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Uniform regularity for the Navier-Stokes-Fourier system in $\mathbb{T}^n$

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Abstract. In this paper, we prove the uniform regularity estimates for the Navier-Stokes-Fourier system in $\mathbb{T}^n$.

§1 Introduction

In this paper, we consider the following Navier-Stokes-Fourier system:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t(\rho e) + \operatorname{div}(\rho u e) + p \operatorname{div} u - k\Delta\theta = (\mu(\nabla u + \nabla u^t) + \lambda \operatorname{div} u \mathbb{I}) : \nabla u, \tag{1.3}$$

$$(\rho, u, \theta)(\cdot, 0) = (\rho_0, u_0, \theta_0)(\cdot) \text{ in } \mathbb{T}^n. \tag{1.4}$$

where ρ denotes the density, u the velocity field and $e := C_V\theta$ is the specific internal energy, $p := R\rho\theta$ is the pressure. C_V and R are the physical constants. λ and μ are two viscosity constants satisfying

$$\mu > 0, \quad \lambda + \frac{2}{n}\mu \geq 0. \tag{1.5}$$

$k > 0$ is the heat conductivity coefficient. We will denote

$$Q(\nabla u) := (\mu(\nabla u + \nabla u^t) + \lambda \operatorname{div} u \mathbb{I}) : \nabla u.$$

Local well-posedness of classical solutions is known in [10, 12–14, 16] without vacuum. Cho and Kim [2] obtained the local existence and uniqueness of strong solutions with some compatible conditions and vacuum.

When $n = 1$, many results on vanishing viscosity limits have been obtained in [6, 18] for the rarefaction wave, in [17] for the shock wave, in [9] for the contact discontinuity, and in [4, 5] for the superposition of two rarefaction waves and a contact discontinuity and the superposition of one shock and one rarefaction wave cases.

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When $n=2$, Li, Wang and Wang [8] (and references therein) studied the vanishing viscosity limit of the isentropic Navier-Stokes equations in the case that the corresponding 2D Euler system has a planar rarefaction wave solution.

The aim of this paper is to prove uniform regularity estimates in (λ,μ,k). We will prove

Theorem 1.1. *Let $0<\mu<1, 0<\lambda+\mu<1, 0<k<1, 0<\frac{1}{C_0}\leq\rho_0,\theta_0\leq C_0$, and $\rho_0,u_0,\theta_0\in H^s(\mathbb{T}^n)$ with $s>1+\frac{n}{2}$. Let (ρ,u,θ) be the unique local smooth solutions to the problem* (1.1)-(1.4). *Then*

$$\|(\rho,u,\theta)\|_{H^s}\leq C \quad in \quad [0,T] \tag{1.6}$$

holds true for some positive constants C and $T_0(\leq T)$ independent of λ,μ and k.

We define

$$\begin{aligned} M(t): \quad = \quad & 1+\sup_{0\leq\tau\leq t}\Big\{\|(\rho,u,\theta)(\cdot,\tau)\|_{H^s}+\|\partial_t u(\cdot,\tau)\|_{L^2}+\|\partial_t\theta(\cdot,\tau)\|_{L^2} \\ & +\left\|\frac{1}{\rho}(\cdot,\tau)\right\|_{L^\infty}+\left\|\frac{1}{\theta}(\cdot,\tau)\right\|_{L^\infty}\Big\}. \end{aligned} \tag{1.7}$$

We can prove

Theorem 1.2. *For any $t\in[0,T_0)$ $(T_0\leq 1)$, we have that*

$$M(t)\leq C_0(M_0)\exp(tC(M)) \tag{1.8}$$

for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

It follows from (1.8) that [1, 3, 11]:

$$M(t)\leq C. \tag{1.9}$$

In the following proofs, we will use the bilinear commutator and product estimates due to Kato-Ponce [7]:

$$\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}}+\|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{1.10}$$

$$\|\Lambda^s(fg)\|_{L^p}\leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}}+\|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{1.11}$$

with $s>0,\Lambda:=(-\Delta)^{\frac{1}{2}}$ and $\dfrac{1}{p}=\dfrac{1}{p_1}+\dfrac{1}{q_1}=\dfrac{1}{p_2}+\dfrac{1}{q_2}$.

