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All entire solutions of Fermat type difference-differential equations of one variable

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Abstract. The main purpose of this paper is to try to find all entire solutions of the Fermat type difference-differential equation

$$[p_1(z)f(z+c)]^2+\left[p_2(z)f(z)+p_3(z)f^{'}(z)\right]^2=p(z),$$

or

$$[p_1(z)f(z)]^2+\left[p_2(z)f^{'}(z)+p_3(z)f(z+c)\right]^2=p(z)$$

or

$$\left[p_1(z)f^{'}(z)\right]^2+[p_2(z)f(z+c)+p_3(z)f(z)]^2=p(z),$$

where c is a nonzero complex number, p_1, p_2 and p_3 are polynomials in $\mathbb{C}$ satisfying $p_1p_3 \not\equiv 0$, and p is a nonzero irreducible polynomial in $\mathbb{C}$.

§1 Introduction and main results

It is known that the Fermat equation $x^m+y^m=1$ (when $m\geq 3$) does not admit nontrivial solutions in rational numbers by Fermat's last theorem [17] [18], while admit nontrivial rational solutions when $m=2$. For a positive integer m, the functional equation $f^m+g^m=1$ can be regarded as the Fermat type equations over function fields. The study of Fermat type functional equation goes back to Cartan [1], Montel [14] and Gross [4]. The entire solutions of the Fermat type functional equation $f^m+g^m=1$ are characterized as follows: (i)for $m=2$, we have $f=\cos p$ and $g=\sin p$, where p is an entire function on $\mathbb{C}$; (ii) for $m>2$, there are no nonconstant entire solutions. For the background, refer to [5].

If replace g by the derivative operator of f, then it is easy to get that the Fermat type equation $f^2+(f^{'})^2=1$ has no nonconstant polynomial solutions, since the degree of $f^{'}$ is lower

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than the degree of f. In 2004, C C Yang and P Li [22, Theorem 1] firstly studied derivative operator $f^{'}$ into Fermat type functional equations. In 2012, K Liu, the present third author and H Z Cao [10] firstly considered difference operator $f(z+c)$ into Fermat type function equations. Later on, many researcher (see [12] [11] [8] for instances) obtained important theorems on the subject of solutions of Fermat type functional equations. Furthermore, solutions of Fermat type functional equations in several variables were also investigated in [15] [9] [19] [13] and others.

The main purpose of this paper is to consider entire solutions $f(z)$ of Fermat type difference-differential equations concerning with both its derivative operator $f^{'}(z)$ and its difference operator $f(z+c)$. That is, we will try to find all entire solutions f of the Fermat type functional equation

$$X^2 + Y^2 = p(z),$$

satisfying one of the following three cases:

(i) $X = p_1(z)f(z+c)$ and $Y = p_2(z)f(z) + p_3(z)f^{'}(z)$;

(ii) $X = p_1(z)f(z)$ and $Y = p_2(z)f^{'}(z) + p_3(z)f(z+c)$;

(iii) $X = p_1(z)f^{'}(z)$ and $Y = p_2(z)f(z+c) + p_3(z)f(z)$,

where c is a nonzero complex number, p_1, p_2 and p_3 are polynomials in $\mathbb{C}$ satisfying $p_1p_3 \not\equiv 0$, and p is a nonzero irreducible polynomial in $\mathbb{C}$.

To state our next theorems, we use the notations

$$\overline{\omega}(z) = \omega(z+c),\ \underline{\omega}(z) = \omega(z-c)$$

for any meromorphic function $\omega(z)$ and a finite nonzero complex number c, which already appeared in [6].

Firstly, we obtain the following result for the case (i).

Theorem 1.1. *Let f be an entire solution of the Fermat type difference-differential equation*

$$[p_1(z)f(z+c)]^2 + [p_2(z)f(z) + p_3(z)f^{'}(z)]^2 = p(z), \tag{1}$$

where the polynomials $p_1, p_2 and p_3$ in $\mathbb{C}$ satisfy $p_1p_3 \not\equiv 0$, and p is a nonzero irreducible polynomial in $\mathbb{C}$. Then f has the form of

$$f(z) = \frac{e^{i\underline{g}} + \underline{p}e^{-i\underline{g}}}{2\underline{p_1}},$$

where g is a constant or a nonconstant linear polynomial $g(z) = Az + B$ such that

$$e^{-iAc} \equiv \frac{k\underline{p_1}^2}{(p_2 - Ap_3)\,\underline{p_1} - ip_3\underline{p_1}'},\ (k = \pm 1).$$

For the special case whenever $p_1 = p_3 = p = 1$ and $p_2 = 0$ in Theorem 1.1, it implies the following corollary which improves [10, Theorem 1.3]. Remark that there is no any assumption on the growth of entire solutions.

Corollary 1.2. *All transcendental entire solutions of*

$$f^{'}(z)^2 + f(z+c)^2 = 1 \tag{2}$$

must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = (2k+1)\pi$, while k is an integer.

Secondly, we obtain the following result for the case (ii).

