Q) \sum\limits_{(w,v)\in K(C_Q)} G_{wv}(x_w, x_v)$ and where $G_{wv}(x_w, x_v)$ is an arbitrary function, Q is the set of all spanning unicyclic graphs of (g, B), $W(Q)$ is the weight of Q, C_Q denotes the directed cycle of Q, and $K(C_Q)$ is the set of arcs in C_Q. In particular, if (g, B) is strongly connected, then $c_w > 0$ for $1 \leq w \leq l$.*

Remark 1. It is clear that in order to prove the existence and global exponential stability of system (1), we only need to prove the existence and global exponential stability of (3)-(6).

§3 The existence of periodic solutions

Lemma 3.1 *If $V(t) = \sum\limits_{w=1}^{l} c_w V_w(t), c_w > 0$ is the cofactor of the with diagonal element of Laplacian matrix of $(g, B), \lambda > 0, N > 0$ are two constants and*

$$\begin{aligned}&\frac{dV_w(t)}{dt}\\ \leq\ &\lambda\sum_{v=1}^{l}\Big\{-\frac{a_w}{l}|u_w^R(t)| + (\frac{a_v}{l}-\alpha)|u_v^R(t)| - \frac{a_w}{l}|u_w^I(t)| + (\frac{a_v}{l}-\alpha)u_v^I(t)| - \frac{a_w}{l}|u_w^J| + (\frac{a_v}{l} -\\ &\alpha)|u_v^J| - \frac{a_w}{l}|u_w^K(t)| + (\frac{a_v}{l}-\alpha)|u_v^K(t)| + \frac{N}{l}\Big\},\end{aligned}$$

then

$$\frac{dV(t)}{dt} \leq \lambda\sum_{w=1}^{l} c_w \sum_{v=1}^{l}\Big[-\alpha(|u_w^R(t)| + |u_w^I(t)| + |u_w^J(t)| + |u_w^K(t)|) + N\Big].$$

Proof. Since

$$\begin{aligned}&\frac{dV_w(t)}{dt}\\ \leq\ &\lambda\sum_{v=1}^{l}\Big\{-\frac{a_w}{l}|u_w^R| + (\frac{a_v}{l}-\alpha)|u_v^R| - \frac{a_w}{l}|u_w^I| + (\frac{a_v}{l}| - \alpha)u_v^I| - \frac{a_w}{l}|u_w^J| + (\frac{a_v}{l}-\alpha)|u_v^J|\\ &-\frac{a_w}{l}\times|u_w^K| + (\frac{a_v}{l}-\alpha)|u_v^K| + \frac{N}{l}\Big\}\\ \leq\ &\lambda\sum_{v=1}^{l}\Big\{-(\frac{a_w}{l}-\alpha)|u_w^R| + (\frac{a_v}{l}-\alpha)|u_v^R| - (\frac{a_w}{l}-\alpha)|u_w^I| + (\frac{a_v}{l}-\alpha)|u_v^I| - (\frac{a_w}{l}-\alpha)|u_w^J|\\ &+(\frac{a_v}{l}-\alpha)|u_v^J| - (\frac{a_w}{l}-\alpha)|u_w^K| + (\frac{a_v}{l}-\alpha)|u_v^K| + \frac{N}{l} - \alpha(|u_w^R| + |u_w^I| + |u_w^J| + |u_w^K|)\\ &+\frac{N}{l}\}. \qquad (7)\end{aligned}$$

Setting $b_{1wv} = 1 (w \neq v), b_{1wv} = 0, w = v, G_{1wv}(|u_w^R|, |u_v^R|) = (\frac{a_v}{l}-\alpha)|u_v^R| - (\frac{a_w}{l}-\alpha)|u_w^R|$, and $\beta_{1w}(|u_w^R|) = (\frac{a_w}{l}-\alpha)|u_w^R|; b_{2wv} = 1 (w \neq v), b_{2wv} = 0, w = v, G_{2wv}(|u_w^I|, |u_v^I|) = (\frac{a_v}{l}-\alpha)|u_v^I| -$

$(\frac{a_w}{l}-\alpha)|u_w^I|, \beta_{2w}(|u_w^I|)=(\frac{a_w}{l}-\alpha)|u_w^I|; b_{3wv}=1(w\neq v), b_{3wv}=0, w=v, G_{3wv}(|u_w^J|,|u_v^J|)$ $=(\frac{a_v}{J}-\alpha)|u_v^J|-(\frac{a_w}{l}-\alpha)|u_w^J|, \beta_{3w}(|u_w^J|)=(\frac{a_w}{l}-\alpha)|u_w^J|; b_{4wv}=1(w\neq v), b_{4wv}=0, w=v,$ $G_{4wv}(|u_w^K|,|u_v^K|)=(\frac{a_v}{l}-\alpha)|u_v^K|-(\frac{a_w}{l}-\alpha)|u_w^K|, \beta_{4w}(|u_w^K|)=(\frac{a_w}{l}-\alpha)|u_w^K|$, then we have from (7) that

$$\begin{aligned}
&\frac{dV_w(t)}{dt}\\
\leq\ & \lambda\Big\{\sum_{v=1}^{l} b_{1wv}G_{1wv}\Big(|u_w^R|,|u_v^R|\Big)+\sum_{v=1}^{l} b_{2wv}G_{2wv}\Big(|u_w^I|,|u_v^I|\Big)+\sum_{v=1}^{l} b_{3wv}G_{3wv}\Big(|u_w^R|,|u_v^R|\Big)+\\
&\sum_{v=1}^{l} b_{4wv}G_{4wv}\Big(|u_w^R|,|u_v^R|\Big)-\alpha\sum_{v=1}^{l}(|u_w^R|+|u_w^I|+|u_w^J|+|u_w^K|)+N\Big\}, \qquad (8)
\end{aligned}$$

where

$$G_{1wv}\Big(|u_w^R|,|u_v^R|\Big)=\beta_q(|u_v^R|)-\beta_w(|u_w^R|), \qquad (9)$$

$$G_{2vw}\Big(|u_w^I|,|u_v^I|\Big)=\beta_q(|u_v^I|)-\beta_w(|u_w^I|), \qquad (10)$$

$$G_{3wv}\Big(|u_w^J|,|u_v^J|\Big)=\beta_v(|u_v^J|)-\beta_w(|u_w^J|), \qquad (11)$$

and

$$G_{4wv}\Big(|u_w^K|,|u_v^K|\Big)=\beta_v(|u_v^K|-\beta_w(|u_w^K|). \qquad (12)$$