We only need to show Theorem 1.2.

§2 Proof of Theorem 1.2

First, testing (1.1) by ρ^{q-1}, we see that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^q\mathrm{d}x=\left(1-\frac{1}{q}\right)\int\rho^q\mathrm{div}\,u\mathrm{d}x\leq\|\mathrm{div}\,u\|_{L^\infty}\int\rho^q\mathrm{d}x,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^q}\leq\|\mathrm{div}\,u\|_{L^\infty}\|\rho\|_{L^q},$$

which gives

$$\|\rho\|_{L^q}\leq\|\rho_0\|_{L^q}\exp\left(\int_0^t\|\mathrm{div}\,u\|_{L^\infty}\mathrm{d}\tau\right). \tag{2.1}$$

Taking $q\to+\infty$, we get

$$\|\rho\|_{L^\infty}\leq\|\rho_0\|_{L^\infty}\exp(tC(M)). \tag{2.2}$$

It follows from (1.1) that

$$\partial_t \frac{1}{\rho} + u \cdot \nabla \frac{1}{\rho} - \frac{1}{\rho} \operatorname{div} u = 0. \tag{2.3}$$

Testing (2.3) by $\left(\frac{1}{\rho}\right)^{q-1}$, we find that

$$\frac{1}{q} \frac{\mathrm{d}}{\mathrm{d}t} \int \left(\frac{1}{\rho}\right)^q \mathrm{d}x = \left(1 + \frac{1}{q}\right) \int \left(\frac{1}{\rho}\right)^q \operatorname{div} u \mathrm{d}x \leq \left(1 + \frac{1}{q}\right) \left\|\frac{1}{\rho}\right\|_{L^q}^q \|\operatorname{div} u\|_{L^\infty},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\|\frac{1}{\rho}\right\|_{L^q} \leq \left(1 + \frac{1}{q}\right) \left\|\frac{1}{\rho}\right\|_{L^q} \|\operatorname{div} u\|_{L^\infty},$$

which gives

$$\left\|\frac{1}{\rho}\right\|_{L^q} \leq \left\|\frac{1}{\rho_0}\right\|_{L^q} \exp\left(\left(1 + \frac{1}{q}\right) \int_0^t \|\operatorname{div} u\|_{L^\infty} \mathrm{d}\tau\right)$$

and we have

$$\left\|\frac{1}{\rho}\right\|_{L^\infty} \leq \left\|\frac{1}{\rho_0}\right\|_{L^\infty} \exp(tC(M)) \tag{2.4}$$

by sending $q \to +\infty$.

Testing (1.3) by θ^{q-1} and using (1.1), we get

$$\begin{aligned} & \frac{C_V}{q} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho \theta^q \mathrm{d}x + k \int \nabla \theta \cdot \nabla \theta^{q-1} \mathrm{d}x \\ = & \int Q \theta^{q-1} \mathrm{d}x - \int p \theta^{q-1} \operatorname{div} u \mathrm{d}x \\ \leq & C(M) \|Q\|_{L^q} \|\rho^{\frac{1}{q}} \theta\|_{L^q}^{q-1} + C \|\operatorname{div} u\|_{L^\infty} \|\rho^{\frac{1}{q}} \theta\|_{L^q}^q, \end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho^{\frac{1}{q}} \theta\|_{L^q} \leq C(M) \|Q\|_{L^q} + C \|\operatorname{div} u\|_{L^\infty} \|\rho^{\frac{1}{q}} \theta\|_{L^q},$$

which, similarly to (2.2), implies

$$\|\theta\|_{L^\infty} \leq C_0(M_0) \exp(tC(M)). \tag{2.5}$$

Multiplying (1.3) by $\frac{1}{\theta^2} \cdot \frac{1}{C_V}$, we deduce

$$\begin{aligned} & \rho \partial_t \frac{1}{\theta} + \rho u \cdot \nabla \frac{1}{\theta} + \frac{k}{C_V} \cdot \frac{1}{\theta^2} \Delta \theta \\ = & \frac{R}{C_V} \rho \frac{\operatorname{div} u}{\theta} - \frac{Q}{\theta^2 C_V} \leq \frac{R}{C_V} \rho \frac{1}{\theta} \operatorname{div} u. \end{aligned} \tag{2.6}$$