Theorem 1.3. *Let f be an entire solution of the Fermat type difference-differential equation*

$$[p_1(z)f(z)]^2 + [p_2(z)f^{'}(z) + p_3(z)f(z+c)]^2 = p(z), \tag{3}$$

where the polynomials $p_1, p_2 and p_3$ in $\mathbb{C}$ satisfy $p_1p_3 \not\equiv 0$, and p is a nonzero irreducible polynomial in $\mathbb{C}$. Then f has the form of

$$f(z) = \frac{e^{ig(z)} + p(z)e^{-ig(z)}}{2p_1(z)},$$

where g is a constant or a nonconstant linear polynomial $g(z) = Az + B$ such that

$$e^{-iAc} \equiv \frac{ip_1^2p_3}{\overline{p_1}(Ap_2p_1 + ip_2 + kp_1^2)}, \; (k = \pm 1).$$

For the special case whenever $p_1 = p_3 = p = 1$ and $p_2 = 0$ in Theorem 1.3, it implies the following corollary which improves Theorem [10, Theorem 1.1]. Remark that there is also no any assumption on the growth of entire solutions.

Corollary 1.4. *All transcendental entire solutions of*

$$f(z)^2 + f(z+c)^2 = 1 \tag{4}$$

must satisfy $f(z) = \sin(Az + B)$, where B is a constant and $A = \frac{(4k+1)\pi}{2c}$ with k an integer.

Finally, we obtain the following result for the case (iii).

Theorem 1.5. *Let p_1, p_2 and p_3 be three polynomials in $\mathbb{C}$ satisfying $p_1p_3 \not\equiv 0$ and $\frac{p_2}{p_3} = M$ where M is a constant, and p be a nonzero irreducible polynomial in $\mathbb{C}$. If f is an entire solution of the Fermat type difference-differential equation*

$$[p_1(z)f^{'}(z)]^2 + [p_2(z)f(z+c) + p_3(z)f(z)]^2 = p(z), \tag{5}$$

then f satisfies

$$f^{'}(z) = \frac{e^{ig(z)} + p(z)e^{-ig(z)}}{2p_1(z)},$$

where g satisfies the following two cases:

(i) whenever $M = 0$ (that is $p_2 \equiv 0$), then g is a constant or a nonconstant linear polynomial

$$g = \frac{1}{k}\frac{p_3}{p_1}z + Constant$$

such that all p, p_1 and p_3 are constant.

(ii) whenever $M \neq 0$, then g is a constant or nonconstant linear polynomial $g(z) = Az + B$ such that

$$e^{-iAc} \equiv \frac{ip_1p_2p_3}{kiAp_1p_3\overline{p_1} - ip_3^2\overline{p_1} - kp_3^{'}p_1\overline{p_1}}.$$

For $M = 0$ (namely, $p_2 = 0$) and $p_1 = p_3 = p = 1$ in Theorem 1.5, it implies the following result which is a special case of [22, Theorem 1].

Corollary 1.6. *All transcendental entire solutions of*

$$f^{'}(z)^2 + f(z)^2 = 1 \tag{6}$$

must satisfy

$$f(z)=\frac{1}{2}\left(Pe^{-iz}+\frac{1}{P}e^{iz}\right)=\sin(z+Ai+\frac{\pi}{2}),$$

where P is nonzero constant and $e^A=P$.

For $M\neq 0$ and $p_1=1$, $-p_2=p_3=1$ and $p=1$ in Theorem 1.5, it implies the following corollary which improves [10, Theorem 1.5]. Remark that there is also no any assumption on the growth of entire solutions.

Corollary 1.7. *The transcendental entire solutions of*

$$f^{'}(z)^2+[f(z+c)-f(z)]^2=1 \tag{7}$$

must satisfy $f(z)=\frac{1}{2}\sin(2z+Bi)$, where $c=(k+\frac{1}{2})\pi$, k is an integer and B is a constant.

We would like to arise a question that it may be interesting to consider all entire solutions of (5) without the assumption of $\frac{p_2}{p_3}$ being a constant in Theorem 1.5.

These theorems in this paper can also be regarded as useful judgment methods to nonexistence of entire functions of the Fermat type difference-differential equations. For instances, in Theorem 1.5, let $p_1(z)=z$, $p_2=0$, $p_3=1$ and $p(z)=z^2$. Then it is easy to check that there are no nonconstant entire solutions of the Fermat type equation

$$(zf^{'}(z))^2+f(z)^2=z^2.$$

If let $p_1(z)=1$, $p_2=0$, $p_3=1$ and $p(z)=z+1$, then it is easy to check that there are no nonconstant entire solutions of the Fermat type equation

$$(f^{'}(z))^2+f(z)^2=z+1.$$

More examples can be easily taken like this way.

The remainder of this paper is organized as follows. In next section, we introduce some basic results on Nevanlinna theory for meromorphic function on the complex plane $\mathbb{C}$ and some lemmas, which play the key role in this paper. We then prove Theorems 1.1, 1.3 and 1.5, respectively.