Because $V(t)=\sum\limits_{w=1}^{l} c_wV_w(t)$, where $c_w>0$ is the cofactor of the with diagonal element of the Laplacian matrix of (g,B), then we have

$$\begin{aligned}
&\frac{dV(t)}{dt}\\
=\ & \sum_{w=1}^{l} c_w\frac{dV_w(t)}{dt}\\
=\ & \lambda\sum_{w=1}^{l} c_w\sum_{v=1}^{l}\Big\{b_{1wv}G_{1wv}(|u_w^R|,|u_v^R|)+b_{2wv}G_{2wv}(|u_w^I|,|u_v^I|)+b_{3wv}G_{3wv}(|u_w^J|,|u_v^J|)+b_{4wv}\\
&\times G_{4wv}(|u_w^K|,|u_v^K|)-\alpha(|u_w^R|+|u_w^I|+|u_w^J|+|u_w^K|)+N\Big\}. \qquad (13)
\end{aligned}$$

From Lemma 2.2, it follows that

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_wb_{1wv}G_{1wv}\Big(|u_w^R|,|u_v^R|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{1wv}\Big(|u_w^R|,|u_v^R|\Big), \qquad (14)$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_wb_{2wv}G_{2wv}\Big(|u_w^I|,|u_v^I|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{2wv}\Big(|u_w^I|,|u_v^R|\Big), \qquad (15)$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_wb_{3wv}G_{3wv}\Big(|u_w^R|,|u_v^R|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{3wv}\Big(|u_w^R|,|u_v^R|\Big), \qquad (16)$$

and

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_w b_{4wv} G_{4wv}\Big(|u_w^I|,|u_v^I|\Big) = \sum_{Q\in\Omega} W(Q) \sum_{(w,v)\in K(C_\Omega)} G_{4wv}\Big(|u_w^I|,|u_v^R|\Big). \tag{17}$$

By (13)-(17), it follows that from a fact $W(Q)>0$,

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_w b_{1wv} G_{1wv}\Big(|u_w^R|,|u_v^R|\Big) = \sum_{Q\in\Omega} W(Q) \sum_{(w,v)\in K(C_\Omega)} \Big[\beta_v(|u_v^R|)-\beta_w(|u_w^R|)\Big] \le 0, \tag{18}$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_w b_{2wv} G_{2wv}\Big(|u_w^I|,|u_v^I|\Big) = \sum_{Q\in\Omega} W(Q) \sum_{(w,v)\in K(C_\Omega)} \Big[\beta_v(|u_v^I|)-\beta_w(|u_w^I|)\Big] \le 0, \tag{19}$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_w b_{3wv} G_{3wv}\Big(|u_w^R|,|u_v^R|\Big) = \sum_{Q\in\Omega} W(Q) \sum_{(w,v)\in K(C_\Omega)} \Big[\beta_v(|u_v^J|)-\beta_w(|u_w^J|)\Big] \le 0, \tag{20}$$

and

$$\sum_{w=1}^{l}\sum_{v=1}^{l} c_w b_{4wv} G_{4wv}\Big(|u_w^K|,|u_v^K|\Big) = \sum_{Q\in\Omega} W(Q) \sum_{(w,v)\in K(C_\Omega)} \Big[\beta_v(|u_v^K|)-\beta_w(|u_w^K|)\Big)\Big] \le 0. \tag{21}$$

Substituting (18)-(21) into (13) gives

$$\frac{dV(t)}{dt} \le \lambda \sum_{w=1}^{l} c_w \sum_{v=1}^{l} \Big[-\alpha(|u_w^R(t)|+|u_w^I(t)|+|u_w^J(t)|+|u_w^K(t)|)+N\Big].$$

This ends the proof of Lemma 3.1.

Lemma 3.2 *We consider the following system for the parameter* $\lambda\in(0,1)$

$$\begin{aligned}
u_w'^R(t) &= \lambda V_{wR}(t,u(t),u(t-\sigma_{w1}(t)),\cdots,u(t-\sigma_{wl}(t))),\\
u_w'^I(t) &= \lambda V_{wI}(t,u(t),u(t-\sigma_{w1}(t)),\cdots,u(t-\sigma_{wl}(t))),\\
u_w'^J(t) &= \lambda V_{wJ}(t,u(t),u(t-\sigma_{w1}(t)),\cdots,u(t-\sigma_{wl}(t))),\\
u_w'^K(t) &= \lambda V_{wK}(t,u(t),u(t-\sigma_{w1}(t)),\cdots,u(t-\sigma_{wl}(t))).
\end{aligned} \tag{22}$$

Suppose that $(h_1)-(h_4)$ *hold. Then there exists a positive constant* M *which is independent of the parameter* λ *such that every periodic solution* $u(t)=\Big(u_1^R(t),u_1^I(t),u_1^J(t),u_1^K(t),u_2^R(t),$ $u_2^I(t),u_2^J(t),u_2^K(t),\cdots,u_l^R(t),u_l^I(t),u_l^J(t),u_l^K(t)\Big)^T$ *of system (22) satisfies* $\|u(t)\|\le M$, *where* $\|u(t)\| = \sum_{w=1}^{l}\max_{t\in[0,\omega]}\Big[|u_w^R(t)|+|u_w^I(t)|+|u_w^J(t)|+|u_w^K(t)|\Big]$.

Proof. Since $4\delta(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}}) < \frac{a_v}{l}$, there exists a positive constant α such that the following inequality holds

(h_5)

$$4\delta(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}}) < \frac{a_v}{l}-\alpha.$$

We construct a suitable Lyapunov functional as $V_w(t)=V_{1w}(t)+V_{2w}(t)$,
where

$$V_{1w}(t)=|u_w^R(t)|+|u_w^I(t)|+|u_w^J(t)|+|u_w^K(t)|,$$

and

$$V_{2w}(t)=4\lambda\delta\sum_{v=1}^{l}\frac{(\overline{c_{wv}^R}+\overline{c_{wv}^I}+\overline{c_{wv}^J}+\overline{c_{wv}^K})}{1-\sigma'_{wv}(t)}\int_{t-\sigma_{wv}(t)}^{t}\Big[|u_v^R(s)|+|u_v^I(s)|+|u_v^J(s)|+|u_v^K(s)|\Big]ds.$$