Similarly to (2.5), testing (2.6) by $\left(\frac{1}{\theta}\right)^{q-1}$, we compute

$$\begin{aligned} \frac{1}{q} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho \left(\frac{1}{\theta}\right)^q \mathrm{d}x & \leq \frac{R}{C_V} \int \rho \left(\frac{1}{\theta}\right)^q \operatorname{div} u \mathrm{d}x \\ & \leq \frac{R}{C_V} \|\operatorname{div} u\|_{L^\infty} \int \rho \left(\frac{1}{\theta}\right)^q \mathrm{d}x, \end{aligned}$$

and thus

$$\left\|\frac{1}{\theta}\right\|_{L^\infty} \leq C_0(M_0) \exp(tC(M)). \tag{2.7}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |u|^2 \mathrm{d}x = 2 \int u \partial_t u \mathrm{d}x \leq 2 \|u\|_{L^2} \|\partial_t u\|_{L^2} \leq C(M),$$

which implies

$$\|u\|_{L^2} \leq C_0(M_0)\exp(tC(M)). \tag{2.8}$$

The equations (1.1)-(1.3) can be rewritten in the following symmetric form:

$$\frac{\theta}{\rho}\partial_t\rho + \frac{\theta}{\rho}u \cdot \nabla\rho + \theta\mathrm{div}\, u = 0, \tag{2.9}$$

$$\rho\partial_t u + \rho u \cdot \nabla u + \rho\nabla\theta + \theta\nabla\rho - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\, u = 0, \tag{2.10}$$

$$\frac{\rho}{\theta}\partial_t\theta + \frac{\rho}{\theta}u \cdot \nabla\theta - \frac{k}{\theta}\Delta\theta + \rho\mathrm{div}\, u = \frac{1}{\theta}Q, \tag{2.11}$$

where we have taken $R = C_V = 1$ for simplicity.

Taking Λ^s to (2.9), testing by $\Lambda^s\rho$, using (1.1), (1.3), (1.10) and (1.11), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\theta}{\rho}(\Lambda^s\rho)^2\mathrm{d}x + \int \theta\Lambda^s\mathrm{div}\, u\Lambda^s\rho\mathrm{d}x \\
= \;& -\int \left[\Lambda^s\left(\frac{\theta}{\rho}\partial_t\rho\right) - \frac{\theta}{\rho}\Lambda^s\partial_t\rho\right]\Lambda^s\rho\mathrm{d}x \\
& -\int \left[\Lambda^s\left(\frac{\theta}{\rho}u\cdot\nabla\rho\right) - \frac{\theta}{\rho}u\cdot\nabla\Lambda^s\rho\right]\Lambda^s\rho\mathrm{d}x \\
& -\int (\Lambda^s(\theta\mathrm{div}\, u) - \theta\Lambda^s\mathrm{div}\, u)\Lambda^s\rho\mathrm{d}x \\
& +\frac{1}{2}\int \partial_t\left(\frac{\theta}{\rho}\right)(\Lambda^s\rho)^2\mathrm{d}x - \int\left(\frac{\theta}{\rho}u\cdot\nabla\Lambda^s\rho\right)\Lambda^s\rho\mathrm{d}x \\
\leq \;& C\left(\left\|\nabla\frac{\theta}{\rho}\right\|_{L^\infty}\|\Lambda^{s-1}\partial_t\rho\|_{L^2} + \|\partial_t\rho\|_{L^\infty}\left\|\Lambda^s\frac{\theta}{\rho}\right\|_{L^2}\right)\|\Lambda^s\rho\|_{L^2} \\
& +C\left(\left\|\nabla\frac{\theta u}{\rho}\right\|_{L^\infty}\|\Lambda^s\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\left\|\Lambda^s\frac{\theta u}{\rho}\right\|_{L^2}\right)\|\Lambda^s\rho\|_{L^2} \\
& +C(\|\nabla\theta\|_{L^\infty}\|\Lambda^s u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2})\|\Lambda^s\rho\|_{L^2} \\
& +C\left\|\partial_t\frac{\theta}{\rho}\right\|_{L^\infty}\|\Lambda^s\rho\|_{L^2}^2 + C\left\|\nabla\frac{\theta u}{\rho}\right\|_{L^\infty}\|\Lambda^s\rho\|_{L^2}^2 \\
\leq \;& C(M)(\|\Lambda^{s-1}\partial_t\rho\|_{L^2} + \|\partial_t\rho\|_{L^\infty}) + C(M) + C(M)\left\|\partial_t\frac{\theta}{\rho}\right\|_{L^\infty} \\
\leq \;& C(M) + \frac{k}{4}\int\frac{1}{\theta}(\Lambda^{s+1}\theta)^2\mathrm{d}x.
\end{aligned} \tag{2.12}$$