§2 Preliminaries on Nevanlinna theory and lemmas

Throughout this paper, a meromorphic function f means meromorphic in the complex plane $\mathbb{C}$. If no poles occur, then f reduces to an entire function. For every real number $x\geq 0$, we define $\log^+ x := \max\{0,\log x\}$. Assume that $n(r,f)$ counts the number of the poles of f in $|z|\leq r$ (counting multiplicity), and if ignoring multiplicity, then denote it by $\overline{n}(r,f)$. The Nevanlinna characteristic function of f is defined by

$$T(r,f):=m(r,f)+N(r,f),$$

where

$$N(r,f):=\int_0^r\frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log r$$

is called the counting function of poles of f and

$$m(r,f):=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\theta})\right|d\theta$$

is called the proximity function of f. The order of f is defined as
$$\rho(f) = \lim\sup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$
It is known [23, Corollary of Theorem 1.5] that f is a transcendental meromorphic function if and only if
$$\liminf_{r\to\infty} \frac{T(r,f)}{\log r} = \infty.$$
The first main theorem in Nevanlinna theory states that
$$T(r, \frac{1}{f-a}) = T(r,f) + o(T(r,f))$$
holds for any value $a \in \mathbb{C}$. The second main theorem says that for any q distinct small functions $a_1, a_2, \ldots, a_q$ with respect to a meromorphic function f (that is, each a_j is a meromorphic functions such that $T(r,a_j) = o(T(r,f))$ possibly outside a set with finite logarithmic measure), we have
$$(q-2)T(r,f) \le \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) + o(T(r,f))$$
for all sufficiently large r possibly outside a set with finite logarithmic measure (see Steinmetz [16]). Second main theorem with reduced form for small functions was affirmed due to K. Yamanoi [20]. For more basic notations and definitions of the Nevanlinna theory, refer to [23] [7].

The first lemma is proved by Clunie.

Lemma 2.1. *[23, Theorem 1.46] Suppose that $f(z)$ is a transcendental meromorphic function and $h(z)$ is a nonconstant entire function. Then*
$$\lim_{r\to\infty} \frac{T(r,f(h))}{T(r,h)} = \infty.$$

Lemma 2.2. *[3, Theorem 1.6 of Charpter 2] Let $f(z)$ be a meromorphic function, and let $f_1 = f(az+b)$ with $a \neq 0$. Then $f(z)$ and $f_1(z)$, as well as $N(r,f)$ and $N(r,f_1)$ are of the same growth category. In fact, by [3, Remark and proof of Theorem 1.6 in Charpter 2], we have*
$$\begin{aligned} T(r, f(z+b)=a) &\ge (1+o(1))T(r-|b|, f(z)=a) \\ &\ge (1+o(1))T(r-2|b|, f(z+b)=a) \end{aligned}$$
and
$$\begin{aligned} N(r, f(z+b)=a) &\ge (1+o(1))N(r-|b|, f(z)=a) \\ &\ge (1+o(1))N(r-2|b|, f(z+b)=a). \end{aligned}$$

Lemma 2.3. *[21] Suppose $f(z)$ is a meromorphic function in the complex plane and*
$$p(z) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n,$$
where $a_0(\not\equiv 0), a_1, \cdots, a_n$ are meromorphic functions satisfying $T(r,a_j) = o(T(r,f))$ $(j = 0,1,\cdots,n)$ for all positive r possibly outside a set E of finite linear measure. Then
$$T(r,p(f)) = nT(r,f) + o(T(r,f)), \quad r \notin E.$$

The following result will be used throughout the proofs of our main theorems.

Lemma 2.4. *Let Aandc be nonzero complex numbers, and let $f(z)$ be a meromorphic function. Then we have*

$$T(r, f(z)) = o(T(r, e^{Af(z+c)})).$$

Proof. Lemma 2.2 shows that $T(r, f(z))$ and $T(r, f(z+c))$ have the same growth category, and Lemma 2.1 implies that the growth category of $T(r, f(z+c))$ is lower than that of $T(r, e^{Af(z+c)})$. Thus the growth category of $T(r, f(z))$ is lower than that of $T(r, e^{Af(z+c)})$. □

The following lemma is obtained by H X Yi.

Lemma 2.5. *[23, Theorem 1.57] Let $f_j (j = 1, 2, 3)$ be meromorphic functions and f_1 is not a constant. If $f_1(z) + f_2(z) + f_3(z) \equiv 1$ and*

$$\sum_{j=1}^{3} \left\{ N(r, \frac{1}{f_j}) + 2\sum_{j=2}^{3} \overline{N}(r, f_j) \right\} < (\lambda + o(1))T(r, f_1), \qquad (r \in I)$$

where $\lambda < 1$ and I is a set of infinite linear measure, then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

At last in this section, we give the following result which will play an important role in the proofs of our main theorems.

Lemma 2.6. *Let c be a nonzero complex number and g be a nonconstant entire function satisfying the equation*

$$A(z)e^{-2ig(z)} + B(z)e^{i(g(z-c)-g(z))} + C(z)e^{-i(g(z-c)+g(z))} \equiv 1, \tag{8}$$

where $A(\not\equiv 0)$, B and $C(\not\equiv$ Constant$)$ are meromorphic functions small with respect to $e^{g(z)}$ (that is, $\max\{T(r, A), T(r, B), T(r, C)\} = o(T(r, e^{g(z)}))$) possibly outside a set with finite logarithmic measure). Then $g(z - c) + g(z)$ is nonconstant.