From system (22), we have

$$\begin{aligned}
&\frac{d[V_{1w}(t)]}{dt}\\
\leq\ & \lambda\Big\{sign[u_w^R(t)]\Big(-a_wu_w^R(t)+\sum_{v=1}^{l}\Big[b_{wv}^R(t)g_v^R-b_{wv}^I(t)g_v^I-b_{wv}^J(t)g_v^J-b_{wv}^K(t)g_v^K\Big]+\sum_{v=1}^{l}\Big[\\
&c_{wv}^R(t)G_v^R-c_{wv}^I(t)G_v^I-c_{wv}^J(t)G_v^J-c_{wv}^K(t)G_v^K\Big]+r_w^R(t)\Big)+sign[u_w^I(t)]\Big(-a_wu_w^I(t)+\sum_{v=1}^{l}\\
&\Big[b_{wv}^I(t)g_v^R+b_{wv}^R(t)g_v^I-b_{wv}^K(t)g_v^J+b_{wv}^J(t)g_v^K\Big]+\sum_{v=1}^{l}\Big[c_{wv}^I(t)G_v^R+c_{wv}^R(t)G_v^I-c_{wv}^K(t)G_v^J+\\
&c_{wv}^J(t)G_v^K\Big]+r_w^I(t)\Big)+sign[u_w^J(t)]\Big(-a_wu_w^J(t)+\sum_{v=1}^{l}\Big[b_{wv}^J(t)g_v^R+b_{wv}^K(t)g_v^I+b_{wv}^R(t)g_v^J-\\
&b_{wv}^I(t)g_v^K\Big]+\sum_{v=1}^{l}\Big[c_{wv}^J(t)G_v^R+c_{wv}^K(t)G_v^I+c_{wv}^R(t)G_v^J-c_{wv}^I(t)G_v^K\Big]+r_w^J(t)\Big)+sign[u_w^K(t)]\\
&\times\Big(-a_wu_w^K(t)\sum_{v=1}^{l}\Big[b_{wv}^K(t)g_v^R-b_{wv}^J(t)g_v^I+b_{wv}^I(t)g_v^J+b_{wv}^R(t)g_v^K\Big]+\sum_{v=1}^{l}\Big[c_{wv}^K(t)G_v^R-c_{wv}^J(t)\\
&\times G_v^I+c_{wv}^I(t)G_v^J+c_{wv}^R(t)G_v^K\Big]+r_w^K(t)\Big)\Big\}\\
\leq\ & \lambda\Big\{-a_w|u_w^R|+\sum_{v=1}^{l}\Big[(\overline{b_{wv}^R}+\overline{c_{wv}^R})|g_v^R(0,0,0,0)|+(\overline{b_{wv}^I}+\overline{c_{wv}^I})|g_v^I(0,0,0,0)|+|g_v^J(0,0,0,0)|\\
&\times(\overline{b_{wv}^J}+\overline{c_{wv}^J})+(\overline{b_{wv}^K}+\overline{c_{wv}^K})|g_v^K(0,0,0,0)|+\delta(\overline{b_{wv}^R}+\overline{b_{wv}^I}+\overline{b_{wv}^J}+\overline{b_{wv}^K})(|u_v^R|+|u_v^I|+|u_v^J|+\\
&|u_v^K|)\Big]+\delta\sum_{v=1}^{l}(\overline{c_{wv}^R}+\overline{c_{wv}^I}+\overline{c_{wv}^J}+\overline{c_{wv}^K})\Big[|u_v^R(t-\sigma_{wv}(t))|+|u_q^I(t-\sigma_{wv}(t))|+|u_v^J(t-\sigma_{wv}(t)\\
&)|+|u_v^K(t-\sigma_{wv}(t))|\Big]+\overline{r_w^R}-a_w|u_w^I|+\sum_{v=1}^{l}\Big[(\overline{b_{wv}^I}+\overline{c_{wv}^I})|g_v^R(0,0,0,0)|+(\overline{b_{wv}^R}+\overline{c_{wv}^R})\\
&\times|g_v^I(0,0,0,0)|+|g_v^J(0,0,0,0)|(\overline{b_{wv}^K}+\overline{c_{wv}^K})+(\overline{b_{wv}^J}+\overline{c_{wv}^J})|g_v^K(0,0,0,0)|\\
&+\delta(\overline{b_{wv}^R}+\overline{b_{wv}^I}+\overline{b_{wv}^J}+\overline{b_{wv}^K})(|u_v^R|+|u_v^I|+|u_v^J|+|u_v^K|)\Big]\\
&+\delta\sum_{v=1}^{l}(\overline{c_{wv}^R}+\overline{c_{wv}^I}+\overline{c_{wv}^J}+\overline{c_{wv}^K})\Big[|u_v^R(t-\sigma_{wv}(t))|+|u_v^I(t-\sigma_{wv}(t))|\\
&+|u_v^J(t-\sigma_{wv}(t))|+|u_v^K(t-\sigma_{wv}(t))|\Big]+\overline{r_w^I}-a_w|u_w^J|+\sum_{v=1}^{l}\Big[(\overline{b_{wv}^J}+\overline{c_{wv}^J})\times|g_v^R(0,0,0,0)|\\
&+(\overline{b_{wv}^K}+\overline{c_{wv}^K})\times|g_v^I(0,0,0,0)|+(\overline{b_{wv}^R}+\overline{c_{wv}^R})|g_q^J(0,0,0,0)|+(\overline{b_{wv}^I}+\overline{c_{wv}^I})\times|g_v^K(0,0,0,0)|
\end{aligned}$$

$$
\begin{aligned}
&+\delta(\overline{b^J_{wv}}+\overline{b^K_{wv}}+\overline{b^K_{wv}}+\overline{b^I_{wv}})[|u^R_v|+|u^I_v|+|u^J_v|+|u^K_v|\Big]+\delta\sum_{v=1}^{l}(\overline{c^J_{wv}}+\overline{c^K_{wv}}+\overline{c^R_{wwv}}+\overline{c^I_{wv}})\\
&\times\Big[|u^R_v(t-\sigma_{wv}(t))|+|u^I_v(t-\sigma_{wv}(t))|+|u^J_v(t-\sigma_{wv}(t)|+|u^K_q(t-\sigma_{wv}(t))|\Big]+\overline{r^J_w}-a_w|u^K_w|\\
&\quad+\sum_{v=1}^{l}\Big[(\overline{b^K_{wv}}+\overline{c^K_{wv}})\times|g^R_v(0,0,0,0)|+(\overline{b^J_{wv}}+\overline{c^J_{wv}})|g^I_v(0,0,0,0,0)|+(\overline{b^I_{wwv}}+\overline{c^I_{wv}})\\
&\quad|g^J_v(0,0,0,0)|+(\overline{b^R_{wv}}+\overline{c^R_{wv}})\times|g^K_v(0,0,0,0)|+\delta(\overline{b^K_{wv}}+\overline{b^I_{wwv}}+\overline{b^J_{wv}}+\overline{b^R_{wv}})\\
&\quad(|u^R_v|+|u^I_v|+|u^J_v|+|u^K_v|)\Big]+\delta\sum_{v=1}^{l}(\overline{c^K_{wv}}+\overline{c^I_{wv}}+\overline{c^J_{wv}}+\overline{c^R_{wv}})\Big[|u^R_v(t-\sigma_{wv}(t))|\\
&\quad+|u^I_v(t-\sigma_{wv}(t))|+|u^J_v(t-\sigma_{wv}(t))|+|u^K_v(t-\sigma_{wv}(t))|\Big]+\overline{r^K_w}\Big\}.
\end{aligned}\tag{23}
$$