Here we have used the estimate [15]:

$$\left\|\Lambda^s\frac{1}{\rho}\right\|_{L^2} \leq C(M)\|\Lambda^s\rho\|_{L^2} \leq C(M).$$

It is obvious that

$$\int_0^t\int|\partial_t u|^2\mathrm{d}x\mathrm{d}\tau \leq t\sup\int|\partial_t u|^2\mathrm{d}x \leq tC(M). \tag{2.13}$$

Applying Λ^{s-1} to (1.2), testing by $\Lambda^{s-1}\partial_t u$, using (1.10) and (1.11), we obtain

$$\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s u|^2\mathrm{d}x + \frac{\lambda+\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\Lambda^{s-1}\mathrm{div}\, u)^2\mathrm{d}x + \int\rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x \\
= \;& -\int\Lambda^{s-1}\nabla p\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x - \int\Lambda^{s-1}(\rho u\cdot\nabla u)\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
& -\int[\Lambda^{s-1}(\rho\partial_t u)-\rho\Lambda^{s-1}\partial_t u]\Lambda^{s-1}\partial_t u\mathrm{d}x\\
\le\ & C\|\Lambda^s p\|_{L^2}\|\Lambda^{s-1}\partial_t u\|_{L^2}+C\|\rho\|_{H^{s-1}}\|u\|_{H^s}^2\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
& +C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
\le\ & C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(M)(\|\Lambda^{s-2}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
\le\ & C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
& +C(M)(\|\partial_t u\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}}+\|\partial_t u\|_{L^2}+\|\partial_t u\|_{L^2}^{\frac{s-1-\frac{n}{2}}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
\le\ & C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(M)(\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}}+\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
\le\ & \frac{1}{2}\int\rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x+C(M),
\end{aligned}
$$

which gives

$$\int_0^t\int|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x\mathrm{d}\tau\le C_0(M_0)\exp(tC(M)). \tag{2.14}$$

Applying Λ^s to (2.10), testing by $\Lambda^s u$, using (1.1), (1.10) and (1.11), we have

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\Lambda^s u|^2\mathrm{d}x+\mu\int|\Lambda^{s+1}u|^2\mathrm{d}x+(\lambda+\mu)\int(\Lambda^s\mathrm{div}\, u)^2\mathrm{d}x\\
& +\int\rho\Lambda^s\nabla\theta\cdot\Lambda^s u\mathrm{d}x+\int\theta\nabla\Lambda^s\rho\cdot\Lambda^s u\mathrm{d}x\\
=\ & -\int(\Lambda^s(\rho\partial_t u)-\rho\Lambda^s\partial_t u)\Lambda^s u\mathrm{d}x-\int(\Lambda^s(\rho u\cdot\nabla u)-\rho u\cdot\nabla\Lambda^s u)\Lambda^s u\mathrm{d}x\\
& -\int(\Lambda^s(\rho\nabla\theta)-\rho\nabla\Lambda^s\theta)\Lambda^s u\mathrm{d}x-\int(\Lambda^s(\theta\nabla\rho)-\theta\nabla\Lambda^s\rho)\Lambda^s u\mathrm{d}x\\
\le\ & C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s u\|_{L^2}\\
& +C(\|\nabla u\|_{L^\infty}\|\Lambda^s(\rho u)\|_{L^2}+\|\nabla(\rho u)\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s u\|_{L^2}\\
& +C(\|\nabla\rho\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s u\|_{L^2}\\
& +C(\|\nabla\theta\|_{L^\infty}\|\Lambda^s\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\Lambda^s\theta\|_{L^2})\|\Lambda^s u\|_{L^2}\\
\le\ & C(M)+C(M)(\|\Lambda^{s-1}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})\\
\le\ & C(M)+\|\Lambda^{s-1}\partial_t u\|_{L^2}^2.
\end{aligned} \tag{2.15}
$$