Proof. Since g is a nonconstant entire function, both $e^{g(z)}$ and $e^{-2ig(z)}$ are transcendental entire functions. If the conclusion is not true, then we may suppose that $g(z - c) + g(z) \equiv K$ where K is a constant. Then it gives by (8) that

$$A(z)e^{-2ig(z)} + B(z)e^{i(g(z-c)-g(z))} \equiv 1 - C(z)e^{-iK}. \tag{9}$$

If $B(z) \equiv 0$, then $e^{-2ig(z)} \equiv \frac{1-C(z)e^{-iK}}{A(z)}$. This contradicts to

$$T\left(r, \frac{1 - C(z)e^{-iK}}{A(z)}\right) = o(T(r, e^{-2ig(z)})) = o(T(r, e^{g(z)})).$$

Now we get that $B(z) \not\equiv 0$.

Note that $C(z)$ is not a constant. Set

$$F := A(z)e^{-2ig(z)} \text{ and } G := B(z)e^{i(g(z-c)-g(z))}.$$

Then by (8) and the Nevanlinna's second main theorem,

$$\begin{aligned} T(r, F) &< N(r, F) + N(r, \frac{1}{F}) + N(r, \frac{1}{F - (1 - C(z)e^{-iK})}) + o(T(r, F)) \\ &= N(r, F) + N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + o(T(r, F)) \\ &\leq N(r, A) + N(r, \frac{1}{A}) + N(r, \frac{1}{B}) + o(T(r, F)) \\ &= o(T(r, F)). \end{aligned}$$

This is a contradiction. Hence, $g(z-c)+g(z)$ must be nonconstant. $\square$

§3 Proof of Theorem 1.1

Let f be an entire solution of (1). Then

$$X^2+Y^2=(X+iY)(X-iY)=p(z),$$

where $X=p_1(z)f(z+c)$ and $Y=p_2(z)f(z)+p_3(z)f^{'}(z)$. Since p is an irreducible polynomial, there exists an entire function g such that

$$\begin{cases} X+iY= & e^{ig}, \\ X-iY= & pe^{-ig}, \end{cases}$$

or

$$\begin{cases} X-iY= & e^{ig}, \\ X+iY= & pe^{-ig}. \end{cases}$$

We get from solving for X and Y that

$$p_1f(z+c)=X=\frac{e^{ig}+pe^{-ig}}{2}, \tag{10}$$

$$p_2f(z)+p_3f^{'}(z)=Y=k\frac{e^{ig}-pe^{-ig}}{2i}, \tag{11}$$

where $k=1$ in the first case and $k=-1$ in the second case. It follows from (10) that

$$f(z)=\frac{e^{i\underline{g}}+\underline{p}e^{-i\underline{g}}}{2\underline{p_1}}, \tag{12}$$

$$f^{'}(z)=\left[\left(\frac{1}{2\underline{p_1}}\right)^{'}+\frac{i\underline{g}^{'}}{2\underline{p_1}}\right]e^{i\underline{g}}+\left[\left(\frac{1}{2\underline{p_1}}\right)^{'}\underline{p}+\frac{\underline{p}^{'}-i\underline{p}\underline{g}^{'}}{2\underline{p_1}}\right]e^{-i\underline{g}}, \tag{13}$$

where $p_1\not\equiv 0$.

Submitting (12) and (13) into (11), we obtain that

$$\alpha e^{ig(z)}=\beta e^{-ig(z)}+\gamma e^{ig(z-c)}+\delta e^{-ig(z-c)}, \tag{14}$$

where

$$\alpha(z)=k\underline{p_1}^2,\ \beta(z)=p\underline{p_1}^2,$$
$$\gamma(z)=-p_3\left[i\underline{p_1}^{'}+\underline{g}^{'}\underline{p_1}\right]+p_2\underline{p_1},$$

and

$$\delta(z)=p_3\left[\underline{p_1}\left(i\underline{p}^{'}+\underline{p}\underline{g}^{'}\right)-i\underline{p_1}^{'}\underline{p}\right]+ip_2\underline{p_1}\underline{p}.$$

Since $p_1,p\not\equiv 0$, we have $\alpha,\beta\not\equiv 0$. Then we get from (14) that

$$\frac{\beta}{\alpha}e^{-2ig(z)}+\frac{\gamma}{\alpha}e^{i(g(z-c)-g(z))}+\frac{\delta}{\alpha}e^{-i(g(z-c)+g(z))}\equiv 1. \tag{15}$$

If g is a constant, then the theorem is proved already. We now assume below that g is a nonconstant entire function. Then both $e^{2ig(z)}$ and $e^{-2ig(z)}$ are transcendental entire functions. The assumptions $p_1,p_3,p\not\equiv 0$ imply that $\underline{g}^{'}$ appears in both γ and δ.

Assume that $\frac{\delta}{\alpha}$ is constant. Then we get that $\underline{g}^{'}$ and thus g must be a polynomial. Then g is nonconstant implies that $g(z-c)+g(z)$ is not constant. Assume that $\frac{\delta}{\alpha}$ is not a constant.