Since

$$
\begin{aligned}
\frac{dV_{2w}(t)}{dt}\quad&=4\lambda\delta\sum_{v=1}^{l}\frac{(\overline{c^R_{wv}}+\overline{c^I_{wv}}+\overline{c^J_{wv}}+\overline{c^K_{wv}})}{1-\sigma'_{wv}(t)}\Big[|u^R_v(t)|+|u^I_v(t)|+|u^J_v(t)|+|u^K_v(t)|\Big]\\
&\quad-4\lambda\delta\sum_{v=1}^{l}(\overline{c^R_{wv}}+\overline{c^I_{wv}}+\overline{c^J_{wv}}+\overline{c^K_{wv}})\Big[|u^R_v(t-\sigma_{wv}(t))|+|u^I_v(t-\sigma_{wv}(t))|\\
&\quad+|u^J_v(t-\sigma_{wv}(t))|+|u^K_v(t-\sigma_{wv}(t))|\Big],
\end{aligned}\tag{24}
$$

from (23) and (24), we have

$$
\begin{aligned}
&\frac{dV_w(t)}{dt}\\
\le\;&\lambda\Big\{-a_w|u^R_w|+\sum_{v=1}^{l}\Big[(\overline{b^R_{wv}}+\overline{c^R_{wv}})|g^R_v(0,0,0,0)|+(\overline{b^I_{wv}}+\overline{c^I_{wv}})|g^I_v(0,0,0,0)|+|g^J_v(0,0,0,0)|\\
&\times(\overline{b^J_{wv}}+\overline{c^J_{wv}})+(\overline{b^K_{wv}}+\overline{c^K_{wv}})|g^K_v(0,0,0,0)|+\delta(\overline{b^R_{wv}}+\overline{b^I_{wv}}+\overline{b^J_{wv}}+\overline{b^K_{wv}})(|u^R_v|+|u^I_v|+|u^J_v|\\
&+|u^K_v|)\Big]+\overline{r^R_w}-a_w|u^I_w|+\sum_{v=1}^{l}\Big[(\overline{b^I_{wv}}+\overline{c^I_{wv}})|g^R_v(0,0,0,0)|+(\overline{b^R_{wv}}+\overline{c^R_{wv}})|g^I_v(0,0,0,0)|+\\
&|g^J_v(0,0,0,0)|(\overline{b^K_{wv}}+\overline{c^K_{wv}})+(\overline{b^J_{wv}}+\overline{c^J_{wv}})|g^K_v(0,0,0,0)|+\delta(\overline{b^R_{wv}}+\overline{b^I_{wv}}+\overline{b^J_{wv}}+\overline{b^K_{wv}})(|u^R_v|\\
&+|u^I_v|+|u^J_v|+|u^K_v|)\Big]+\overline{r^I_w}-a_w|u^J_w|+\sum_{v=1}^{l}\Big[(\overline{b^J_{wv}}+\overline{c^J_{wv}})|g^R_v(0,0,0,0)|+(\overline{b^K_{wv}}+\overline{c^K_{wv}})\times\\
&|g^I_v(0,0,0,0)|+(\overline{b^R_{wv}}+\overline{c^R_{wv}})|g^J_v(0,0,0,0)|+(\overline{b^I_{wv}}+\overline{c^I_{wv}})|g^K_v(0,0,0,0)|+\delta(\overline{b^J_{wv}}+\overline{b^K_{wv}}+\\
&\overline{b^K_{wv}}+\overline{b^I_{wv}})[|u^R_v|+|u^I_v|+|u^J_v|+|u^K_v|\Big]+\overline{r^J_w}-a_w|u^K_w|+\sum_{v=1}^{l}\Big[(\overline{b^K_{wv}}+\overline{c^K_{wv}})|g^R_v(0,0,0,0)|\\
&+(\overline{b^J_{wv}}+\overline{c^J_{wwv}})|g^I_v(0,0,0,0,0)|+(\overline{b^I_{wv}}+\overline{c^I_{wv}})|g^J_v(0,0,0,0)|+(\overline{b^R_{wv}}+\overline{c^R_{wv}})|g^K_v(0,0,0,0)|\\
&+\delta(\overline{b^K_{wv}}+\overline{b^I_{wv}}+\overline{b^J_{wv}}+\overline{b^R_{wv}})(|u^R_v|+|u^I_v|+|u^J_v|+|u^K_v|)\Big]+\overline{r^K_w}+4\delta\times\\
&\sum_{v=1}^{l}\frac{(\overline{c^R_{wv}}+\overline{c^I_{wv}}+\overline{c^J_{wv}}+\overline{c^K_{wv}})}{1-\sigma'_{wv}(t)}\Big[|u^R_v|+|u^I_v|+|u^J_v|+|u^K_v|\Big]+N\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\Big\{-a_w(|u_w^R|+|u_w^I|+|u_w^J|+|u_w^K|)+\delta\sum_{v=1}^{l}(\overline{b_{wv}^R}+\overline{b_{wv}^I}+\overline{b_{wv}^J}+\overline{b_{wv}^K})(|u_v^R|+|u_v^I|+|u_v^J|+\\
&\quad |u_v^K|)+4\delta\sum_{v=1}^{l}\frac{(\overline{c_{wv}^R}+\overline{c_{wv}^I}+\overline{c_{wv}^J}+\overline{c_{wv}^K})}{1-\sigma'_{wv}(t)}\Big[|u_v^R|+|u_v^I|+|u_v^J|+|u_v^K|\Big]+N\Big\}\\
&\le \lambda\Big\{-a_w(|u_w^R|+|u_w^I|+|u_w^J|+|u_w^K|)+4\delta\sum_{v=1}^{l}b_{wv}(|u_v^R|+|u_v^I|+|u_v^J|+|u_v^K|)+16\delta\times\\
&\quad \sum_{v=1}^{l}\frac{c_{wv}}{1-\sigma'_{wv}(t)}\Big[|u_v^R|+|u_v^I|+|u_v^J|+|u_v^K|\Big]+N\Big\}\\
&= \lambda\Big\{-a_w|u_w^R|+4\delta\sum_{v=1}^{l}(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^R|-a_w|u_w^I|+4\delta\sum_{v=1}^{l}(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^I|\\
&\quad -a_w|u_w^J|+4\delta\sum_{v=1}^{l}(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^J|-a_w|u_w^K|+4\delta\sum_{v=1}^{l}(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^K|+N\Big\}, \quad (25)
\end{aligned}
$$

where N is a positive constant.