Applying Λ^{s-1} to (1.3), testing by $\Lambda^{s-1}\partial_t\theta$, using (1.10) and (1.11), we have

$$
\begin{aligned}
& \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\Lambda^s\theta)^2\mathrm{d}x+\int\rho|\Lambda^{s-1}\partial_t\theta|_{L^2}^2\mathrm{d}x\\
=\ & -\int\Lambda^{s-1}(p\mathrm{div}\, u)\Lambda^{s-1}\partial_t\theta\mathrm{d}x-\int\Lambda^{s-1}(\rho u\cdot\nabla\theta)\Lambda^{s-1}\partial_t\theta\mathrm{d}x\\
& -\int[\Lambda^{s-1}(\rho\partial_t\theta)-\rho\Lambda^{s-1}\partial_t\theta]\Lambda^{s-1}\partial_t\theta\mathrm{d}x\\
& \qquad\qquad (\text{where we took } C_V=1)\\
\le\ & \|\Lambda^{s-1}(p\mathrm{div}\, u)\|_{L^2}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+\|\Lambda^{s-1}(\rho u\cdot\nabla\theta)\|_{L^2}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
& +C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2})\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
\le\ & C(M)\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+C(M)(\|\Lambda^{s-2}\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty})\|\Lambda^{s-1}\partial_t\theta\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(M)\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+C(M)(\|\partial_t\theta\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}^{\frac{s-2}{s-1}}+\|\partial_t\theta\|_{L^2}\\
&\quad+\|\partial_t\theta\|_{L^2}^{\frac{s-1-\frac{n}{2}}{s-1}}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
&\leq \frac{1}{2}\int\rho|\Lambda^{s-1}\partial_t\theta|^2\mathrm{d}x+C(M),
\end{aligned}
$$

which leads to

$$
\int_0^t\int|\Lambda^{s-1}\partial_t\theta|^2\mathrm{d}x\mathrm{d}\tau\leq C_0(M_0)\exp(tC(M)). \tag{2.16}
$$