By Lemma 2.1 and Lemma 2.4 we know that

$$\max\{T(r,\frac{\beta}{\alpha}),T(r,\frac{\gamma}{\alpha}),T(r,\frac{\delta}{\alpha})\}=o(T(e^{-2ig(z)})).$$

Then by Lemma 2.6 we also get that $g(z-c)+g(z)$ is not constant. Hence, now we get that both $e^{-2ig(z)}$ and $e^{-i(g(z-c)+g(z))}$ are transcendental. We divide four cases as follows.

Case 1. Assume that $\gamma\equiv\delta\equiv 0$. Then equation (15) reduces to $e^{-2ig(z)}\equiv\frac{\alpha}{\beta}$, which contradicts to the fact $T\left(r,\frac{\alpha}{\beta}\right)=o(T(r,e^{-2ig(z)}))$.

Case 2. Assume $\gamma\equiv 0$ and $\delta\not\equiv 0$. Then (15) reduces to

$$\frac{\beta}{\alpha}e^{-2ig(z)}+\frac{\delta}{\alpha}e^{-i(g(z-c)+g(z))}\equiv 1. \tag{16}$$

By Lemma 2.1 and Lemma 2.4,

$$\max\{T(r,\alpha),T(r,\beta),T(r,\delta)\}=o\left(T(r,e^{-2ig(z)}\right).$$

Set $F:=\frac{\beta}{\alpha}e^{-2ig(z)}$ and $G:=\frac{\delta}{\alpha}e^{-i(g(z-c)+g(z))}$. Then by (16) and the Nevanlinna's second main theorem,

$$\begin{aligned}T(r,F) &< N(r,F)+N(r,\frac{1}{F})+N(r,\frac{1}{F-1})+o(T(r,F))\\ &= N(r,F)+N(r,\frac{1}{F})+N(r,\frac{1}{G})+o(T(r,F))\\ &\le N(r,\frac{1}{\alpha})+N(r,\frac{1}{\beta})+N(r,\frac{1}{\delta})+o(T(r,F))\\ &= o(T(r,F)).\end{aligned}$$

This is a contradiction.

Case 3. Assume $\gamma\not\equiv 0$ and $\delta\equiv 0$. Then (15) reduces to

$$\frac{\beta}{\alpha}e^{-2ig(z)}+\frac{\gamma}{\alpha}e^{i(g(z-c)-g(z))}\equiv 1. \tag{17}$$

By Lemma 2.1 and Lemma 2.4,

$$\max\{T(r,\alpha),T(r,\beta),T(r,\gamma)\}=o\left(T(r,e^{-2ig(z)}\right).$$

Set $F:=\frac{\beta}{\alpha}e^{-2ig(z)}$ and $G=\frac{\gamma}{\alpha}e^{i(g(z-c)-g(z))}$. Then by (17) and the Nevanlinna's second main theorem,

$$\begin{aligned}T(r,F) &< N(r,F)+N(r,\frac{1}{F})+N(r,\frac{1}{F-1})+o(T(r,F))\\ &= N(r,F)+N(r,\frac{1}{F})+N(r,\frac{1}{G})+o(T(r,F))\\ &\le N(r,\frac{1}{\alpha})+N(r,\frac{1}{\beta})+N(r,\frac{1}{\gamma})+o(T(r,F))\\ &= o(T(r,F)).\end{aligned}$$

This is a contradiction.

Case 4. Assume $\delta\not\equiv 0$ and $\gamma\not\equiv 0$. By Lemma 2.1 and Lemma 2.4,

$$\begin{aligned}&\max\left\{N(r,F),N(r,G),N(r,H),N\left(r,\frac{1}{F}\right),N\left(r,\frac{1}{G}\right),N\left(r,\frac{1}{H}\right)\right\}\\ \le\ &\max\{T(r,\alpha),T(r,\beta),T(r,\gamma),T(r,\delta)\}\\ =\ &o\left(T(r,F)\right),\end{aligned}$$

where $F := \frac{\beta}{\alpha}e^{-2ig(z)}$, $G := \frac{\gamma}{\alpha}e^{i(g(z-c)-g(z))}$ and $H := \frac{\delta}{\alpha}e^{-i(g(z-c)+g(z))}$. Then applying Lemma 2.5 into (15), we get either

$$G = \frac{\gamma}{\alpha}e^{i(g(z-c)-g(z))} \equiv 1$$

or

$$H = \frac{\delta}{\alpha}e^{-i(g(z-c)+g(z))} \equiv 1.$$

Submitting either $G \equiv 1$ or $H \equiv 1$ into (15) gives either

$$e^{i(g(z-c)-g(z))} \equiv \frac{\alpha}{\gamma} \equiv -\frac{\delta}{\beta}$$

or

$$e^{-i(g(z-c)+g(z))} \equiv \frac{\alpha}{\delta} \equiv -\frac{\gamma}{\beta},$$

respectively. Thus we have

$$\delta\gamma + \alpha\beta \equiv 0,$$

that is

$$p_3^2\underline{p_1}^2\underline{p}(\underline{g}')^2 + \left(Dp_3\underline{p_1} - Cp_3\underline{p_1p}\right)\underline{g}' - \left(CD + k\underline{p_1}^4p\right) \equiv 0,$$