From (25), we have by using (h_5)

$$
\begin{aligned}
&\frac{dV_w(t)}{dt}\\
&\le \lambda\sum_{v=1}^{l}\Big\{-\frac{a_w}{l}|u_w^R|+4\delta(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^R|-\frac{a_w}{l}|u_w^I|+4\delta(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^I|-\frac{a_w}{l}|u_w^J|\\
&\quad +4\delta(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^J|-\frac{a_w}{l}|u_w^K|+4\delta(b_{wv}+\frac{4c_{wv}}{1-\sigma'_{wv}})|u_v^K|+\frac{N}{l}\Big\}\\
&\le \lambda\sum_{v=1}^{l}\Big\{-\frac{a_w}{l}|u_w^R|+(\frac{a_v}{l}-\alpha)|u_v^R|-\frac{a_w}{l}|u_w^I|+(\frac{a_v}{l}|-\alpha)u_v^I|-\frac{a_w}{l}|u_w^J|+(\frac{a_v}{l}-\alpha)|u_v^J|\\
&\quad -\frac{a_w}{l}\times|u_w^K|+(\frac{a_v}{l}-\alpha)|u_v^K|+\frac{N}{l}\Big\}.
\end{aligned}
$$

By Lemma 3.1, we have:

$$
\frac{dV(t)}{dt}\le\lambda\sum_{w=1}^{l}c_w\sum_{v=1}^{l}\Big[-\alpha(|u_w^R(t)|+|u_w^I(t)|+|u_w^J(t)|+|u_w^K(t)|)+N\Big]. \quad (26)
$$

From (26), it follows that there exists a positive constant M such that when $\|u(t)\|\ge M$, $\frac{dV(t)}{dt}<0$. Hence, there exists a positive constant M such that every periodic solution $u(t)$ of system (7) satisfies $\|u(t)\|\le M$(If the periodic solution $u(t)$ satisfies $\|u(t)\|>M$, then we have $\frac{dV(t)}{dt}<0$. Since $V(t)$ is a periodic function, while $\frac{dV(t)}{dt}<0$, this leads to a contradiction). Hence, the proof of Lemma 3.2 is complete.

Lemma 3.3. *If $c_w>0$ is the cofactor of the wth diagonal element of the Laplacian matrix* *of* (g, B) *and*

$$\begin{aligned}
&\frac{dV(t)}{dt}\\
\leq\ &\sum_{w=1}^{l} c_w \sum_{v=1}^{l} e^{\xi t}\Big\{ -(\frac{a_w}{l}-\xi)|U_w^R(t)| + (\frac{a_v}{l}-\xi)|U_v^R(t)| - (\frac{a_w}{l}-\xi)|U_w^I(t)|\\
&+(\frac{a_v}{l}-\xi)|U_v^I(t)| - (\frac{a_w}{l}-\xi)|U_w^J(t)| + (\frac{a_v}{l}-\xi)|U_v^J(t)| - (\frac{a_w}{l}-\xi)|U_w^K(t)|\\
&+(\frac{a_v}{l}-\xi)|U_v^K(t)| - \xi(|U_w^R(t)| + |U_w^I(t)| + |U_w^J(t)| + |U_w^K(t)|)\Big\},
\end{aligned}$$

then

$$\frac{dV(t)}{dt} \leq e^{\xi t}\sum_{w=1}^{l} c_w \sum_{v=1}^{l}\Big[-\xi(|U_w^R(t)| + |U_w^I(t)| + |U_w^J(t)| + |U_w^K(t)|)\Big].$$

Proof. Since

$$\begin{aligned}
&\frac{dV(t)}{dt}\\
\leq\ &\sum_{w=1}^{l} c_w \sum_{v=1}^{l} e^{\xi t}\Big\{ -(\frac{a_w}{l}-\xi)|U_w^R| + (\frac{a_v}{l}-\xi)|U_v^R| - (\frac{a_w}{l}-\xi)|U_w^I| + (\frac{a_v}{l}-\xi)|U_v^I|\\
&-(\frac{a_w}{l}-\xi)|U_w^J| + (\frac{a_v}{l}-\xi)|U_v^J| - (\frac{a_w}{l}-\xi)|U_w^K| + (\frac{a_v}{l}-\xi)|U_v^K| - (|U_w^R| + |U_w^I|\\
&+|U_w^J| + |U_w^K|)\Big\}, \qquad (27)
\end{aligned}$$

setting $b_{1wv} = 1 (w \neq v), b_{1wv} = 0, w = v, G_{1wv}(|U_w^R|, |U_v^R|) = (\frac{a_v}{l}-\xi)|U_v^R| - (\frac{a_w}{l}-\xi)|U_w^R|$, and $\beta_{1w}(|U_w^R|) = (\frac{a_w}{l}-\xi)|U_w^R|; b_{2wv} = 1(w \neq v), b_{2wv} = 0, w = v, G_{2wv}(|U_w^I|, |U_v^I|) = (\frac{a_v}{l}-\xi)|U_v^I| - (\frac{a_w}{l}-\xi)|U_w^I|, \beta_{2w}(|U_w^I|) = (\frac{a_w}{l}-\xi)|U_w^I|; b_{3wv} = 1(w \neq v), b_{3wv} = 0, w = v, G_{3wv}(|U_w^J|, |U_v^J|) = (\frac{a_v}{J}-\xi)|U_v^J| - (\frac{a_w}{l}-\xi)|U_w^J|, \beta_{3w}(|U_w^J|) = (\frac{a_w}{l}-\xi)|U_w^J|; b_{4wv} = 1(w \neq v), b_{4wv} = 0, w = v, G_{4wv}(|U_w^K|, |U_v^K|) = (\frac{a_v}{l}-\xi)|U_v^K| - (\frac{a_w}{l}-\xi)|U_w^K|, \beta_{4w}(|U_w^K|) = (\frac{a_w}{l}-\xi)|U_w^K|$, then we have from (27):

$$\begin{aligned}
&\frac{dV(t)}{dt}\\
\leq\ & e^{\xi t}\sum_{w=1}^{l} c_w\Big\{\sum_{v=1}^{l} b_{1wv}G_{1wv}\Big(|U_w^R|, |U_v^R|\Big) + \sum_{v=1}^{l} b_{2wv}G_{2wv}\Big(|U_w^I|, |U_v^I|\Big) + \sum_{v=1}^{l} b_{3wv}G_{3wv}\times\\
&\Big(|U_w^R|, |U_v^R|\Big) + \sum_{v=1}^{l} b_{4wv}G_{4wv}\Big(|U_w^R|, |U_v^R|\Big) - \xi\sum_{v=1}^{l}(|U_w^R| + |U_w^I| + |U_w^J| + |U_w^K|)\Big\}, \quad (28)
\end{aligned}$$

$$G_{1wv}\Big(|U_w^R|, |U_v^R|\Big) = \beta_v(|U_v^R|) - \beta_w(|U_w^R|), \qquad (29)$$

$$G_{2wv}\Big(|U_w^I|, |U_v^I|\Big) = \beta_v(|U_v^I|) - \beta_w(|U_w^I|), \qquad (30)$$

$$G_{3wv}\Big(|U_w^J|, |U_v^J|\Big) = \beta_v(|U_v^J|) - \beta_w(|U_w^J|), \qquad (31)$$

and

$$G_{4wv}\Big(|U_w^K|, |U_v^K|\Big) = \beta_v(|U_v^K| - \beta_w(|U_w^K|). \qquad (32)$$

From Lemma 2.2, it follows that

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{1wv}G_{1wv}\Big(|U_w^R|,|U_v^R|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{1wv}\Big(|U_w^R|,|U_v^R|\Big), \tag{33}$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{2wv}G_{2wv}\Big(|U_w^I|,|U_v^I|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{2wv}\Big(|U_w^I|,|U_v^R|\Big), \tag{34}$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_p b_{3wv}G_{3wv}\Big(|U_w^R|,|U_v^R|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{3wv}\Big(|U_w^R|,|U_v^R|\Big), \tag{35}$$

and

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{4wv}G_{4wv}\Big(|U_w^I|,|U_v^I|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}G_{4wv}\Big(|U_w^I|,|U_v^R|\Big). \tag{36}$$