Taking Λ^s to (2.11), testing by $\Lambda^s\theta$, using (1.10) and (1.11), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\rho}{\theta}(\Lambda^s\theta)^2\mathrm{d}x+k\int\frac{1}{\theta}(\Lambda^{s+1}\theta)^2\mathrm{d}x+\int\rho\Lambda^s\mathrm{div}\,u\Lambda^s\theta\mathrm{d}x\\
=\ &k\int\frac{\nabla\theta}{\theta^2}\Lambda^s\theta\nabla\Lambda^s\theta\mathrm{d}x+k\int\left[\Lambda^s\left(\frac{1}{\theta}\Delta\theta\right)-\frac{1}{\theta}\Delta\Lambda^s\theta\right]\Lambda^s\theta\mathrm{d}x\\
&-\int\left[\Lambda^s\left(\frac{\rho}{\theta}\partial_t\theta\right)-\frac{\rho}{\theta}\Lambda^s\partial_t\theta\right]\Lambda^s\theta\mathrm{d}x+\frac{1}{2}\int\partial_t\left(\frac{\rho}{\theta}\right)(\Lambda^s\theta)^2\mathrm{d}x\\
&-\int\left[\Lambda^s\left(\frac{\rho u}{\theta}\nabla\theta\right)-\frac{\rho u}{\theta}\nabla\Lambda^s\theta\right]\Lambda^s\theta\mathrm{d}x-\int\frac{\rho u}{\theta}\nabla\Lambda^s\theta\cdot\Lambda^s\theta\mathrm{d}x\\
&-\int(\Lambda^s(\rho\mathrm{div}\,u)-\rho\Lambda^s\mathrm{div}\,u)\Lambda^s\theta\mathrm{d}x+\int\Lambda^s\left(\frac{Q}{\theta}\right)\Lambda^s\theta\mathrm{d}x\\
\leq\ &\frac{k}{8}\int\frac{1}{\theta}(\Lambda^{s+1}\theta)^2\mathrm{d}x+C(M)\|\nabla\theta\|_{L^\infty}^2\|\Lambda^s\theta\|_{L^2}^2\\
&+kC\left(\left\|\frac{\nabla\theta}{\theta^2}\right\|_{L^\infty}\|\Lambda^{s+1}\theta\|_{L^2}+\|\Delta\theta\|_{L^\infty}\left\|\Lambda^s\frac{1}{\theta}\right\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
&+C\left(\left\|\nabla\frac{\rho}{\theta}\right\|_{L^\infty}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty}\left\|\Lambda^s\frac{\rho}{\theta}\right\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
&+C\left\|\partial_t\frac{\rho}{\theta}\right\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}^2+C\left(\left\|\nabla\frac{\rho u}{\theta}\right\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\left\|\Lambda^s\frac{\rho u}{\theta}\right\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
&+C\left\|\nabla\frac{\rho u}{\theta}\right\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}^2+C(\|\nabla\rho\|_{L^\infty}\|\Lambda^s u\|_{L^2}+\|\mathrm{div}\,u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s\theta\|_{L^2}\\
&+C\left(\left\|\frac{1}{\theta}\right\|_{L^\infty}\|\Lambda^sQ\|_{L^2}+\|Q\|_{L^\infty}\left\|\Lambda^s\frac{1}{\theta}\right\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
\leq\ &\frac{k}{4}\int\frac{1}{\theta}(\Lambda^{s+1}\theta)^2\mathrm{d}x+C(M)+C(M)(\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty})\\
&+\frac{\mu}{2}\|\Lambda^{s+1}u\|_{L^2}^2+\frac{\lambda+\mu}{2}\|\Lambda^s\mathrm{div}\,u\|_{L^2}^2\\
\leq\ &\frac{k}{4}\int\frac{1}{\theta}(\Lambda^{s+1}\theta)^2\mathrm{d}x+\|\Lambda^{s-1}\partial_t\theta\|_{L^2}^2+C(M)+\frac{\mu}{2}\|\Lambda^{s+1}u\|_{L^2}^2+\frac{\lambda+\mu}{2}\|\Lambda^s\mathrm{div}\,u\|_{L^2}^2.
\end{aligned} \tag{2.17}
$$

Summing up (2.12), (2.15) and (2.17), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{\theta}{\rho}(\Lambda^s\rho)^2+\rho|\Lambda^s u|^2+\frac{\rho}{\theta}(\Lambda^s\theta)^2\right)\mathrm{d}x+\frac{\mu}{2}\int|\Lambda^{s+1}u|^2\mathrm{d}x\\
&+\frac{\lambda+\mu}{2}\int(\Lambda^s\mathrm{div}\,u)^2\mathrm{d}x+\frac{k}{2}\int\frac{1}{\theta}(\Lambda^{s+1}\theta)^2\mathrm{d}x\\
=\ &-\int\theta\Lambda^s\mathrm{div}\,u\Lambda^s\rho\mathrm{d}x-\int\rho\Lambda^s\nabla\theta\cdot\Lambda^s u\mathrm{d}x-\int\theta\Lambda^s\nabla\rho\cdot\Lambda^s u\mathrm{d}x-\int\rho\Lambda^s\mathrm{div}\,u\Lambda^s\theta\mathrm{d}x\\
&+\|\Lambda^{s-1}\partial_t u\|_{L^2}^2+\|\Lambda^{s-1}\partial_t\theta\|_{L^2}^2+C(M)
\end{aligned}
$$

$$
\begin{aligned}
&= \int \Lambda^s u \Lambda^s \rho \nabla\theta \mathrm{d}x + \int \Lambda^s u \Lambda^s \theta \nabla\rho \mathrm{d}x + \|\Lambda^{s-1}\partial_t u\|_{L^2}^2 + \|\Lambda^{s-1}\partial_t\theta\|_{L^2}^2 + C(M) \\
&\leq C(M) + \|\Lambda^{s-1}\partial_t u\|_{L^2}^2 + \|\Lambda^{s-1}\partial_t\theta\|_{L^2}^2. \qquad (2.18)
\end{aligned}
$$