where $C := p_2\underline{p_1} - ip_3\underline{p_1}'$ and $D = ip_3\underline{p_1p}' - ip_3\underline{p_1}'\underline{p} + ip_2\underline{p_1p}$. Assume that $\underline{g}'$ is transcendental, then it follows from Lemma 2.3 that

$$2T(r,\underline{g}') = T\left(r, p_3^2\underline{p_1}^2\underline{p}(\underline{g}')^2 + \left(Dp_3\underline{p_1} - Cp_3\underline{p_1p}\right)\underline{g}' - \left(CD + k\underline{p_1}^4p\right)\right) = T(r,0) = O(\log r),$$

which implies that

$$p_3^2\underline{p_1}^2\underline{p} \equiv Dp_3\underline{p_1} - Cp_3\underline{p_1p} \equiv CD + k\underline{p_1}^4p \equiv 0.$$

This is a contradiction, since p, p_1 and $p_3 \not\equiv 0$. Hence, $\underline{g}'$ and thus g is a polynomial.

Now since g is a polynomial, it follows from $e^{i(g(z-c)-g(z))} \equiv \frac{\alpha}{\gamma}$ that g must be a linear polynomial, say a polynomial $g = Az + B$. Then $e^{i(g(z-c)-g(z))} \equiv \frac{\alpha}{\gamma}$ gives

$$e^{-iAc} \equiv \frac{k\underline{p_1}^2}{(p_2 - Ap_3)\,\underline{p_1} - ip_3\underline{p_1}'}.$$

Or, it follows from $e^{-i(g(z-c)+g(z))} \equiv \frac{\alpha}{\delta}$ and g is a polynomial that $g(z-c)+g(z)$, and thus g must be a constant. Therefore, the theorem is proved.

§4 Proof of Theorem 1.3

Let f be an entire solution of (3). Using similar discussion as in the proof of Theorem 1.1, we get that since p is an irreducible polynomial, there exists an entire function g such that

$$p_1f(z) = \frac{e^{ig} + pe^{-ig}}{2}, \tag{18}$$

$$p_2f'(z) + p_3f(z+c) = k\frac{e^{ig} - pe^{-ig}}{2i}, \tag{19}$$

where $k = \pm 1$. By (18), we get that

$$\begin{aligned} f'(z) &= \left(\frac{ig'(z)}{2p_1(z)} - \frac{1}{2p_1^2(z)}\right)e^{ig(z)} \\ &\quad + \left(\frac{p'(z) - ip(z)g'(z)}{2p_1(z)} - \frac{p(z)}{2p_1^2(z)}\right)e^{-ig(z)}, \end{aligned}$$

and

$$f(z+c)=\frac{e^{i\overline{g}}+\overline{p}e^{-i\overline{g}}}{2\overline{p_1}}.$$

Submitting these equations into (19), we have

$$\alpha e^{ig(z+c)}=\beta e^{-ig(z+c)}+\gamma e^{ig(z)}+\delta e^{-ig(z)}, \tag{20}$$

where

$$\alpha(z)=-ip_1^2p_3,\ \beta(z)=ip_1^2p_3\overline{p},$$
$$\gamma(z)=\overline{p_1}\left\{p_2\left[-g^{'}p_1-i\right]-kp_1^2\right\},$$

and

$$\delta(z)=\overline{p_1}\left\{p_3\left[\left(ip^{'}+pg^{'}\right)p_1-ip\right]-kpp_1^2\right\}.$$

The assumption p_1,p_3 and $p\not\equiv 0$ implies that $\alpha,\beta\not\equiv 0$, and that δ depends on $g^{'}(z-c)$. We rewrite (20) to be

$$\frac{\beta}{\alpha}e^{-2ig(z+c)}+\frac{\gamma}{\alpha}e^{i(g(z)-g(z+c))}+\frac{\delta}{\alpha}e^{-i(g(z)+g(z+c))}=1. \tag{21}$$

If g is a constant, then the theorem is proved already. We now assume below that g is a nonconstant entire function. Then both $e^{g(z)}$ and $e^{-2ig(z+c)}$ are transcendental entire functions. The assumptions p_1,p_3 and $p\not\equiv 0$ imply that the $g^{'}(z)$ appears in δ.

Using as the same as in the proof of Theorem 1.1, we get that $g(z)+g(z+c)$ is not constant. Hence, both $e^{-2ig(z+c)}$ and $e^{-i(g(z)+g(z+c))}$ are transcendental. And also by the same discussion as in the proof of Theorem 1.1, we get contradictions whenever at least one of δ and γ is identical equal to zero; while whenever $\delta\not\equiv 0$ and $\gamma\not\equiv 0$, we get that either

$$e^{i(g(z)-g(z+c))}\equiv\frac{\alpha}{\gamma}\equiv-\frac{\delta}{\beta}$$

or

$$e^{-i(g(z)+g(z+c))}\equiv\frac{\alpha}{\delta}\equiv-\frac{\gamma}{\beta}.$$

Thus

$$\delta\gamma+\alpha\beta\equiv 0,$$

that is

$$p_1^2\overline{p_1}^2p_2p_3p(g^{'})^2+(C\overline{p_1}p_3p_1p+Dp_1\overline{p_1}p_2)\,g^{'}+\left(CD-p_1^4p_3^2\overline{p}\right)\equiv 0,$$

where $C:=i\overline{p_1}p_2+kp_1^2\overline{p_1}$ and $D=i\overline{p_1}p_3(p^{'}p_1-p)-kpp_1^2\overline{p_1}$.