By (29)-(36), it follows that from a fact $W(Q)>0$,

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{1wv}G_{1wv}\Big(|U_w^R|,|U_v^R|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}\Big[\beta_v(|U_v^R|)-\beta_w(|U_w^R|)\Big]\le 0, \tag{37}$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{2wv}G_{2wv}\Big(|U_w^I|,|U_v^I|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}\Big[\beta_v(|U_v^I|)-\beta_w(|U_w^I|)\Big]\le 0, \tag{38}$$

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{3wv}G_{3wv}\Big(|U_w^R|,|U_v^R|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}\Big[\beta_v(|U_v^J|)-\beta_w(|U_w^J|)\Big]\le 0, \tag{39}$$

and

$$\sum_{w=1}^{l}\sum_{v=1}^{l}c_w b_{4wv}G_{4wv}\Big(|U_w^K|,|U_v^K|\Big)=\sum_{Q\in\Omega}W(Q)\sum_{(w,v)\in K(C_\Omega)}\Big[\beta_v(|U_v^K|)-\beta_w(|U_w^K|\Big)\Big]\le 0. \tag{40}$$

Substituting (37)-(40) into (28) gives

$$\frac{dV(t)}{dt}\le e^{\xi t}\sum_{w=1}^{l}c_w\sum_{v=1}^{l}\Big[-\xi(|U_w^R(t)|+|U_w^I(t)|+|U_w^J(t)|+|U_w^K(t)|)\Big]<0.$$

This finishes the proof of Lemma 3.3.

Theorem 3.1 *Assume that $(h_1)-(h_4)$ hold. Then system (1), i.e., system (3)-(6) has at least one T periodic solution.*

Proof. Let

$$\begin{aligned}X_1&=Z_1\\&=\Big\{u:u=(u_1^R,u_1^I,u_1^J,u_1^K,u_2^R,u_2^I,u_2^J,u_2^K,\cdots,u_l^R,u_l^I,u_l^J,u_l^K)^T\in C(R,R^{4l}),\\&\qquad u(\omega+t)=u(t)\Big\},\end{aligned}$$

$$\|u\|=\sum_{w=1}^{l}\max_{t\in[0,\omega]}(|u_w^R(t)|+|u_w^I(t)|+|u_w^J(t)|+|u_w^K(t)|).$$

Then X_1 and Z_1 are two Banach spaces when they are endowed with the norm $\|\|$.

Set $L_1 : DomL_1 = \{u \in C^1(R, R^{4l}) : u, u' \in X_1\} \subset X_1 \to X_1$ by

$$(L_1u)(t) = u'(t).$$

Obviously, L_1 is a linear operator. Furthermore, it is easy to see that $KerL_1 = \{0\}$ and $ImL_1 = X_1$. Hence, L_1 is one-to-one. Define L_1^{-1} the inverse of L_1, we have

$$L_1^{-1} : ImL_1 \to DomL_1,$$

$$(L_1^{-1})u(t) = \int_0^t u(s)ds - 0.5\int_0^\omega u(s)ds, t \in R.$$

Define the operator $N_1 : X_1 \to X_1$ by

$$(N_1u)(t) = \Big((N_{11}u)(t), (N_{12}u)(t), \cdots, (N_{1l}u)(t)\Big)^T,$$

where for $w = 1, 2, \cdots, l$,

$$\begin{aligned}(N_{1w}u)(t) &= \Big((N_{1w}^Ru)(t), (N_{1w}^Iu)(t), (N_{1w}^Ju)(t), (N_{1w}^Ku)(t)\Big)\\ &= \Big(V_{wR}(t, u(t), u(t-\sigma_{w1}(t), \cdots, u(t-\sigma_{wl}(t))),\\ &\quad V_{wI}(t, u(t), u(t-\sigma_{w1}(t), \cdots, u(t-\sigma_{wl}(t))),\\ &\quad V_{wJ}(t, u(t), u(t-\sigma_{w1}(t), \cdots, u(t-\sigma_{wl}(t))),\\ &\quad V_{wK}(t, u(t), u(t-\sigma_{w1}(t), \cdots, u(t-\sigma_{wl}(t)))\Big).\end{aligned}$$

Clearly, N_1 is continuous. Setting $K_1 = L_1^{-1}N_1$, by applying the Arzela-Ascoli theorem, we can prove that K_1 is compact.

Corresponding to the operator $L_1u = \lambda N_1u$, we have system (1). From Lemma 3.2, there exists a positive constant M such that every periodic solution of $L_1u = \lambda N_1u$ satisfies $\|u(t)\| \leq M$. Letting $\Omega = \{u(t) : u(t) \in X_1, \|u(t)\| \leq M + H\}, H$ is a positive constant such that $M + H > \frac{lN\max_{1\leq w\leq l}\{c_w\}}{\alpha^*\min_{1\leq w\leq l}\{c_w\}}$. Thus, by Lemma 2.1 we conclude that $L_1u = N_1u$ has at least one solution in X_1. By Lemma 3.2, system (1) has at least one ω periodic solution.

§4 Exponential stability of periodic solutions

Theorem 4.1 *Assume that all conditions in Theorem 3.1 hold. Then the unique ω periodic solution of system (1) is globally exponentially stable.*

Proof. By Theorem 3.1, system (1) has at least an ω periodic solution, say, $u^*(t) = \Big(u_1^{R*}(t),$ $u_1^{I*}(t), u_1^{J*}(t), u_1^{K*}(t), u_2^{R*}(t), u_2^{I*}(t), u_2^{J*}(t), u_2^{K*}(t), \cdots, u_l^{R*}(t), u_l^{I*}(t), u_l^{J*}(t), u_l^{K*}(t)\Big)^T$. Let

$$u(t) = \Big(u_1^R(t), u_1^I(t), u_1^J(t), u_1^K(t), u_2^R(t), u_2^I(t), u_2^J(t), u_2^K(t), \cdots, u_l^R(t), u_l^I(t), u_l^J(t), u_l^K(t)\Big)^T$$

is an arbitrary solution of system (1) with the initial value condition.