Using (2.14) and (2.16), we have

$$
\begin{aligned}
&\|\Lambda^s(\rho,u,\theta)(\cdot,t)\|_{L^2} + \sqrt{\mu}\|\Lambda^{s+1}u\|_{L^2(0,t;L^2)} \\
&+\sqrt{\lambda+\mu}\|\Lambda^s \mathrm{div}\, u\|_{L^2(0,t;L^2)} + \sqrt{k}\|\Lambda^{s+1}\theta\|_{L^2(0,t;L^2)} \\
\leq\ & C_0(M_0)\exp(tC(M)). \qquad (2.19)
\end{aligned}
$$

On the other hand, it follows from (1.2) that

$$
\begin{aligned}
\|\partial_t u\|_{L^2} &= \left\| \frac{1}{\rho}(\mu\Delta u + (\lambda+\mu)\nabla \mathrm{div}\, u - \nabla p - \rho u\cdot\nabla u) \right\|_{L^2} \\
&\leq C_0(M_0)\exp(tC(M)). \qquad (2.20)
\end{aligned}
$$

Similarly, we have

$$
\|\partial_t\theta\|_{L^2} \leq C_0(M_0)\exp(tC(M)). \qquad (2.21)
$$

Combining (2.4), (2.7), (2.8), (2.19), (2.20), and (2.21), we conclude that (1.8) holds true. This completes the proof.

□

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] T Alazard. *Low Mach number limit of the full Navier-Stokes equations*, Arch Ration Mech Anal, 2006, 180: 1-73.

[2] Y Cho, H Kim. *Existence results for viscous polytropic fluids with vacuum*, J Differ Equ, 2006, 228(2): 377-411.

[3] C Dou, S Jiang, Y Ou. *Low Mach number limit of full Navier-Stokes equations in a 3D bounded domain*, J Differ Equ, 2014, 258(2): 379-398.

[4] F Huang, Y Wang, T Yang. *Fluid dynamic limit to the Riemann solutions of Euler equations: I. Superposition of rarefaction waves and contact discontinuity*, Kinet Relat Models, 2010, 3(4): 685-728.

[5] F Huang, Y Wang, T Yang. *Vanishing viscosity limit of the compressible Navier-Stokes equations for solutions to a Riemann problem*, Arch Ration Mech Anal, 2012, 203: 379-413.

[6] S Jiang, G Ni, W Sun. *Vanishing viscosity limit to rarefaction waves for the Navier-Stokes equations of one-dimensional compressible heat-conducting fluids*, SIAM J Math Anal, 2006, 38(2): 368-384.

[7] T Kato, G Ponce. *Commutator estimates and the Euler and Navier-Stokes equations*, Comm Pure Appl Math, 1988, 41: 891-907.

[8] L Li, D Wang, Y Wang. *Vanishing viscosity limit to the planar rarefaction wave for the two-dimensional compressible Navier-Stokes equations*, Comm Math Phys, 2020, 376: 353-384.

[9] S Ma. *Zero dissipation limit to strong contact discontinuity for the 1-D compressible Navier-Stokes equations*, J Differ Equ, 2020, 248(1): 95-110.

[10] A Matsumura, T Nishida. *Initial boundary value problems for the equations of motion of compressible and heat-conductive fluids*, Comm Math Phys, 1983, 89: 445-464.

[11] G Métivier, S Schochet. *The incompressible limit of the non-isentropic Euler equations*, Arch Ration Mech Anal, 2001, 158: 61-90.

[12] J Nash. *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull Soc Math France, 1962, 90: 487-497.

[13] J Serrin. *On the uniqueness of compressible fluid motions*, Arch Ration Mech Anal, 1959, 3: 271-288.

[14] A Tani. *On the first initial-boundary value problem of compressible viscous fluids motion*, Publ Res Inst Math Sci Kyoto Univ, 1971, 13: 193-253.

[15] H Triebel. *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[16] A Valli, W M Zajaczkowski. *Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case*, Comm Math Phys, 1986, 103: 259-296.

[17] Y Wang. *Zero dissipation limit of the compressible heat-conducting Navier-Stokes equations in the presence of the shock*, Acta Math Sci Ser B, 2008, 28(4): 727-748.

[18] Z Xin, H Zeng. *Convergence to rarefaction waves for the nonlinear Boltzmann equation and compressible Navier-Stokes equations*, J Differ Equ, 2010, 249(4): 827-871.

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