Assume that $g^{'}$ is transcendental, then by Lemma 2.3 we get that

$$2T(r,g^{'})=T\left(r,p_1^2\overline{p_1}^2p_2p_3p(g^{'})^2+(C\overline{p_1}p_3p_1p+Dp_1\overline{p_1}p_2)\,g^{'}+\left(CD-p_1^4p_3^2\overline{p}\right)\right)$$
$$=T(r,0)=O(\log r)$$

which implies

$$p_1^2\overline{p_1}^2p_2p_3p\equiv C\overline{p_1}p_3p_1p+Dp_1\overline{p_1}p_2\equiv CD-p_1^4p_3^2\overline{p}\equiv 0.$$

Then $p_1^2\overline{p_1}^2p_2p_3p\equiv 0$ shows that $p_2\equiv 0$, since p,p_1 and $p_3\not\equiv 0$. Thus, $C\overline{p_1}p_3p_1p+Dp_1\overline{p_1}p_2\equiv 0$ reduces to $kp_1^3\overline{p_1}^2p_3p\equiv 0$, which is impossible. Hence, $g^{'}$ and thus g, is a polynomial.

Now since g is a polynomial, it follows from $e^{i(g(z-c)-g(z))}\equiv\frac{\alpha}{\gamma}$ that g must be a linear polynomial, say a polynomial $g = Az + B$. Then $e^{i(g(z)-g(z+c))} \equiv \frac{\alpha}{\gamma}$ gives

$$e^{-iAc} \equiv \frac{ip_1^2 p_3}{\overline{p_1}(Ap_2p_1 + ip_2 + kp_1^2)}.$$

Or it follows from $e^{-i(g(z)+g(z+c))} \equiv \frac{\alpha}{\delta}$ and g is a polynomial that $g(z) + g(z+c)$, and thus g, must be a constant. Therefore, the theorem is proved.

§5 Proof of Theorem 1.5

Let f be an entire solution of (5). Using similar discussion as in the proof of Theorem 1.1, we get that since p is an irreducible polynomial, there exists an entire function g such that

$$p_1(z)f^{'}(z) = \frac{e^{ig(z)} + p(z)e^{-ig(z)}}{2}, \tag{22}$$

$$p_3(z)\left(Mf(z+c) + f(z)\right) = k\frac{e^{ig(z)} - p(z)e^{-ig(z)}}{2i}, \tag{23}$$

where $k = \pm 1$. By (22), we get that

$$f^{'}(z+c) \quad = \quad \frac{e^{ig(z+c)} + p(z+c)e^{-ig(z+c)}}{2p_1(z+c)}. \tag{24}$$

Differentiating both sides of (23), and then according to (22) and (24), we have

$$\alpha e^{ig(z+c)} = \beta e^{-ig(z+c)} + \gamma e^{ig(z)} + \delta e^{-ig(z)}, \tag{25}$$

where

$$\alpha(z) = \frac{iMp_3(z)}{p_1(z+c)}, \ \ \beta(z) = -\frac{iMp_3(z)p(z+c)}{p_1(z+c)},$$

$$\gamma(z) = kig^{'}(z) - \frac{ip_3(z)}{p_1(z)} - \frac{kp_3^{'}(z)}{p_3(z)},$$

and

$$\delta(z) = k(ip(z)g^{'}(z) - p^{'}(z)) + \frac{kp_3^{'}(z)p(z)}{p_3(z)} - \frac{ip_3(z)p(z)}{p_1(z)}.$$

Note that p_1, p_3 and $p \not\equiv 0$ implies that γ and δ depend on $g^{'}(z)$.

(i) It is clear that if $M = 0$ (that is $p_2 \equiv 0$), then $\alpha \equiv \beta \equiv 0$. Then (25) reduces to

$$\gamma e^{2ig(z)} \equiv -\delta.$$

If g is a constant, the theorem is already proved. Now assume that g is not a constant. Then it must be

$$\gamma \equiv \delta \equiv 0$$

by estimating the Nevanlinna's characteristic function of both sides of the above equation. So,

$$kig^{'} - i\frac{p_3}{p_1} - k\frac{p_3^{'}}{p_3} \equiv 0$$

and

$$kig^{'} - i\frac{p_3}{p_1} + k\frac{p_3^{'}}{p_3} - k\frac{p^{'}}{p} \equiv 0.$$