Since $4\delta(b_{wv} + \frac{4c_{wv}}{1-\sigma'_{wv}}) < \frac{a_v}{l}$, then there exists a positive constant ξ such that (h_6)

$$4\delta(b_{wv} + \frac{4c_{wv}e^{\xi\sigma}}{1-\sigma'_{wv}}) < \frac{a_v}{l} - \xi.$$

We construct a Lyapunov functional as follows

$$\begin{aligned} & V(t) \\ = \; & \sum_{w=1}^{l} c_w e^{\xi t}\Big[|U_w^R(t)|+|U_w^I(t)|+|U_w^J(t)|+|U_w^K(t)|\Big]+4\delta\sum_{w=1}^{l}\sum_{v=1}^{l}\frac{(\overline{c_{wv}^R}+\overline{c_{wv}^I}+\overline{c_{wv}^J}+\overline{c_{wv}^K})}{1-\sigma_{wv}'(t)}\times \\ & \int_{t-\sigma_{wv}(t)}^{t} e^{\xi(s+\sigma)}\Big[|U_v^R(s)|+|U_v^I(s)|+|U_v^J(s)|+|U_v^K(s)|\Big]ds, \end{aligned} \tag{41}$$

where

$U_w^R(t)=u_w^R(t)-u_w^{R*}(t), U_w^I(t)=u_w^I(t)-u_w^{I*}(t), U_w^J(t)=u_w^J(t)-u_w^{J*}(t), U_w^K(t)=u_w^K(t)-u_w^{K*}(t)$.

Denote

$$g^{\Delta R}=g_v^R-g_v^{R*}, g^{\Delta I}=g_v^I-g_v^{I*}, g^{\Delta J}=g_v^J-g_v^{J*}, g^{\Delta K}=g_v^K-g_v^{K*},$$
$$G^{\Delta R}=G_v^R-G_v^{R*}, G^{\Delta I}=G_v^I-G_v^{I*}, G^{\Delta J}=G_v^J-G_v^{J*}, G^{\Delta K}=G_v^K-G_v^{K*},$$

where

$$g_v^{r*}=g_v^r\Big(u_v^{R*}(t),u_v^{I*}(t),u_v^{J*}(t),u_v^{K*}(t)\Big), \quad r=R,I,J,K,$$
$$G_v^{r*}=G_v^r\Big(u_v^{R*}(t-\sigma_{wv}(t)),u_v^{I*}(t-\sigma_{wv}(t)),u_v^{J*}(t-\sigma_{wv}(t)),u_v^{K*}(t-\sigma_{wv}(t))\Big).$$

Then it follows that by calculating the time derivative of $V(t)$ along the trajectories of system (3)-(6)

$$\begin{aligned} & \frac{dV(t)}{dt} \\ = \; & \sum_{w=1}^{l} c_w\xi e^{\xi t}\Big[|U_w^R(t)|+|U_w^I(t)|+|U_w^J(t)|+|U_w^K(t)|\Big]+\sum_{w=1}^{l} c_w e^{\xi t}\Big\{sign[U_w^R(t)]\Big(-a_wU_w^R(t) \\ & +\sum_{v=1}^{l}\Big[b_{wv}^R(t)g_v^{\Delta R}-b_{wv}^I(t)g_v^{\Delta I}-b_{wv}^J(t)g_v^{\Delta J}-b_{wv}^K(t)g_v^{\Delta K}\Big]+\sum_{v=1}^{l}\Big[c_{wv}^R(t)G_v^{\Delta R}-c_{wv}^I(t)G_v^{\Delta I} \\ & -c_{wv}^J(t)G_v^{\Delta J}-c_{wv}^K(t)G_v^{\Delta K}\Big]\Big)+sign[U_w^I(t)]\Big(-a_wU_w^I(t)+\sum_{v=1}^{l}\Big[b_{wv}^I(t)g_v^{\Delta R}+b_{wv}^R(t)g_v^{\Delta I}- \\ & b_{wv}^K(t)g_v^{\Delta J}+b_{wv}^J(t)g_v^{\Delta K}\Big]+\sum_{v=1}^{l}\Big[c_{wv}^I(t)G_v^{\Delta R}+c_{wv}^R(t)G_v^{\Delta I}-c_{wv}^K(t)G_v^{\Delta J}+c_{wv}^J(t)G_v^{\Delta K}\Big]\Big)+ \\ & sign[U_w^J(t)]\Big(-a_wU_w^J(t)+\sum_{v=1}^{l}\Big[b_{wv}^J(t)g_v^{\Delta R}+b_{wv}^K(t)g_v^{\Delta I}+b_{wv}^R(t)g_v^{\Delta J}-b_{wv}^I(t)g_v^{\Delta K}\Big]+ \\ & \sum_{v=1}^{l}\Big[c_{wv}^J(t)G_v^{\Delta R}+c_{wv}^K(t)G_v^{\Delta I}+c_{wv}^R(t)G_v^{\Delta J}-c_{wv}^I(t)G_v^{\Delta K}\Big]\Big)+sign[U_w^K(t)]\Big(-a_wU_w^K(t)+ \\ & \sum_{v=1}^{l}\Big[b_{wv}^K(t)g_v^{\Delta R}-b_{wv}^J(t)g_v^{\Delta I}+b_{wv}^I(t)g_v^{\Delta J}+b_{wv}^R(t)g_v^{\Delta K}\Big]+\sum_{v=1}^{l}\Big[c_{wv}^K(t)G_v^{\Delta R}-c_{wv}^J(t)G_v^{\Delta I} \\ & +c_{wv}^I(t)G_v^{\Delta J}+c_{wv}^R(t)G_v^{\Delta K}\Big]\Big)\Big\}+4\delta\sum_{w=1}^{l}\sum_{v=1}^{l}\frac{4c_{wv}e^{\xi(t+\sigma)}}{1-\sigma_{wv}'}\Big[|U_v^R(t)|+|U_v^I(t)|+|U_v^J(t)|+ \end{aligned}$$

$$
\begin{aligned}
&|U_v^K(t)|\Big] - 4\delta\sum_{w=1}^{l}\sum_{v=1}^{l} e^{\xi t}(\overline{c_{wv}^R} + \overline{c_{wv}^I} + \overline{c_{wv}^J} + \overline{c_{wv}^K})\Big[|U_q^R(t-\sigma_{wv}(t))| + |U_v^I(t-\sigma_{wv}(t))| + \\
&|U_v^J(t-\sigma_{wv}(t))| + |U_v^K(t-\sigma_{wv}(t))|\Big] \\
\leq\ & \sum_{w=1}^{l} c_w \sum_{v=1}^{l} e^{\xi t}\Big\{ -\frac{a_w}{l}|U_w^R| + 4\delta(b_{wv} + \frac{4e^{\xi\sigma}c_{wv}}{1-\sigma'_{wv}})|U_v^R| - \frac{a_w}{l}|U_w^I| + 4\delta(b_{wv} + \frac{4e^{\xi\sigma}c_{wv}}{1-\sigma'_{wv}})|U_v^I| \\
&-\frac{a_w}{l}|U_w^J| + 4\delta(b_{wv} + \frac{4e^{\xi\sigma}c_{wv}}{1-\sigma'_{wv}})|U_v^J| - \frac{a_w}{l}|U_w^K| + 4\delta(b_{wv} + \frac{4e^{\xi\sigma}c_{wv}}{1-\sigma'_{wv}})|U_v^K|\Big\} \\
\leq\ & \sum_{w=1}^{l} c_w \sum_{v=1}^{l} e^{\xi t}\Big\{ -\frac{a_w}{l}|U_w^R| + (\frac{a_v}{l} - \xi)|U_v^R| - \frac{a_w}{l}|U_w^I| + (\frac{a_v}{l}| - \xi)U_v^I| - \frac{a_w}{l}|U_w^J| + (\frac{a_v}{l} - \\
&\xi)|U_v^J| - \frac{a_w}{l}|U_w^K| + (\frac{a_v}{l} - \xi)|U_v^K|\Big\} \\
\leq\ & \sum_{w=1}^{l} c_w \sum_{v=1}^{l} e^{\xi t}\Big\{ -(\frac{a_w}{l} - \xi)|U_w^R| + (\frac{a_v}{l} - \xi)|U_v^R| - (\frac{a_w}{l} - \xi)|U_w^I| + (\frac{a_v}{l} - \xi)|U_v^I| - (\frac{a_w}{l} \\
&-\xi)|U_w^J| + (\frac{a_v}{l} - \xi)|U_v^J| - (\frac{a_w}{l} - \xi)|U_w^K| + (\frac{a_v}{l} - \xi)|U_v^K| - \xi(|U_w^R| + |U_w^I| + |U_w^J| \\
&+|U_w^K|)\Big\}.
\end{aligned}
$$