Since p, p_1 and p_3 are polynomials, $g^{'}$ and thus g must be a polynomial by the above equations. Combing the two equations gives $\frac{p^{'}}{p} = 2\frac{p_3^{'}}{p_3}$. Furthermore, since g' is entire, it follows from the first equation that every zero with multiplicity ℓ of p_3 must be a simple pole of $\frac{p_3^{'}}{p_3}$, and thus is a zero of p_1 with multiplicity $\ell+1$, and vice versa. Thus if p_3 is not a constant, then p_3 and p_1 must have the same zeros with multiplicities ℓ and $\ell+1$ respectively, and for all these zeros we have $\deg(p_1)=\deg(p_3)+\overline{n}(\frac{1}{p_3})$, where $\overline{n}(\frac{1}{p_3})$ denotes the number of distinct zeros of p_3. Therefore, if p_3 is a nonconstant and g' is a nonzero polynomial, then the degrees on both sides of the equation $kip_1p_3g^{'}\equiv ip_3^2+kp_1p_3^{'}$ are not equal, that is, we have

$$\begin{aligned} & \deg g^{'}+2\deg p_3+\overline{n}(\frac{1}{p_3})=\deg(kip_1p_3g^{'}) \\ > \quad & \deg(ip_3^2+kp_1p_3^{'})=\max\{2\deg p_3, 2\deg p_3+\overline{n}(\frac{1}{p_3})-1\}=2\deg p_3+\overline{n}(\frac{1}{p_3})-1. \end{aligned}$$

This contradiction implies that if p_3 is not a constant, then $g^{'}\equiv 0$, and g is a constant. Hence, we know that p_3 is a constant, so all p and p_1 are constants, and $g^{'}=\frac{1}{k}\frac{p_3}{p_1}$. Therefore, g is a constant, or a nonconstant linear polynomial $g=\frac{1}{k}\frac{p_3}{p_1}z+Constant$.

(ii) We now assume that $M\neq 0$. The assumption p_1,p_3 and $p\not\equiv 0$ gives $\alpha,\beta\not\equiv 0$. Then we rewrite (25) to be

$$\frac{\beta}{\alpha}e^{-2ig(z+c)}+\frac{\gamma}{\alpha}e^{i(g(z)-g(z+c))}+\frac{\delta}{\alpha}e^{-i(g(z)+g(z+c))}=1. \tag{26}$$

If g is a constant, then the theorem is proved already. We now assume below that g is a nonconstant entire function. Then both $e^{g(z)}$ and $e^{-2ig(z+c)}$ are transcendental entire functions. The assumptions p_1,p_3 and $p\not\equiv 0$ imply that the $g^{'}(z)$ appears in γ and δ.

Using as the same as in the proof of Theorem 1.1, we get that $g(z)+g(z+c)$ is not constant. Hence, both $e^{-2ig(z+c)}$ and $e^{-i(g(z)+g(z+c))}$ are transcendental. And also by the same discussion as in the proof of Theorem 1.1, we get contradictions whenever at least one of δ and γ is identical equal to zero; while whenever $\delta\not\equiv 0$ and $\gamma\not\equiv 0$, we get that either

$$e^{i(g(z)-g(z+c))}\equiv\frac{\alpha}{\gamma}\equiv-\frac{\delta}{\beta}$$

or

$$e^{-i(g(z)+g(z+c))}\equiv\frac{\alpha}{\delta}\equiv-\frac{\gamma}{\beta}.$$

Thus

$$\delta\gamma+\alpha\beta\equiv 0,$$

that is

$$p\overline{p_1}^2(g^{'})^2+ki\overline{p_1}^2\left(Cp+D\right)g^{'}-\left(CD\overline{p_1}^2+M^2p_3^2\overline{p}\right)\equiv 0,$$

where $C:=\frac{ip_3}{p_1}+\frac{kp_3^{'}}{p_3}$ and $D=kp^{'}-\frac{kpp_3^{'}}{p_3}+\frac{ipp_3}{p_1}$.

Assume that $g^{'}$ is transcendental, then by Lemma 2.3 we get that

$$2T(r,g^{'})=T\left(r,p\overline{p_1}^2(g^{'})^2+ki\overline{p_1}^2\left(Cp+D\right)g^{'}-\left(CD\overline{p_1}^2+M^2p_3^2\overline{p}\right)\right)=T(r,0)=O(\log r),$$

which implies

$$p\overline{p_1}^2\equiv ki\overline{p_1}^2\left(Cp+D\right)\equiv CD\overline{p_1}^2+M^2p_3^2\overline{p}\equiv 0.$$

This is obviously a contradiction, since p,p_1 and $p_3\not\equiv 0$. Hence, $g^{'}$ and thus g, is a polynomial.

Now since g is a polynomial, it follows from $e^{i(g(z-c)-g(z))}\equiv\frac{\alpha}{\gamma}$ that g must be a linear polynomial, say a polynomial $g = Az + B$. Then $e^{i(g(z)-g(z+c))} \equiv \frac{\alpha}{\gamma}$ gives

$$e^{-iAc} \equiv \frac{ip_1p_2p_3}{kiAp_1p_3\overline{p_1} - ip_3^2\overline{p_1} - kp_3'p_1\overline{p_1}}.$$

Or it follows from $e^{-i(g(z)+g(z+c))} \equiv \frac{\alpha}{\delta}$ and g is a polynomial that $g(z)+g(z+c)$, and thus g must be a constant. Therefore, the theorem is proved.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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