According to Lemma 3.3, from (42), we have

$$
\frac{dV(t)}{dt} \leq e^{\xi t}\sum_{w=1}^{l} c_w \sum_{v=1}^{l}\Big[-\xi(|U_w^R(t)| + |U_w^I(t)| + |U_w^J(t)| + |U_w^K(t)|)\Big] < 0.
$$

The rest proof is the same as that of the corresponding part of Theorem 4.1 in [19] and it is omitted. This proof of Theorem 4.1 is complete.

Remark 2. In [19], by combining the continuation theorem of Mawhin's coincidence degree theory with the priori estimate method of periodic solutions, the existence of periodic solutions for QVCNNS was obtained. While in our article, the existence of periodic solutions for QVCNNS is obtained by combining the continuation theorem of Mawhin's coincidence degree theory with Graph theory as well as Lyapunov functional approach for QVCNNS. In the result of the existence of periodic solutions in [19], the conditions (H_1) and (H_3) are kept in our paper, the condition (H_2) is replaced with the condition (h_4) in our paper*and* the (H_4) is replaced with (h_3) in our paper. Namely the assumptions that the activation functions are bounded and the values of the activation functions in origin point are zero are replaced with new conditions.

Remark 3. In the result of the globally exponential stability in [19], the condition (P_1) is replaced with (h_4) in our paper. Novel sufficient conditions on the existence and global exponential stability of periodic solutions for system (1) are established by removing the assumptions for boundedness on the activation functions and the assumptions of the values of the activation functions at origin being zero in [19]. Hence the results of our paper are less conservative than those obtained in [19] and our results are quite novel.

Remark 4. In our article, the separation method is used to study the periodic solutions for QVCNNS, namely, by separating the quaternion-valued system (1) into a real-valued system of four one-order differential equations (3)-(6), and by studying the periodic solutions of (3)-(6), the periodic solutions of system (1) is studied. The advantages of this method are that by transforming the quaternion-valued system into real-valued system, the periodic solutions of quaternion-valued system are investigated by studying the periodic solutions of the real-valued system, which enriches the theoretical results in the QVCNNS. The disadvantages of real-valued method are that without applying the quaternion-valued method, the novel works cannot be obtained for QVCNNS.

§5 Numerical example

In this section, we give an example for showing our result.

Example 5.1 Consider the following QVCNN

$$u'_w(t) = -a_w u_w(t) + \sum_{v=1}^{2} b_{wv}(t) g_v(u_v(t)) + \sum_{v=1}^{2} c_{wv}(t) g_v(u_v(t-\sigma_{wv}(t))) + r_w(t), w = 1, 2, \quad (42)$$

where $u'_w(t) = u'^R_w(t) + iu'^I_w(t) + ju'^J_w(t) + ku'^K_w(t) \in Q, g_v(u_v) = 0.01(|u^R_v| + |u^I_v| + |u^J_v| + |u^K_v|) + 0.01i(|u^R_v| + |u^I_v| + |u^J_v| + |u^K_v|) + 0.01j(|u^R_v| + |u^I_v| + |u^J_v| + |u^K_v|) + 0.005k(|u^R_v| + |u^I_v| + |u^J_v| + |u^K_v|) - 0.01sint + icost + 0.1jsint - 0.09kcost, v = 1, 2$ and $\sigma_{wv}(t) = 2 - 0.5sint, \sigma'_{wv} = -0.5cost, b^R_{wv}(t) = b^I_{wv}(t) = b^J_{wv}(t) = b^K_{wv}(t) = 3sint, c^R_{wv}(t) = c^I_{wv}(t) = c^J_{wv}(t) = c^K_{wv}(t) = 4cost, v = 1, 2$. Then in Theorem 4.1, $\delta = 0.01, \sigma'_{wv} = 0.5, b_{wv} = 3, c_{wv} = 4, l = 2, a_v = 4$. Since the activation functions in system (42) are not bounded, while the activation functions in [19] were assumed to be bounded, hence, the global exponential stability of periodic solutions of system (42) cannot be verified by the results obtained in [19]. It is easy to prove that the conditions $(h_1) - (h_3)$ in Theorem 4.1 are satisfied. On the other hand, we can prove that (h_4) also holds:

$$4\delta(b_{wv} + \frac{4c_{wv}}{1 - \sigma'_{wv}}) < \frac{a_v}{l}.$$

Hence the all conditions in Theorem 4.1 are satisfied. By Theorem 4.1, system (42) has one 2π periodic solution which is globally exponential stable. Curves of state variables in Example 5.1 is shown in Figure 1.

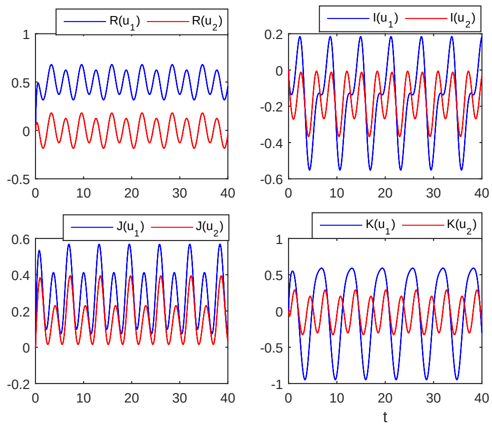

Figure 1. Curves of u_1 and u_2 in Example 5.1.

§6 Conclusion

By combining graph theory with coincidence degree theory as well as Lyapunov functional method, sufficient conditions to guarantee the existence and exponential stability of periodic solutions for quaternion-valued cellular neural networks with time-varying delays are obtained. Our results improve partly those obtained in [19] by removing the assumptions for the boundedness on the activation functions and the assumptions that the values of the activation functions are zero at origin.

Declarations

Conflict of interest The authors declare no conflict of interest.

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